Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(3): 2268–2286.
DOI: 10.3934/math.2021137
Received: 09 October 2020
Accepted: 07 December 2020
Published: 14 December 2020

Research article

Fuzzy subnear-semirings and fuzzy soft subnear-semirings

Abdelghani Taouti[1,*] **and Waheed Ahmad Khan**[2]

[1] ETS-Maths and NS Engineering Division, HCT, University City P.O. Box 7947, Sharjah, United Arab Emirates

[2] Department of Mathematics, University of Education, Lahore, Attock Campus 43600, Pakistan

* **Correspondence:** Email: ganitaouti@yahoo.com.au.

Abstract: Our purpose in this paper is to initiate and study the notions of fuzzy subnear-semirings and fuzzy soft subnear-semirings. We study few of their elementary properties by providing suitable examples. Moreover, we present the characterizations of zero symmetric near-semirings (seminearrings) through their fuzzy ideals and fuzzy soft ideals. Fuzzy soft anti-homomorphism of fuzzy soft near-semirings and fuzzy soft R-homomorphisms of fuzzy soft R-subsemigroups are also introduced and discussed.

Keywords: fuzzy soft subnear-semirings; fuzzy soft anti-homomorphisms; fuzzy soft left (right) near-semiring; fuzzy soft left (right) ideals
Mathematics Subject Classification: 16Y30, 16Y99, 03G25

1. Introduction

Problems that arise in daily life cannot be solved by using conventional mathematical theories. There are established concepts like fuzzy sets [35], rough sets [28] and vague sets [13] which can be considered as mathematical tools. Unfortunately, all of the above theories have their intrinsic difficulties which were indicated by Molodtsov [26]. Recently, many researchers are working on soft sets theory and the theory is developing speedily. Applications of soft sets towards decision making theory are described in [24]. Recently, the application of soft sets towards multi-attribute decision-making theory is described in [22]. Currently, algebraic soft structures have been studied progressively, soft rings [3], soft semirings [12], soft nearrings [32] have been introduced and discussed in the literature. Recently, the authors (with A. Rehman) have introduced the concept of soft near-semirings [18].

Alternatively, Zadeh [35] introduced most suitable theory to deal with uncertainties. Fuzzy rings, Fuzzy subnear-rings [1] and fuzzy semirings [4] have been explored in the literature but we can hardly find the discussion about fuzzy near-semirings.

Fuzzy soft sets have been introduced and discussed in [23]. Several Applications of fuzzy soft sets in decision theory were presented in [31]. At the same time, N. Çağman et. al [8] redefined fuzzy soft sets and provided its applications in decision theory and the problems containing uncertainties. Moreover, the notion of generalized fuzzy soft sets along with its applications in decision theory was discussed in [25]. Later, the notion of generalised multi-fuzzy soft sets along with its applications in the theory of decision making and also towards uncertainties was presented in [10]. Alternatively, fuzzy soft algebraic structures like fuzzy soft groups [7] fuzzy soft rings ([16,33]), fuzzy soft subnear-rings [27], fuzzy soft semirings [29] have been introduced in the literature.

On the other hand, near-semiring is the generalization of both nearring and semiring and was first introduced by Rootselaar [30]. Hoorn and Rootselaar [15] further introduced few related concepts of seminearrings. By near-semirings we mean a structure $(R, +, \cdot)$ such that $(R, +)$ is a monoid, $(R, \ \cdot)$ is a semigroup and satisfying only one (left or right) distributive law with 0 as a one sided absorbing element. Moreover, if $0 \in R$ such that $a + 0 = 0 + a = a$, $a \ \cdot 0 = 0 \ \cdot a = 0$, then we call R a zero-symmetric near-semiring (or seminearring). Near-semirings arises naturally from the mappings on monoids to itself under componentwise addition and composition of mappings. Near-semirings have a strong relationship with various kinds of algebras. Due to this, special type of near-semirings i.e., balanced near-semirings were introduced in [9]. Moreover, the ideals of near-semirings are different from the typical ideals of rings (resp., semirings, nearrings) and thus many results in rings (resp., semirings, nearrings) are dissimilar from near-semirings using solely ideals. Recently, Some special types of ideals of near-semirings are introduced in [19].

Near-semirings have a lot of applications in modern technologies. Krishna and Chatterjee [21] presented the applications of near-semirings towards linear sequential machines. Recently, the applications of near-semirings towards automata and formal languages were explored in [17].

In this note, we present the notions of fuzzy subnear-semirings and fuzzy soft near-semirings along with their few of substructures. Basically, we shift some classical terms related to near-semirings such as R-sub-semigroup, R-homomorphism, R-antihomomorphism towards fuzzy sets and fuzzy soft set theories. We incorporate the concepts of fuzzy (soft) R-subsemigroup, fuzzy (soft) R-subsemigroup, fuzzy near-semiring R-homomorphism and fuzzy soft near-semiring anti-homomorphism. We present few characterizations of regular seminearrings, strongly idempotent seminearring, distributively generated seminearrings through their fuzzy ideals. We interrelate R-semigroups and fuzzy R-semigroups of near-semiring through R-epimorphism. Applying fuzzy R-homomorphism map, we investigate that the image of fuzzy R-subsemigroup of near-semiring is a fuzzy R-subsemigroup. Similarly, we conclude that under fuzzy soft R-isomorphism of fuzzy soft near-semiring the image of fuzzy soft R-subsemigroup is a fuzzy soft R-subsemigroup. While utilizing fuzzy soft anti-isomorphism, we find that the image of fuzzy soft right (left) near-semiring is a fuzzy soft left (right) near-semiring. Throughout, we mainly emphasis on the algebraic properties of fuzzy subnear-semirings and fuzzy soft near-semirings.

Futher to the above, our contribution through this work serves as a bridging link between some classical algebraic structures and the fuzzy (soft) algebraic structures. Since near-semiring is the structure lying between nearring and semiring, the outcome of our present study similarly links the existing gap between fuzzy soft subnear-rings [27] and fuzzy soft semirings [29]. Near-semirings have proven to be useful in studying sequential machine, automata and formal languages. On the other hand, the notions of automata and formal languages have been generalized and extensively studied in a fuzzy framework. Thus one may expect that fuzzy (soft) subnear-semirings and fuzzy soft near-semirings may prove useful in studying fuzzy automata and sequential machines.

2. Preliminaries

In this section we present few terminologies which are useful in our forthcoming discussions.

Definition 1. *[30] An algebraic structure $(R, +, \cdot)$ is a left (resp., right) near-semiring if*
(i) *R is a monoid w.r.t +"*
(ii) *R is a semigroup w.r.t ·"*
(iii) $c \cdot (a + b) = c \cdot a + c \cdot b$ *(resp.,* $(a + b) \cdot c = a \cdot c + b \cdot c$*), for each a, ¸$c $in R.*

Moreover, if $0 \in R$ such that $a + 0 = 0 + a = a$, $a \cdot 0 = 0 \cdot a = 0$ then R is said to be a zero-symmetric near-semiring (or seminearring).

A mapping $\vartheta : R \rightarrow R'$ between two near-semirings R and R' is said to be a near-semiring homomorphism if for all $a, b \in R$,

$$\vartheta(a + b) = \vartheta(a) + \vartheta(b), \vartheta(ab) = \vartheta(a)\vartheta(b).$$

Similarly, if R and R' are two near-semirings then the map $\chi : R \rightarrow R'$ defined by $\chi(x + y) = \chi(x) + \chi(y)$, $\chi(xy) = \chi(y)\chi(x)$, $\chi(0) = 0$, for all $x, y \in R$, is a near-semiring anti-homomorphism. An ideal of a seminearring (zero-symmetric near-semiring) is the kernel of a semi-nearring homomorphism [15]. J. Ahsan [5] presented the generalization of an ideal of seminearrings i.e., a subset I of a seminearring R is said to be a right (left) S-ideal of R if (i) $a + b \in I$, for all $a, b \in I$ and (ii) $ar(ra) \in I$, for all $a \in I$ and $r \in R$.

