AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(3): 2244–2267.
DOI:10.3934/math.2021136
Received: 06 September 2020
Accepted: 19 November 2020
Published: 11 December 2020

Research article

Finite-time fuzzy output-feedback control for p-norm stochastic nonlinear systems with output constraints

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Abstract: This paper investigates the finite-time control problem of p-norm stochastic nonlinear systems subject to output constraint. Combining a tan-type barrier Lyapunov function (BLF) with the adding a power integrator technique, a fuzzy state-feedback controller is constructed. Then, an output-feedback controller design scheme is developed by the constructed state-feedback controller and a reduce-order observer. Finally, both the rigorous analysis and the simulation results demonstrate that the designed output-feedback controller not only guarantees that the output constraint is not violated, but also ensures that the system is semi-global finite-time stable in probability (SGFSP).

Keywords: finite-time control; stochastic nonlinear systems; output constraint; output-feedback controller; barrier Lyapunov function
Mathematics Subject Classification: 37F15, 34D09

1. Introduction

Over the past decades, a variety of control design strategies have been proposed for different nonlinear systems [1–7]. Especially, many approximated-based control schemes have been developed for uncertain nonlinear systems by using neural networks (NNs) or the fuzzy logic systems (FLSs) [8–19]. Among these studies, the research of stochastic systems is much more attracted (see, e.g. [16–19] and the references therein), due to their wide application. It is worth noting that the aforementioned NNs-based or FLSs-based control strategies haven't taken output constraint into account. In fact, many practical systems are usually required to satisfy an output constraint in the operation for considering the performance specifications or safety [20, 21]. It is well known that, the BLF-based approaches are useful tools to settle controller design problems of output-constrained nonlinear systems, see references [22–27] for instances. In the latest research progress of constrained control, many kinds of adaptive neural or fuzzy control design methods have been presented by combining the different BLFs with NNs or FLSs approximators for various stochastic nonlinear systems subject to output constraint and unknown nonlinearities in [28–32]. Nevertheless, the above-mentioned works have mainly considered strict-feedback stochastic systems whose fractional powers are all equal to one, rather than p-norm stochastic nonlinear systems in which the fractional powers are the positive odd rational numbers and at least one of the fractional powers is greater than one.

As an important class of nonlinear systems, the p-norm stochastic nonlinear systems which are more general and complex, have received increased attention in recent years. For such a kind of systems, many control design problems have been well solved by utilizing the adding a power integrator technique under some strong or weaker growth conditions [33–37]. Subsequently, references [38–40] have taken these growth conditions away and considered the adaptive NNs control for switched and large-scale stochastic high-order nonlinear systems. Meanwhile, the control problems have been investigated for p-norm nonlinear systems with output/states constraints under some nonlinear growth conditions in a few literatures, such as [41–44]. In these works, different state-feedback controllers have been mainly constructed and finite-time stability has been obtained. Worth noting that an output-feedback controller has been designed for high-order planar deterministic nonlinear systems in [44]. On these basics, references [45–48] have further taken both unknown nonlinearities and constraints into accounts, and designed some fuzzy controllers for p-norm stochastic nonlinear systems with output/states constraints. However, it should be pointed out that the above-mentioned results of p-norm stochastic nonlinear systems have mainly focused on addressing the asymptotical convergence rather than the finite-time convergence in the case of unknown nonlinearities. As a matter of fact, the finite-time convergence has only been investigated for p-norm stochastic nonlinear systems with known nonlinearities satisfying some nonlinear growth conditions. On the other hand, it can be observed that the existing constrained controllers of p-norm stochastic nonlinear systems are mainly based on the assumption that full-state measurements are available. In other words, when only the system output can be accurately measured, the problem of finite-time output-feedback control for p-norm stochastic nonlinear systems with unknown nonlinearities and output constraints, to our best knowledge, has never been considered in the literature.

Motivated by above discussions, this paper will investigate how to design the finite-time output-feedback controller for a kind of p-norm stochastic nonlinear systems subject to output constraint. The main contributions can be summarized as follows: 1) *It is first time to consider p-norm stochastic nonlinear systems with output constraints and unmeasurable states.* Note that the existing studies have mainly addressed the constrained control problems under the the assumption that all the states are measurable (e.g., [33–37, 41, 43]), while this paper investigates the constrained control design in the case of that the states are all unmeasurable except the system output. What's more, the system nonlinearities are completely unknown. Thus, this work will extend and develop the existing control design theory for p-norm stochastic nonlinear systems; 2) *A finite-time controller is designed.* The developed scheme not only ensures the system output is constrained in a given compact set, but also enables the closed-loop system is semi-global finite-time stable in probability (SGFSP).

2. Problem and Preliminaries

In this paper, we consider the following class of p-norm stochastic nonlinear systems

$$
\begin{aligned}
dx_i &= x_{i+1}^p dt + \phi_i(\bar{x}_i)dt + g_i^T(\bar{x}_i)d\omega, i = 1, \cdots, n-1, \\
dx_n &= u^p dt + \phi_n(x)dt + g_n^T(x)d\omega, \\
y &= x_1,
\end{aligned}
\tag{2.1}
$$

where ω is a r-dimension standard Wiener process; $x = (x_1, \cdots, x_n)^T \in R^n$ is system state vector; $u \in R$ and $y \in R$ are respectively control input and output; the fractional power $p_\in R_{odd}^{\geq 1} := \{m/k|m \geq k,\ m$ and k are positive odd integers$\}$; for $i = 1, \cdots, n,\ \bar{x}_i = (x_1, \cdots, x_i)^T \in R^i$; $\phi_i : R^i \to R$ and $g_i : R^i \to R^r$ are unknown continuous functions satisfying $\phi_i(0) = 0,\ g_i(0) = 0$. The system output $y = x_1$ is measurable and constrained in $\Pi_1 = \{y(t) \in R, |y(t)| < \varepsilon\}$ with a constant $\varepsilon > 0$, while the other states $x_2, \cdots, x_n$ are all unmeasurable.

The objective of this paper is to design a finite-time fuzzy output-feedback controller for system (2.1) such that: 1) the output don't violate the given constrained boundary; 2) all the signals of the closed-loop system converge to a small compact of the original point in finite-time in probability in presence of unknown nonlinearities and unmeasured states $x_i (i = 2, \cdots, n)$.

Firstly, some concepts and lemmas are presented for preliminaries. Consider the following stochastic system

$$
dx = \phi(x)dt + g(x)d\omega, \tag{2.2}
$$

where $\phi(x)$ and $g(x)$ are continuous functions with satisfying $\phi(0) = g(0) = 0$.

Definition 1. *[1] For any given $V(x) \in C^2(R^n)$, associated with system (2.2), the second-order differential operator ℓ is defined as follows:*

$$
\ell V = \frac{\partial V}{\partial x}\phi(x) + \frac{1}{2} tr \left\{ g^T(x) \frac{\partial^2 V}{\partial x^2} g(x) \right\}. \tag{2.3}
$$

Definition 2. *[17] The equilibrium $x = 0$ of stochastic nonlinear system (2.2) is semi-global finite-time stable in probability (SGFSP) if for all $x(t_0, \omega) = x(0)$, there exist a constant $c > 0$ and a settling time $T^*(c, x_0, \omega) < \infty$ to make $E[\|x(t, \omega)\|] < c$, for all $t \geq t_0 + T^*$.*

Lemma 1. *[17] Consider the stochastic system (2.2) and assume that $f(0)$, $h(0)$ are bounded uniformly in t. If there exist a C^2 Lyapunov function $V : R^n \to R^+$, functions $\varrho_1, \varrho_2 \in \mathcal{K}_\infty$, and constants $\bar{\mu}_0 > 0$, $0 < \bar{\varsigma}_0 < 1$ and $\bar{\nu}_0 > 0$ such that*

$$
\begin{cases} \varrho_1(|x|) \leq V(x) \leq \varrho_2(|x|) \\ \ell V(x) \leq -\bar{\mu}_0 V(x)^{\bar{\varsigma}_0} + \bar{\nu}_0 \end{cases}, \forall x \in R^n, \tag{2.4}
$$

Then, the stochastic nonlinear system (2.2) is SGFSP.

Remark 1. *As stated in [17], the Eq (2.4) implies that there exists the stochastic setting time function $T^*(x, \omega) = \frac{1}{l_0 \bar{\mu}_0 (1-\bar{\varsigma}_0)} \left[E[V^{1-\bar{\varsigma}_0}(x(0))] - \left(\frac{\bar{\nu}_0}{(1-l_0)\bar{\mu}_0} \right)^{\frac{\bar{\varsigma}_0}{1-\bar{\varsigma}_0}} \right]$, such that $E[V^{\bar{\varsigma}_0}(x)] \leq \frac{\bar{\nu}_0}{(1-l_0)\bar{\mu}_0}$ for all $t \geq t_0 + T^*(x, \omega)$, where $0 < l_0 \leq 1$ is an arbitrary constant.*

Lemma 2. *[7] Let $a, b \in R^+$* with $a \geq 1$. *For any $\zeta, \eta \in R$, the following inequalities hold:*

(i) $|\zeta^a - \eta^a| \leq a(2^{a-2} + 2)|\zeta - \eta|(|\zeta - \eta|^{a-1} + \eta^{a-1})$,

(ii) $\mid \zeta^{\frac{b}{a}} - \eta^{\frac{b}{a}} \mid \leq 2^{1-\frac{1}{a}} \mid \lceil\zeta\rceil^{b} - \lceil\eta\rceil^{b} \mid^{\frac{1}{a}}$,
(iii) $(|\zeta| + |\eta|)^{\frac{1}{a}} \leq |\zeta|^{\frac{1}{a}} + |\eta|^{\frac{1}{a}} \leq 2^{1-\frac{1}{a}}(|\zeta| + |\eta|)^{\frac{1}{a}}$.

