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Research article

An interesting approach to the existence of coupled fixed point

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Abstract: Configure a coupled fixed point result on a nonempty set engaging a partial order and induced with a quasi-metric in the sense of Kunzi [12] in the framework of $\mathcal{G}$-metric spaces. Our result is supported by an illustrative example.

Keywords: $\mathcal{G}$-metric space; Quasi-metric space; coupled fixed point; $\mathcal{G}$-Cauchy sequence; partial order
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1. Introduction and Preliminaries

S. Banach in his famous work [5] established a contraction which is known as Banach contraction. After that, a large number of authors established various fixed point results by extending the contraction mapping principle. Metric spaces have been extended in various directions by various authors and $\mathcal{G}$-metric space is one of such direction introduced by Mustafa and Sims [13]. Some more results on that spaces may be noted as [1, 2, 6, 9–11, 15, 20, 21]. Recently, various authors [4, 16–19] established coupled fixed point sresults for non-linear contractive operators in partially order $\mathcal{G}$-metric spaces.

The main features of the present work are
(i) We have established a coupled fixed point result (will be represented as CFP form now on).
(ii) The space where we establish the result is $\mathcal{G}$-metric space induced with quasi metric $\varsigma_{\mathcal{G}}$.
(iii) Use of control functions to derive the results.

(iv) Mixed monotone property is also used to derive the result.
(v) An explicit discussion on an example is also provided to validate our theorem.

Some important definitions and mathematical preliminaries are given below.

Definition 1.1 (See [3, 8, 12]). *A quasi-metric on a non-empty set $\mathcal{X}$ is a function* $\mathrm{q} : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *satisfying the following properties:*
(q1) $\mathrm{q}(\sigma, \nu) = 0$ *if and only if* $\sigma = \nu$*;*
(q2) $\mathrm{q}(\sigma, \nu) \leq \mathrm{q}(\sigma, \kappa) + \mathrm{q}(\kappa, \nu)$*, for all* $\sigma, \nu, \kappa \in \mathcal{X}$*.*
In such a case, the ordered pair $(\mathcal{X}, \mathrm{q})$ *is called a quasi-metric space.*

Example 1.1 (See [3]). *Let $\mathcal{X}$ be a subset of $\mathbb{R}$ containing* $[0, 1]$ *and define, for all* $\sigma, \nu \in \mathcal{X}$*,*

$$\mathrm{q}(\sigma, \nu) = \begin{cases} \sigma - \nu; \sigma \geq \nu, \\ 1; \ \textit{otherwise}. \end{cases}$$

Then $(\mathcal{X}, \mathrm{q})$ *is a quasi-metric space.*

For more terms like symmetry, convergence, Cauchy sequence, completeness, continuity in quasi-metric spaces see Agarwal et al. [3].

Definition 1.2 (See [13]). *Let $\mathcal{X}$ be a non-empty set,* $\mathcal{G} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ *be a function satisfying the following properties:*
(G1) $\mathcal{G}(\sigma, \nu, \kappa) = 0$ *if* $\sigma = \nu = \kappa$*,*
(G2) $0 < \mathcal{G}(\sigma, \sigma, \nu)$ *for all* $\sigma, \nu \in \mathcal{X}$ *with* $\sigma \neq \nu$*,*
(G3) $\mathcal{G}(\sigma, \sigma, \nu) \leq \mathcal{G}(\sigma, \nu, \kappa)$ *for all* $\sigma, \nu, \kappa \in \mathcal{X}$ *with* $\nu \neq \kappa$*,*
(G4) $\mathcal{G}(\sigma, \nu, \kappa) = \mathcal{G}(\sigma, \kappa, \nu) = \mathcal{G}(\nu, \kappa, \sigma) = \ldots$ *(symmetry in all three variables),*
(G5) $\mathcal{G}(\sigma, \nu, \kappa) \leq \mathcal{G}(\sigma, a, a) + \mathcal{G}(a, \nu, \kappa)$ *for all* $\sigma, \nu, \kappa, a \in \mathcal{X}$*(rectangle inequality).*
Then the function $\mathcal{G}$ is known as a generalized metric, or, more precisely, a $\mathcal{G}$-metric on $\mathcal{X}$ and the pair $(\mathcal{X}, \mathcal{G})$ *is called a $\mathcal{G}$-metric space.*

Example 1.2 (See [14]). *Let* $(\mathcal{X}, \varsigma)$ *be a metric space. The function* $\mathcal{G} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0, +\infty)$*, defined by either*

$$\mathcal{G}(\sigma, \nu, \kappa) = \max\{\varsigma(\sigma, \nu), \varsigma(\nu, \kappa), \varsigma(\kappa, \sigma)\}$$

or

$$\mathcal{G}(\sigma, \nu, \kappa) = \varsigma(\sigma, \nu) + \varsigma(\nu, \kappa) + \varsigma(\kappa, \sigma)$$

for all $\sigma, \nu, \kappa \in \mathcal{X}$*, is a $\mathcal{G}$ - metric on $\mathcal{X}$.*

For more detail discussions about symmetric $\mathcal{G}$-metric, $\mathcal{G}$-Cauchy sequence, continuity of $\mathcal{G}$-function, $\mathcal{G}$-completeness, one may refer to paper Mustafa et al. [13].