If R is a near-semiring then we call a semigroup $(\Gamma, +)$ having zero 0_Γ an R-semigroup if there exists a composition $(x, \gamma) \rightarrowtail x\gamma$ of $R \times \Gamma$ satisfying: (i) $(x + y)\gamma = x\gamma + y\gamma$; (ii) $(xy)\gamma = x(y\gamma)$; and (iii) $0_R\gamma = 0_\Gamma$, for all $x, y \in R$ and $\gamma \in \Gamma$. An R-subsemigroup Δ of an R-semigroup Γ is a subsemigroup if $R\Delta \subset \Delta$. If Γ and Γ' are two R-semigroups of a near-semiring R, then the mapping $\omega : \Gamma \rightarrow \Gamma'$ is said to be an R-morphism between R-semigroups Γ and Γ' if it satisfies: (i) $\omega(x + y) = \omega(x) + \omega(y)$, (ii) $\omega(rx) = r\omega(x)$, for all $x, y \in \Gamma$, $r \in R$ and (iii) $\omega(0_\Gamma) = 0_{\Gamma'}$. We consult ([15, 20, 34]) for further about near-semirings and R-semigroups of near-semirings.

Fuzzy subset λ in a set R is the function $\lambda : R \rightarrow [0, 1]$ [35]. For any $t \in [0, 1]$, the set $R^t_\lambda = \{x \in R : \lambda(x) \geq t\}$ is said to be a level subset of a fuzzy set λ of R. If ζ and ξ are two fuzzy subsets of R then $(\zeta \wedge \xi)(a) = \zeta(a) \wedge (\zeta)(a) = min\{(\xi)(a), \zeta(a))\}$ and $(\xi \vee \zeta)(a) = \xi(a) \vee (\zeta)(a) = max\{\xi(a), \zeta(a)\}$.

Throughout by R we mean a near-semiring (or seminearring) unless otherwise specified, U the initial universe, E the set of parameters related to the entities in U, $\wp(U)$ denotes the family consists of the subsets of U, $\mathcal{F}(U)$ represents the fuzzy power set of U and a set $A \neq \emptyset$ is contained in E.

Definition 2. *[11]* Fuzzy subset ξ of a ring S is said to be a fuzzy subring of S if
(i) $\xi(a - b) > min\{\xi(a), \xi(b)\}$
(ii) $\xi(ab) > min\{\xi(a), \xi(b)\}$ for all $a, b \in S$.

Definition 3. *[1]* A fuzzy subset ζ of a near-ring N is a fuzzy subnear-ring of N if
(i) $\zeta(a - b) > min\ \{\zeta(a), \zeta(b)\}$
(ii) $\zeta(ab) > min\{\zeta(a), \zeta(b)\}$ for all $a, b \in N$.

Definition 4. *[26] Let η be the map from $A \rightarrow \wp(U)$. Then we call (η, A) is a soft set over U.*

Definition 5. *[23] Let F* be the mapping given by $F : A \rightarrow \mathcal{F}(U)$. *Then the pair* (F, A) is a fuzzy soft set over U.

The operation of intersection between two fuzzy soft sets was first introduced in [23] but for general case it was presented in [2].

Definition 6. *[2]* Intersection of the given two fuzzy soft sets represented by $(F, A) \tilde{\sqcap} (G, B)$ *is a fuzzy soft set (H, C)*, where $C = A \cap B \neq \emptyset$; $H(e) = F(e) \wedge G(e)$, $\forall\, e \in C$.

Definition 7. *[23]* (i) (F, A) is said to be a fuzzy soft subset of (G, B) over U, if $A \subset B$ and $F(a) \subset G(a)$ for every $a \in A$. We denote it by $(F, A) \tilde{\subseteq} (G, B)$.
(ii) Union of two fuzzy soft sets (F, A) and (G, B) is a fuzzy soft set (H, C) where $C = A \cup B$ defined by

$$H(x) = \begin{cases} F(x), & \text{if } x \in A - B \\ G(x), & \text{if } x \in B - A \\ F(x) \cup G(x), & \text{if } x \in A \cap B \end{cases}$$

We denote it by $(F, A) \tilde{\cup} (G, B) = (H, C)$.
(iii) The operation-*AND* between two fuzzy soft sets (F, A) AND (G, B) is symbolized by $(F, A) \tilde{\wedge} (G, B)$ and defined by $(F, A) \tilde{\wedge} (G, B) = (H, A \times B)$, where $H(a, b) = F(a) \cup G(b)$ for all $a, b \in (A \times B)$.
(iv) The operation-*OR* between two fuzzy soft sets (F, A) *OR* (G, B) is symbolized by $(F, A) \tilde{\vee} (G, B)$ and defined by $(F, A) \tilde{\vee} (G, B) = (H, A \times B)$, where $H(a, b) = F(a) \cap G(b)$ for all $a, b \in (A \times B)$.

Definition 8. *[14]* If (F, A) and (G, B) are two fuzzy soft sets then their cartesian product denoted by $(F, A) \tilde{\times} (G, B)$ *is the fuzzy soft set* (H, C) *where* $C = A \times B$ and $H(a, b) = F(a) \times G(b)$, $\forall\ (a, b) \in A \times B$.

3. Fuzzy subnear-semirings and fuzzy soft subnear-semirings

This section comprises of two subsections. In the first subsection, we initiate the concepts of fuzzy subnear-semirings while in the second subsection we introduce and discuss the notion of fuzzy soft subnear-semirings.

3.1. Fuzzy subnear-semirings

Fuzzy subnear-rings [1] and fuzzy semirings [4] have been introduced in the literature. We introduce the notion of fuzzy subnear-semirings and study few characterizations of the seminearrings (zero symmetric near-semiring) at the base of its fuzzy ideals. By an ideal of seminearrings, we mean S-ideal introduced in [5]. We mainly consult [5, 6] for the classical terminologies related to near-semirings (seminearrings). We begin with the definition.

Definition 9. Let λ be a fuzzy subset of a near-semiring R. *Then* λ is said to be a fuzzy subnear-semiring of R if *is fuzzy submonoid under* " + "and is a fuzzy subsemigroup under " · ".

Lemma 1. *A fuzzy subset λ is a fuzzy subnear-semiring of a near-semiring R if and only if,*
(*i*) $\lambda(x + y) \geq min(\lambda(x), \lambda(y))$ *and*
(*ii*) $\lambda(xy) \geq min(\lambda(x), \lambda(y))$ *for all x, y in R.*

Proof. Immediate. □

Example 1. Consider a right near-semiring $R = \{0, a, b, c, d\}$ with the operations tables given below.

Tables set 1

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

·	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	d
c	0	a	b	b	d
d	0	a	b	b	d

We define a fuzzy set $\lambda : R \rightarrow [0, 1]$ by $\lambda(0) = 0.9$, $\lambda(a) = 0.8$, $\lambda(b) = \lambda(d) = 0.6$, $\lambda(c) = 0.4$. Obviously, λ is a fuzzy subnear-semiring of R.

Definition 10. Let λ be a fuzzy subset of near-semiring R. Then λ is a fuzzy left (right) ideal of R if it satisfies the followings.
(*i*) $\lambda(a + b) \geq \min\{\lambda(a), \lambda(b)\}$
(*ii*) $\lambda(ab) \geq \lambda(b)$(resp., $\lambda(ab) \geq \lambda(a)$) for all $a, b \in R$.

Example 2. Let $R = \{0, a, b, c, d\}$ be a (left) near-semiring with the operations tables given below.

Tables set 2

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	d	d
d	d	d	d	d	d

·	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	b	b	b	b
c	0	a	a	a	a
d	0	a	a	a	a

Defining a fuzzy set $\lambda : R \rightarrow [0, 1]$ as $\lambda(0) = 0.9$, $\lambda(a) = 0.8$, $\lambda(b) = 0.7$, $\lambda(c) = 0.5$, $\lambda(d) = 0.7$. Clearly, λ is a fuzzy right ideal of R however λ is not a fuzzy left ideal as $\lambda(b \cdot a) \ngeq \lambda(a)$.

For each S-ideal I of a seminearring R, if $I = < I^2 >$ then R is said to be a strongly idempotent (or SI-seminearring) [5]. Similarly, if for each $a \in R$ there exist $b \in R$ such that $aba = a$, then R is said to be a regular seminearring [5]. Also, if R is a regular seminearring then it is a strongly idempotent with $I^2 = I$ for each S-ideal I of R [5]. Similarly, a fuzzy ideal λ of R is idempotent if $\lambda\lambda = \lambda^2 = \lambda$.