Lemma 3. *[42] For any constants $k_1, k_2, \vartheta, \varsigma \in R^+$ and any variables $\zeta_1, \zeta_2 \in R$, we have the following inequality*

$$\vartheta|\zeta_1|^{k_1}|\zeta_2|^{k_2} \leq \varsigma\frac{k_1}{k_1 + k_2}|\zeta_1|^{k_1+k_2} + \frac{k_1}{k_1 + k_2}\vartheta^{\frac{k_1+k_2}{k_2}}\varsigma^{-\frac{k_1}{k_2}}|\zeta_2|^{k_1+k_2}.$$

Lemma 4. *[8] Let $p \in (0, \infty)$, for any $\zeta_i \in R, i = 1, \cdots, n$, one has*

$$(|\zeta_1| + \cdots + |\zeta_n|)^p \leq b(|\zeta_1|^p + \cdots + |\zeta_n|^p),$$

where $b = \max\{n^{p-1}, 1\}$.

Lemma 5. *[36] If $\zeta, \eta \in R$ and $p > 1$ is an odd number, then*

$$-(\zeta - \eta)(\zeta^p - \eta^p) \leq -\frac{1}{2^{p-1}}(\zeta - \eta)^{p+1}.$$

Lemma 6. *[12] With any $a_0 > 0, c_0 > 0$, and $\zeta(t) > 0$, for $\dot{\eta}(t) = a_0\zeta(t) - c_0\eta(t)$, if $\eta(0) \geq 0$ can be satisfied, then one has $\eta(t) \geq 0$ for $\forall t \geq 0$.*

In this paper, the nonlinear functions $\phi_i(\cdot)$ and $g_i(\cdot)$ are all unknown. The unknown functions will be approximated by the FLSs based on the following presented lemma.

Lemma 7. *[49] Let $F(X)$ be a continuous function defined on a compact set Π_0. Then, for a given desired level of accuracy $\delta > 0$, there exists a fuzzy logic system $\Upsilon^T\Psi(X)$ such that*

$$\sup_{X\in\Pi_0} |F(X) - \Upsilon^T\Psi(X)| \leq \delta,$$

$\Upsilon = (\upsilon_1, \cdots, \upsilon_N)^T$ is the ideal constant weight vector, and $\Psi(X) = \frac{(\psi_1(X), \cdots, \psi_N(X))^T}{\sum_{j=1}^{N}\psi_j(X)}$ is the basis function vector, with $N > 1$ being the number of the fuzzy rules and $\psi_j(X)$ being chosen as Gaussian functions, i.e., for $j = 1, \cdots, N$

$$\psi_j(X) = \exp\left[-\frac{(X - \lambda_j)^T(X - \lambda_j)}{\vartheta_j^2}\right]$$

where $\lambda_j = (\lambda_{j1}, \cdots, \lambda_{jn})^T$ and ϑ_j respectively denote the center vector and the width of the Gaussian function.

Remark 2. *In view of Lemma 7, any function $F(X)$ which is defined and continuous on a compact set Π_0, can be approximated by*

$$F(X) = \Upsilon^T\Psi(X) + \epsilon(X),$$

where $\epsilon(X)$ is the FLS approximation error satisfying $|\epsilon(X)| < \delta$.

3. Main results

3.1. State-feedback controller design

In this section, a fuzzy state-feedback controller will be explicitly designed for system (2.1) by combining a tan-type BLF and the FLSs into the adding a power integrator technique.

First of all, we introduce a coordinate transformation as follow

$$\chi_1 = x_1, \chi_i = \frac{x_i}{H^{q_i}}, i = 2, \cdots, n, \nu = \frac{u}{H^{q_{n+1}}}, \tag{3.1}$$

where $q_1 = 0$, $q_j = \frac{q_{j-1}+1}{p}(j = 2, \cdots, n + 1)$, and $H > 1$ is a constant to be determined later. Based on (3.1) , system (2.1) turns into

$$\begin{aligned} d\chi_i &= H\chi_{i+1}^p dt + f_i(\bar{\chi}_i)dt + h_i^T(\bar{\chi}_i)d\omega, i = 1, \cdots, n-1, \\ d\chi_n &= H\nu^p dt + f_n(\chi)dt + h_n^T(\chi)d\omega, \\ y &= \chi_1. \end{aligned} \tag{3.2}$$

where $\bar{\chi}_i = (\chi_1, \cdots, \chi_i)$, $f_1 = \phi_1$, $h_1 = g_1$, $f_j = \frac{\phi_j}{H^{q_j}}$, and $h_j = \frac{g_j}{H^{q_j}}, (j = 2, \cdots, n)$.

In what follows, a fuzzy state-feedback controller will be designed through n steps based on the equivalent system (3.2).

Define

$$\xi_1 = \chi_1, \xi_i = \chi_i - \beta_{i-1}, i = 2, \ldots, n, \tag{3.3}$$

where β_i's are the virtual signals being constructed later.

Step 1. From (3.1), we can get

$$d\xi_1 = d\chi_1 = (H\chi_2^p + f_1)dt + h_1^T d\omega. \tag{3.4}$$

Choose the first Lyapunov function

$$V_1 = \frac{\varepsilon^4}{2\pi}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right) + \frac{1}{2b_1}\tilde{\alpha}_1^2 \triangleq V_B(\xi_1) + \frac{1}{2b_1}\tilde{\alpha}_1^2,$$

where $b_1 > 0$ is an adjustment parameter, $\tilde{\alpha}_1 = \alpha_1 - \hat{\alpha}_1$ is the estimate error and $\hat{\alpha}_1$ is the estimator of the parameter α_1.

Remark 3. *Clearly, $V_B(\xi_1) = \frac{\varepsilon^4}{2\pi}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right)$ is a tan-type BLF adopted to deal with the system output constraint. Compared to the log-type BLF, $V_B(\xi_1)$ possesses the following characteristic:*

$$\lim_{\varepsilon\to\infty} V_B(\xi_1) = \frac{\varepsilon^4}{2\pi}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right) = \frac{\xi_1^4}{4},$$

which implies that the proposed method is also applicable to the system without output constraints.

Then, one can easily get from the definition of $V_B(\xi_1)$ that

$$\frac{\partial V_B}{\partial \xi_1} = S_1(\xi_1)\xi_1^3, \tag{3.5}$$

$$\frac{\partial^2 V_B}{\partial \xi_1^2} = 3S_1(\xi_1)\xi_1^2 + \frac{4\pi}{\varepsilon^4}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right)S_1(\xi_1)\xi_1^6. \tag{3.6}$$

where $S_1(\xi_1) = \sec^2\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right)$.

In view of (2.3), (3.5) and (3.6), it is not hard to gain

$$\begin{aligned}\ell V_1 &= \frac{\partial V_B}{\partial \xi_1}(H\chi_2^p + f_1) + \frac{1}{2}\frac{\partial^2 V_B}{\partial \xi_1^2}h_1^T h_1 - \frac{1}{b_1}\tilde{\alpha}_1\dot{\hat{\alpha}}_1\\ &= S_1(\xi_1)\xi_1^3(H\chi_2^p + f_1) - \frac{1}{b_1}\tilde{\alpha}_1\dot{\hat{\alpha}}_1 + \frac{3}{2}S_1(\xi_1)\|h_1\|^2\xi_1^2\\ &+ \frac{2\pi}{\varepsilon^4}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right)S_1(\xi_1)\|h_1\|^2\xi_1^6.\end{aligned}$$

From Lemma 3, one obtains

$$\frac{3}{2}S_1(\xi_1)\|h_1\|^2\xi_1^2 \le \frac{3}{4}S_1(\xi_1)^2\|h_1\|^4\xi_1^4 + \frac{3}{4}.$$

Then, we have

$$\begin{aligned}\ell V_1 &\le HS_1(\xi_1)\xi_1^3\chi_2^p + HS_1(\xi_1)\xi_1^3F_1(Z_1) - \frac{\varrho_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right)\\ &- \frac{1}{b_1}\tilde{\alpha}_1\dot{\hat{\alpha}}_1 + \frac{3}{4},\end{aligned} \tag{3.7}$$

where $Z_1 = \chi_1$,

$$\begin{aligned}F_1(Z_1) &= \frac{1}{H}\left[f_1 + \frac{3}{4}S_1(\xi_1)^2\|h_1\|^4\xi_1\right] + \frac{1}{H}\frac{2\pi}{\varepsilon^4}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right)S_1(\xi_1)\|h_1\|^2\xi_1^3\\ &+ \frac{\varrho_1\varepsilon^4\sin\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right)}{2H\pi\xi_1^3}\cos\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right).\end{aligned}$$

and $\varrho_1 > 0$ is an adjustment parameter.

Thus, by Lemma 7, one can approximate $F_1(Z_1)$ by

$$F_1(Z_1) = \Upsilon_1^T\Psi_1(Z_1) + \epsilon_1(Z_1), \tag{3.8}$$

where $|\epsilon_1(Z_1)| \le \delta_1$ and $\delta_1 > 0$ is a given constant.

Since $\Psi_1^T(\cdot)\Psi_1(\cdot) \le 1$, it is easily obtained from Lemma 3 that

$$\begin{aligned}S_1(\xi_1)\xi_1^3F_1(Z_1) &\le S_1(\xi_1)|\xi_1|^3(\|\Upsilon_1\|\|\Psi_1\| + \delta_1)\\ &\le \frac{3\sigma_{11}\alpha_1}{p+3}(S_1(\xi_1))^{\frac{p+3}{3}}\xi_1^{p+3}\\ &+ \frac{3}{p+3}(S_1(\xi_1))^{\frac{p+3}{3}}\xi_1^{p+3} + \frac{p}{p+3}\sigma_{11}^{-\frac{3}{p}} + \frac{p}{p+3}\delta_1^{\frac{p+3}{p}},\end{aligned} \tag{3.9}$$

where $\alpha_1 = \|\Upsilon_1\|^{\frac{p+3}{3}}$ and $\sigma_{11} > 0$ is an adjustment parameter.