Definition 1.3 (See [7]). *Let $\mathcal{X}$ be a nonempty set and* $\mathcal{F} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ *be a given mapping. We say that* $(\sigma, \nu) \in \mathcal{X} \times \mathcal{X}$ *is a coupled fixed point of $\mathcal{F}$ if* $\mathcal{F}(\sigma, \nu) = \sigma$ *and* $\mathcal{F}(\nu, \sigma) = \nu$*.*

Definition 1.4 (See [7]). *Let* $(\mathcal{X}, \preceq)$ *be a partially ordered set and* $\mathcal{F} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ *be a given mapping. We say that $\mathcal{F}$ has the mixed monotone property if*

$$\sigma_1, \sigma_2 \in \mathcal{X}, \sigma_1 \preceq \sigma_2 \Rightarrow \mathcal{F}(\sigma_1, \nu) \preceq \mathcal{F}(\sigma_2, \nu), \textit{for all } \nu \in \mathcal{X}$$

and $\nu_1, \nu_2 \in \mathcal{X}, \nu_1 \preceq \nu_2 \Rightarrow \mathcal{F}(\sigma, \nu_2) \preceq \mathcal{F}(\sigma, \nu_1)$*, for all* $\sigma \in \mathcal{X}$*.*

2. Main result

In this section we have established a coupled fixed point result in partially ordered $\mathcal{G}$ - metric spaces. The existence of CFP is shown in the context of $\mathcal{G}$- metric spaces induced with the quasi metric.

We denote Ψ, the family of continuous and monotone non-decreasing functions $\chi : [0, \infty) \to [0, \infty)$ such that
i) $\chi(t) = 0$, iff $t = 0$
ii) $\chi(t + s) \leq \chi(t) + \chi(s)$, for all $t, s \in [0, \ \infty)$
and Φ denote the family of continuous non decreasing functions $\varrho : [0, \infty) \to [0, \infty)$ with $\varrho(0) = 0$.

Throughout this section, we assume that $(\mathcal{X}, \mathcal{G})$ is a $\mathcal{G}$-metric space and define $\varsigma_{\mathcal{G}} : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by $\varsigma_{\mathcal{G}}(\sigma, \ \nu) = \mathcal{G}(\sigma, \ \nu, \ \nu)$. Using Lemma 3.3.1 of [3], every $\mathcal{G}$-metric $\mathcal{G}$ induces a quasi-metric $\varsigma_{\mathcal{G}}$ in the sense of Kunzi [12] in such a way that $\tau(\mathcal{G}) = \tau(\varsigma_{\mathcal{G}})$, where τ is a topology on $\mathcal{X}$.

Theorem 2.1. *Let $(\mathcal{X}, \ \preceq)$ be a partially ordered set induced with quasi-metric $\varsigma_{\mathcal{G}}$ such that $(\mathcal{X}, \ \mathcal{G})$ be a $\mathcal{G}$-complete $\mathcal{G}$-metric space. Let $\mathcal{F} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a $\mathcal{G}$-continuous mapping having mixed monotone property on $\mathcal{X}$ and satisfies the following*

$$\chi[\mathcal{G}(\mathcal{F}(\sigma, \ \nu), \ \mathcal{F}(\beta, \ \lambda), \ \mathcal{F}(\theta, \ \kappa))] \leq \frac{1}{2}\chi[\mathcal{G}(\sigma, \ \beta, \ \theta) + \mathcal{G}(\nu, \ \lambda, \ \kappa)] - \varrho[\mathcal{G}(\sigma, \ \beta, \ \theta) + \mathcal{G}(\nu, \ \lambda, \ \kappa)] \quad (2.1)$$

for all $\sigma, \ \nu, \ \kappa, \ \beta, \ \lambda, \ \theta \in \mathcal{X}$ with $\sigma \succeq \beta \succeq \theta$, $\nu \preceq \lambda \preceq \kappa$, $\varrho \in \Phi, \chi \in \Psi$ and either $\nu \neq \kappa$ or $\beta \neq \theta$.
If there exist $\sigma_0, \ \nu_0 \in \mathcal{X}$ such that $\sigma_0 \preceq \mathcal{F}(\sigma_0, \ \nu_0)$ and $\nu_0 \succeq \mathcal{F}(\nu_0, \ \sigma_0)$, then $\mathcal{F}$ has a CFP in $\mathcal{X}$, that is, there exist $\sigma, \ \nu \in \mathcal{X}$ such that $\sigma = \mathcal{F}(\sigma, \ \nu)$ and $\nu = \mathcal{F}(\nu, \ \sigma)$.