Theorem 1. *Let R be a regular seminearring. Then the following statements are equivalent.*
(1) *R is a strongly idempotent*
(2) *Each fuzzy S-ideal in R is a strongly idempotent*

Proof. (1) ⇒ (2) Assume that λ be a fuzzy S-ideal of a seminearring R. Then, for every $x \in I$,

$$\begin{aligned}
\lambda^2(x) &= (\lambda\lambda)(x)\\
&= \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [\bigwedge_{1\le i\le n} [\lambda(y_i)\wedge\lambda(z_i)]]\\
&\le \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [\bigwedge_{1\le i\le n} [\lambda(y_i z_i)\wedge\lambda(y_i z_i)]]\\
&= \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [[\bigwedge_{1\le i\le n} \lambda(y_i z_i)]\wedge[\bigwedge_{1\le i\le n} \lambda(y_i z_i)]]\\
&\le \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [\lambda(x)\wedge\lambda(x)]\\
&= \lambda(x).
\end{aligned}$$

since each ideal is idempotent which means $(x) = (x)^2\ \forall\ x \in R$. As $x \in (x) \Rightarrow x \in (x)^2 = x \in (x)^2 = RxRRxR$. Thus, $x = \sum_{i=1}^{n} r_i x_i r_i' s_i x s_i'$, where $r_i,\ s_i,\ r_i',\ s_i' \in R$. Consider, $\lambda(x) = \lambda(x) \wedge \lambda(x) \le \lambda(r_i x_i r_i') \wedge \lambda(s_i x_i s_i')$. Consequently, we have

$$\begin{aligned}
\lambda(x) &\le \bigwedge_{1\le i\le n} [\lambda(r_i x_i r_i') \wedge \lambda(s_i x_i s_i')]\\
&\le \bigvee_{x=\sum_{i=1}^{n} r_i x_i r_i' s_i x s_i'} [\bigwedge_{1\le i\le n} [\lambda(r_i x_i r_i') \wedge \lambda(s_i x_i s_i')]]\\
&\le \bigvee_{x=\sum_{j=1}^{n} y_j z_j} [\bigwedge_{1\le j\le n} [\lambda(y_j) \wedge \lambda(z_j)]]\\
&= (\lambda\lambda)(x)\\
&= \lambda^2(x).
\end{aligned}$$

Hence $\lambda^2 = \lambda$.

(2) ⇒ (1) Consider an S-ideal I of R and hence λ_I the characteristic function of I is the fuzzy S-ideal of R. Thus, $\lambda_I = (\lambda_I)^2$. Hence, $\lambda_I\lambda_I = \lambda_I \Rightarrow \lambda_{I^2} = \lambda_I$, it implies $I = I^2$. □

Theorem 2. *Let I be a left (right) S-ideal of a seminearring R. Then, for any $t \in (0, 1)$ there is a fuzzy left (right) S-ideal λ of R such that $\lambda_t = I$.*

Proof. Let us consider a fuzzy set $\lambda : R \to [0, 1]$ defined by

$$\lambda(a) = \begin{cases} t, & \text{if } a \in I\\ 0, & \text{if } a \notin I \end{cases}$$

where $t \in (0, 1)$. Then $\lambda_t = I$. Evidently, $\lambda(a + b) \ge \min\{\lambda(a), \lambda(b)\}$ and $\lambda(ab) \ge \lambda(b)$ (resp. $\lambda(ab) \ge \lambda(a)$) for all $a, b \in R$. □

It is well known that an element a of a ring S is a central idempotent if it is idempotent and also lying in the center $Z(S)$ of a ring S. We present the characterization of seminearring having central idempotents with respect to their fuzzy ideal as below.

Theorem 3. *Let R be a seminearring containing central idempotents. Then the following statements are equivalent.*
(1) *R is strongly idempotent*
(2) *For each fuzzy S-ideal ζ and fuzzy S-ideal ξ, $\zeta\xi = \zeta \wedge \xi$.*

Proof. (1) $\Rightarrow$ (2) Let us consider that ζ and ξ are fuzzy S-ideals of R. For every $x \in R$

$$\begin{aligned}(\zeta\xi)(x) &= \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [\bigwedge_{1\le i\le n} (\zeta(y_i) \wedge \xi(z_i))] \\ &\le \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [\bigwedge_{1\le i\le n} [\zeta(y_i z_i) \wedge \xi(y_i z_i)]] \\ &= \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [[\bigwedge_{1\le i\le n} \zeta(y_i z_i)] \wedge [\bigwedge_{1\le i\le n} \xi(y_i z_i)]] \\ &\le \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [[\zeta(x)] \wedge [\xi(x)]] \\ &= \zeta(x) \wedge \xi(x) = (\zeta \wedge \xi)(x).\end{aligned}$$

Again, since R is strongly idempotent i.e., $(x) = (x)^2$ for any $x \in R$. Hence, following the pattern of the proof of part (1) of Theorem1, we have $(\zeta \wedge \xi)(x) =$

$$\begin{aligned}(\zeta \wedge \xi)(x) &= \zeta(x) \wedge \xi(x) \\ &\le \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [\bigwedge_{1\le i\le n} (\zeta(y_i) \wedge \xi(z_i))] \\ &= \zeta\xi(x).\end{aligned}$$

Hence, $(\zeta \wedge \xi)(x) = \zeta\xi(x)$.

(2) $\Rightarrow$ (1) Let ζ and ξ be the two fuzzy S-ideals of R with $(\zeta \wedge \xi)(x) = \zeta\xi(x)$. By taking $\zeta = \xi$, we have $\zeta \wedge \zeta = \zeta^2$, where ζ is any fuzzy S-ideal of R. □

Remark 1. Let R be a seminearring with central idempotents. If R is regular then for each fuzzy left S-ideal λ and fuzzy right S-ideal μ, $\lambda \circ \mu = \lambda \wedge \mu$.

Let R be a right seminearring, by left distributive element of a right seminearring we mean an element $a \in R$ such that for all $x, y \in R$, we have $a(x + y) = ax + ay$. Similarly, right seminearring R is said to be a distributively generated (or d.g) if $(R, +)$ is generated by a multiplicative subsemigroup D of left distributive elements.

Theorem 4. *Let R be a distributively generated right seminearring. Then the followings are equivalent.*
(1) *R is strongly idempotent*
(2) *The set Ω consists of all fuzzy right S-ideals of R (ordered by $\le$) form a distributive lattice under operations of addition and intersection with $\xi \wedge \zeta = \xi\zeta$, where $\xi, \zeta \in \Omega$.*

Proof. (1) $\Rightarrow$ (2) Obviously, the set Ω is a lattice under the operations addition and intersection of fuzzy ideals. Since R is strongly idempotent seminearring, by Theorem 3 we have $\xi\zeta = \xi \wedge \zeta$ for any pair of fuzzy ideals ξ, ζ of R. Finally, it is remaining to show R is a distributive i.e., for fuzzy ideals ξ, ζ, and ρ of R, $[(\xi \wedge \zeta) + \rho] = [(\xi + \rho) \wedge (\zeta + \rho]$. So for this, let $x \in R$,

$$\begin{aligned}
[(\xi \wedge \zeta) + \rho](x) &= \bigvee_{x=y+z} [(\xi \wedge \zeta)(y) \wedge \rho(z)] \\
&= \bigvee_{x=y+z} [\xi(y) \wedge (\zeta)(y) \wedge \rho(z)] \\
&= \bigvee_{x=y+z} [\xi(y) \wedge (\zeta)(y) \wedge \rho(z) \wedge \rho(z)] \\
&= \bigvee_{x=y+z} [[\xi(y) \wedge \rho(z)] \wedge [(\zeta)(y) \wedge \rho(z)] \\
&\leq \bigvee_{x=y+z} [(\xi + \rho)(x) \wedge (\rho + \rho)(x)] \\
&= [(\xi + \rho) \wedge (\zeta + \rho)](x).
\end{aligned}$$

Alternatively,

$$\begin{aligned}
[(\xi + \rho) \wedge (\zeta + \rho)](x) &= [(\xi + \rho)(\zeta + \rho)](x) \\
&= \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [\bigwedge_{1\leq i\leq n} (\xi + \rho)(y_i)(\zeta + \rho)(z_i)] \\
&= \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [\bigwedge_{1\leq i\leq n} [\bigvee_{\substack{y_i = r_i s_i \\ z_i = t_i u_i}} [[(\xi(r_i) + \rho(s_i)] \wedge [(\zeta)(t_i) + \rho(u_i)]]] \\
&= \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [\bigwedge_{1\leq i\leq n} [\bigvee_{\substack{y_i = r_i s_i \\ z_i = t_i u_i}} [(\xi(r_i) \wedge \rho(s_i) \wedge \rho(s_i) \wedge \zeta(t_i) \wedge \rho(u_i)]] \\
&\leq \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [\bigwedge_{1\leq i\leq n} [\bigvee_{\substack{y_i = r_i s_i \\ z_i = t_i u_i}} [(\xi(r_i t_i) \wedge \zeta(r_i t_i) \wedge \rho(s_i t_i) \wedge \rho(s_i u_i) \wedge \rho(r_i u_i)]] \\
&\leq \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [\bigwedge_{1\leq i\leq n} [\bigvee_{\substack{y_i = r_i s_i \\ z_i = t_i u_i}} [(\xi \wedge \zeta)(r_i t_i) \wedge \rho(s_i t_i + s_i u_i + r_i u_i)]] \\
&\leq \bigvee_{x=\sum_{i=1}^{n} y_i z_i} [\bigwedge_{1\leq i\leq n} [(\xi \wedge \zeta) + \rho)(y_i z_i)]] \\
&= [(\xi \wedge \zeta) + \rho)(x)].
\end{aligned}$$

Hence $[(\xi + \rho) \wedge (\zeta + \rho)](x) = [(\xi \wedge \zeta) + \rho](x)$.

