Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(3): 2287–2303.
DOI: 10.3934/math.2021138
Received: 09 November 2020
Accepted: 11 December 2020
Published: 15 December 2020

Research article

Impulsive control strategy for the Mittag-Leffler synchronization of fractional-order neural networks with mixed bounded and unbounded delays

Ivanka Stamova[1,*] **and Gani Stamov**[2]

[1] Department of Mathematics, University of Texas at San Antonio, San Antonio, TX 78249, USA
[2] Department of Mathematics, Technical University of Sofia, Bulgaria

* **Correspondence:** Email: ivanka.stamova@utsa.edu.

Abstract: In this paper we apply an impulsive control method to keep the Mittag-Leffler stability properties for a class of Caputo fractional-order cellular neural networks with mixed bounded and unbounded delays. The impulsive controls are realized at fixed moments of time. Our results generalize some known criteria to the fractional-order case and provide a design method of impulsive control law for the impulse free fractional-order neural network model. Examples are presented to demonstrate the effectiveness of our results.

Keywords: global Mittag-Leffler synchronization; neural networks; fractional-order derivatives; mixed delays; impulsive control
Mathematics Subject Classification: 26A33, 34K37, 34K45, 34K25, 34K60

1. Introduction

Recently, the interest of researchers to fractional-order models is greatly increased because of their numerous opportunities for applications in population dynamics [1–4], bioengineering and neuroscience [5–8], economics [9, 10], and in general, in every area of science, engineering and technology [11–15]. Indeed, fractional differentiation provides neurons with a fundamental and general computation ability that can contribute to efficient information processing, stimulus anticipation and frequency-independent phase shifts of oscillatory neuronal firing [6].

The advantages of fractional-order derivatives have generated much recent interest to the study of fractional neural networks and have resulted in a large amount of papers devoted to the dynamical properties of such networks. See, for example, [16–22] and the references therein. The numerous applications of fractional-order neural networks to solve optimization, associative memory, pattern recognition and computer vision problems heavily depend on the dynamic behavior of networks;

therefore, the qualitative analysis of these dynamic behaviors is an essential step in the workable design of such neural networks. It is worth mentioning that most of the existing studies on delayed fractional-order neural network models considered bounded delays. However, the corresponding investigations on fractional-order delayed neural networks with distributed delays are relatively less. In addition, to the best of our knowledge, unbounded delays are not examined in the existing literature.This interesting open problem is one of the motivations of our study.

The control theory of systems is an important part of their qualitative theory. Among the numerous excellent results on control methods in the literature, we will refer to [23], where a very good comprehensive discussion on several important for neural network models control strategies is given. Recently, the type of control called Impulsive Control, attracts the attention of researchers. This type of control arises naturally in a wide variety of applications and it allows the control action only at some discrete instances which may reduce the amount of information to be transmitted and increase the robustness against the disturbance, see [24–33]. The impulsive control strategies are also applied to some fractional-order neural networks [34–38]. However, there are still many challenging open questions in this direction. The stability behavior of fractional neural networks with unbounded delays under an impulsive control is not investigated previously, which is the next motivation for the present paper.

On the other side, the concept of Mittag-Leffler stability for fractional-order systems has been introduced in [39]. The importance of the notion lies in the fact that it generalizes the exponential stability concept to the fractional-order case. Since, for integer-order neural network models the exponential convergence is the most desirable behavior, the Mittag-Leffler stability notion of fractional-order systems received a great deal of interest among scientists [35–37, 39–42]. However, the concept has not been applied to fractional-order neural networks with unbounded delays, and we plan to fill the gap. The contribution of our paper is in following aspects:

1. Unbounded delays are introduced intro a fractional-order neural network model, and their effects on the qualitative behavior of the solutions are examined.

2. An impulsive control approach is applied to achieve a synchronization between fractional neural networks with mixed bounded and unbounded delays.

3. The concept of global Mittag-Leffler synchronization is adapted to fractional neural network models with mixed delays under impulsive control.

4. New global Mittag-Leffler synchronization criteria are established for the model under consideration.

The rest of the paper is structured as follows. In Section 2, we introduce our fractional-order neural network model with bounded and unbounded delays. The corresponding impulsive control strategy and the impulsive control system are also developed. The concept of global Mittag-Leffler synchronization is proposed for the fractional-order master and controlled systems. Some basic fractional Lyapunov function method nomenclatures and lemmas are presented. In Section 3, by using the fractional Lyapunov method and designing an appropriate impulsive controller, we obtain two main global Mittag-Leffler synchronization criteria. The synchronization results can be easily applied to the stability analysis of such systems. Our results generalize and complement some existing results on integer-order and fractional-order neural networks. Two examples are given to illustrate the efficiency of the results established in Section 4. The conclusion is given in Section 5.

2. Preliminaries

In this paper we will use the standard notation $\mathbb{R}^n$ for the n-dimensional Euclidean space. Let $||x|| = \sum_{i=1}^{n} |x_i|$ denote the norm of $x \in \mathbb{R}^n$, $\mathbb{R}_+ = [0, \infty)$, $t_0 \in \mathbb{R}_+$, and let Γ denotes the Gamma function.

Consider a q, $0 < q < 1$. Following [14], the *Caputo fractional derivative of order q* for a function $g \in C^1[[t_0, b], R]$, $b > t_0$, is given as

$$ {}_{t_0}^{C}D_t^q g(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{g'(\xi)}{(t-\xi)^q} d\xi, \ t \geq t_0. $$

2.1. Model introduction. Impulsive control

Since distributed and unbounded delays are more realistic, in this research we will consider the following fractional-order neural network with bounded and unbounded delays

$$ \begin{cases} {}_{t_0}^{C}D_t^q x_i(t) = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t - \tau_j(t))\right) \\ \qquad + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_j(t,s) f_j\left(x_j(s)\right) ds + I_i, \ t \geq t_0, \end{cases} \tag{2.1} $$

where $i = 1, 2, ..., n$, $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T$ and $x_i(t)$ is the state of ith neuron at time t, $f_j(x_j(t))$ denotes the activation function of the jth unit at time t, a_{ij}, b_{ij},c_{ij} denote the interconnection strength representing the weight coefficient of the neurons, $d_i(t)$ is a continuous function that represents the self-feedback connection weight of the ith unit at time t, $d_i(t) > 0$, $t \geq t_0$, the bounded time delays $\tau_j(t)$ are such that $0 \leq \tau_j(t) \leq \tau$, I_i is the external input on the ith neuron, the delay kernel $K_j(t, s) = K_j(t-s)$, $(j = 1, 2, ..., n)$ is of convolution type with an initial condition

$$ x(s) = \bar{\varphi}_0(s), \ s \in (-\infty, 0], \ \bar{\varphi}_0 \in C[(-\infty, 0], \mathbb{R}^n]. $$

The main objective in this paper is to control the stability behavior of the fractional-order network (2.1) by adding an impulsive controller to its nodes such that the trajectories of all nodes can be synchronized onto that of system (2.1).