Substituting (3.9) into (3.7) gets

$$\begin{aligned}\ell V_1 \leq\ & HS_1(\xi_1)\xi_1^3(x_2^p - \beta_1^p) + HS_1(\xi_1)\xi_1^3\beta_1^p - \frac{\varrho_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right)\\ &+ \frac{3H}{p+3}S_1(\xi_1)^{\frac{p+3}{3}}\left[\sigma_{11}\hat{\alpha}_1 + 1\right]\xi_1^{p+3} + \frac{pH}{p+3}\sigma_{11}^{-\frac{3}{p}}\\ &+ H\tilde{\alpha}_1\left[\frac{3\sigma_{11}}{p+3}(S_1(\xi_1))^{\frac{p+3}{3}}\xi_1^{p+3} - \frac{1}{b_1H}\dot{\hat{\alpha}}_1\right] + \frac{3}{4} + \frac{pH}{p+3}\delta_1^{\frac{p+3}{p}}.\end{aligned} \tag{3.10}$$

Then, one could design

$$\beta_1 = -M_1^{\frac{1}{p}}\xi_1 \triangleq -\varphi_1\xi_1 \tag{3.11}$$

and

$$\dot{\hat{\alpha}}_1 = \frac{3Hb_1\sigma_{11}}{p+3}(S_1(\xi_1))^{\frac{p+3}{3}}\xi_1^{p+3} - d_1\hat{\alpha}_1, \tag{3.12}$$

where $M_1 \geq \frac{3}{p+3}(S_1(\xi_1))^{\frac{p}{3}}[\sigma_{11}\hat{\alpha}_1 + 1] + \frac{\rho_1}{S_1(\xi_1)} + \varrho_1 > 0$; $\varrho_1, d_1 > 0$ are adjustment parameters; and the value of $\rho_1 > 0$ will be given in the next step.

Substituting (3.11) and (3.12) into (3.10), gets

$$\begin{aligned}\ell V_1 \leq\ & -\frac{\varrho_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right) - H(\rho_1 + \varrho_1)\xi_1^{p+3} + HS_1(\xi_1)\xi_1^3(\chi_2^p - \beta_1^p)\\ &+ \frac{d_1}{b_1}\tilde{\alpha}_1\hat{\alpha}_1 + \frac{3}{4} + \frac{pH}{p+3}\sigma_{11}^{-\frac{3}{p}} + \frac{pH}{p+3}\delta_1^{\frac{p+3}{p}}.\end{aligned}$$

In addition, it is easily obtained that

$$\frac{d_1}{b_1}\tilde{\alpha}_1\hat{\alpha}_1 = \frac{d_1}{b_1}(\alpha_1 - \tilde{\alpha}_1)\tilde{\alpha}_1 \leq -\frac{d_1\tilde{\alpha}_1^2}{2b_1} + \frac{d_1\alpha_1^2}{2b_1}.$$

Therefore, we can get

$$\begin{aligned}\ell V_1 -\ & \frac{\varrho_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right) - \frac{d_1\tilde{\alpha}_1^2}{2b_1} - H(\rho_1 + \varrho_1)\xi_1^{p+3}\\ &+ Q_1 + HS_1(\xi_1)\xi_1^3(x_2^p - \beta_1^p),\end{aligned} \tag{3.13}$$

where $Q_1 = \frac{3}{4} + \frac{pH}{p+3}\sigma_{11}^{-\frac{3}{p}} + \frac{pH}{p+3}\delta_1^{\frac{p+3}{p}} + \frac{d_1\alpha_1^2}{2b_1}$.

Remark 4. *Notice that*

$$\lim_{\xi_1\to 0}\frac{\varepsilon^4\sin\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right)\cos\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right)}{2\pi\xi_1^3} = \lim_{\xi_1\to 0}\frac{\varepsilon^4\frac{\pi\xi_1^4}{2\varepsilon^4}\cos\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right)}{2\pi\xi_1^3} = 0,$$

which means that the continuity of $F_1(Z_1)$ is ensured.

Remark 5. *According to Lemma 6, one gets* $\hat{\alpha}_1 \geq 0$, *for* $\forall t \geq 0$. *In each design step, this characteristic will be always applied.*

Step 2. From (3.3) and Itô's formula, we have

$$d\xi_2 = (H\chi_3^p + f_2 - \ell\beta_1)dt + \left(h_2 - \frac{\partial \beta_1}{\partial \chi_1} h_1\right)^T d\omega, \tag{3.14}$$

where $\ell\beta_1 = \frac{\partial\beta_1}{\partial\chi_1}(H\chi_2^p + f_1) + \frac{\partial\beta_1}{\partial\hat{\alpha}_1}\dot{\hat{\alpha}}_1 + \frac{1}{2}\frac{\partial^2\beta_1}{\partial\chi_1^2}h_1^T h_1$. Combining the definition of β_1 with the properties of $f_1(\chi_1)$ and $h_1(\chi_1)$, implies that $\ell\beta_1$ is valid and continuous.

Choose the second Lyapunov function as

$$V_2 = V_1 + \Lambda_2 \tag{3.15}$$

with

$$\Lambda_2 = \frac{1}{4}\xi_2^4 + \frac{1}{2b_2}\tilde{\alpha}_2^2, \tag{3.16}$$

where $b_2 > 0$ is an adjustment parameter, $\tilde{\alpha}_2 = \alpha_2 - \hat{\alpha}_2$ is the estimate error and $\hat{\alpha}_2$ is the estimator of the parameter α_2.

Applying (2.3), (3.14) and (3.16), it can be gotten that

$$\ell\Lambda_2 = \xi_2^3(H\chi_3^p + f_2 - \ell\beta_1) - \frac{1}{b_2}\tilde{\alpha}_2\dot{\hat{\alpha}}_2 + \frac{3}{2}\left\|h_2 - \frac{\partial\beta_1}{\partial\chi_1}h_1\right\|^2 \xi_2^2. \tag{3.17}$$

Besides, applying Lemma 3 renders

$$\frac{3}{2}\left\|h_2 - \frac{\partial\beta_1}{\partial\chi_1}h_1\right\|^2 \xi_2^2 \leq \frac{3}{4}\left\|h_2 - \frac{\partial\beta_1}{\partial\chi_1}h_1\right\|^4 \xi_2^4 + \frac{3}{4}.$$

On the other hand, we gets

$$\begin{aligned} S_1(\xi_1)\xi_1^3(\chi_2^p - \beta_1^p) &\leq S_1(\xi_1)|\xi_1|^3|\chi_2^p - \beta_1^p| \\ &\leq DS_1(\xi_1)|\xi_1|^3\left(|\xi_2|^p + \varphi_1^p|\xi_1|^{p-1}\cdot|\xi_2|\right) \\ &\leq \rho_1\xi_1^{p+3} + \tau_2\xi_2^{p+3} + \frac{p\sigma_{12}}{p+3}\xi_2^{p+3}, \end{aligned} \tag{3.18}$$

where $D = (2^{p-2}+2)p$; $\rho_1 = \frac{1}{p+3}\left[3(DS_1(\xi_1))^{\frac{p+3}{3}}\sigma_{12}^{-\frac{p}{3}} + (p+2)S_1(\xi_1)\right]$, $\tau_2 = \frac{S_1(\xi_1)\varphi_1^{p(p+3)}}{p+3}$, and $\sigma_{12} > 0$ is an adjustment parameter.

Thus, from (3.13), (3.17) and (3.18), we have

$$\begin{aligned} \ell V_2 &= \ell V_1 + \ell\Lambda_2 \\ &\leq -\frac{\varrho_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right) - \frac{d_1}{2b_1}\tilde{\alpha}_1^2 - H\varrho_1\xi_1^{p+3} - \frac{1}{b_2}\tilde{\alpha}_2\dot{\hat{\alpha}}_2 \\ &+ H\xi_2^3(\chi_3^p - \beta_2^p) + H\xi_2^3\beta_2^p + H\xi_2^3F_2(Z_2) + \frac{Hp\sigma_{12}}{p+3}\xi_2^{p+3} + Q_1 + \frac{3}{4}, \end{aligned} \tag{3.19}$$

where $Z_2 = (\bar{x}_2, \hat{\alpha}_1)^T$,

$$F_2(Z_2) = \frac{1}{H}[f_2 - \ell\beta_1] + \tau_2\xi_2^p + \frac{3}{4H}\left\|h_2 - \frac{\partial\beta_1}{\partial\chi_1}h_1\right\|^4 \xi_2.$$

Obviously, the continuity of $F_2(Z_2)$ can be directly proved by the fact that the functions $f_2(\bar{\chi}_2)$, $h_2(\bar{\chi}_2)$ and $\ell\beta_1$ are all continuous. Thus, $F_2(Z_2)$ can be approximated as

$$F_2(Z_2) = \Upsilon_2^T\Psi_2(Z_2) + \epsilon_2(Z_2), \tag{3.20}$$

where $|\epsilon_2(Z_2)| \leq \delta_2$ and $\delta_2 > 0$ is a given constant.

In view of the fact $\Psi_2^T(\cdot)\Psi_2(\cdot) \leq 1$ and Lemma 3, we obtain

$$\begin{aligned} \xi_2^3 F_2(Z_2) &\leq |\xi_2|^3(\|\Upsilon_2\|\|\Psi_2\| + \delta_2) \\ &\leq \frac{3\sigma_{21}\alpha_2}{p+3}\xi_2^{p+3} + \frac{p}{p+3}\sigma_{21}^{-\frac{3}{p}} + \frac{3}{p+3}\xi_2^{p+3} + \frac{p}{p+3}\delta_2^{\frac{p+3}{p}}, \end{aligned} \tag{3.21}$$

where $\alpha_2 = \|\Upsilon_2\|^{\frac{p+3}{3}}$ and $\sigma_{21} > 0$ is an adjustment parameter.