Proof. By the statement there exist $\sigma_0, \ \nu_0 \in \mathcal{X}$ such that $\sigma_0 \preceq \mathcal{F}(\sigma_0, \ \nu_0)$ and $\nu_0 \succeq \mathcal{F}(\nu_0, \ \sigma_0)$.
We define $\sigma_1, \ \nu_1 \in \mathcal{X}$ such that $\sigma_1 = \mathcal{F}(\sigma_0, \ \nu_0) \succeq \sigma_0$ and $\nu_1 = \mathcal{F}(\nu_0, \ \sigma_0) \preceq \nu_0$. In the same manner and utilizing the mixed monotone property of $\mathcal{F}$ we construct, $\sigma_2 = \mathcal{F}(\sigma_1, \ \nu_1) \succeq \mathcal{F}(\sigma_0, \ \nu_1) \succeq \mathcal{F}(\sigma_0, \ \nu_0) = \sigma_1$ and $\nu_2 = \mathcal{F}(\nu_1, \ \sigma_1) \preceq \mathcal{F}(\nu_1, \ \sigma_0) \preceq \mathcal{F}(\nu_0, \ \sigma_0) = \nu_1$.
Continuing the iteration, we obtain two sequences $\{\sigma_n\}$ and $\{\nu_n\}$ in $\mathcal{X}$ such that

$$\sigma_{n+1} = \mathcal{F}(\sigma_n, \ \nu_n) \text{ and } \nu_{n+1} = \mathcal{F}(\nu_n, \ \sigma_n) \text{ for all } n \geq 0. \quad (2.2)$$

Thus for all $n \geq 0$,

$$\sigma_0 \preceq \mathcal{F}(\sigma_0, \ \nu_0) = \sigma_1 \preceq \mathcal{F}(\sigma_1, \ \nu_1) = \sigma_2 \preceq \cdots \preceq \mathcal{F}(\sigma_n, \ \nu_n) = \sigma_{n+1} \preceq \cdots \quad (2.3)$$

and

$$\nu_0 \succeq \mathcal{F}(\nu_0, \ \sigma_0) = \nu_1 \succeq \mathcal{F}(\nu_1, \ \sigma_1) = \nu_2 \succeq \cdots \succeq \mathcal{F}(\nu_n, \ \sigma_n) = \nu_{n+1} \succeq \cdots . \quad (2.4)$$

In view of (2.3) and (2.4), we have $\{\sigma_n\}$ is an increasing sequence and $\{\nu_n\}$ is a decreasing sequence. Further from (2.1), (2.3) and (2.4), we have

$$\begin{aligned} \chi[\varsigma_{\mathcal{G}}(\sigma_{n+1}, \ \sigma_n)] &= \chi[\varsigma_{\mathcal{G}}(\mathcal{F}(\sigma_n, \ \nu_n), \ \mathcal{F}(\sigma_{n-1}, \ \nu_{n-1}))] \quad (2.5) \\ &\leq \frac{1}{2}\chi[\varsigma_{\mathcal{G}}(\sigma_n, \ \sigma_{n-1}) + \varsigma_{\mathcal{G}}(\nu_n, \ \nu_{n-1})] \\ &\quad -\varrho[\varsigma_{\mathcal{G}}(\sigma_n, \ \sigma_{n-1}) + \varsigma_{\mathcal{G}}(\nu_n, \ \nu_{n-1})] \end{aligned}$$

and

$$\begin{aligned}\chi[\varsigma_{\mathcal{G}}(\nu_{n+1},\ \nu_n)] &= \chi[\varsigma_{\mathcal{G}}(\mathcal{F}(\nu_n,\ \sigma_n),\ \mathcal{F}(\nu_{n-1},\ \sigma_{n-1}))] \\ &\leq \frac{1}{2}\chi[\varsigma_{\mathcal{G}}(\nu_n,\ \nu_{n-1}) + \varsigma_{\mathcal{G}}(\sigma_n,\ \sigma_{n-1})] \\ &\quad -\varrho[\varsigma_{\mathcal{G}}(\nu_n,\ \nu_{n-1}) + \varsigma_{\mathcal{G}}(\sigma_n,\ \sigma_{n-1})].\end{aligned} \tag{2.6}$$

Letting $a_n = \varsigma_{\mathcal{G}}(\sigma_{n+1},\ \sigma_n)$ and $b_n = \varsigma_{\mathcal{G}}(\nu_{n+1},\ \nu_n)$, using the above inequalities and assumptions of the statement, we have

$$\begin{aligned}\chi(a_n + b_n) &\leq \chi(a_n) + \chi(b_n) \\ &\leq \chi[\varsigma_{\mathcal{G}}(\sigma_n,\ \sigma_{n-1}) + \varsigma_{\mathcal{G}}(\nu_n,\ \nu_{n-1})] \\ &\quad -2\varrho[\varsigma_{\mathcal{G}}(\sigma_n,\ \sigma_{n-1}) + \varsigma_{\mathcal{G}}(\nu_n,\ \nu_{n-1})] \\ &= \chi(a_{n-1} + b_{n-1}) - 2\varrho(a_{n-1} + b_{n-1})\end{aligned} \tag{2.7}$$

Now from the above relation and monotone property of χ, we have $(a_n + b_n) \leq (a_{n-1} + b_{n-1})$. Hence we conclude that the sequence $\{a_n + b_n\}$ is a monotonic decreasing sequence of non-negative real numbers.
Hence there exists $\ell \geq 0$ such that $\lim_{n\to\infty}(a_n + b_n) = \ell$.