Let the points

$$ 0 = t_0 < t_1 < t_2 < ... < t_k < ... $$

are such that $\lim_{k\to\infty} t_k = \infty$, and let the functions P_{ik} are defined on $\mathbb{R}$ for any $i = 1, 2, ..., n$ and $k = 1, 2, ...$

Now we consider the corresponding fractional impulsive control system

$$ \begin{cases} {}_{t_0}^{C}D_t^q y_i(t) = -d_i(t)y_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(y_j(t)\right) + \sum_{j=1}^{n} b_{ij} f_j\left(y_j(t - \tau_j(t))\right) \\ \qquad + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_j(t,s) f_j\left(y_j(s)\right) ds + I_i, \ t \neq t_k, \ t \geq t_0, \\ \Delta y_i(t_k) = y_i(t_k^+) - y_i(t_k) = P_{ik}(y_i(t_k)), \ k = 1, 2, ..., \end{cases} \tag{2.2} $$

where t_k, $k = 1, 2, ...$ are the impulsive control instants and the numbers $y_i(t_k) = y_i(t_k^-)$ and $y_i(t_k^+)$ are, respectively, the states of the ith unit before and after an impulse input at the moment t_k. The functions P_{ik} are the impulsive functions that measure the rate of changes of the states $y_i(t)$ at the impulsive moments t_k.

The impulsive control model (2.2) generalizes many existing fractional-order neural network systems with bounded and distributed delays [15, 36], as well as integer-order neural networks to the fractional-order case. See, for example, [25–33] and the references therein. By means of appropriate impulsive functions P_{ik} we can impulsively control the qualitative behavior of the states of type (2.1) fractional models.

The initial conditions associated with system (2.2) are given in the form:

$$\begin{cases} y(t; t_0, \varphi) = \varphi(t - t_0), \ -\infty < t \leq t_0, \\ y(t_0^+; t_0, \varphi) = \varphi(0), \end{cases} \tag{2.3}$$

where the initial function $\varphi \in \mathbb{R}^n$ is bounded and piecewise continuous on $(-\infty, 0]$ with points of discontinuity of the first kind at which it is continuous from the left. The set of all such functions will be denoted by $PCB[(-\infty, 0], \mathbb{R}^n]$, and the norm of a function $\varphi \in PCB[(-\infty, 0], \mathbb{R}^n]$ is defined as $||\varphi||_\infty = \sup_{s\in(-\infty,0]} ||\varphi(s)|| = \sup_{t\in(-\infty,t_0]} ||\varphi(t - t_0)||$.

According to the theory of impulsive control systems [15, 28, 29, 32–37], the solution $y(t) = y(t; t_0, \varphi) \in R^n$ of the initial value problem (2.2), (2.3) is a piecewise continuous function with points of discontinuity of the first kind t_k, $k = 1, 2, ...$, where it is continuous from the left, i.e. the following relations are valid

$$y_i(t_k^-) = y_i(t_k), \ y_i(t_k^+) = y_i(t_k) + P_{ik}(y_i(t_k)), \ i = 1, 2, ..., n, \ k = 1, 2, ...$$

We introduce some assumptions for the system parameters in (2.2) which will be used in the proofs of our main results. These assumptions also guarantee the existence and uniqueness of the solutions of (2.2):

A2.1. There exist constants $L_i > 0$ such that

$$|f_i(u) - f_i(v)| \leq L_i|u - v|$$

for all $u, v \in \mathbb{R}$, $i = 1, 2, ..., n$.

A2.2. There exist constants $M_i > 0$ such that for all $u \in \mathbb{R}$ and $i = 1, 2, ..., n$

$$|f_i(u)| \leq M_i < \infty.$$

A2.3. The delay kernel $K_i : \mathbb{R}^2 \to \mathbb{R}_+$ is continuous, and there exist positive numbers κ_i such that

$$\int_{-\infty}^{t} K_i(t, s)\, ds \leq \kappa_i < \infty$$

for all $t \geq t_0$, $t \neq t_k$, $k = 1, 2, ...$ and $i = 1, 2, ..., n$.

A2.4. The functions P_{ik} are continuous on $\mathbb{R}$, $i = 1, 2, ..., n$, $k = 1, 2, ...$

A2.5. $t_0 < t_1 < t_2 < ...t_k < t_{k+1} < ...$ and $t_k \to \infty$ as $k \to \infty$.

Remark 2.1. The assumptions A2.1 and A2.2 are used in the existence and uniqueness results for the solutions of fractional-order neural network models of type (2.1). See, for example, [15–17, 19]. The consideration of more general activation functions that are discontinuous and satisfy a nonlinear growth property as in [18, 21] can motivate the researchers for future results on the topic.

To realize the impulsive control and synchronization between the fractional-order neural network systems (2.1) and (2.2) we will use the following impulsive controller:

$$P_{ik}(y_i(t)) = Q_{ik}(y_i(t) - x_i(t)),\ i = 1, 2, ..., n,\ t = t_k,\ k = 1, 2, ... \tag{2.4}$$

Define the synchronization error as $e(t) = (e_1(t), e_2(t), ..., e_n(t))^T$, $e_i(t) = y_i(t) - x_i(t)$, $i = 1, 2, ..., n$, $t > 0$.