Substituting (3.21) into (3.19) yields

$$\begin{aligned} \ell V_2 \leq& -\frac{\varrho_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right) - \frac{d_1}{2b_1}\tilde{\alpha}_1^2 - H\varrho_1\xi_1^{p+3} \\ &+ H\xi_2^3(\chi_3^p - \beta_2^p) + H\xi_2^3\beta_2^p + \frac{3H}{p+3}[\sigma_{21}\hat{\alpha}_2 + 1]\xi_2^{p+3} + \frac{Hp\sigma_{12}}{p+3}\xi_2^{p+3} \\ &+ H\tilde{\alpha}_2\left[\frac{3\sigma_{21}}{p+3}\xi_2^{p+3} - \frac{1}{b_2H}\dot{\hat{\alpha}}_2\right] + Q_1 + \frac{3}{4} + \frac{pH}{p+3}\sigma_{21}^{-\frac{3}{p}} + \frac{pH}{p+3}\delta_2^{\frac{p+3}{p}}. \end{aligned} \tag{3.22}$$

Then, one can design

$$\beta_2 = -M_2^{\frac{1}{p}}\xi_2 \triangleq -\varphi_2\xi_2 \tag{3.23}$$

and

$$\dot{\hat{\alpha}}_2 = \frac{3Hb_2\sigma_{21}}{p+3}\xi_2^{p+3} - d_2\hat{\alpha}_2, \tag{3.24}$$

where $M_2 \geq \frac{3}{p+3}[\sigma_{21}\hat{\alpha}_2 + 1] + \frac{p\sigma_{12}}{p+3} + \rho_2 + \varrho_2 > 0$; $\varrho_2 > 0,\ d_2 > 0$ are adjustment parameters. The value of $\rho_2 > 0$ will be given in the next step.

In addition, it is evident that

$$\frac{d_2}{b_2}\hat{\alpha}_2\tilde{\alpha}_2 \leq -\frac{d_2\tilde{\alpha}_2^2}{2b_2} + \frac{d_2}{2b_2}\hat{\alpha}_2^2. \tag{3.25}$$

Substituting (3.25) into (3.22), one gets

$$\begin{aligned} \ell V_2 \leq& -\frac{\varrho_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{2}\frac{d_j}{2b_j}\tilde{\alpha}_1^2 - H\sum_{j=1}^{2}\varrho_j\xi_j^{p+3} \\ &+ H\xi_2^3(\chi_3^p - \beta_2^p) + \sum_{j=1}^{2}Q_j, \end{aligned} \tag{3.26}$$

where $Q_2 = \frac{3}{4} + \frac{pH}{p+3}\sigma_{12}^{-\frac{3}{p}} + \frac{pH}{p+3}\delta_2^{\frac{p+3}{p}} + \frac{d_2\alpha_2^2}{2b_2}$.
Inductive Step $(3 \le k \le n)$**.** In view of above two steps, we can deduce the following similar property whose proof can be found in the Appendix.

Proposition 1. *For the kth Lyapunov function* $V_k : \Pi_k \to R^+$ *as*

$$V_k = V_{k-1} + \Lambda_k \tag{3.27}$$

with

$$\Lambda_k = \frac{1}{4}\xi_k^4 + \frac{1}{2b_k}\tilde{\alpha}_k^2, \tag{3.28}$$

there exists a virtual controller β_k *and the adaptive law of* $\hat{\alpha}_k$ *of the following forms*

$$\beta_k = -M_k^{\frac{1}{p}}\xi_k \triangleq -\varphi_k\xi_k, \tag{3.29}$$

$$\dot{\hat{\alpha}}_k = \frac{3Hb_k\sigma_{k1}}{p+3}\xi_k^{p+3} - d_k\hat{\alpha}_k, \tag{3.30}$$

such that

$$\begin{aligned}\ell V_k \le & -\frac{\varrho_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{k}\frac{d_j\tilde{\alpha}_j^2}{2b_j} \\ & - H\rho_k\xi_k^{p+3} - H\sum_{j=1}^{k}\varrho_j\xi_j^{p+3} + \sum_{j=1}^{k}Q_j + H\xi_k^3(\chi_{k+1}^p - \beta_k^p),\end{aligned} \tag{3.31}$$

where $M_k \ge \frac{3}{p+3}[\sigma_{k1}\hat{\alpha}_k + 1] + \frac{p\sigma_{k-1\,2}}{p+3} + \rho_k + \varrho_k > 0$*;* $b_k, \varrho_k, d_k > 0$ *are adjustment parameters; and the value of* $\rho_k > 0$ *will be given in the next step.*

Step n According to above steps, there exist a series of virtual controllers and adaptive parameter laws $(\beta_k, \hat{\alpha}_k)(k = 1, \cdots, n)$ make that Eq (3.31) holds when $k = n$ with $\chi_{n+1} = \nu$. Therefore, the fuzzy adaptive control law can be designed as

$$\beta_n = -M_n^{\frac{1}{p}}\xi_n \triangleq -\varphi_n\xi_n, \tag{3.32}$$

$$\dot{\hat{\alpha}}_n = \frac{3Hb_n\sigma_{n1}}{p+3}\xi_n^{p+3} - d_n\hat{\alpha}_n. \tag{3.33}$$

Apparently, one can further get

$$\begin{aligned}\ell V_n \le & -\frac{\varrho_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{n}\frac{d_j\tilde{\alpha}_j^2}{2b_j} \\ & - H\sum_{j=1}^{n}\varrho_j\xi_j^{p+3} + \sum_{j=1}^{n}Q_j + H\xi_n^3(\nu^p - \beta_n^p).\end{aligned} \tag{3.34}$$

By the definitions of $\xi_k(k = 1, \cdots, n)$, one gets the fuzzy state-feedback controller

$$\beta_n = -(\bar{\varphi}_1\chi_1 + \cdots + \bar{\varphi}_n\chi_n), \tag{3.35}$$

where $\bar{\varphi}_k = \prod_{j=k}^{n}\varphi_j$ for $k = 1, \ldots, n$.

3.2. Output-feedback controller design

In this subsection, a fuzzy output-feedback controller will be designed by combining the above-constructed state-feedback controller with the state-observer constructed later. Since $\chi_2, \cdots, \chi_n$ are unmeasurable, a reduced-order observer is required. Firstly, define a series of new variables as follows:

$$z_i = \chi_i - \gamma_i \ldots \gamma_2 \chi_1, \ i = 2, \cdots, n.$$

Then, it directly gets

$$\begin{aligned} dz_i &= H(\chi_{i+1}^p - \gamma_i \ldots \gamma_2 \chi_2^p)dt + (f_i - \gamma_i \ldots \gamma_2 f_1)dt \\ &\quad + [h_i - \gamma_i \ldots \gamma_2 h_1]^T d\omega, i = 2, \cdots, n-1, \\ dz_n &= H(v^p - \gamma_n \ldots \gamma_2 \chi_2^p)dt + (f_n - \gamma_n \ldots \gamma_2 f_1)dt \\ &\quad + [h_n - \gamma_n \ldots \gamma_2 h]^T d\omega, \end{aligned}$$

where $\gamma_i \geq 1(i = 2, \cdots, n)$ are gain parameters to be determined. Hence, the $(n-1)$-dimensional observer can be constructed as [36]

$$\begin{aligned} \dot{\hat{z}}_i &= H(\hat{z}_i + \gamma_i \ldots \gamma_2 \chi_1)^p - H\gamma_i \ldots \gamma_2(\hat{z}_2 + \gamma_2 \chi_1)^p, i = 2, \cdots, n-1, \\ \dot{\hat{z}}_n &= Hv^p - H\gamma_n \ldots \gamma_2(\hat{z}_2 + \gamma_2 \chi_1)^p. \end{aligned} \tag{3.36}$$

According to (3.36), the estimate $\hat{\chi}_i$ of χ_i can be gotten by

$$\hat{\chi}_i = \hat{z}_i + \gamma_i \ldots \gamma_2 \chi_1, i = 2, \cdots, n. \tag{3.37}$$

Thus, one constructs the implementable controller of system (2.4) by using Eq (3.35) and the certainty equivalence principle as below:

$$v = -(\bar{\varphi}_1 \hat{\chi}_1 + \cdots + \bar{\varphi}_n \hat{\chi}_n). \tag{3.38}$$

Therefore, the output-feedback controller of origin system (2.1) is

$$u = H^{q_{n+1}} v = -H^{q_{n+1}}(\bar{\varphi}_1 \hat{\chi}_1 + \cdots + \bar{\varphi}_n \hat{\chi}_n). \tag{3.39}$$

3.3. Selection of the observer gains

In this section, we will analyze the appropriate values of the gains $\gamma_i (i = 2, \cdots, n)$ and some constant parameters in output-feedback controller.

To determine the observer gains $\gamma_2, \cdots, \gamma_n$, we first define the error dynamics

$$e_i \triangleq \chi_i - \hat{\chi}_i, \ i = 2, \cdots, n. \tag{3.40}$$

Further, the following coordinate transformation is introduced

$$\tilde{e}_2 = e_2, \tilde{e}_3 = e_3 - \gamma_3 e_2, \cdots, \tilde{e}_n = e_n - \gamma_n e_{n-1}. \tag{3.41}$$

It can easily infer from (3.40) and (3.41) that

$$\begin{aligned} d\tilde{e}_i &= H[(\chi_{i+1}^p - \hat{\chi}_{i+1}^p) - \gamma_i(\chi_i^p - \hat{\chi}_i^p)]dt \\ &\quad + [f_i - \gamma_i f_{i-1}]dt + [h_i - \gamma_i h_{i-1}]^T d\omega, i = 2, \cdots, n-1 \\ d\tilde{e}_n &= -H\gamma_n(\chi_n^p - \hat{\chi}_n^p)dt + [f_n - \gamma_n f_{n-1}]dt + [h_n - \gamma_n h_{n-1}]^T d\omega. \end{aligned} \tag{3.42}$$

Now, a proposition is provided for helping to determine gain constants, whose proof will be given in Appendix.