(2) ⇒ (1) Suppose (2) holds true. Then, for any fuzzy S-ideal of R, we have $\xi^2 = \xi = \xi \wedge \xi = glb\{\xi, \xi\} = \xi$. Hence R is strongly idempotent. □

Following [6], an ideal I of a distributively generated seminearring R is a direct summand of R if there exists an ideal I' which we call a cosummand of I and that $I + I' = R$ and $I \cap I' = (0)$.

Theorem 5. *Let R be a distributively generated and strongly idempotent right seminearring. Then, any pair of fuzzy right S-ideals λ and λ' such that $\lambda + \lambda' = 1$, $\lambda \wedge \lambda' = 0$ form a boolean sublattice of R under the sum and intersection of fuzzy right S-ideals.*

Proof. Result follows from Theorem4. □

Definition 11. *[6]* An S-ideal P of a seminearring R is a prime S-ideal if $IJ \subset P \Rightarrow I \subset P$ or $J \subset P$. Similarly, an S-ideal P is a strongly irreducible S-ideal if $I \cap J \subset P \Rightarrow I \subset P$ or $J \subset P$.

In the setting of fuzzy theory, we introduce the definition of fuzzy prime S-ideal.

Definition 12. A fuzzy S-ideal γ of a seminearring R is called a fuzzy prime S-ideal if for fuzzy S-ideals ζ, ξ of R, $\zeta\xi \leq \gamma \Rightarrow \zeta \leq \gamma, \xi \leq \gamma$. Similarly, γ is a fuzzy strongly irreducible S-ideal if $\zeta \wedge \xi \leq \gamma \Rightarrow \zeta \leq \gamma, \xi \leq \gamma$.

Theorem 6. *Let R be a strongly idempotent seminearring and λ be its any fuzzy S-ideal. Then the followings are equivalent.*
(1) γ *is a fuzzy prime S-ideal*
(2) γ *is a fuzzy strongly irreducible S-ideal*

Proof. (1) $\Rightarrow$ (2) Let γ be a fuzzy prime S-ideal. We have to prove that γ is fuzzy strongly irreducible i.e., there exist fuzzy S-ideals ζ and ξ such that $\zeta \wedge \xi \leq \gamma \Rightarrow \zeta \leq \gamma, \xi \leq \gamma$. Since $\zeta\xi \leq \gamma \Rightarrow \zeta \leq \gamma$, $\xi \leq \gamma$ and R is strongly idempotent seminearring, it follows from Theorem 4, $\gamma \wedge \zeta = \gamma\zeta$. Thus, $\zeta\xi \leq \gamma \Rightarrow \zeta \wedge \xi \leq \gamma$. Since R is a strongly idempotent then the set of fuzzy S-ideals of R (ordered by $\leq$) is a distributive lattice under addition and intersection of fuzzy S-ideals by Theorem 4. Hence, $\zeta \wedge \xi \leq \gamma \Rightarrow (\zeta \wedge \xi) + \gamma = \gamma$, and by using distributivity we have, $\gamma = (\zeta \wedge \xi) + \gamma = (\zeta + \gamma) \wedge (\xi + \gamma)$. Since γ is a fuzzy prime S-ideal it implies that either $\zeta + \gamma = \gamma$ or $\xi + \gamma = \gamma$, which implies $\zeta \leq \gamma$ or $\xi \leq \gamma$. Hence γ is a fuzzy strongly irreducible S-ideal.

(2) $\Rightarrow$ (1) Let γ be a fuzzy strongly irreducible S-ideal. We have to prove that γ is fuzzy prime i.e., there exist fuzzy S-ideals ζ and ξ such that $\zeta\xi \leq \gamma \Rightarrow \zeta \leq \gamma, \xi \leq \gamma$. Since $\zeta \wedge \xi \leq \gamma \Rightarrow \zeta \leq \gamma, \xi \leq \gamma$. Also R is a strongly idempotent then the set of fuzzy S-ideals of R are ordered by $\leq$ and is a distributive lattice with $\gamma \wedge \zeta = \gamma\zeta$ by Theorem 4. Thus, $\zeta \wedge \xi \leq \gamma \Rightarrow \zeta\xi \leq \gamma \Rightarrow \zeta \leq \gamma, \xi \leq \gamma$. □

Fuzzy R-subsemigroup

We continue our study parallel to the classical substructures of near-semirings. We define fuzzy R-subsemigroup of an R-semigroup of seminearring M as follows.

Definition 13. A fuzzy subset μ of R-semigroup M is said to be a fuzzy left (resp., right) R-subsemigroup of M if
(i) μ is a fuzzy subsemigroup of M
(ii) $\mu(rx) \geq \mu(x)$ (resp., $\mu(xr) \geq \mu(x)$) for all $x, \in M$ *and* $r \in R$. *If* μ is both left and right fuzzy R-subsemigroup of M then we call it a fuzzy R-subsemigroup of M.

Example 3. Let us consider a near-semiring presented in example*1*. Consider a fuzzy subset μ of R-semigroup $M = \{0, a, b, d\}$ defined by, $\mu(0) = 0.9$, $\mu(a) = 0.8$, $\mu(c) = 0.5$, $\mu(b) = 0.7$, $\mu(d) = 0.7$. One can easily verify that μ is a fuzzy R-subsemigroup of R-semigroup M.

Remark 2. Let $(R, +)$ be an R-semigroup of a near-semiring $(R, +, .)$. Then, the fuzzy R-subsemigroup of $(R, +)$ will be the fuzzy ideal of near-semiring R.

Theorem 7. *Let M and N are two R-semigroups and $f : M \to N$ be an R-epimorphism. Let μ be a fuzzy R-subsemigroup of M, then $f(\mu)$ is a fuzzy R-subsemigroup of N.*

Proof. Let $x, y \in M$ and $x', y' \in N$ such that $f(x) = x'$, $f(y) = y'$. Also μ be a fuzzy R-subsemigroup of M, Then

$$\begin{aligned}
[f(\mu)](x' + y') &= \bigvee_{f(z)=x'+y'} \mu(z) \\
&= \bigvee_{f(x+y)=x'+y'} \mu(x + y) \\
&\geq \bigvee_{f(x)=x',\ f(y)=y'} [\mu(x) \wedge \mu(y)] \\
&\geq [\bigvee_{f(x)=x'} \mu(x)] \wedge [\bigvee_{f(y)=y'} \mu(y)] \\
&\geq [f(\mu)](x') \wedge [f(\mu)](y').
\end{aligned}$$

Let $x' \in N$, $r \in R$, then there exists $x \in M$ such that $f(x) = x'$ and $f(rx) = rx'$. Then,

$$\begin{aligned}
[f(\mu)](rx') &= \bigvee_{f(x)=rx'} \mu(x) \\
&= \bigvee_{f(rx)=rx'} \mu(rx) \\
&= \bigvee_{rf(x)=rx'} \mu(rx) \\
&\geq \bigvee_{f(x)=x'} \mu(x) \\
&= [f(\mu)](x').
\end{aligned}$$

Hence, $f(\mu)$ is a fuzzy R-subsemigroup of N. □

Proposition 1. *Let R be a seminearring. Then, K is a left (resp., right) R-subsemigroup of an R-semigroup M if and only if λ_K (characteristics function) is a fuzzy left (resp., right) R-subsemigroup of M.*

Proof. Straightforward. □

In order to study relationship between two fuzzy R-subsemigroups of nearsemirings, we introduce the notion of fuzzy R-homomorphism.