Therefore, the error fractional dynamical system is given as:

$$\begin{cases} {}_{t_0}^{C}D_t^q e_i(t) = -d_i(t)e_i(t) + \sum_{j=1}^{n} a_{ij}\bar{f}_j\left(e_j(t)\right) + \sum_{j=1}^{n} b_{ij}\bar{f}_j\left(e_j(t - \tau_j(t))\right) \\ \qquad + \sum_{j=1}^{n} c_{ij}\int_{-\infty}^{t} K_j(t, s)\bar{f}_j\left(e_j(s)\right)ds,\ t \neq t_k,\ t \geq t_0, \\ e_i(t_k^+) = e_i(t_k) + Q_{ik}(e_i(t_k)),\ k = 1, 2, ..., \end{cases} \tag{2.5}$$

where $\bar{f}_j(e_j(t)) = f_j(y_j(t)) - f_j(x_j(t))$, $\bar{f}_j(e_j(t-\tau_j(t))) = f_j(y_j(t-\tau_j(t))) - f_j(x_j(t-\tau_j(t)))$, and $\bar{f}_j(e_j(s)) = f_j(y_j(s)) - f_j(x_j(s))$, $-\infty < s \leq t, t \geq t_0$, $j = 1, 2, ..., n$, the impulsive control functions Q_{ik} are continuous on $\mathbb{R}$ and $Q_{ik}(0) = 0$ for all $i = 1, 2, ..., n$, $k = 1, 2, ...,$

$$e(s) = \varphi_0(s) - \bar{\varphi}_0(s),\ s \in (-\infty, 0],\ e(t_0^+) = \varphi_0(0) - \bar{\varphi}_0(0).$$

2.2. Mittag-Leffler synchronization concept

Since we will investigate the Mittag-Leffler synchronization opportunities of the impulsive control approach, we will use the standard Mittag-Leffler function [14] given as

$$E_q(z) = \sum_{\kappa=0}^{\infty} \frac{z^\kappa}{\Gamma(q\kappa + 1)},$$

where $q > 0$ in our analysis.

In addition, the following definition of global Mittag-Leffler synchronization between fractional-order systems, which is inspired by the Mittag-Leffler stability definition introduced in [39], will be important.

Definition 2.2. The controlled system (2.2) is said to be globally Mittag-Leffler synchronized onto the system (2.1), if for $\bar{\varphi}_0 \in PCB[(-\infty, 0], \mathbb{R}^n]$ there exist constants $\lambda > 0$ and $d > 0$ such that

$$||e(t)|| \leq \left\{\mathcal{M}(\bar{\varphi}_0)E_q(-\lambda t^q)\right\}^d,\ t > t_0,$$

where $\mathcal{M}(0) = 0$, $\mathcal{M}(\bar{\varphi}_0) \geq 0$, and $\mathcal{M}(\bar{\varphi}_0)$ is Lipschitzian with respect to $\bar{\varphi}_0 \in PCB[(-\infty, 0], \mathbb{R}^n]$.

In other words, the global Mittag-Leffler synchronization between fractional-order systems (2.1) and (2.2) is equivalent to a global Mittag-Leffler stability of the zero solution of the error system (2.5).

We note that, for integer-order systems where $q = 1$, the concept of global Mittag-Leffler synchronization reduces to that of global exponential synchronization, which is a particular type of global asymptotic synchronization. This particular type of synchronization provides the fastest convergent rate and that is why it attracts the high interest. It is also well known, [39] that for fractional-order systems the Mittag-Leffler stability demonstrates a faster convergence speed than the exponential stability near the origin. This is the main motivation of our research.

2.3. Lyapunov method nomenclatures. Auxiliary Lemmas

Introduce the following notations:
$G_k = (t_{k-1}, t_k) \times \mathbb{R}^n,\ k = 1, 2, ...;\ G = \cup_{k=1}^{\infty} G_k.$

Further on we will apply the Lyapunov approach and to this end we will use piecewise continuous Lyapunov functions $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$.

Definition 2.3. We say that the function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$, belongs to the class V_0 if the following conditions are fulfilled:

1. The function V is continuous in $\cup_{k=1}^{\infty} G_k$ and $V(t, 0) = 0$ for $t \in [t_0, \infty)$.
2. The function V satisfies locally the Lipschitz condition with respect to e on each of the sets G_k.
3. For each $k = 1, 2, ...$ and $e \in \mathbb{R}^n$ there exist the finite limits

$$V(t_k^-, e) = \lim_{\substack{t \to t_k \\ t < t_k}} V(t, e),\ \ V(t_k^+, e) = \lim_{\substack{t \to t_k \\ t > t_k}} V(t, e).$$

4. For each $k = 1, 2, ...$ and $e \in \mathbb{R}^n$ the following equalities are valid

$$V(t_k^-, e) = V(t_k, e).$$

We will also need the following auxiliary lemma, which is an impulsive generalization of Lemma 2.1 in [42] for systems with unbounded delays. The proof is identical to the proof of (21) in [42] and we omit it. Similar comparison results are proved in [15, 36, 38] and some of the references therein.

Lemma 2.4. *Assume that the function $V \in V_0$ is such that for $t \in [t_0, \infty)$, $\varphi \in PCB[(-\infty, 0], \mathbb{R}^n]$,*

$$V(t^+, e(t^+) + \Delta(e(t))) \leq V(t, e(t)),\ t = t_k,\ k = 1, 2, ...,$$

and the inequality

$${}_{t_0}^{C}D_t^q V(t, \varphi(t)) \leq MV(t, \varphi(0)),\ t \neq t_k,\ k = 1, 2, ...$$

is valid whenever $V(t + s, \varphi(s)) \leq V(t, \varphi(0))$ for $-\infty < s \leq 0$, where $M \in \mathbb{R}$.

Then $sup_{-\infty<s\leq 0} V(s, \varphi_0(s)) \leq V(t, \varphi_0(0))$ implies

$$V(t, e(t)) \leq \sup_{-\infty<s\leq 0} V(t_0^+, e(s)) E_q(Mt^q),\ t \in [t_0, \infty).$$

In the case, when the Lyapunov-like function is of the type $V(t,e) = \sum_{i=1}^{n} e_i^2(t)$ we will use the following result from [43].

Lemma 2.5. *Let $e_i(t) \in \mathbb{R}$ be a continuous and differentiable function, $i = 1, 2, ..., n$. Then, for $t \geq t_0$ and $0 < q < 1$, the following inequality holds*

$$\frac{1}{2}{}_{t_0}^{C}D_t^q e_i^2(t) \leq e_i(t){}_{t_0}^{C}D_t^q e_i(t),\ i = 1, 2, ..., n.$$

3. Results

In this section, sufficient conditions for the global Mittag-Leffler synchronization of the controlled system (2.2) onto the system (2.1), which imply global asymptotic stability of the zero equilibrium of error system (2.5) are derived.