Proposition 2. *For the Lyapunov function*

$$U_n = \frac{1}{4}\gamma \sum_{i=2}^{n} \tilde{e}_i^4,$$

by utilizing Lemmas 2-5, Eqs (2.3) *and* (3.42), *it is not difficult to obtain*

$$\begin{aligned} \ell U_n \leq & -H\left[\sum_{i=2}^{n} \frac{\gamma\gamma_i}{2^{p-1}}\tilde{e}_i^{p+3} + \sum_{i=2}^{n} \bar{c}_i \tilde{e}_i^{p+3} + \sum_{i=2}^{n} \frac{6\gamma^{\frac{p+3}{3}}}{(p+3)H^{\frac{p+3}{3}}}\tilde{e}_i^{p+3}\right] \\ & + \frac{(p-1)H}{p+3} + \widetilde{F}_1(\chi), \end{aligned} \tag{3.43}$$

where $\gamma > 0$ *is an adjustment constant;* $\bar{c}_i = \bar{c}_i(\gamma_{i+1}, \cdots, \gamma_n) > 0, (i = 2, \cdots, n-1)$ *are constants independent of* H*; and* $\bar{c}_n$ *is a constant independent of* H *and* $\gamma_i (i = 2, \cdots, n)$*; and* $\widetilde{F}_1(\chi) = H\sum_{i=2}^{n}[\frac{1}{4}(f_i - \gamma_i f_{i-1})^4 + \frac{3\gamma^{\frac{2}{3}}}{4H^{\frac{2}{3}}}\|h_i - \gamma_i h_{i-1}\|^4 + \frac{2p-1}{p+3}\chi_i^{p+3}]$ *is an unknown continuous function.*

Besides, we can obtain from Lemmas 2-4 that

$$\begin{aligned} & |\xi_n^3(\nu^p - \beta_n^p)| \\ & \leq D|\xi_n|^3|\nu - \beta_n|[|\nu - \beta_n|^p + |\beta_n|^p] \\ & \leq D(n-1)^p|\xi_n|^3 \cdot \sum_{i=2}^{n} \bar{\varphi}_i^p |e_i|^p + D|\xi_n|^3 \cdot \left(\sum_{i=2}^{n} \bar{\varphi}_i |e_i|\right) \cdot \varphi_n^{p-1}|\xi_n|^{p-1} \\ & \leq \frac{p+1}{p+3}\sum_{i=2}^{n} e_i^{p+3} + \tau_n \xi_n^{p+3} \\ & \leq \tilde{\iota}_i \sum_{i=2}^{n} \tilde{e}_i^{p+3} + \widetilde{F}_2(\xi_n), \end{aligned} \tag{3.44}$$

where $\widetilde{F}_2(\xi_n) = \tau_n \xi_n^{p+3}$; $\tau_n = \frac{3}{p+3}\sum_{i=2}^{n}\left[D(n-1)^p \bar{\varphi}_i^p\right]^{\frac{p+3}{3}} + \frac{p+2}{p+3}\sum_{i=2}^{n} 3\left[D(n-1)^p \varphi_n^{p-1}\bar{\varphi}_i\right]^{\frac{p+3}{p+2}}$; $\tilde{\iota}_n = n^{p+2}\frac{p+1}{p+3}$ and $\tilde{\iota}_i = \tilde{\iota}_i(\gamma_{i+1}, \cdots, \gamma_n)(i = 2, \cdots, n-1)$ are positive constants.

Then, substituting (3.44) into (3.34) yields

$$\begin{aligned} \ell V_n \leq & -\frac{\varrho_1 \varepsilon^4}{2\pi}\tan\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{n} \frac{d_j \tilde{\alpha}_j^2}{2b_j} \\ & - H\sum_{j=1}^{n} \varrho_j \xi_j^{p+3} + \sum_{j=1}^{n} Q_j + H\sum_{i=2}^{n} \tilde{\iota}_i \tilde{e}_i^{p+3} + \widetilde{F}_2(\xi_n). \end{aligned} \tag{3.45}$$

Then, define the entire Lyapunov function $V = V_n + U_n$. Apparently, it directly infers from Proposition 2 and Eq (3.45) that

$$\begin{aligned} \ell V = & \ell V_n + \ell U_n \\ \leq & -\frac{\varrho_1 \varepsilon^4}{2\pi}\tan\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{n} \frac{d_j \tilde{\alpha}_j^2}{2b_j} - H\sum_{j=1}^{n} \varrho_j \xi_j^{p+3} + \widetilde{F}(\cdot) + \sum_{j=1}^{n} Q_j \\ & - H\sum_{i=2}^{n}\left[\frac{\gamma\gamma_i}{2^{p-1}} - \bar{c}_i - \tilde{\iota}_i\right]\tilde{e}_i^{p+3} + H\sum_{i=2}^{n} \frac{6\gamma^{\frac{p+3}{3}}}{(p+3)H^{\frac{p+3}{3}}}\tilde{e}_i^{p+3} + \frac{(p-1)H}{p+3}, \end{aligned} \tag{3.46}$$

where $\widetilde{F}(\cdot) = \widetilde{F}_1(\cdot) + \widetilde{F}_2(\cdot)$.

Using FLS to deal with the unknown function $\widetilde{F}(\cdot)$, one deduces from Lemma 7 that $\widetilde{F}(\cdot) = \Upsilon_0^T \Psi_0 + \epsilon_0(\cdot) \leq \|\Upsilon_0\| \|\Psi_0\| + \delta_0 \leq \frac{3}{p+3}\alpha_0 + \frac{p}{p+3} + \delta_0$ where $\alpha_0 = \|\Upsilon_0\|^{\frac{p+3}{3}}$.

Therefore, the observer gains $\gamma_2, \cdots, \gamma_n$ and constant H can be chosen in the following recursive manner

$$\begin{aligned}
\gamma_n &\geq \max\left\{\frac{2^{p-1}}{\gamma}(\bar{c}_n + \tilde{\iota}_n + 1 + \theta_n), 1\right\},\\
\gamma_{n-1} &\geq \max\left\{\frac{2^{p-1}}{\gamma}(\bar{c}_{n-1}(\gamma_n) + \tilde{\iota}_{n-1}(\gamma_n) + 1 + \theta_{n-1}), 1\right\},\\
&\vdots\\
\gamma_2 &\geq \max\left\{\frac{2^{p-1}}{\gamma}(\bar{c}_2(\gamma_n, \cdots, \gamma_3) + \tilde{\iota}_2(\gamma_n, \cdots, \gamma_3) + 1 + \theta_2), 1\right\},\\
H &\geq \max\left\{1, \left(\frac{6}{p+3}\right)^{\frac{3}{p+3}} \gamma\right\}.
\end{aligned} \tag{3.47}$$

Then, Eq (3.46) turns into

$$\begin{aligned}
\ell V \leq & -\frac{\varrho_1 \varepsilon^4}{2\pi} \tan\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{n} \frac{d_j \tilde{\alpha}_j^2}{2b_j}\\
& - H\sum_{i=1}^{n} \varrho_i \xi_i^{p+3} - H\sum_{i=2}^{n} \theta_i \tilde{e}_i^{p+3} + Q,
\end{aligned} \tag{3.48}$$

where $\theta_i (i = 2, \cdots, n)$ are positive constants, and $Q = \sum_{i=1}^{n} Q_j + \frac{(p-1)H}{p+3} + \frac{3}{p+3}\alpha_0 + \frac{p}{p+3} + \delta_0$.

On the other hand, it can be verified from Lemma 3 that

$$\begin{aligned}
\tilde{e}_i^4 &\leq \frac{4}{p+3}\tilde{e}_i^{p+3} + \frac{p-1}{p+3},\\
\xi_i^4 &\leq \frac{4}{p+3}\xi_i^{p+3} + \frac{p-1}{p+3},
\end{aligned} \tag{3.49}$$

which renders

$$\begin{aligned}
-H\sum_{i=2}^{n} \theta_i \tilde{e}_i^{p+3} &\leq -\sum_{i=2}^{n} \frac{\bar{\theta}_i}{4}\tilde{e}_i^4 + \frac{p-1}{4}H\sum_{i=2}^{n}\theta_i,\\
-H\sum_{i=2}^{n} \varrho_i \xi_i^{p+3} &\leq -\sum_{i=2}^{n} \frac{\bar{\varrho}_i}{4}\xi_i^4 + \frac{p-1}{4}H\sum_{i=2}^{n}\varrho_i,
\end{aligned} \tag{3.50}$$

where $i = 2, \cdots, n$, $\bar{\theta}_i = (p+3)H\theta_i$ and $\bar{\varrho}_i = (p+3)H\varrho_i$. Further, we get

$$\ell V \leq -\frac{\varrho_1 \varepsilon^4}{2\pi} \tan\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{n} \frac{d_j \tilde{\alpha}_j^2}{2b_j} - \sum_{i=2}^{n} \frac{\bar{\varrho}_i}{4}\xi_i^4 - \sum_{i=2}^{n} \frac{\bar{\theta}_i}{4}\tilde{e}_i^4 + \bar{Q}, \tag{3.51}$$

where $\bar{Q} = Q + \frac{p-1}{4}H\sum_{i=2}^{n}\theta_i + \frac{p-1}{4}H\sum_{i=2}^{n}\varrho_i$.

3.4. Stability analysis

To state the main result, the following theorem is presented.