Definition 14. Let R be a near-semiring with H and K are both R-subsemigroups. A fuzzy mapping f from H to K i.e., $f : H \to K$ is said to be a fuzzy R-homomorphism if

(i) $f(h_1 + h_2, k) \geq \sup\{\inf\{f(h_1, k_1), f(h_2, k_2)\} : k = k_1 + k_2; k_1, k_2 \in K\}\}$

(ii) $f(nh, k) \geq \sup\{\inf\{ f(n, k_1), f(h, k_2)\} : k = k_1k_2, k_1, k_2 \in K, n, h \in H\}$.

Lemma 2. *Let H and K be the two R-semigroups of a near-semiring R. Let μ be a fuzzy subset of an R-semigroup H and let f be an R-homomorphism from H onto K. Now, if μ is a fuzzy R-subsemigroup of H then $f(\mu)$ is a fuzzy R-subsemigroup of K.*

3.2. Fuzzy soft subnear-semirings

Fuzzy soft subnear-rings [27] and fuzzy soft semirings [29] have been discussed in the literature. In this sub-section, we present and explain in detail the notions of fuzzy soft near-semirings, fuzzy soft subnear-semirings, fuzzy soft ideals of near-semirings and fuzzy soft R-subsemigroups of near-semirings R.

Definition 15. A fuzzy soft set (f, A) over R is a fuzzy subnear-semiring of R if $f(x)$ is a fuzzy subnear-semiring of R for all $x \in A$.

We may redefine fuzzy soft subnear-semiring as follows.

Definition 16. A non-null fuzzy soft set over R (f, A) is said to be a fuzzy subnear-semiring of R if $f(a) = f_a$ for each $a \in A$, that is, $f_a(x + y) \geq f_a(x) \wedge f_a(y)$, $f_a(xy) \geq f_a(x) \wedge f_a(y)$ for all $x, y \in R$.

Example 4. Let $R = \{0, a, b, c, d\}$ be a (right) near-semiring with the operations tables given in example*1*. Let $A = \{e_1, e_2, e_3\}$ and $f : A \to R$ be the set-valued function defined as follows.

$$\begin{aligned} f(e_1) &= \{(0,\ 0.9),\ (a,\ 0.8),\ (b,\ 0.6),\ (c,\ 0.4),\ (d,\ 0.6)\} \\ f(e_2) &= \{(0,\ 0.8),\ (a,\ 0.5),\ (b,\ 0.3),\ (c,\ 0.1),\ (d,\ 0.3)\} \\ f(e_3) &= \{(0,\ 0.9),\ (a,\ 0.6),\ (b,\ 0.5),\ (c,\ 0.4),\ (d,\ 0.5)\} \end{aligned}$$

Here (f, A) is a fuzzy soft set over R which is also a fuzzy subnear-semiring of R for all $x \in A$. Hence, (f, A) is a fuzzy soft near-semiring over R.

Theorem 8. *Let (f, A) and (g, B) are both the fuzzy soft (right) near-semirings over R. Then $(f, A) \tilde{\wedge} (g, B)$ is a fuzzy soft (right) near-semiring over R.*

Proof. Since $(f, A) \tilde{\wedge} (g, B) = (h, C)$, where $C = A \times B$, and $h : C \to (R)$ is a mapping $h(x, y) = f(x) \cap g(y)$ for all $(x, y) \in C$. As $f(x)$ and $g(y)$ are fuzzy subnear-semirings of R, $f(x) \cap g(y)$ is a fuzzy subring of R. Hence $h(x, y) = f(x) \cap g(y)$ for all $(x, y) \in C$. Therefore, $(h, C) = (f, A) \tilde{\wedge} (g, B)$ is a fuzzy soft near-semiring over R. □

Example 5. Let $R = \{0, a, b, c, d\}$ be a (right) near-semiring with the operations tables given in example*1*. Let $A = \{e_1, e_3\}$, $B = \{e_1, e_2\}$ and $f : A \to (R)$, $g : B \to (R)$ are the corresponding set-valued functions defined as follows.

$$\begin{aligned} f(e_1) &= \{(0,\ 0.7),\ (a,\ 0.6),\ (b,\ 0.4),\ (c,\ 0.2),\ (d,\ 0.4)\} \\ f(e_3) &= \{(0,\ 0.8),\ (a,\ 0.5),\ (b,\ 0.2),\ (c,\ 0.1),\ (d,\ 0.2)\} \\ &\text{and} \\ g(e_1) &= \{(0,\ 0.5),\ (a,\ 0.4),\ (b,\ 0.3),\ (c,\ 0.2),\ (d,\ 0.3)\} \\ g(e_2) &= \{(0,\ 0.4),\ (a,\ 0.3),\ (b,\ 0.2),\ (c,\ 0.1),\ (d,\ 0.2)\} \end{aligned}$$

Clearly, (f, A) and (g, B) are fuzzy soft sets over R. We also observe that $f(x)$ and $g(x)$ are fuzzy subnear-semirings of R for all $x \in A, B$. Hence, (f, A) and (g, B) are fuzzy soft near-semirings over R. Consider $(f, A) \tilde{\wedge} (g, B) = (h, C)$ where $C = A \times B$ and $h : C \to (R)$ defined as $h(x, y) = f(x) \cap g(y)$.

We have,

$$\begin{aligned} h(e_1, e_1) &= \{(0, 0.5), (a, 0.4), (b, 0.3), (c, 0.2), (d, 0.3)\} \\ h(e_1, e_2) &= \{(0, 0.4), (a, 0.3), (b, 0.2), (c, 0.1), (d, 0.2)\} \\ h(e_3, e_1) &= \{(0, 0.5), (a, 0.4), (b, 0.2), (c, 0.1), (d, 0.2)\} \\ h(e_3, e_2) &= \{(0, 0.4), (a, 0.3), (b, 0.2), (c, 0.1), (d, 0.2)\} \end{aligned}$$

(h, C) is a fuzzy soft set over R. We also observe that $h(x, y)$ is a fuzzy subnear-semiring of R, for all $x \in A$, $y \in B$. Hence, $(f, A) \tilde{\wedge} (g, B) = (h, C)$ is a fuzzy soft near-semiring over R.

Proposition 2. *Let $(f_i, A_i)_{i\in I}$ be the non-empty family of fuzzy soft near-semirings over a near-semiring R. Then $\tilde{\wedge}_{i\in I}(f_i, A_i)$ is a fuzzy soft near-semiring over R.*

Proof. Since $\tilde{\wedge}_{i\in I}(f_i, A_i) = (g, B)$, where $B = \prod\limits_{i\in I} A_i$, and $g(x) = \cap_{i\in I} f_i\ (x_i)$ for all $x = (x_i)_{i\in I} \in B$. Thus, the non-empty set $f_i(x_i)$ is a fuzzy subnear-semiring of R for all $i \in I$. Hence, $g(x)$ is a fuzzy subnear-semiring of R, and $\tilde{\wedge}_{i\in I}\ (\zeta_i, A_i) = (\eta, B)$ is a fuzzy subnear-semiring of R. Finally, $\tilde{\wedge}_{i\in I}\ (f_i, A_i) = (g, B)$ is a fuzzy soft near-semiring over R. □

Now we are ready to introduce the notions of fuzzy soft subnear-semirings and fuzzy soft ideals.

Definition 17. Let (f, A) and (g, B) be the two non-null fuzzy soft set over R. Then, (g, B) is a fuzzy soft subnear-semiring of (f, A) if
(i) $B \subset A$
(ii) $g(x)$ is a fuzzy subnear-semiring of $f(x)$, for all $x \in B$.
It is denoted by $(g, B) \tilde{\leq} (f, A)$.

Example 6. Let $R = \{0, a, b\}$ be a (right) near-semiring with the operations tables given below.