Theorem 3.1. *Assume that assumptions A2.1–A2.5 hold and there exist positive constants $\underline{d}_i$ such that $\underline{d}_i \leq d_i(t)$ for $t \in \mathbb{R}_+$.*

Then, the neural network system (2.2) is globally Mittag-Leffler synchronized onto the system (2.1) under the impulsive controller (2.4), if the following conditions simultaneously hold:

$$\min_{1\leq i\leq n}\left(\underline{d}_i - L_i\sum_{j=1}^{n}|a_{ji}|\right) > \max_{1\leq i\leq n}\left(L_i\left(\sum_{j=1}^{n}|b_{ji}| + \kappa_i\sum_{j=1}^{n}|c_{ji}|\right)\right) > 0 \tag{3.1}$$

$$Q_{ik}(e_i(t_k)) = -\gamma_{ik}(y_i(t_k) - x_i(t_k)), \quad 0 < \gamma_{ik} < 2, \tag{3.2}$$

$i = 1, 2, ..., n,\ k = 1, 2, ...$

Proof. For the error system (2.5) we construct a Lyapunov function as:

$$V(t,e) = \sum_{i=1}^{n}|e_i(t)|.$$

Case 1. Consider first the case when $t \geq t_0$ and $t \in [t_{k-1}, t_k)$. Then, by the definition of the Caputo fractional derivative of order q, we have [14, 15, 42]

$${}_{t_0}^{C}D_t^q|e_i(t)| = sgn(e_i(t)){}_{t_0}^{C}D_t^q e_i(t)\ i = 1, 2, ..., n.$$

Using A2.1 and A2.3 and the assumptions of Theorem 3.1, we obtain

$${}_{t_0}^{C}D_t^q|e_i(t)| \leq -\underline{d}_i|e_i(t)| + \sum_{j=1}^{n}L_j|a_{ij}||e_j(t)| + \sum_{j=1}^{n}L_j|b_{ij}||e_j(t-\tau_j(t))| + \sum_{j=1}^{n}L_j\kappa_j|c_{ij}|\sup_{-\infty<s\leq t}|e_j(s)|,$$

for $i = 1, 2, ..., n$.

After taking the sum for all $i = 1, 2, ..., n$, we have

$$
{}_{t_0}^{C}D_t^q V(t,e(t)) \le -\sum_{i=1}^{n}\left[\underline{d}_i - L_i\sum_{j=1}^{n}|a_{ji}|\right]|e_i(t)| + \sum_{j=1}^{n}\sum_{i=1}^{n}L_j|b_{ij}||e_j(t-\tau_j(t))|
$$

$$
+\sum_{j=1}^{n}\sum_{i=1}^{n}L_j\kappa_j|c_{ij}|\sup_{-\infty<s\le t}|e_j(s)| \le -\nu_1 V(t,e(t)) + \nu_2\sup_{-\infty<s\le t}V(s,e(s)),
$$

where

$$
\nu_1 = \min_{1\le i\le n}\left(\underline{d}_i - L_i\sum_{j=1}^{n}|a_{ji}|\right) > 0,
$$

$$
\nu_2 = \max_{1\le i\le n}\left(L_i\left(\sum_{j=1}^{n}|b_{ji}| + \kappa_i\sum_{j=1}^{n}|c_{ji}|\right)\right) > 0.
$$

According to the above estimate for any solution $e(t)$ of (2.5) such that

$$
V(s,e(s)) \le V(t,e(t)), \quad -\infty < s \le t,
$$

we have

$$
{}_{t_0}^{C}D_t^q V(t,e(t)) \le -(\nu_1-\nu_2)V(t,e(t)),\ t\ne t_k,\ k=1,2,....
$$

Condition (3.1) of Theorem 3.1 implies the existence of a real number $\lambda > 0$ such that

$$
\nu_1 - \nu_2 \ge \lambda,
$$

and, it follows that

$$
{}_{t_0}^{C}D_t^q V(t,e(t)) \le -\lambda V(t,e(t)),\ t\ne t_k,\ t\ge t_0. \tag{3.3}
$$

Case 2. For $t = t_k$, $k = 1,2,...$, from condition (3.2), we obtain

$$
V(t_k^+, e(t_k)+\Delta(e(t_k))) = \sum_{i=1}^{n}|e_i(t_k)+Q_{ik}(e(t_k))|
$$

$$
= \sum_{i=1}^{n}|y_i(t_k)-x_i(t_k)-\gamma_{ik}(y_i(t_k)-x_i(t_k))| = \sum_{i=1}^{n}|1-\gamma_{ik}||y_i(t_k)-x_i(t_k)|
$$

$$
< \sum_{i=1}^{n}|y_i(t_k)-x_i(t_k)| = V(t_k,e(t_k)),\ k=1,2,.... \tag{3.4}
$$

Thereupon, using (3.3) and (3.4), it follows from Lemma 2.4 that

$$
V(t,e(t)) \le \sup_{-\infty<s\le 0}V(t_0^+,e(s))E_q(-\lambda t^q),\ t\in[t_0,\infty).
$$

Therefore, we have

$$
\|e(t)\| = \sum_{i=1}^{n}|e_i(t)| \le \|\varphi_0-\bar{\varphi}_0\|_\infty E_q(-\lambda t^q), \quad t\ge t_0.
$$

Denote by $\mathcal{M} = \|\varphi_0 - \bar{\varphi}_0\|_\infty$. Hence,

$$\|e(t)\| \leq \mathcal{M} E_q(-\lambda t^q), \quad t > t_0,$$

where $\mathcal{M} \geq 0$ and $\mathcal{M} = 0$ holds only if $\varphi_0(s) = \bar{\varphi}_0(s)$ for $s \in (-\infty, 0]$, which implies that the closed-loop neural network (2.2) is globally Mittag-Leffler synchronized onto system (2.1) under the designed impulsive control law (2.4). □

Theorem 3.2. *Assume that assumptions of Theorem 3.1 hold, and (3.1) is replaced by*

$$\min_{1\leq i\leq n}\left(2\underline{d}_i - \sum_{j=1}^{n}\Big(L_j(|a_{ij}| + |b_{ij}| + \kappa_j|c_{ij}|) + L_i|a_{ji}|\Big)\right) > \max_{1\leq i\leq n}\left(L_i\left(\sum_{j=1}^{n}|b_{ji}| + \kappa_i\sum_{j=1}^{n}|c_{ji}|\right)\right) > 0 \tag{3.5}$$

Then, the neural network system (2.2) is globally Mittag-Leffler synchronized onto the system (2.1) under the impulsive controller (2.4).

Proof. The proof of Theorem 3.2 is similar to those of Theorem 3.1. We will mention just some main points.