First assume that $\ell > 0$. Taking the limit as $n \to \infty$ in (2.7) and using the property of χ, we have $\chi(\ell) \leq \chi(\ell) - 2\varrho(\ell)$, which is a contradiction unless $\ell = 0$.
Hence, we have

$$\lim_{n\to\infty}[\varsigma_{\mathcal{G}}(\sigma_n,\ \sigma_{n-1}) + \varsigma_{\mathcal{G}}(\nu_n,\ \nu_{n-1})] = 0.$$

It implies

$$\lim_{n\to\infty}\varsigma_{\mathcal{G}}(\sigma_n,\ \sigma_{n-1}) = 0 \tag{2.8}$$

and

$$\lim_{n\to\infty}\varsigma_{\mathcal{G}}(\nu_n,\ \nu_{n-1}) = 0. \tag{2.9}$$

Next we have to show that $\{\sigma_n\}$ and $\{\nu_n\}$ are $\mathcal{G}$-Cauchy sequences.
If otherwise, there exists $\epsilon > 0$ for which there are integers $\{m_k\}$ and $\{n_k\}$ such that $n(k) > m(k) > k$ for which

$$\mathcal{G}_k = \varsigma_{\mathcal{G}}(\sigma_{m(k)},\ \sigma_{n(k)}) + \varsigma_{\mathcal{G}}(\nu_{m(k)},\ \nu_{n(k)}) \geq \epsilon \tag{2.10}$$

We choose $m(k)$ be the smallest positive integer for which (2.10) holds. Then we have,

$$\mathcal{G}_k = \varsigma_{\mathcal{G}}(\sigma_{m(k)-1},\ \sigma_{n(k)}) + \varsigma_{\mathcal{G}}(\nu_{m(k)-1},\ \nu_{n(k)}) < \epsilon \tag{2.11}$$

Then from (2.10) and (2.11), we have,

$$\begin{aligned}\epsilon \leq \mathcal{G}_k &= \varsigma_{\mathcal{G}}(\sigma_{m(k)},\ \sigma_{n(k)}) + \varsigma_{\mathcal{G}}(\nu_{m(k)},\ \nu_{n(k)}) \\ &\leq \varsigma_{\mathcal{G}}(\sigma_{m(k)},\ \sigma_{m(k)-1}) + \varsigma_{\mathcal{G}}(\sigma_{m(k)-1},\ \sigma_{n(k)})\end{aligned}$$

$$
\begin{aligned}
&+\varsigma_{\mathcal{G}}(\nu_{m(k)},\ \nu_{m(k)-1})+\varsigma_{\mathcal{G}}(\nu_{m(k)-1},\ \nu_{n(k)}) \\
<\ &\epsilon+a_{m(k)-1}+b_{m(k)-1}
\end{aligned}
$$

Taking $k\to\infty$ in the above inequality and using (2.8) and (2.9), we get,

$$
\lim_{k\to\infty}\mathcal{G}_k=\epsilon \tag{2.12}
$$

Now from (2.1), (2.3), (2.4) and (2.10), for all $k\geq 0$, we have,

$$
\begin{aligned}
\chi[\varsigma_{\mathcal{G}}(\sigma_{m(k)+1},\ \sigma_{n(k)+1})] &= \chi[\varsigma_{\mathcal{G}}(\mathcal{F}(\sigma_{m(k)},\ \nu_{m(k)}),\ \mathcal{F}(\sigma_{n(k)},\nu_{n(k)}))] \\
&\leq \frac{1}{2}\chi[(\varsigma_{\mathcal{G}}(\sigma_{m(k)},\ \sigma_{n(k)})+\varsigma_{\mathcal{G}}(\nu_{m(k)},\ \nu_{n(k)})] \\
&\quad -\varrho[(\varsigma_{\mathcal{G}}(\sigma_{m(k)},\ \sigma_{n(k)})+\varsigma_{\mathcal{G}}(\nu_{m(k)},\ \nu_{n(k)})] \\
&= \frac{1}{2}\chi(\mathcal{G}_k)-\varrho(\mathcal{G}_k)
\end{aligned} \tag{2.13}
$$

Also from (2.1), (2.3), (2.4) and (2.10), for all $k\geq 0$, we have,

$$
\begin{aligned}
\chi[\varsigma_{\mathcal{G}}(\nu_{m(k)+1},\ \nu_{n(k)+1})] &= \chi[\varsigma_{\mathcal{G}}(\mathcal{F}(\nu_{m(k)},\sigma_{m(k)}),\ \mathcal{F}(\nu_{n(k)},\sigma_{n(k)}))] \\
&\leq \frac{1}{2}\chi[(\varsigma_{\mathcal{G}}(\nu_{m(k)},\ \nu_{n(k)})+\varsigma_{\mathcal{G}}(\sigma_{m(k)},\ \sigma_{n(k)})] \\
&\quad -\varrho[(\varsigma_{\mathcal{G}}(\nu_{m(k)},\ \nu_{n(k)})+\varsigma_{\mathcal{G}}(\sigma_{m(k)},\ \sigma_{n(k)})] \\
&= \frac{1}{2}\chi(\mathcal{G}_k)-\varrho(\mathcal{G}_k).
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
\chi(\mathcal{G}_{k+1}) &= \chi[\varsigma_{\mathcal{G}}(\sigma_{m(k)+1},\ \sigma_{n(k)+1})+\varsigma_{\mathcal{G}}(\nu_{m(k)+1},\ \nu_{n(k)+1})] \\
&\leq \chi[\varsigma_{\mathcal{G}}(\sigma_{m(k)+1},\ \sigma_{n(k)+1})]+\chi[\varsigma_{\mathcal{G}}(\nu_{m(k)+1},\ \nu_{n(k)+1})], \\
&\qquad\qquad\qquad\qquad \text{[By the property (ii) of } \chi\text{-function]} \\
&= \frac{1}{2}\chi(\mathcal{G}_k)+\frac{1}{2}\chi(\mathcal{G}_k)-2\varrho(\mathcal{G}_k)\ =\ \chi(\mathcal{G}_k)-2\varrho(\mathcal{G}_k).
\end{aligned} \tag{2.14}
$$