Theorem 1. *For the p-norm stochastic nonlinear system* (2.1) *and a given constant, there exists a finite-time fuzzy output-feedback controller* (3.39) *together with the parameter adaptive laws* (3.12), (3.24), (3.30), *and* (3.33) *such that*

i) the system output isn't violated in the sense of probability, i.e., $P\{|y(t)| < \varepsilon\} = 1$.
ii) all the signals in the closed-loop stochastic nonlinear system (2.1) *are SGFSP.*

Proof. i) Let $\mu_0 = \min\{\varrho_1, d_1, \cdots, d_n, \bar{\varrho}_2, \cdots \bar{\varrho}_n, \frac{\bar{\theta}_2}{\gamma}, \cdots \frac{\bar{\theta}_n}{\gamma}\}$ and $\pi_0 = \bar{Q}$. Then, Eq (3.51) can be expressed as

$$\ell V \leq -\mu_0 V + \pi_0. \tag{3.52}$$

We can easily get from Eq (3.52) that

$$EV(t) \leq V(t_0)e^{-\mu_0 t} + \frac{\pi_0}{\mu_0}. \tag{3.53}$$

For $x(0) = (x_1(t_0), \cdots, x_n(t_0))^T$ satisfying $x_1(t_0) \in \Pi_1$, it easily obtains that the mean of $V(t)$ is bounded, which implies that V is bounded in probability. It can be directly deduced from the definition of V that

$$P\{V_B(\xi_1) < \infty\} = 1. \tag{3.54}$$

Consequently, it is clear that $P\{|y(t)| < \varepsilon\} = P\{|\xi_1(t)| < \varepsilon\} = 1$, which demonstrates that the output constraint of system (2.1) is not violated in the sense of probability.
ii) For $\forall\ 0 < \bar{\varsigma}_0 < 1$, it is easy to get from Lemma 3 that

$$V^{\bar{\varsigma}_0} \leq \bar{\varsigma}_0 V + (1 - \bar{\varsigma}_0).$$

Further, one has

$$-\mu_0 V \leq -\frac{\mu_0}{\bar{\varsigma}_0} V^{\bar{\varsigma}_0} + \frac{(1 - \bar{\varsigma}_0)\mu_0}{\bar{\varsigma}_0}. \tag{3.55}$$

Then, substituting (3.55) into (3.52) drives

$$\ell V_{\leq} - \bar{\mu}_0 V^{\bar{\varsigma}_0} + \bar{\pi}_0, \tag{3.56}$$

where $\bar{\mu}_0 = \frac{\mu_0}{\bar{\varsigma}_0}$ and $\bar{\pi}_0 = \frac{(1-\bar{\varsigma}_0)\mu_0}{\bar{\varsigma}_0} + \pi_0$.

Let $T^* = \frac{1}{l_0\bar{\mu}_0(1-\bar{\varsigma}_0)}\left[E\left(V^{1-\bar{\varsigma}_0}(\chi(0), \tilde{e}(0), \hat{\alpha}(0))\right) - \left(\frac{\bar{\pi}_0}{\bar{\mu}_0(1-\bar{\varsigma}_0)}\right)^{\frac{1-\bar{\varsigma}_0}{\bar{\varsigma}_0}}\right]$ where $\chi(0) = (\chi_1(t_0), \cdots, \chi_n(t_0))^T$, $\tilde{e}(0) = (\tilde{e}_2(t_0), \cdots, \tilde{e}_n(t_0))^T$, $\hat{\alpha}(0) = (\hat{\alpha}_1(t_0), \cdots, \hat{\alpha}_n(t_0))$, $0 < l_0 < 1$ is a constant. Then it follows from Lemma 1 that for $\forall t \geq t_0 + T^*$, $E\left(V^{1-\varsigma}(\chi, \tilde{e}, \hat{\alpha})\right) \leq \frac{\bar{\pi}_0}{\bar{\mu}_0(1-\bar{\varsigma}_0)}$, which means that all the signals in the closed-loop systems are semi-global finite-time stable in probability. ■

Remark 6. *In this paper, we construct an output-feedback controller rather than the designed state-feedback controllers in existing results about output constraints. On the other hand, it should be pointed out the considered constraint is symmetric rather than asymmetric, which leads that the proposed scheme can not be directly employed or further extended to the case of asymmetric constraints. However, a control scheme based on a new BLF can be developed for asymmetric output constraints in a similar way to this paper. In addition, another limitation is that all of the fractional powers are equal to p. If p_i's are taken different values, the proposed strategy seems not applicable. In the future, we will address the two issues.*

4. Simulation example

The validation of the proposed strategy will be testified by the following system.

$$\begin{cases} dx_1 = x_2^{\frac{13}{5}}dt + 2\ln(1+x_1^2)dt + 4x_1^2 d\omega, \\ dx_2 = u^{\frac{13}{5}}dt + x_1x_2^2dt + x_2^2 d\omega, \\ y = \chi_1, \end{cases} \tag{4.1}$$

where the output $y = x_1$ is measurable and constrained by $\Pi_1 = \{y(t) \in R, |y(t)| < 1\}$, and the state x_2 is unmeasurable.

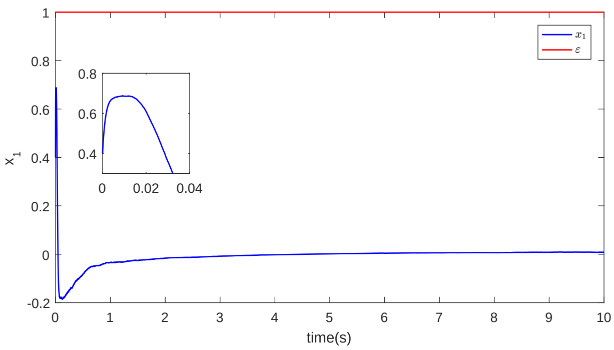

Figure 1. Trajectory of $x_1(t)$ with $\varepsilon = 1$.

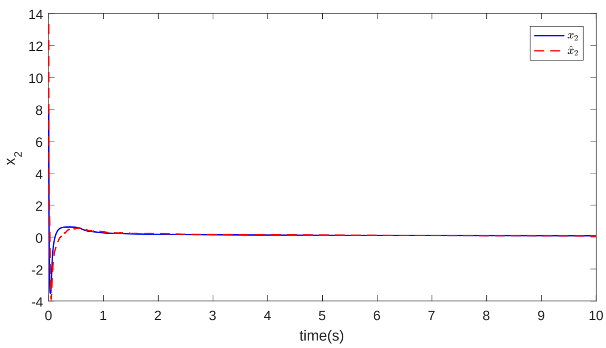

Figure 2. Trajectory of $x_2(t)$ with $\varepsilon = 1$.

According to the controller design procedure, we can respectively design the finite-time output-feedback controller, the adaptive laws and the observer as follows:

$$\begin{aligned} u &= -H^{\frac{90}{169}}(M_1^{\frac{5}{13}}x_1 + M_2^{\frac{5}{13}}\hat{\chi}_2), \\ \dot{\hat{\alpha}}_1 &= \frac{15Hb_1\sigma_{11}}{28}(S_1(x_1))^{\frac{28}{15}}x_1^{\frac{28}{5}} - d_1\hat{\alpha}_1, \\ \dot{\hat{\alpha}}_2 &= \frac{15Hb_2\sigma_{21}}{28}\xi_2^{\frac{28}{5}} - d_2\hat{\alpha}_2, \\ \dot{\hat{z}}_2 &= H\nu^{\frac{13}{5}} - H\gamma_2(\hat{z}_2 + \gamma_2 x_1)^{\frac{13}{5}}, \\ \hat{\chi}_2 &= \hat{z}_2 + \gamma_2 x_1, \end{aligned} \tag{4.2}$$

where $M_1 = \frac{15}{28}(S_1(x_1))^{\frac{13}{15}}[\sigma_{11}\hat{\alpha}_1 + 1] + \frac{\rho_1}{S_1(x_1)} + \varrho_1$ and $M_2 = \frac{15}{28}[\sigma_{21}\hat{\alpha}_2 + 1] + \frac{13\sigma_{12}}{28} + \rho_2 + \varrho_2$.

Now, we choose the related parameters as $\gamma = 1.5, \sigma_{11} = 22$, $\sigma_{21} = 20$, $\sigma_{12} = 1$, $\theta_2 = 3$, $b_1 = b_2 = 40$, $d_1 = d_2 = 2$, $\varrho_1 = \varrho_2 = 1$ and infer $H = 1.6$, $\gamma_2 = 8.1$. Next, the initial states and adaptive parameters are selected as $[x_1(0), x_2(0), \hat{x}_2(0), \hat{\alpha}_1, \hat{\alpha}_2]^T = [0.4, 8, 8, 10, 10]^T$ the simulation results are displayed in Figures 1–4.

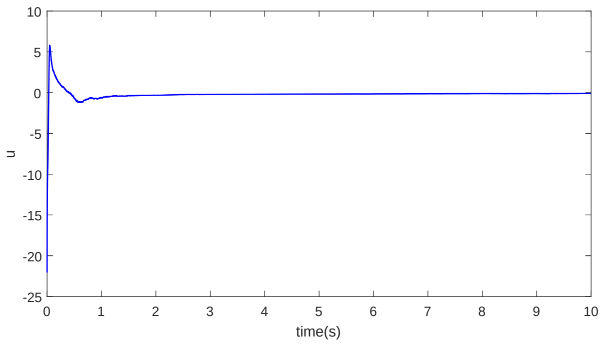

Figure 3. Trajectory of u.

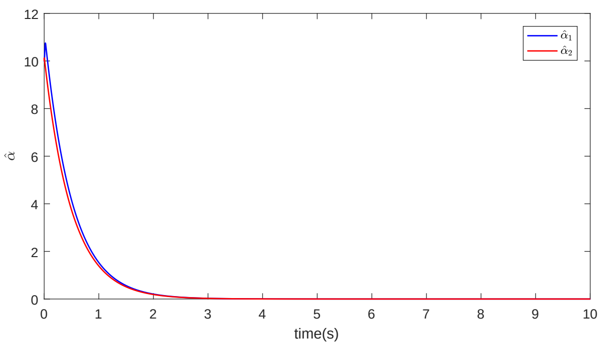

Figure 4. Trajectory of $\hat{\alpha}$.