For the error system (2.5) we construct a Lyapunov function as:

$$V(t,e) = \sum_{i=1}^{n} e_i^2(t).$$

First, for $t = t_k$, $k = 1, 2, ...$, from condition (3.2), we obtain

$$V(t_k^+, e(t_k) + \Delta(e(t_k))) = \sum_{i=1}^{n}(e_i(t_k) + Q_{ik}(e(t_k)))^2$$

$$= \sum_{i=1}^{n}(y_i(t_k) - x_i(t_k) - \gamma_{ik}(y_i(t_k) - x_i(t_k)))^2 = \sum_{i=1}^{n}(1-\gamma_{ik})^2(y_i(t_k) - x_i(t_k))^2$$

$$< \sum_{i=1}^{n}(y_i(t_k) - x_i(t_k))^2 = V(t_k, e(t_k)),\ k = 1, 2, ... \tag{3.6}$$

Secondly, for $t \geq t_0$ and $t \in [t_{k-1}, t_k)$, $k = 1, 2, ...$, we apply again A2.1, A2.3 and the assumptions of Theorem 3.1 to get

$${}_{t_0}^{C}D_t^q V(t,e(t)) \leq \sum_{i=1}^{n} 2|e_i(t)|sgn(e_i(t))\Bigg[-d_i(t)e_i(t) + \sum_{j=1}^{n} a_{ij}\bar{f}_j\left(e_j(t)\right) + \sum_{j=1}^{n} b_{ij}\bar{f}_j\left(e_j(t-\tau_j(t))\right)$$

$$+ \sum_{j=1}^{n} c_{ij}\int_{-\infty}^{t} K_j(t,s)\bar{f}_j\left(e_j(s)\right)ds\Bigg]$$

$$\leq \sum_{i=1}^{n}\Bigg[-2\underline{d}_i e_i^2(t) + 2\sum_{j=1}^{n} L_j|a_{ij}||e_i(t)||e_j(t)| + 2\sum_{j=1}^{n} L_j|b_{ij}||e_i(t)||e_j(t-\tau_j(t))|$$

$$+2\sum_{j=1}^{n} L_j\kappa_j|c_{ij}||e_i(t)| \sup_{s\in(-\infty,t]} |e_j(s)|\Bigg]$$

$$\leq \sum_{i=1}^{n}\Bigg[-2\underline{d}_i e_i^2(t) + \sum_{j=1}^{n} L_j|a_{ij}|(e_i^2(t)+e_j^2(t)) + \sum_{j=1}^{n} L_j|b_{ij}|(e_i^2(t)+e_j^2(t-\tau_j(t)))$$

$$+\sum_{j=1}^{n} L_j\kappa_j|c_{ij}|(e_i^2(t) + \sup_{s\in(-\infty,t]} e_j^2(s))\Bigg]$$

$$= -\sum_{i=1}^{n}\left[2\underline{d}_i + \sum_{j=1}^{n}\Big(L_j(|a_{ij}|+|b_{ij}|+\kappa_j|c_{ij}|) + L_i|a_{ji}|\Big)\right] e_i^2(t)$$

$$+\sum_{j=1}^{n}\sum_{i=1}^{n}\left(L_j|b_{ij}|e_j^2(t-\tau_j(t)) + L_j\kappa_j|c_{ij}| \sup_{s\in(-\infty,t]} e_j^2(s))\right)$$

$$\leq -\bar{\nu}_1 V(t,e(t)) + \nu_2 \sup_{-\infty<s\leq t} V(s,e(s)),$$

where

$$\bar{\nu}_1 = \min_{1\leq i\leq n}\left(2\underline{d}_i - \sum_{j=1}^{n}\Big(L_j(|a_{ij}|+|b_{ij}|+\kappa_j|c_{ij}|) + L_i|a_{ji}|\Big)\right) > 0,$$

$$\nu_2 = \max_{1\leq i\leq n}\left(L_i\left(\sum_{j=1}^{n}|b_{ji}| + \kappa_i\sum_{j=1}^{n}|c_{ji}|\right)\right) > 0.$$

The rest of the proof repeats the end of the proof of Theorem 3.1 using (3.5) instead of (3.1). □

Remark 3.3. Theorems 3.1 and 3.2 offer an impulsive control law with control gains γ_{ik} that can be applied to the Mittag-Leffler synchronization of the fractional neural network model (2.2) into the model (2.1). Since the synchronization is directly related to the stability and stabilization problems of systems, the proposed impulsive control strategy is a power tool for studying the stability dynamics of fractional-order neural networks.

Remark 3.4. Impulsive control approaches are applied to various integer-order neural networks models in the existing literature. See, for example, [26–33] and some of the references therein. Indeed, it is well known that such control strategies have important advantages. In this paper, we generalize the impulsive control approach to the fractional-order case. In fact, the impulsive controllers (2.4) are designed so that to achieve global Mittag-Leffler synchronization between the fractional master system (2.1) and the control fractional system (2.2).

Remark 3.5. In spite of the great possibilities of applications, the theory of the fractional neural networks with distributed delays is still in the initial stage [22]. To the best of our knowledge, there has not been any work so far considering unbounded delays in fractional-order neural network models, which is very important in theories and applications and also is a very challenging problem. Hence, our results extend the existent ones in the literature adding the effect of unbounded delays.

Remark 3.6. A review of the existing results shows that impulsive control strategies have been applied to some fractional-order neural network models [34–38]. With the presented research we complement the existing works by incorporating unbounded delays into the fractional-order neural network model. The results in [36] can be considered as special cases of the proposed results here, when the terms with unbounded delays are removed in (2.1) and (2.2). Hence, as compared to [34–38], the proposed results in this paper are more general. To the best of our knowledge, this is the first paper on fractional-order neural network models with unbounded delays and subject to impulsive control law.

Remark 3.7. The concept of Mittag-Leffler synchronization between fractional-order neural networks is investigated in some papers. See, for example [36, 37, 40, 41]. In this paper, we extended the existing results incorporating unbounded delays and impulsive perturbations into fractional-order neural networks and dealt with an impulsive synchronization analysis for the addressed more general models.

4. Examples and simulations

In this section, examples are presented to illustrate the main results.