Tables set 3

+	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

.	0	a	b
0	0	a	b
a	a	a	b
b	b	a	b

Let $A = \{e_1, e_2, e_3\}$ and $f : A \to (R)$ be the set-valued function defined as follows.

$$\begin{aligned} f(e_1) &= \{(0,\ 0.9),\ (a,\ 0.8),\ (b,\ 0.7)\} \\ f(e_2) &= \{(0,\ 0.9),\ (a,\ 0.8),\ (b,\ 0.4)\} \\ f(e_3) &= \{(0,\ 0.8),\ (a,\ 0.7),\ (b,\ 0.7)\} \end{aligned}$$

Obviously, (f, A) is a fuzzy soft set over R, and is easy to verify that $f(x)$ is a fuzzy subnear-semiring of R for all $x \in A$. Hence (f, A) is a fuzzy soft near-semiring over R. Again consider $B = \{e_1, e_3\}$ and $g : B \to (R)$ be the set-valued function defined as follows.

$$\begin{aligned} g(e_1) &= \{(0,\ 0.8),\ (a,\ 0.6),\ (b,\ 0.4)\} \\ g(e_3) &= \{(0,\ 0.5),\ (a,\ 0.3),\ (b,\ 0.3)\} \end{aligned}$$

Clearly, (g, B) is a fuzzy soft subnear-semiring of a fuzzy soft near-semiring (f, A) i.e., $(g, B) \tilde{\leq} (f, A)$.

After introducing fuzzy soft subnear-semiring we present the definition of fuzzy soft ideal of near-semiring.

Definition 18. Let (f, A) be a fuzzy soft near-semiring over R. A fuzzy soft set (g, I) over R is called a fuzzy soft ideal of (f, A), if it satisfies the following conditions.
(i) $I \subset A$
(ii) $g(x)$ is a fuzzy ideal of $f(x)$ for all $x \in I$.
We will denote it by $(g, I) \widetilde{\trianglelefteq} (f, A)$.

Example 7. We consider a fuzzy soft near-semiring described in Example 4. Let $I = \{e_1, e_3\}$ and a map $g : I \to (R)$ defined by

$$\begin{aligned} g(e_1) &= \{(0,\ 0.7),\ (a,\ 0.6),\ (b,\ 0.4),\ (c,\ 0.2),\ (d,\ 0.4)\} \\ g(e_3) &= \{(0,\ 0.8),\ (a,\ 0.5),\ (b,\ 0.2),\ (c,\ 0.1),\ (d,\ 0.2)\}. \end{aligned}$$

Then, (g, I) is a fuzzy soft set and it is easy to verify that $g(x)$ is a fuzzy ideal of $f(x)$ for all $x \in B$. Hence, (g, I) is a fuzzy soft ideal of (f, A), that is $(g, I)\widetilde{\trianglelefteq}(f, A)$.

Remark 3. Every fuzzy soft ideal of a fuzzy soft near-semiring is a fuzzy soft subnear-semiring but the converse doesn't hold true. For instance in Example 6, (g, B) is a fuzzy soft subnear-semiring of (f, A). One can easily verify by doing simple calculations that (g, B) is not a fuzzy soft ideal of a fuzzy soft near-semiring (f, A).

Example 8. Let $R = \{0, a, b, c, d\}$ be a (right) near-semiring with the operations tables given in Example 2. Let $A = \{e_1, e_2, e_3\}$ and $f : A \to R$ be the set-valued function defined as follows.

$$\begin{aligned} f(e_1) &= \{(0,\ 0.9),\ (a,\ 0.8),\ (b,\ 0.6),\ (c,\ 0.4),\ (d,\ 0.6)\} \\ f(e_2) &= \{(0,\ 0.8),\ (a,\ 0.5),\ (b,\ 0.3),\ (c,\ 0.1),\ (d,\ 0.3)\} \\ f(e_3) &= \{(0,\ 0.9),\ (a,\ 0.6),\ (b,\ 0.5),\ (c,\ 0.4),\ (d,\ 0.5)\} \end{aligned}$$

Clearly, (f, A) is a fuzzy soft set over R. We also observe that $f(x)$ is a fuzzy subnear-semiring of R for all $x \in A$. Hence, (f, A) is a fuzzy soft near-semiring over R. Let $I = \{e_1\}$ and a mapping $g : I \to (R)$ defined by$g(e_1) = \{(0, 0.8), (a, 0.7), (b, 0.4), (c, 0.2), (d, 0.4)\}$. Obviously, (g, I) is a fuzzy soft set over R. We see that $g(x)$ is a fuzzy subnear-semiring of R for all $x \in I$. Hence, (g, I) is a fuzzy soft near-semiring over R. Finally, $I \subset A$ and $g(x)$ is a fuzzy right ideal of $f(x)$ for all $x \in I$. Thus, (g, I) is a fuzzy right ideal of (f, A).

Let S is a non-empty subset of a near-semiring R, the characteristic function on S is the function $\chi_S : R \to [0, 1]$ defined by $\chi_S(x) = 1$, if $x \in S$ and $\chi_S(x) = 0$, if $x \notin S$. Similarly, A fuzzy soft characteristic function (f^S, A) over R is defined by $(f^S)_{e_1} = \chi_S$ (i.e., $(f^S)(e_1) = \chi_S$) for all $e_i \in A$.

Lemma 3. *Let I be a subset of a near-semiring R and (f^I, A) be a fuzzy soft characteristic function over R. Then, I is left (resp., right) ideal of R if and only if (α^I, A) is a fuzzy soft left (resp., right) ideal over R.*

Proof. The proof is easy and hence omitted. □

Fuzzy soft R-subsemigroup and fuzzy soft R-morphism

We introduce the notion of fuzzy soft R-subsemigroup of a near-semiring. We also shift the terminology of R-morphism of classical near-semiring theory towards fuzzy soft set theory. A subsemigroup H of R is called a left R-subsemigroup of R if $RH \subseteq H$ and H is called a right R-subsemigroup of R if $HR \subseteq H$. We begin with the definition.

Definition 19. Let M be an R-semigroup of a near-semiring R. A fuzzy soft set (F, A) over M is called a fuzzy soft left (resp., right) R-subsemigroup over M if $F(x)$ is a fuzzy left (resp., right) R-subsemigroup of M for all $x \in A$. We will denote a fuzzy soft left (resp., right) R-subsemigroup over M by $F_A <^l_M M$ (resp., $F_A <^r_M M$).

Example 9. Let us consider a near-semiring R with the operations tables given in example*1*. Let $M = \{0, a, b, d\}$ be a right R-semigroup of a near-semiring R. We define a fuzzy set $\mu : R \to [0, 1]$ by $\mu(c) \leq \mu(b) = \mu(d) \leq \mu(a) \leq \mu(0)$. Let $A = \{e_1, e_2, e_3\}$ and $F : A \to \mu(M)$ be the set-valued function defined by

$$\begin{aligned} F(e_1) &= \{(0,\ 0.9),\ (a,\ 0.8),\ (b,\ 0.6),\ (d, 0.6)\} \\ F(e_2) &= \{(0, 0.8),\ (a,\ 0.5),\ (b,\ 0.3),\ (d, 0.3)\} \\ F(e_3) &= \{(0,\ 0.9), (a,\ 0.6),\ (b,\ 0.5),\ (d,\ 0.5)\}. \end{aligned}$$

One can easily check that it is a fuzzy soft right R-subsemigroup over M.

Remark 4. Every fuzzy soft left (resp., right) ideal of a near-semiring R is a fuzzy soft left (resp., right) R-subsemigroup over R.

Definition 20. Let M and N be the two left (resp., right) R-semigroups of a near-semiring R. Assume that$F_A <^l_M M$ (resp., $F_A <^r_M M$) and $G_B <^l_N N$ (resp., $G_B <^r_N N$). Then, their cartesian product defined as $F_A \tilde{\times} G_B = H_{A\times B}$, where $H_{A\times B}(x, y) = F_A(x) \tilde{\times} G_B(y)$ for all $(x, y) \in A\times B$ is a fuzzy soft left (resp., right) R-subsemigroup over $M \times N$.