Figure 1 provides the trajectory of $x_1(t)$, which indicates that the system output constraint is not violated under controller (4.2). Meanwhile, the trajectories of $x_2(t)$ and $\hat{x}_2(t)$ are given in Figure 2, which shows that $x_2(t)$ is well estimated by $\hat{x}_2(t)$. Moreover, the trajectory of the controller u is displayed in Figure 3. Finally, Figure 4 expresses the curves of the adaptive parameter vector under the developed strategy. Also, one could evidently observe from these figures that all the signals of system (4.1) are semi-global finite-time stable in probability under controller (4.2).

5. Conclusion

In this paper, the output-feedback controller design problem is investigated for a class of p-norm stochastic nonlinear systems with output constraints. Through using a tan-type BLF, an adaptive fuzzy state-feedback controller is proposed by the adding a power integrator technique. Then, a finite-time fuzzy output-feedback controller is constructed by combining the proposed state-feedback controller and a reduced-order observer. Both rigorous proof and the simulation example verify that the designed controller can ensure the achievement of the system output constraint and semi-global finite-time stability of all the signals in probability. In the future, we will consider the situations of asymmetric constraints, different fractional powers, or multi-input multi-output stochastic nonlinear systems.

Acknowledgments

This work was supported by supported by the National Science Foundation of China under Grant 61973142, the Jiangsu Natural Science Foundation for Distinguished Young Scholars under Grant BK20180045, the Priority Academic Program Development of Jiangsu Higher Education Institutions, and the Project of Anhui Province Outstanding Young Talent Support Program under Grant gxyq2018089.

Conflict of interest

The authors declare that there are no conflicts of interest.

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Appendix

Proof of Proposition 1. Firstly, suppose there exist $\beta_{k-1}(3 \leq k \leq n)$ such that

$$\begin{aligned} \ell V_{k-1} \leq & -\frac{\varrho_1 \varepsilon^4}{2\pi} \tan\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{k-1} \frac{d_j \tilde{\alpha}_j^2}{2b_j} - H\rho_{k-1}\xi_{k-1}^{p+3} \\ & - H\sum_{j=1}^{k-1} \varrho_j \xi_j^{p+3} + \sum_{j=1}^{k-1} Q_j + H\xi_{k-1}^3(\chi_k^p - \beta_{k-1}^p). \end{aligned} \tag{A.1}$$

At the same time, from Itô's formula and (3.1), one gets

$$d\xi_k = (H\chi_{k-1}^p + f_k - \ell\beta_{k-1})dt + \left(h_k - \sum_{j=1}^{k-1} \frac{\partial \beta_{k-1}}{\partial \chi_j} h_j\right)^T d\omega, \tag{A.2}$$

where $\ell\beta_{k-1} = \sum_{j=1}^{k-1} \frac{\partial \beta_{k-1}}{\partial \chi_j}(H\chi_{j+1}^p + f_j) + \Sigma_{j=1}^{k-1} \frac{\partial \beta_{k-1}}{\partial \hat{\alpha}_j}\dot{\hat{\alpha}}_j + \frac{1}{2}\sum_{j,l=1}^{k-1} \frac{\partial^2 \beta_{k-1}}{\partial \chi_j \partial \chi_l} h_j^T h_l$. Clearly, $\ell\beta_{k-1}$ is valid and continuous, which can be illustrated in a similar way by following the lines to obtain $\ell\beta_1$.

We choose the Lyapunov function as

$$V_k = V_{k-1} + \Lambda_k \tag{A.3}$$

with

$$\Lambda_k = \frac{1}{4}\xi_k^4 + \frac{1}{2b_k}\tilde{\alpha}_k^2, \tag{A.4}$$

where $b_k > 0$ is an adjustment parameter, $\tilde{\alpha}_k = \alpha_k - \hat{\alpha}_k$ is the parameter error, and $\hat{\alpha}_k$ is the estimation of the unknown parameter α_k.

Applying (2.3), (A.2) and (A.4), one can obtain

$$\begin{aligned}\ell\Lambda_k &= \xi_k^3(H\chi_{k+1}^p + f_k - \ell\beta_{k-1}) - \frac{1}{b_k}\tilde{\alpha}_k\dot{\hat{\alpha}}_k \\ &+ \frac{3}{2}\left\|h_k - \sum_{j=1}^{k-1}\frac{\partial\beta_{k-1}}{\partial\chi_j}h_j\right\|^2\xi_k^2.\end{aligned} \tag{A.5}$$

It is not difficult to get from Lemma 3 that

$$\frac{3}{2}\left\|h_k - \sum_{j=1}^{k-1}\frac{\partial\beta_{k-1}}{\partial\chi_j}h_j\right\|^2\xi_k^2 \le \frac{3}{4}\left\|h_k - \sum_{j=1}^{k-1}\frac{\partial\beta_{k-1}}{\partial\chi_j}h_j\right\|^4\xi_k^4 + \frac{3}{4}. \tag{A.6}$$

On the other hand, from Lemmas 2 and 3, one can verify

$$\begin{aligned}\xi_{k-1}^3\left(\chi_k^p - \beta_{k-1}^p\right) &\le D|\xi_{k-1}|^3\left[|\xi_k|^p + \varphi_{k-1}^{p-1}|\xi_{k-1}|^{p-1}\cdot|\xi_k|\right] \\ &\le \rho_{k-1}\xi_{k-1}^{p+3} + \tau_k\xi_k^{p+3} + \frac{p\sigma_{k-1\ 2}}{p+3}\xi_k^{p+3},\end{aligned} \tag{A.7}$$

where $\rho_{k-1} = \frac{1}{p+3}\left[3D^{\frac{p+3}{3}}\sigma_{12}^{-\frac{p}{3}} + p + 2\right]$, $\tau_k = \frac{\varphi_1^{(p-1)(p+3)}}{p+3}$, and $\sigma_{k-1\ 2} > 0$ is an adjustment parameter.

From (A.1)-(A.7), it can be deduced that

$$\begin{aligned}\ell V_k &= \ell V_{k-1} + \ell\Lambda_k \\ &\le -\frac{\varrho_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\xi_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{k-1}\frac{d_j\tilde{\alpha}_j^2}{2b_j} - H\sum_{j=1}^{k-1}\varrho_j\xi_j^{p+3} + \sum_{j=1}^{k-1}Q_j + \frac{3}{4} \\ &+ H\frac{p\sigma_{k-1\ 2}}{p+3}\xi_k^{p+3} + H\xi_k^3F_k(Z_k) + H\xi_k^3(\chi_{k+1}^p - \beta_k^p) + H\xi_k^3\beta_k^p\end{aligned} \tag{A.8}$$

where $Z_k = (\bar{\chi}_k^T, \bar{\hat{\alpha}}_{k-1}^T)^T$, $\bar{\hat{\alpha}}_{k-1} = (\hat{\alpha}_1, \cdots, \hat{\alpha}_{k-1})^T$ and

$$F_k(Z_k) = \frac{1}{H}(f_k - \ell\beta_{k-1}) + \tau_k\xi_k^p + \frac{3}{4}\left\|h_k - \sum_{j=1}^{k-1}\frac{\partial\beta_{k-1}}{\partial\chi_j}h_j\right\|^4\xi_k.$$

Similar to the first two steps, $F_k(Z_k)$ can also be approximated as

$$F_k(Z_k) = \|\Upsilon_k^T\Psi_k(Z_k)\| + \epsilon_k(Z_k), \tag{A.9}$$

where $|\epsilon_k(Z_k)| \le \delta_k$ and $\delta_k > 0$ is a given constant.

One directly obtains from Eq (A.9) and Lemma 3 that

$$\begin{aligned}\xi_k^3F_k(Z_k) &\le |\xi_k|^3(\|\Upsilon_k\|\|\Psi_k\| + \delta_k) \\ &\le \frac{3\sigma_{k1}\alpha_k}{p+3}\xi_k^{p+3} + \frac{p}{p+3}\sigma_{k1}^{-\frac{3}{p}} + \frac{3}{p+3}\xi_k^{p+3} + \frac{p}{p+3}\delta_k^{\frac{p+3}{p}},\end{aligned} \tag{A.10}$$

where $\alpha_k = \|\Upsilon_k\|^{\frac{p+3}{3}}$ and $\sigma_{k1} > 0$ is an adjustment parameter.

Substituting (A.10) into (A.8) renders

$$\begin{aligned}\ell V_k \le & -\frac{\varrho_1 \varepsilon^4}{2\pi}\tan\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{k-1}\frac{d_j\tilde{\alpha}_j^2}{2b_j} - H\sum_{j=1}^{k-1}\varrho_j\xi_j^{p+3} + H\xi_k^3(\chi_{k+1}^p - \beta_k^p) \\ & + \frac{3H}{p+3}\left[\sigma_{k1}\hat{\alpha}_k + 1\right]\xi_k^{p+3} + H\frac{p\sigma_{k-1\ 2}}{p+3}\xi_k^{p+3} + H\xi_k^3\beta_k^p + \sum_{j=1}^{k-1}Q_j \\ & + H\tilde{\alpha}_k\left[\frac{3\sigma_{k1}}{p+3}\xi_k^{p+3} - \frac{1}{b_k H}\dot{\hat{\alpha}}_k\right] + \frac{3}{4} + \frac{pH}{p+3}\sigma_{k1}^{-\frac{3}{p}} + \frac{pH}{p+3}\delta_k^{\frac{p+3}{p}}.\end{aligned} \tag{A.11}$$

Then, we can design

$$\begin{aligned}\beta_k &= -M_k^{\frac{1}{p}}\xi_k \triangleq -\varphi_k\xi_k, \\ \dot{\hat{\alpha}}_k &= \frac{3Hb_k\sigma_{k1}}{p+3}\xi_k^{p+3} - d_k\hat{\alpha}_k,\end{aligned} \tag{A.12}$$

where $M_k \ge \frac{3}{p+3}\left[\sigma_{k1}\hat{\alpha}_k + 1\right] + \frac{p\sigma_{k-1\ 2}}{p+3} + \rho_k + \varrho_k$; $\varrho_k, d_k > 0$ are adjustment parameters; and the value of $\rho_k > 0$ will be given in $(k+1)$th step.