Example 4.1. Consider a two-dimensional Caputo fractional neural network of type (2.1) with mixed delays as follows

$$\begin{cases} {}_{t_0}^{C}D_t^q x_i(t) = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t-\tau_j(t))\right) \\ \qquad + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_j(t,s) f_j\left(x_j(s)\right) ds + I_i,\ t \geq t_0, \end{cases} \tag{4.1}$$

where $n = 2$, $t_0 = 0$, $I_1 = 1$, $I_2 = 1$, $d_1(t) = d_2(t) = 3$, $f_i(x_i) = \frac{1}{2}(|x_i + 1| - |x_i - 1|)$, $0 \leq \tau_i(t) \leq \tau$ $(\tau = 1)$, $K_i(s) = e^{-s}$, $i = 1, 2$,

$$(a_{ij})_{2\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.6 \\ -0.5 & 1 \end{pmatrix},$$

$$(b_{ij})_{2\times 2} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0.9 & -0.2 \\ -0.04 & 0.07 \end{pmatrix},$$

$$(c_{ij})_{2\times 2} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 0.5 & -0.6 \\ -0.03 & 0.08 \end{pmatrix}.$$

Then, we have that $\underline{d}_1 = \underline{d}_2 = 3$, $L_1 = L_2 = 1$, and the assumption A2.3 is satisfied, since $\int_0^\infty e^{-s} ds = 1$.

We consider the following impulsive control system

$$\begin{cases} {}_{t_0}^{C}D_t^q y_i(t) = -d_i(t)y_i(t) + \sum_{j=1}^{2} a_{ij} f_j\left(y_j(t)\right) + \sum_{j=1}^{2} b_{ij} f_j\left(y_j(t-\tau_j(t))\right) \\ \qquad + \sum_{j=1}^{2} c_{ij} \int_{-\infty}^{t} K_j(t,s) f_j\left(y_j(s)\right) ds + I_i,\ t \neq t_k,\ t \geq 0, \end{cases} \tag{4.2}$$

with an impulsive control rule defined as

$$\begin{cases} \Delta y_1(t_k) = -\dfrac{3}{4}(y_1(t_k) - x_1(t_k)), \ k = 1, 2, ..., \\ \\ \Delta y_2(t_k) = -\dfrac{1}{2}(y_2(t_k) - x_2(t_k)), \ k = 1, 2, ..., \end{cases} \tag{4.3}$$

where the impulsive moments are such that $0 < t_1 < t_2 < ...$, and $\lim_{k\to\infty} t_k = \infty$.

Hence, the condition (3.1) of Theorem 3.1 is satisfied for $\nu_1 = 1.4$, $\nu_2 = 0.97$. We also have that the impulsive control gains satisfy condition (3.2), since

$$0 < \gamma_{1k} = \frac{3}{4} < 2, \qquad 0 < \gamma_{2k} = \frac{1}{2} < 2.$$

According to Theorem 3.1, the master system (4.1) and response system (4.2) with impulsive control (4.3) are globally Mittag-Leffler (globally asymptotically) synchronized. Time responses of the state variables of the synchronization error system are given on Figure 1, for $t_k = 5k$, $k = 1, 2, ...$

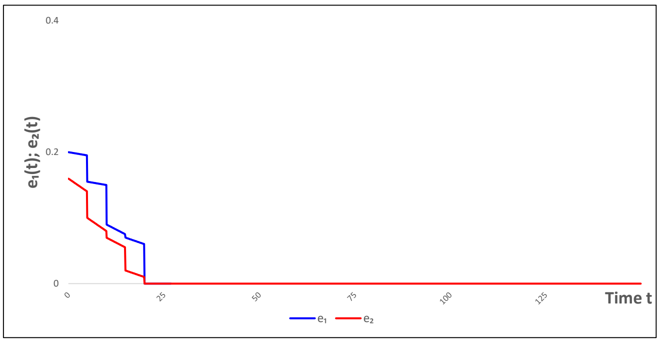

Figure 1. Time responses of the state variables of the synchronization error system between (4.1) and (4.2) with impulsive control (4.3) in Example 4.1 for $t_k = 5k$, $k = 1, 2, ...$

If we consider again the system (4.2) but with impulsive perturbations of the form

$$\begin{cases} \Delta y_1(t_k) = -3(y_1(t_k) - x_1(t_k)), \ k = 1, 2, ..., \\ \\ \Delta y_2(t_k) = -\dfrac{1}{2}(y_2(t_k) - x_2(t_k)), \ k = 1, 2, ..., \end{cases} \tag{4.4}$$

the impulsive control rule (3.2) is not satisfied, and there is nothing we can say about the impulsive synchronization between systems (4.1) and (4.2) with impulsive perturbations (4.4), since $\gamma_{1k} = 3 > 2$. The state variable $e_1(t)$ of the error system is demonstrated on Figure 2 in the case when $t_k = 5k$, $k = 1, 2, ...$ It is interesting to see that the impulses cannot synchronize both systems in this case.

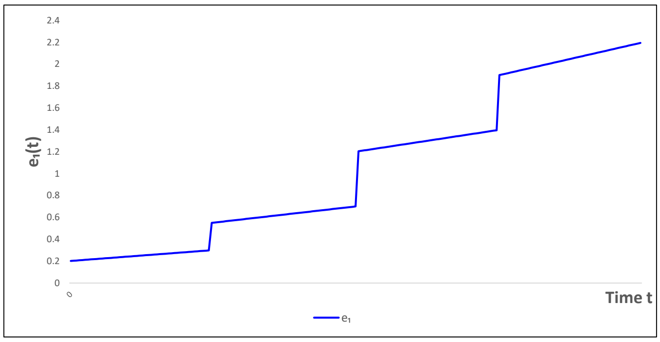

Figure 2. Time response of the state variable e_1 of the error system between (4.1) and (4.2) with impulses (4.4) in Example 4.1 for $t_k = 5k$, $k = 1, 2, ...$

Remark 4.2. Note that the impulsive functions Q_{ik}, $i = 1, 2, ..., n$, $k = 1, 2, ...$ in the impulsive control rule (2.4) may be of different type. In this paper, we investigate the case, when the controller (2.4) has the form given in (3.2). By means of Example 1, we demonstrate the efficiency of the controllers of type (3.2). We show that, if the impulsive controller is designed according to (3.2), it contributes to the global Mittag-Leffler synchronization between the fractional-order models (4.1)–(4.3). We also illustrate the case, when the impulsive controller does not satisfy (3.2), and the synchronization between (4.1) and (4.2), (4.4) fails, i.e., the zero solution of the error system is globally Mittag-Leffler unstable.

Example 4.3. Consider a two-dimensional Caputo fractional neural network of type (4.1) with mixed delays, where $t_0 = 0$; $I_1 = 1$, $I_2 = 1$; $d_1(t) = d_2(t) = 4 + t$, $f_i(x_i) = \frac{1}{2}(|x_i + 1| - |x_i - 1|)$, $0 \le \tau_i(t) \le \tau$ ($\tau = 1$), $K_i(s) = e^{-s}$, $i = 1, 2$,

$$(a_{ij})_{2\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix},$$

$$(b_{ij})_{2\times 2} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0.1 & -0.2 \\ -0.15 & 0.1 \end{pmatrix},$$

$$(c_{ij})_{2\times 2} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 0.1 & -0.2 \\ -0.15 & 0.2 \end{pmatrix}.$$

Now, we have that $\underline{d}_1 = \underline{d}_2 = 4$, $L_1 = L_2 = 1$.