Theorem 9. *If $F_A <^l_M M$(resp., ($F_A <^r_M M$) and $G_B <^l_N N$ (resp., $G_B <^r_N N$), then $F_A \tilde{\times} G_B <^l_{M\times N} M\times N$ (resp., $F_A \tilde{\times} G_B <^r_{M\times N} M\times N$).*

Proof. Let $F_A <^l_M M$ and $G_B <^l_N N$. Since M and N are left R-semigroups, it is easily seen that $M\times N$ is a left R-semigroup of $R \times R$. Since, $F_A \tilde{\times} G_B = H_{A\times B}$, where $H_{A\times B}(x) = F_A(x) \tilde{\times} G_B(y)$ for all $(x, y) \in A \times B$. Then, for all $(x_1, y_1), (x_2, y_2) \in A \times B$, $(r_1, r_2) \in R\times R$ we have,

$$\begin{aligned} (i)\ H_{A\times B}((x_1,\ y_1) + (x_2,\ y_2)) &= H_{A\times B}(x_1 + x_2,\ y_1 + y_2) \\ &= F_A(x_1 + x_2) \times G_B(y_1 + y_2) \\ &\supseteq (F_A(x_1) \cap F_A(x_2)) \times (G_B(y_1) \cap G_B(y_2)) \\ &= (F_A(x_1) \times G_B(y_1)) \cap (F_A(x_2) \times G_B(y_2)) \\ &= H_{A\times B}(x_1,\ y_1) \cap H_{A\times B}(x_2,\ y_2) \\ (ii)\ H_{A\times B}((r_1,\ r_2)(x_1,\ y_1)) &= H_{A\times B}(r_1x_1,\ r_2y_1) = F_A(r_1x_1) \times G_B(r_2y_1) \\ &\supseteq F_A(x_1) \times G_B(y_1) \\ &= H_{A\times B}(x_1,\ y_1) \end{aligned}$$

Hence, $F_A \tilde{\times} G_B <^l_{M\times N} M\times N$. Similarly, one can prove $F_A \tilde{\times} G_B <^r_{M\times N} M\times N$. □

In a sequel, we introduce fuzzy soft R-homomorphism between two fuzzy soft R-subsemigroups with an illustrative example.

Definition 21. Let R_1 and R_2 be near-semirings. Let (F, A) be a fuzzy soft left (resp., right) R_1-subsemigroup over R_1-semigroup M and (G, B) be a fuzzy soft left (resp., right) R_2-semigroup over R_2-semi-group N. Let $f : M \to N$ and $g : A \to B$ be the two mappings. Then, the pair (f, g) is called a fuzzy soft R-homomorphism if it satisfies the following conditions.
(i) f is an R-epimorphism
(ii) g is a surjective mapping
(iii) $f(F(x)) = G(g(x))$ for all $x \in A$.

We say that (F, A) is fuzzy soft R-homomorphic to (G, B), If there exists a fuzzy soft R-homomorphism between (F, A) and (G, B). Similarly, if f is an R-isomorphism of R-subemigroups and g is a bijective mapping then we call (f, g) a fuzzy soft R-isomorphism, we say that (F, A) is a fuzzy soft R-isomorphic to (G, B) denoted by $(F, A) \simeq_R (G, B)$.

Example 10. Let us take the near-semiring $R = \{0, a, b, c, d\}$ defined in example*1*. Let (F, A) be a fuzzy soft left (resp., right) R_1-subsemigroup over an R_1-semigroup M and (G, B) be a fuzzy soft left (resp., right) R_2-subsemigroup over an R_2-semigroup N. We assume that $M = N = \{0, a, b\}$ and let $f : M \to N$ be the map defined by $f(x) = x$. One can easily verify that f is an R-isomorphism. Let $A = B = \{e_1, e_2, e_3\}$ and a map $g : A \to B$ defined by $g(x) = x$ is clearly a bijective map. Let $F : A \to \wp(M)$ be the map defined by

$$\begin{aligned} F(e_1) &= \{(0,\ 0.9), (a,\ 0.7),\ (b,\ 0.6)\} \\ F(e_2) &= \{(0, 0.8),\ (a,\ 0.5),\ (b,\ 0.3)\} \\ F(e_3) &= \{(0,\ 0.9), (a,\ 0.6),\ (b,\ 0.5)\}. \end{aligned}$$

Then, (F, A) is a fuzzy soft R-subsemigroup over R-semigroup M. Again, let $B = \{e_1, e_2, e_3\}$ and the map $G : B \to \wp(N)$ defined by

$$\begin{aligned} G(e_1) &= \{(0,\ 0.9), (a,\ 0.7),\ (b,\ 0.6)\} \\ G(e_2) &= \{(0,\ 0.8),\ (a,\ 0.5),\ (b,\ 0.3)\} \\ G(e_3) &= \{(0,\ 0.9), (a,\ 0.6),\ (b,\ 0.5)\}. \end{aligned}$$

(G, B) is also a fuzzy soft R-subsemigroup over R-semigroup N. Finally, consider $f(F(x)) = G(g(x))$ for all $x \in A$. Hence $(F, A) \simeq_R (G, B)$.

Theorem 10. *Let M, N be two R-subsemigroups of a near-semiring R. Let F_A and G_B be the fuzzy soft sets over M and N, respectively. Let Ψ be a fuzzy soft R-isomorphism from M to N. If F_A is a fuzzy soft R-subsemigroup over M, then $\Psi(F_A)$ is a fuzzy soft R-subsemigroup over N.*

Proof. Let $n_1, n_2 \in N$. Since, Ψ is surjective there exists $m_1, m_2 \in M$ such that $\Psi(m_1) = n_1$ and $\Psi(m_2) = n_2$. Thus,

$$
\begin{aligned}
\Psi((F_A)(n_1+n_2)) &= \bigvee\{F_M(m) : m \in M,\ \Psi(m) = (n_1+n_2)\} \\
&= \bigvee\{F_M(m) : m \in M,\ m = \Psi^{-1}(n_1+n_2\} \\
&= \bigvee\{F_M(m) : m \in M,\ m = \Psi^{-1}(\Psi(m_1+\ m_2)) = m_1+\ m_2\} \\
&= \bigvee\{F_M(m_1+m_2) :\ n_i \in N,\ \Psi(m_i) = n_i,\ i = 1,\ 2\} \\
&\supseteq \bigvee\{F(m_1) \cap F(m_2) :\ m_i \in M,\ \Psi(m_i) = n_i,\ i = 1,\ 2\} \\
&= (\bigvee\{F(m_1) : n_1 \in N,\ \Psi(m_1) = n_1\}) \wedge (\bigvee\{F(m_2) : n_2 \in N,\ \Psi(m_2) = n_2\}) \\
&= (\Psi(F_M))(m_1) \wedge (\Psi(F_M))(m_2).
\end{aligned}
$$

Now, let $r \in R$ and $n \in N$. Since, Ψ is surjective, there exists $m' \in M$ such that $\Psi(m') = n$. Then,

$$
\begin{aligned}
(\Psi(F_M)(rn) &= \bigvee\{F_M(m) : m \in M,\ \Psi(m) = rn\} \\
&= \bigvee\{F_M(m) : m \in M,\ m = \Psi^{-1}(rn)\} \\
&= \bigvee\{F_M(m) : m \in M,\ m = \Psi^{-1}(r\Psi(m'))\} \\
&= \bigvee\{F_M(m) : m \in M,\ m = \Psi^{-1}(\Psi(rm')) = rm'\} \\
&= \bigvee\{F_M(rm') : m' \in M,\ \Psi(m') = n\} \\
&\supseteq \bigvee\{F_M(m') : m' \in M, \Psi(m') = n\} \\
&= (\Psi(F_M))(n).
\end{aligned}
$$

Hence, $\Psi(F_M)$ is a fuzzy soft R-subsemigroup of an R-semigroup N. □

Fuzzy soft anti-homomorphism of near-semirings

Here, we introduce the notion of anti-homomorphism of near-semirings in the setting of fuzzy soft set theory. We start with the definition.

Definition 22. Let (F, A) be a soft right near-semiring over the right near-semiring R_1 and (G, B) be the soft left near-semiring over the left near-semiring R_2. Let $f : R_1 \to R_2$ and $g : A \to B$ be the two mappings. Then, the pair (f, g) is called a fuzzy soft near-semiring anti-homomorphism if it satisfies the following conditions.
(i) f is an anti-epimorphism of near-semirings
(ii) g is a surjective mapping.
(iii) $f(F(x)) = G(g(x))$ for all $x \in A$.

If there exists a fuzzy soft near-semiring anti-homomorphism between (F, A) and (G, B), we say that (F, A) is soft anti-homomorphic to (G, B). If f is an anti-isomorphism of near-semirings and g is a bijective mapping between A and B then (F, A) and (G, B) are soft anti-isomorphic under the fuzzy soft anti-isomorphism (f, g), we will write it as $(F, A) \simeq (G, B)$.

Theorem 11. *Fuzzy soft anti-homomorphic image of fuzzy soft right* (*left*) *near-semiring is a fuzzy soft left* (*right*) *near-semiring.*

Example 11. Consider a right near-semiring R_1 defined in the below tables set 4 and left near-semiring R_2 defined in the tables set 5.