Taking $k\to\infty$ in (2.14) and continuity of χ function, we have,

$$
\chi(\epsilon)\leq\chi(\epsilon)-2\varrho(\epsilon)
$$

which is a contradiction.
As a consequence we have, $\{\sigma_n\}$ and $\{\nu_n\}$ both are $\mathcal{G}$-Cauchy sequences in $\mathcal{X}$. As $\mathcal{X}$ is $\mathcal{G}$-complete, we have

$$
\sigma_n\to\sigma\in\mathcal{X},\ \text{as } n\to\infty. \tag{2.15}
$$

and

$$
\nu_n\to\nu\in\mathcal{X},\ \text{as } n\to\infty. \tag{2.16}
$$

Now we have to prove that $\sigma=\mathcal{F}(\sigma,\ \nu)$ and $\nu=\mathcal{F}(\nu,\sigma)$.

By (2.5) and (2.14), we have,

$$\begin{aligned}\chi[\varsigma_{\mathcal{G}}(\mathcal{F}(\sigma,\ \nu),\sigma_n)] &= \varsigma_{\mathcal{G}}[\mathcal{F}(\sigma,\ \nu),\mathcal{F}(\sigma_{n-1},\ \nu_{n-1})]\\ &\leq \frac{1}{2}\chi[\varsigma_{\mathcal{G}}(\sigma,\ \sigma_{n-1})+\varsigma_{\mathcal{G}}(\nu,\ \nu_{n-1})]\\ &\quad -\varrho[\varsigma_{\mathcal{G}}(\sigma,\ \sigma_{n-1})+\varsigma_{\mathcal{G}}(\nu,\ \nu_{n-1})]\end{aligned} \tag{2.17}$$

Taking $n \to \infty$ and using the $\mathcal{G}$-continuity of $\mathcal{F}$, we have,

$$\lim_{n\to\infty}\chi[\varsigma_{\mathcal{G}}(\mathcal{F}(\sigma,\ \nu),\ \sigma_n)] \leq 0,$$

that is,

$$\lim_{n\to\infty}\varsigma_{\mathcal{G}}(\mathcal{F}(\sigma,\ \nu),\ \sigma_n) \leq 0,\ [\text{ as } \chi \text{ is non-decreaing}]$$

which is only possible when $\sigma = \mathcal{F}(\sigma,\ \nu)$.
Similarly, we can prove that $\nu = \mathcal{F}(\nu,\ \sigma)$. □

3. Applications

In this section, we give some consequences of our main result.
Considering χ as an identity function we have the following result.

Corollary 3.1. *Let $(\mathcal{X},\ \preceq)$ be a partially ordered set induced with quasi-metric $\varsigma_{\mathcal{G}}$ such that $(\mathcal{X},\ \mathcal{G})$ be a $\mathcal{G}$-complete $\mathcal{G}$-metric space. Let $\mathcal{F}:\mathcal{X}\times\mathcal{X}\to\mathcal{X}$ be a $\mathcal{G}$-continuous mapping having mixed monotone property on $\mathcal{X}$ and satisfies the following*

$$\mathcal{G}(\mathcal{F}(\sigma,\ \nu),\ \mathcal{F}(\beta,\ \lambda),\ \mathcal{F}(\theta,\ \kappa)] \leq \frac{1}{2}[\mathcal{G}(\sigma,\ \beta,\ \theta)+\mathcal{G}(\nu,\ \lambda,\ \kappa)] - \varrho[\mathcal{G}(\sigma,\ \beta,\ \theta)+\mathcal{G}(\nu,\ \lambda,\ \kappa)]$$

for all $\sigma,\ \nu,\ \kappa,\ \beta,\ \lambda,\ \theta \in \mathcal{X}$ with $\sigma \succeq \beta \succeq \theta$, $\nu \preceq \lambda \preceq \kappa$, $\varrho \in \Phi, \chi \in \Psi$ and either $\nu \neq \kappa$ or $\beta \neq \theta$. If there exist $\sigma_0,\ \nu_0 \in \mathcal{X}$ such that $\sigma_0 \preceq \mathcal{F}(\sigma_0,\ \nu_0)$ and $\nu_0 \succeq \mathcal{F}(\nu_0,\ \sigma_0)$, then $\mathcal{F}$ has a CFP in $\mathcal{X}$, that is, there exist $\sigma,\ \nu \in \mathcal{X}$ such that $\sigma = \mathcal{F}(\sigma,\ \nu)$ and $\nu = \mathcal{F}(\nu,\ \sigma)$.