If the impulsive controllers (2.4) are chosen so that

$$\begin{cases} P_{1k}(y_1(t_k)) = -\frac{4}{5}(y_1(t_k) - x_1(t_k)), \ k = 1, 2, ..., \\ P_{2k}(y_2(t_k)) = -\frac{2}{3}(y_2(t_k) - x_2(t_k)), \ k = 1, 2, ..., \end{cases} \tag{4.5}$$

where the impulsive moments are such that $0 < t_1 < t_2 < ...$, and $\lim_{k\to\infty} t_k = \infty$, then the corresponding impulsive control system is

$$\begin{cases} {}_{t_0}^{C}D_t^q y_i(t) = -d_i(t)y_i(t) + \sum_{j=1}^{2} a_{ij}f_j\left(y_j(t)\right) + \sum_{j=1}^{2} b_{ij}f_j\left(y_j(t-\tau_j(t))\right) \\ \qquad + \sum_{j=1}^{2} c_{ij}\int_{-\infty}^{t} K_j(t,s)f_j\left(y_j(s)\right)ds + I_i, \ t \neq t_k, \ t \geq t_0, \\ \Delta y_1(t_k) = -\frac{4}{5}(y_1(t_k) - x_1(t_k)), \ k = 1, 2, ..., \\ \Delta y_2(t_k) = -\frac{2}{3}(y_2(t_k) - x_2(t_k)), \ k = 1, 2, ... \end{cases} \tag{4.6}$$

Hence, the condition (3.5) of Theorem 3.2 is satisfied for $\nu_1 = 1.4$, $\nu_2 = 0.7$. We also have that the impulsive control gains satisfy condition (3.2), since

$$0 < \gamma_{1k} = \frac{4}{5} < 2, \qquad 0 < \gamma_{2k} = \frac{2}{3} < 2.$$

Therefore, we obtain by Theorem 3.2, that the master system (4.1) and response system (4.6) are globally Mittag-Leffler (globally asymptotically) synchronized. Time responses of the state variables of the synchronization error system are represented on Figure 3, for $t_k = 5k$, $k = 1, 2, ...$

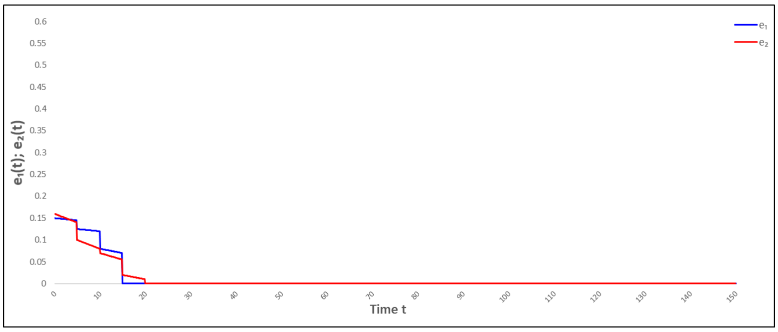

Figure 3. Time responses of the state variables of the synchronization error system between (4.1) and (4.6) in Example 4.2 for $t_k = 5k$, $k = 1, 2, ...$

Remark 4.4. We presented two examples to demonstrate the efficiency of both theorems that ensure the global Mittag-Leffler synchronization of fractional-order neural networks by employing different Lyapunov function candidates, and hence using different conditions of type (3.1) and (3.5). For both synchronization criteria we apply a unified impulsive control rule (3.2). However, the established criteria can help to check the synchronization of impulsive fractional-order neural networks with different systems parameters.

5. Conclusions

In this paper, we have investigated the global Mittag-Leffler synchronization problem between two fractional-order neural networks with bounded and unbounded delays. Applying an impulsive control approach, criteria are obtained by constructing appropriate Lyapunov functions and the fractional Lyapunov approach. The obtained results are new and complement the existing global Mittag-Leffler synchronization results for fractional neural networks under impulsive controls. Examples are demonstrated to illustrate the obtained results. Since, the impulsive control approach is preferable in many real-world applications, our results can be extended to fractional-order models with reaction-diffusion terms. The discrete cases are also a subject to future investigations.

Acknowledgments

This work was supported in part by the European Regional Development Fund through the Operational Program "Science and Education for Smart Growth" under contract UNITe No BG05M2OP001–1.001–0004 (2018–2023).

Conflict of interest

The authors declare no conflict of interest.

References

1. S. Abbas, M. Banerjee, S. Momani, Dynamical analysis of fractional-order modified logistic model, *Comput. Math. Appl.,* **62** (2011), 1098–1104.
2. S. Das, P. K. Gupta, A mathematical model on fractional Lotka-Volterra equations, *J. Theor. Biol.,* **277** (2011), 1–6.
3. H. L. Li, L. Zhang, C. Hu, Y. L. Jiang, Z. Teng, Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge, *J. Appl. Math. Comput.,* **54** (2017), 435–449.
4. G. Stamov, I. Stamova, Modelling and almost periodic processes in impulsive Lasota-Wazewska equations of fractional order with time-varying delays, *Quaest. Math.,* **40** (2017), 1041–1057.
5. E. H. Dulf, D. C. Vodnar, A. Danku, C. I. Muresan, O. Crisan, Fractional-order models for biochemical processes, *Fractal Fract.,* **4** (2020), 12.
6. B. N. Lundstrom, M. H. Higgs, W. J. Spain, A. L. Fairhall, Fractional differentiation by neocortical pyramidal neurons, *Nat. Neurosci.,* **11** (2008), 1335–1342.
7. R. L. Magin, *Fractional calculus in bioengineering*, Redding: Begell House, 2006.
8. W. Teka, T. M. Marinov, F. Santamaria, Neuronal spike timing adaptation described with a fractional leaky integrate-and-fire model, *PLoS Comput. Biol.,* **10** (2014), e1003526.
9. K. Sayevand, A study on existence and global asymptotical Mittag-Leffler stability of fractional Black-Scholes equation for a European option pricing equation, *J. Hyperstruct.,* **3** (2014), 126–138.

10. J. J. Nieto, G. T. Stamov, I. M. Stamova, A fractional-order impulsive delay model of price fluctuations in commodity markets: Almost periodic solutions, *Eur. Phys. J. Spec. Top.,* **226** (2017), 3811–3825.