Moreover, one has

$$\frac{d_k}{b_k}\hat{\alpha}_k\tilde{\alpha}_k \le -\frac{d_k}{2b_k}\tilde{\alpha}_k^2 + \frac{d_k}{2b_k}\hat{\alpha}_k^2. \tag{A.13}$$

From (A.1)-(A.13), it can be deduced that

$$\begin{aligned}\ell V_k \le & -\frac{\varrho_1 \varepsilon^4}{2\pi}\tan\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{k}\frac{d_j\tilde{\alpha}_j^2}{2b_j} - H\sum_{j=1}^{k}\varrho_j\xi_j^{p+3} \\ & - H\tau_k\xi_k^{p+3} + \sum_{j=1}^{k}Q_j + H\xi_k^3(\chi_{k+1}^p - \beta_k^p),\end{aligned} \tag{A.14}$$

where $Q_k = \frac{3}{4} + \frac{pH}{p+3}\sigma_{k1}^{-\frac{3}{p}} + \frac{pH}{p+3}\delta_k^{\frac{p+3}{p}} + \frac{d_k\alpha_k^2}{2b_k}$. The proof of Proposition 1 is completed.

Proof of Proposition 2. By the definition of U_n, one has

$$\begin{aligned}\ell U_n = & H\gamma\sum_{i=2}^{n-1}\tilde{e}_i^3(\chi_{i+1}^p - \hat{\chi}_{i+1}^p) - H\gamma\sum_{i=2}^{n}\tilde{e}_i^3\gamma_i(\chi_i^p - \hat{\chi}_i^p) \\ & + \gamma\sum_{i=2}^{n}\tilde{e}_i^3(f_i - \gamma_i f_{i-1}) + \frac{3}{2}\gamma\sum_{i=2}^{n}\tilde{e}_i^2\|h_i - \gamma_i h_{i-1}\|^2 \\ = & -H\sum_{i=2}^{n}\gamma\gamma_i\tilde{e}_i^3[(\hat{\chi}_i + \tilde{e}_i)^p - \hat{\chi}_i^p] - H\sum_{i=2}^{n}\gamma\gamma_i\tilde{e}_i^3[\chi_i^p - (\hat{\chi}_i + \tilde{e}_i)^p] \\ & + H\sum_{i=3}^{n}\gamma\tilde{e}_{i-1}^3(\chi_i^p - \hat{\chi}_i^p) + H\sum_{i=2}^{n}\frac{\gamma}{H}\tilde{e}_i^3(f_i - \gamma_i f_{i-1}) + H\sum_{i=2}^{n}\frac{3\gamma}{2H}\tilde{e}_i^2\|h_i - \gamma_i h_{i-1}\|^2.\end{aligned} \tag{A.15}$$

By Lemma 5, one can infer

$$\begin{aligned}-\gamma\gamma_i\tilde{e}_i^3[(\hat{\chi}_i+\tilde{e}_i)^p-\hat{\chi}_i^p] &= -\gamma\gamma_i\tilde{e}_i^2(\hat{\chi}_i+\tilde{e}_i-\hat{\chi}_i)[(\hat{\chi}_i+\tilde{e}_i)^p-\hat{\chi}_i^p]\\ &\le -\frac{\gamma\gamma_i}{2^{p-1}}\tilde{e}_i^{p+3}.\end{aligned} \tag{A.16}$$

Since $\chi_i-\hat{\chi}_i-\tilde{e}_i=\sum_{j=2}^{i-1}\gamma_i\cdots\gamma_{j+1}\tilde{e}_j$, we can get that through applying Lemmas 2-5

$$\begin{aligned}&|-\gamma\gamma_i\tilde{e}_i^3[\chi_i^p-(\hat{\chi}_i+\tilde{e}_i)^p]|\\ &\le \gamma\gamma_i|\tilde{e}_i|^3|\chi_i-\hat{\chi}_i-\tilde{e}_i||(\chi_i-\hat{\chi}_i-\tilde{e}_i)^{p-1}+\chi_i^{p-1}|\\ &\le \gamma_i|\tilde{e}_i|^3\left(\sum_{j=2}^{i-1}\gamma_i\cdots\gamma_{j+1}|\tilde{e}_j|\right)^p\\ &+\gamma_i|\tilde{e}_i|^3\left[\frac{1}{p}\left(\sum_{j=2}^{i-1}\gamma_i\cdots\gamma_{j+2}|\tilde{e}_j|\right)^p+\frac{p-1}{p}|\chi_i|^p\right]\\ &\le \frac{3\gamma\gamma_i}{p+3}\tilde{e}_i^{p+3}+\frac{p\gamma\gamma_i}{p+3}\left(\sum_{j=2}^{i-1}\gamma_i\cdots\gamma_{j+1}|\tilde{e}_j|\right)^{p+3}+\frac{3\gamma\gamma_i}{p(p+3)}\tilde{e}_i^{p+3}\\ &+\frac{\gamma\gamma_i}{p+3}\left(\sum_{j=2}^{i-1}\gamma_i\cdots\gamma_{j+2}|\tilde{e}_j|\right)^{p+3}+\frac{3(p-1)(\gamma\gamma_i)^{\frac{p+3}{3}}}{p(p+3)}\tilde{e}_i^{p+3}+\frac{p-1}{p+3}\chi_i^{p+3}\\ &\le \sum_{j=2}^{i}\tilde{a}_{ij}\tilde{e}_j^{p+3}+\frac{p-1}{p+3}\chi_i^{p+3},\end{aligned} \tag{A.17}$$

where $\tilde{a}_{ij}=\tilde{a}_{ij}(\gamma_i,\cdots,\gamma_{j+1})$ is a constant independent of H.

Noting that $e_i=\sum_{j=2}^{i-1}\gamma_i\cdots\gamma_{j+1}\tilde{e}_j+\tilde{e}_i$, one has

$$\begin{aligned}&|\gamma\tilde{e}_{i-1}^3(\chi_i^p-\hat{\chi}_i^p)|=\gamma|\tilde{e}_{i-1}|^3|\chi_i^p-(\chi_i-e_i)^p|\\ &\le 2^p\gamma|\tilde{e}_{i-1}|^3(|\chi_i|^p+|e_i|^p)\\ &\le \frac{3(2^p\gamma)^{\frac{p+3}{3}}}{p+3}\tilde{e}_{i-1}^{p+3}+\frac{p}{p+3}\chi_i^{p+3}+\frac{3(2^p\gamma)^{\frac{p+3}{3}}}{p+3}\tilde{e}_{i-1}^{p+3}+\frac{p}{p+3}e_i^{p+3}\\ &\le \frac{6(2^p\gamma)^{\frac{p+3}{3}}}{p+3}\tilde{e}_{i-1}^{p+3}+\frac{p2^{p+2}}{p+3}[\xi_i^{p+3}+\varphi_{i-1}^{p+3}\xi_{i-1}^{p+3}]\\ &+\frac{pi^{p+2}}{p+3}[\tilde{e}_i^{p+3}+\sum_{j=2}^{i-1}(\gamma_i\cdots\gamma_{j+1})^{p+3}\tilde{e}_j^{p+3}]\\ &\le \sum_{j=2}^{i}\tilde{c}_{ij}(\gamma_i,\cdots,\gamma_{j+1})\tilde{e}_j^{p+3}+\frac{p}{p+3}\chi_i^{p+3},\end{aligned} \tag{A.18}$$

where $\tilde{c}_{ij}=\tilde{c}_{ij}(\gamma_i,\cdots,\gamma_{j+1})$ is a constant independent of H.

In addition, we have

$$
\begin{aligned}
&\frac{\gamma}{H}\tilde{e}_i^3(f_i-\gamma_i f_{i-1})+\frac{3}{2}\frac{\gamma}{H}\tilde{e}_i^2\|h_i-\gamma_i h_{i-1}\|^2\\
&\le \frac{3\gamma^{\frac{4}{3}}}{2H^{\frac{4}{3}}}\tilde{e}_i^4+\frac{1}{4}(f_i-\gamma_i f_{i-1})^4+\frac{\gamma^{\frac{2}{3}}}{4H^{\frac{2}{3}}}\|h_i-\gamma_i h_{i-1}\|^4\\
&\le \frac{6\gamma^{\frac{p+3}{3}}}{(p+3)H^{\frac{p+3}{3}}}\tilde{e}_i^{p+3}+\frac{p-1}{p+3}+\frac{1}{4}(f_i-\gamma_i f_{i-1})^4+\frac{3\gamma^{\frac{2}{3}}}{4H^{\frac{2}{3}}}\|h_i-\gamma_i h_{i-1}\|^4.
\end{aligned}
\tag{A.19}
$$

Substituting (A.16)-(A.19) into (A.15) yields

$$
\begin{aligned}
\ell U_n \le& -H\left[\sum_{i=2}^{n}\frac{\gamma\gamma_i}{2^{p-1}}\tilde{e}_i^{p+3}-\sum_{i=2}^{n}\bar{c}_i\tilde{e}_i^{p+3}-\frac{6\gamma^{\frac{p+3}{3}}}{(p+3)H^{\frac{p+3}{3}}}\tilde{e}_i^{p+3}\right]+\frac{(p-1)H}{p+3}\\
&+\sum_{i=2}^{n}\frac{(2p-1)H}{p+3}\chi_i^{p+3}+\sum_{i=2}^{n}H\left[\frac{1}{4}(f_i-\gamma_i f_{i-1})^4+\frac{3\gamma^{\frac{2}{3}}}{4H^{\frac{2}{3}}}\|h_i-\gamma_i h_{i-1}\|^4\right],
\end{aligned}
$$

where $\bar{c}_i=\sum_{j=2}^{i}(\tilde{a}_{ji}+\tilde{c}_{ji})$. The proof of Proposition 2 is completed.