If we take $\varrho(t) = \dfrac{(1-t)k}{2}, k \in [0,\ 1)$ in Corollary 3.1, we get the main result of Choudhury et al. [9].

Corollary 3.2. *Let $(\mathcal{X},\ \preceq)$ be a partially ordered set and $\mathcal{G}$ be a $\mathcal{G}$-metric on $\mathcal{X}$ such that $(\mathcal{X},\ \mathcal{G})$ is a complete $\mathcal{G}$ -metric space. Let $\mathcal{F}:\mathcal{X}\times\mathcal{X}\to\mathcal{X}$ be a continuous mapping having the mixed monotone property on $\mathcal{X}$. Assume that there exists a $k \in [0,\ 1)$ such that for $\sigma,\ \nu,\ \kappa,\ \beta,\ \lambda,\ \theta \in \mathcal{X}$, the following holds:*

$$\mathcal{G}(\mathcal{F}(\sigma,\ \nu),\ \mathcal{F}(\beta,\ \lambda),\ \mathcal{F}(\theta,\ \kappa)] \leq \frac{k}{2}[\mathcal{G}(\sigma,\ \beta,\ \theta)+\mathcal{G}(\nu,\ \lambda,\ \kappa)]$$

for all $\sigma,\ \nu,\ \kappa,\ \beta,\ \lambda,\ \theta \in \mathcal{X}$ with with $\sigma \succeq \beta \succeq \theta$, $\nu \preceq \lambda \preceq \kappa$ and either $\nu \neq \kappa$ or $\beta \neq \theta$.
If there exist $\sigma_0,\ \nu_0 \in \mathcal{X}$ such that $\sigma_0 \preceq \mathcal{F}(\sigma_0,\ \nu_0)$ and $\nu_0 \succeq \mathcal{F}(\nu_0,\ \sigma_0)$, then $\mathcal{F}$ has a CFP in $\mathcal{X}$, that is, there exist $\sigma,\ \nu \in \mathcal{X}$ such that $\sigma = \mathcal{F}(\sigma,\ \nu)$ and $\nu = \mathcal{F}(\nu,\ \sigma)$.

4. Illustration

Now, we present the following non-trivial example which satisfies our main result.

Example 4.1. *Let* $\mathcal{X} = [0, \infty)$ *and* $\mathcal{G}(\sigma,\ \nu,\ \kappa) = \max\{|\sigma - \nu|, |\nu - \kappa|, |\kappa - \sigma|\}$ *for all* $\sigma, \nu, \kappa \in \mathcal{X}$. *Then* $(\mathcal{X},\ \mathcal{G})$ *is a complete* $\mathcal{G}$*- metric space. Define the mapping*

$$\mathcal{F}(\sigma,\ \nu) = \begin{cases} \frac{|\sigma-\nu|}{4}\ , & if\ \ \sigma \geq \nu \\ 0\ , & if\ \ \sigma < \nu. \end{cases}$$

Clearly $\mathcal{F}$ *is a continuous mapping.*
Also let $\Phi : [0, \infty) \to [0, \infty)$ *define by* $\varrho(\sigma) = \frac{1}{8}\ \sqrt{\sigma}$ *is a continuous non-decreasing function and* $\chi(t) = \dfrac{1}{2}\sqrt{|t|},\quad \forall\ t \geq 0.$

Explanation. *Taking* $\chi(t) = \dfrac{1}{2}\sqrt{|t|}$, *we have*
(i) $\chi(t) = 0$, *if* $t = 0$ *and*
(ii) $\chi(t+s) = \dfrac{1}{2}\sqrt{|t+s|}, \chi(t) = \dfrac{1}{2}\sqrt{|t|}, \chi(s) = \dfrac{1}{2}\sqrt{|s|}.$
Therefore, $\chi(t+s) = \chi(t) + \chi(s)$.
Without loss of generality and by symmetry, taking $\sigma \geq \nu \geq \kappa$ *and* $\beta \leq\ \lambda \leq \theta$.
We have the following three cases:
Case-I:
Let $\mathcal{G}(\sigma,\ \nu,\ \kappa) = \max\{|\sigma - \nu|, |\nu - \kappa|, |\kappa - \sigma|\} = |\sigma - \nu|$.
Now,

$$\begin{aligned} \chi[\mathcal{G}\{\mathcal{F}(\sigma,\ \nu), \mathcal{F}(\beta,\ \lambda), \mathcal{F}(\theta,\ \kappa)\}] &= \chi[\max\{|\mathcal{F}(\sigma,\ \nu) - \mathcal{F}(\beta,\ \lambda)|, |\mathcal{F}(\beta,\ \lambda) - \mathcal{F}(\theta,\ \kappa)|, \\ &\qquad |\mathcal{F}(\theta,\ \kappa) - \mathcal{F}(\sigma,\ \nu)|\}] \\ &= \chi[|\mathcal{F}(\sigma,\ \nu) - \mathcal{F}(\beta,\ \lambda)|] \\ &= \chi[|\frac{\sigma - \nu}{4} - 0|] \\ &= \chi[|\frac{\sigma - \nu}{4}|] \\ &= \frac{1}{4}\sqrt{|\sigma - \nu|}. \end{aligned}$$