11. A. Atangana, Fractal-fractional differentiation and integration: Connecting fractal calculus and fractional calculus to predict complex system, *Chaos Solitons Fractals,* **102** (2017), 396–406.

12. D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Fractional calculus: Models and numerical methods*, Hackensack: World Scientific, 2012.

13. R. Hilfer, *Application of fractional calculus in physics*, River Edge: World Scientific, 2000.

14. I. Podlubny, *Fractional differential equations*, San Diego: BAcademic Press, 1999.

15. I. M. Stamova, G. T. Stamov, *Functional and impulsive differential equations of fractional order: Qualitative analysis and applications*, Boca-Raton: CRC Press, 2017.

16. H. Bao, J. H. Park, J. Cao, Non-fragile state estimation for fractional-order delayed memristive BAM neural networks, *Neural Networks,* **119** (2019), 190–199.

17. H. L. Li, H. Jiang, J. Cao, Global synchronization of fractional-order quaternion-valued neural networks with leakage and discrete delays, *Neurocomputing,* **385** (2020), 211–219.

18. X. Peng, H. Wu, J. Cao, Global nonfragile synchronization in finite time for fractional-order discontinuous neural networks with nonlinear growth activations, *IEEE Trans. Neural Networks Learn. Syst.,* **30** (2019), 2123–2137.

19. R. Rakkiyappan, C. Velmurugan, J. Cao, Stability analysis of memristor-based fractional-order neural networks with different memductance functions, *Cogn. Neurodynamics,* **9** (2015), 145–177.

20. I. M. Stamova, S. Simeonov, Delayed reaction-diffusion cellular neural networks of fractional order: Mittag-Leffler stability and synchronization, *J. Comput. Nonlinear Dyn.,* **13** (2018), 1–7.

21. X. Wang, H. Wu, J. Cao, Global leader-following consensus in finite time for fractional-order multi-agent systems with discontinuous inherent dynamics subject to nonlinear growth, *Nonlinear Anal: Hybrid Syst.,* **37** (2020), 10088.

22. H. Zhang, R. Ye, S. Liu, J. Cao, A. Alsaedi, X. Li, LMI-based approach to stability analysis for fractional-order neural networks with discrete and distributed delays, *Int. J. Syst. Sci.,* **49** (2018), 537–545.

23. D. Yang, X. Li, J. Qiu, Output tracking control of delayed switched systems via state-dependent switching and dynamic output feedback, *Nonlinear Anal: Hybrid Syst.,* **32** (2019), 294–305.

24. F. Cacace, V. Cusimano, P. Palumbo, Optimal impulsive control with application to antiangiogenic tumor therapy, *IEEE Trans. Control Syst. Technol.,* **28** (2020), 106–117.

25. J. Hu, G. Sui, X. Lu, X. Li, Fixed-time control of delayed neural networks with impulsive perturbations, *Nonlinear Anal. Model. Control,* **23** (2018), 904–920.

26. X. Li, J. Shen, R. Rakkiyappan, Persistent impulsive effects on stability of functional differential equations with finite or infinite delay, *Appl. Math. Comput.,* **329** (2018), 14–22.

27. X. Li, S. Song, Stabilization of delay systems: Delay-dependent impulsive control, *IEEE Trans. Autom. Control,* **62** (2016), 406–411.

28. X. Li, J. Wu, Sufficient stability conditions of nonlinear differential systems under impulsive control with state-dependent delay, *IEEE Trans. Autom. Control,* **63** (2018), 306–311.

29. X. Li, X. Yang, J. Cao, Event-triggered impulsive control for nonlinear delay systems, *Automatica,* **117** (2020), 108981.
30. X. Li, X. Yang, T. Huang, Persistence of delayed cooperative models: Impulsive control method, *Appl. Math. Comput.,* **342** (2019), 130–146.
31. I. M. Stamova, G. T. Stamov, Impulsive control on global asymptotic stability for a class of bidirectional associative memory neural networks with distributed delays, *Math. Comput. Model.,* **53** (2011), 824–831.
32. T. Yang, *Impulsive control theory*, Berlin: Springer, 2001.
33. X. Yang, D. Peng, X. Lv, X. Li, Recent progress in impulsive control systems, *Math. Comput. Simul.,* **155** (2019), 244–268.
34. M. Bohner, I. M. Stamova, G. T. Stamov, Impulsive control functional differential systems of fractional order: Stability with respect to manifolds, *Eur. Phys. J. Spec. Top.,* **226** (2017), 3591–3607.
35. A. Pratap, R. Raja, J. Alzabut, J. Dianavinnarasi, J. Cao, G. Rajchakit, Finite-time Mittag-Leffler stability of fractional-order quaternion-valued memristive neural networks with impulses, *Neural Process. Lett.,* **51** (2020), 1485–1526.
36. I. M. Stamova, Global Mittag-Leffler stability and synchronization of impulsive fractional-order neural networks with time-varying delays, *Nonlinear Dyn.,* **77** (2014), 1251–1260.
37. I. M. Stamova, G. T. Stamov, Mittag-Leffler synchronization of fractional neural networks with time-varying delays and reaction-diffusion terms using impulsive and linear controllers, *Neural Networks,* **96** (2017), 22–32.
38. R. Tuladhar, F. Santamaria, I. Stamova, Fractional Lotka-Volterra-type cooperation models: Impulsive control on their stability behavior, *Entropy,* **22** (2020), 970.
39. Y. Li, Y. Chen, I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, *Comput. Math. Appl.,* **59** (2010), 1810–1821.
40. J. Chen, C. Li, X. Yang, Global Mittag-Leffler projective synchronization of nonidentical fractional-order neural networks with delay via sliding mode control, *Neurocomputing,* **313** (2018), 324–332.
41. A. Wu, Z. Zeng, Global Mittag-Leffler stabilization of fractional-order memristive neural networks, *IEEE Trans. Neural Networks Learn. Syst.,* **28** (2017), 206–217.
42. R. Ye, X. Liu, H. Zhang, J. Cao, Global Mittag-Leffler synchronization for fractional-order BAM neural networks with impulses and multiple variable delays via delayed-feedback control strategy, *Neural Process. Lett.,* **49** (2019), 1–18.
43. N. Aguila-Camacho, M. Duarte-Mermoud, J. Gallegos, Lyapunov functions for fractional order systems, *Commun. Nonlinear Sci. Numer. Simul.,* **19** (2014), 2951–2957.