Tables set 4

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	d
c	0	a	b	b	d
d	0	a	b	b	d

Tables set 5

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	b	b
d	0	a	d	d	d

Let $A = \{e_1, e_2\}$ and $F : A \to (R_1)$ be the set-valued function defined by $F(e_1) = \{(0, 0.9)\ (a, 0.7), (b, 0.5)\ (c, 0.3)\ (d, 0.5)\}$ and $F(e_2) = \{(0, 0.8)\ (a, 0.6), (b, 0.4)\ (c, 0.2)\ (d, 0.4)\}$. Clearly, (F, A) is a fuzzy soft right near-semiring over a right near-semiring R_1. Again, consider $B = \{e'_1, e'_2\}$ and $G : B \to (R_2)$ defined by $G(e'_1) = \{(0, 0.9)\ (a, 0.7), (b, 0.5)\ (c, 0.3)\ (d, 0.5)\}$ and $G(e'_2) = \{(0, 0.8)\ (a, 0.6), (b, 0.4)\ (c, 0.2)\ (d, 0.4)\}$. Clearly, (G, B) is a fuzzy soft left near-semiring over a left near-semiring R_2. The map $f : R_1 \to R_2$ defined by $f(x) = x$ is an anti-epimorphism of near-semirings, which is easy to verify that $f(b.d)= f(d).f(b) = d$. However, we see that $f(b.d) = d \neq f(b).f(d) = b$, so f is purely an anti-homomorphism. And $g : A \to B$ defined by $g(x) = x$ is a surjective map. Finally, one can easily prove that $f(F(x)) = G(g(x))$ for all $x \in A$. Hence, fuzzy soft right near-semiring (F, A) is soft anti-isomorphic to fuzzy soft left near-semiring (G, B).

Remark 5. Fuzzy soft anti-homomorphic image of fuzzy soft right (left) ideal of near-semiring is a fuzzy soft left (right) ideal.

4. Conclusions

In this manuscript, we have introduced and discussed the notions of fuzzy subnear-semirings and fuzzy soft subnear-semirings along with their substructures. We also presented few of their characterizations. For the purpose of reaching the highest possible level of accurateness in this pursue, a different types of operations are applied to these structures. Classical terms of R-homomorphisms and R-antihomomorphisms are also shifted towards fuzzy (soft) R-homomorphisms and fuzzy soft R-antihomomorphism of fuzzy soft near-semirings. As a result, we developed some relationships among fuzzy (-soft) substructures of near-semirings under these maps.

In due course, we examined some properties of near-semirings (seminearrings) at the base of its newly established fuzzy (soft) substructures such as if R be a distributively generated strongly idempotent right seminearring then the collection of all fuzzy right ideals form a lattice under the operations of addition and intersection. In the same setting, one can further discuss the balanced near-semirings and affine near-semirings on the ground of fuzzy (soft) set theory. Furthermore, our developed structures and their corresponding results can be referred for future work towards intuitionistic fuzzy soft theory by introducing intuitionistic fuzzy soft near-semirings.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. S. Abou Zaid, On fuzzy subnear-rings and ideals, *Fuzzy Set. Syst.,* **44** (1991), 139–146.
2. B. Ahmad, A. Kharal, On Fuzzy soft sets, *Adv. Fuzzy Syst.,* **2009** (2009), 1–6.
3. U. Acar, B. Koyuncu, B. Tanay, Soft sets and soft rings, *Comput. Math. Appl.,* **59** (2010), 3458–3463.
4. J. Ahsan, K. Saifullah, M. Farid Khan, Fuzzy semirings, *Fuzzy Set. Syst.,* **60** (1993), 309–320.
5. J. Ahsan, Seminear-rings characterized by their S-ideals I, *Proc. Japan Acad. Ser. A, Math. Sci.,* **71** (1995), 101–103.
6. J. Ahsan, Seminear-rings characterized by their S-ideals II, *Proc. Japan Acad. Ser. A, Math. Sci.,* **71** (1995), 111–113.
7. A. Aygünoglu, H. Aygün, Introduction to fuzzy soft groups, *Comput. Math. Appl.,* **58** (2009), 1279–1286.
8. N. Çağman, S. Enginoğlu, F. Çitak, Fuzzy soft set theory and its applications, *Iran. J. Fuzzy Syst.,* **8** (2011), 137–147.
9. I. Chajda, H. Langer, Near-semirings and semirings with involution, *Miskolc Math. Notes,* **17** (2016), 801–810.
10. A. Dey, M. Pal, Generalised multi-fuzzy soft set and its application in decision making, *Pac. Sci. Rev. A: Nat. Sci. Eng.,* **17** (2015), 23–28.
11. V. N. Dixit, R. Kumar, N. Ajmal, On fuzzy rings, *Fuzzy Set. Syst.,* **49** (1992), 205–213.
12. F. Feng, Y. B. Jun, X. Zhao, Soft semirings, *Comput. Math. Appl.,* **56** (2008), 2621–2628.
13. W. L. Gau, D. J. Buehrer, Vague sets, *IEEE Trans. Syst., Man Cybern.,* **23** (1993), 610–614.
14. Z. Haiyan, J. Jingjing, *Fuzzy soft relation and its application in decision making,* 7th International Conference on Modelling, Identification and Control (ICMIC), IEEE, 2015.
15. W. G. V. Hoorn, B. Van Rootselaar, Fundamental notions in the theory of seminearrings, *Compositio Math.,* **18** (1967), 65–78.
16. E. İnan, M. A. Öztürk, Fuzzy soft rings and fuzzy soft ideals, *Neural Comput. Appl.,* **21** (2012), 1–8.

17. C. Jenila, P. Dheena, Ideal theory in near-semirings and its applications to automata, *Adv. Math.: Sci. J.,* **9** (2020), 4293–4302.

18. W. A. Khan, A. Rehman, A. Taouti, Soft near-semirings, *AIMS Math.,* **5** (2020), 6464–6478.

19. K. Koppula, K. B. Srinivas, K. S. Prasad, On prime strong ideals of a seminearring, *Mat. Vesnik,* **72** (2020), 243–256.

20. K. V. Krishna, *Near-semirings, theory and application,* Ph.D thesis, IIT Delhi, New Delhi, India, 2005.

21. K. V. Krishna, N. Chatterjee, A necessary condition to test the minimality of generalized linear sequential machines using the theory of near-semirings, *Algebra Discrete Math.,* **4** (2005), 30–45.

22. A. Z. Khameneh, A. Kilman, Multi-attribute decision-making based on soft set theory: A systematic review, *Soft Comput.,* **23** (2019), 6899–6920.

23. P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft sets, *J. Fuzzy Math.,* **9** (2001), 589–602.

24. P. K. Maji, A. R. Roy, R. Biswas, An application of soft sets in a decision making problem, *Comput. Math. Appl.,* **44** (2002), 1077–1083.

25. P. Majumdar, S. K. Samanta, Generalised fuzzy soft sets, *Comput. Math. Appl.,* **59** (2010), 1425–1432.

26. D. Molodtsov, Soft set theory-First results, *Comput. Math. Appl.,* **37** (1999), 19–31.

27. M. A. Öztürk, E. İnan, Fuzzy soft subnear-rings and $(\in, \in \vee q)$-fuzzy soft subnear-rings, *Comput. Math. Appl.,* **63** (2012), 617–628.

28. Z. Pawlak, Rough sets, *Int. J. Inform. Comput. Sci.,* **11** (1982), 341–356.

29. M. M. K. Rao, B. Venkateswarlu, Fuzzy soft k-ideals over semirings and fuzzy soft semiring homomorphism, *J. Hyperstructures,* **4** (2015), 93–116.

30. B. V. Rootselaar, Algebraische kennzeichnung freier wortarithmetiken, *Compos. Math.,* **15** (1963), 156–168.

31. A. R. Roy, P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, *J. Comput. Appl. Math.,* **203** (2007), 412–418.

32. A. Sezgin, A. O. Atagün, E. Aygün, A note on soft near-rings and idealistic soft near-rings, *Filomat,* **25** (2011), 53–68.

33. B. P. Varol, A. Aygunöglu, H. Aygün, On fuzzy soft rings, *J. Hyperstructures,* **1** (2012), 1–15.

34. H. J. Weinert, *Semi-nearrings, semi-nearfields and their semigroup-theoretical background,* Semigroup Forum, **24** (1982), 231–254.

35. L. A. Zadeh, *Fuzzy sets,* Infor. Control, **8** (1965), 338–353.

AIMS Press