Again ,

$$\begin{aligned} \frac{1}{2}\chi[\mathcal{G}(\sigma,\ \beta,\ \theta) + \mathcal{G}(\nu,\ \lambda,\ \kappa)] - \varrho[\mathcal{G}(\sigma,\ \beta,\ \lambda) + \mathcal{G}(\nu,\ \lambda,\ \kappa)] &= \frac{1}{2}\chi[|\sigma - \beta| + |\nu - \lambda|] - \varrho[|\sigma - \beta| + |\nu - \lambda|] \\ &= \frac{1}{4}[\ \sqrt{|\sigma - \beta| + |\nu - \lambda|}] - \frac{1}{8}[\ \sqrt{|\sigma - \beta| + |\nu - \lambda|}] \\ &= \frac{1}{4}[\ \sqrt{|\sigma - \beta| + |\nu - \lambda|}] \end{aligned}$$

Now,

$$\frac{1}{4}|\sigma - \nu| \leq \frac{1}{4}|(\sigma - \nu) - (\beta - \lambda)|\ as\ (\beta - \lambda) \leq 0$$

$$= \frac{1}{4}|(\sigma-\beta)-(\nu-\lambda)|$$
$$\leq \frac{1}{4}[|\sigma-\beta|+|\nu-\lambda|]$$

So,

$$\frac{1}{4}\sqrt{|\sigma-\nu|} \leq \frac{1}{4}[\sqrt{|\sigma-\beta|+|\nu-\lambda|}]$$

Therefore,

$$\chi[\mathcal{G}\{\mathcal{F}(\sigma,\ \nu),\mathcal{F}(\beta,\ \lambda),\mathcal{F}(\theta,\ \kappa)\}] \ \leq \ \frac{1}{2}\chi[\mathcal{G}(\sigma,\ \beta,\ \theta)+\mathcal{G}(\nu,\ \lambda,\ \kappa)]-\varrho[\mathcal{G}(\sigma,\ \beta,\ \theta)+\mathcal{G}(\nu,\ \lambda,\ \kappa)]$$

Case-II:
Let $\mathcal{G}(\sigma,\ \nu,\ \kappa)=\max\{|\sigma-\nu|,|\nu-\kappa|,|\kappa-\sigma|\}=|\nu-\kappa|$

$$\begin{aligned}\chi[\mathcal{G}(\mathcal{F}(\sigma,\ \nu),\ \mathcal{F}(\beta,\ \lambda),\ \mathcal{F}(\theta,\ \kappa))] &= \chi[\max\{|\mathcal{F}(\sigma,\ \nu)-\mathcal{F}(\beta,\ \lambda)|,|\mathcal{F}(\beta,\ \lambda)-\mathcal{F}(\theta,\ \kappa)|,|\mathcal{F}(\theta,\ \kappa)-\mathcal{F}(\sigma,\ \nu)|\}]\\ &= \chi[|\mathcal{F}(\beta,\ \lambda)-\mathcal{F}(\theta,\ \kappa)|]\\ &= \begin{cases}\frac{1}{4}\sqrt{|w-\kappa|}\ ,\ if\ w\geq\kappa\\ 0\ ,\quad if\ \theta<\kappa\end{cases}\end{aligned} \tag{4.1}$$

Now,

$$\begin{aligned}\frac{1}{2}\chi[\mathcal{G}(\sigma,\ \beta,\ \theta)+\mathcal{G}(\nu,\ \lambda,\ \kappa)]-\varrho[\mathcal{G}(\sigma,\ u,\ w)+\mathcal{G}(\nu,\ \nu,\ \kappa)] &= \frac{1}{2}\chi[|\beta-\theta|+|\lambda-\kappa|]-\varrho[||\beta-\theta|+|\lambda-\kappa|]\\ &= \frac{1}{4}[\sqrt{|\beta-\theta|+|\lambda-\kappa|}]-\frac{1}{8}[\sqrt{|\beta-\theta|+|\lambda-\kappa|}]\\ &= \frac{1}{4}[\sqrt{|\beta-\theta|+|\lambda-\kappa|}]\end{aligned} \tag{4.2}$$

Therefore, if $\theta\geq\kappa$*, then*

$$\begin{aligned}\frac{1}{4}|\theta-\kappa| &\leq \frac{1}{4}|(\theta-\kappa)-(\beta-\lambda)|\ as\ \beta\leq\lambda\\ &= \frac{1}{4}|\beta-\lambda-\theta+\kappa|\\ &= |(\beta-\theta)-(\lambda-\kappa)|\\ &\leq \frac{1}{4}[|\beta-\theta|+|\lambda-\kappa|].\end{aligned}$$

So,

$$\frac{1}{4}\sqrt{|\theta-\kappa|} \leq \frac{1}{4}[\sqrt{|\beta-\theta|+|\lambda-\kappa|}]. \tag{4.3}$$

If $\theta < \kappa$, then the case is obvious.
From (3.18), (3.19) *and*(3.20)*, we get*

$$\chi[\mathcal{G}(\mathcal{F}(\sigma,\ \nu),\ \mathcal{F}(\beta,\ \lambda),\ \mathcal{F}(\theta,\ \kappa))] \leq \frac{1}{2}\chi[\mathcal{G}(\sigma,\ \beta,\ \lambda) + \mathcal{G}(\nu,\ \lambda,\ \theta)] - \varrho[\mathcal{G}(\sigma,\ \beta,\ \theta) + \mathcal{G}(\nu,\ \lambda,\ \kappa)]$$

Case III:
Let $\mathcal{G}(\sigma,\ \nu,\ \kappa) = \max\{|\sigma-\nu|, |\nu-\kappa|, |\kappa-\sigma|\} = |\kappa-\sigma|$.

$$\begin{aligned}\chi[\mathcal{G}(\mathcal{F}(\sigma,\nu),\ \mathcal{F}(\beta,\ \lambda),\ \mathcal{F}(\theta,\ \kappa))] &= \chi[\max\{|\mathcal{F}(\sigma,\ \nu)-\mathcal{F}(\beta,\ \lambda)|, |\mathcal{F}(\beta,\ \lambda)-\mathcal{F}(\theta,\ \kappa)|, |\mathcal{F}(\theta,\ \kappa)-\mathcal{F}(\sigma,\ \nu)|\}]\\ &= \begin{cases} \frac{1}{4}\sqrt{|\theta-\kappa-\sigma+\nu|} & ,if\ \theta \geq \kappa \\ \frac{1}{4}\sqrt{|\sigma-\nu|} & ,if\ \theta < \kappa \end{cases}\end{aligned}$$

Now,

$$\begin{aligned}\frac{1}{2}\chi[\mathcal{G}(\sigma,\ \beta,\ \theta) + \mathcal{G}(\nu,\ \lambda,\ \kappa)] - \varrho[\mathcal{G}(\sigma,\ \beta,\ \theta) + \mathcal{G}(\nu,\ \lambda,\ \kappa)] &= \frac{1}{2}\chi[|\theta-\sigma| + |\kappa-\nu|] - \varrho[|\theta-\sigma| + |\kappa-\nu|]\\ &= \frac{1}{2}[\sqrt{|w-\sigma| + |\kappa-\nu|}] - \frac{1}{8}[\sqrt{|w-\sigma| + |\kappa-\nu|}]\\ &= \frac{1}{4}[\sqrt{|w-\sigma| + |\kappa-\nu|}].\end{aligned}$$

If $w \geq \kappa$, then,

$$\frac{1}{4}\sqrt{|\theta-\kappa-\sigma+\nu|} = \frac{1}{4}\sqrt{|(\theta-\sigma)-(\kappa-\nu)|} \leq \frac{1}{4}[\sqrt{|\theta-\sigma| + |\kappa-\nu|}].$$

If $w < \kappa$, then,

$$\frac{1}{4}\sqrt{|\sigma-\nu|} \leq \frac{1}{4}\sqrt{|(\sigma-\nu)-(\theta-\kappa)|} = \frac{1}{4}\sqrt{|(w-\sigma)-(\kappa-\nu)|} \leq \frac{1}{4}\sqrt{|w-\sigma| + |\kappa-\nu|}.$$

Therefore,

$$\chi[\mathcal{G}\{\mathcal{F}(\sigma,\ \nu), \mathcal{F}(\beta,\ \lambda), \mathcal{F}(\theta,\ \kappa)\}] \leq \frac{1}{2}\chi[\mathcal{G}(\sigma,\ \beta,\ \theta) + \mathcal{G}(\nu,\ \lambda,\ \theta)] - \varrho[\mathcal{G}(\sigma,\ \beta,\ \theta) + \mathcal{G}(\nu,\ \lambda,\ \theta)].$$

Here, $(0,\ 0)$ *is a coupled fixed point of* $\mathcal{F}$.

Remark 4.1. *A $\mathcal{G}$-metric naturally induces a metric $\varsigma_{\mathcal{G}}$ given by $\varsigma_{\mathcal{G}}(\sigma,\nu) = \mathcal{G}(\sigma,\nu,\nu) + \mathcal{G}(\sigma,\sigma,\nu)$ (see [13]). Due to the condition that either $\nu \neq \kappa$ or $\lambda \neq \theta$, the given inequality of the paper does not reduce to any metric inequality with the metric $\varsigma_{\mathcal{G}}$. Hence our results do not reduce to fixed point problems in the corresponding metric space $(\mathcal{X}, \varsigma_{\mathcal{G}})$.*

5. Conclusions

In this paper, we obtain a coupled fixed point result using mixed monotone property in the setting of $\mathcal{G}$-metric spaces induced with quasi metric $\varsigma_{\mathcal{G}}$. Using control functions our result generalizes the various results in the literature. Also, our results do not reduce to fixed point problem in the corresponding metric space.

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Conflict of interest

All authors declare that they have no conflict of interests.

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