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Research article

Ricci curvature of contact CR-warped product submanifolds in generalized Sasakian space forms admitting nearly Sasakian structure

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Abstract: The objective of this paper is to achieve the inequality for Ricci curvature of a contact CR-warped product submanifold isometrically immersed in a generalized Sasakian space form admitting a nearly Sasakian structure in the expressions of the squared norm of mean curvature vector and warping function. In addition, the equality case is likewise discussed. Later, we proved that under a certain condition the base manifold $N_T^{n_1}$ is isometric to a n_1-dimensional sphere $S^{n_1}(\frac{\lambda_1}{n_1})$ with constant sectional curvature $\frac{\lambda_1}{n_1}$.

Keywords: contact CR-submanifolds; generalized Sasakian space form; Ricci curvature; Laplacian; warped product
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1. Introduction

Due to applications of warped product manifolds in Physics and theory of relativity [10], the study of warped product manifolds has been a fascinating topic of research. The warped products provide many basic solutions to the Einstein field equations [10]. One of the most important example of warped product manifolds is the modeling of space time near black holes in the universe. The Robertson-Walker model is a warped product which represents a cosmological model to the model of universe as a space time [27].

Bishop and Neill [11] explored the geometry of Riemannian manifolds of negative curvature and introduced the notion of warped product for these manifolds (see the definition in section 2). The warped product manifolds are the natural generalization of Riemannian product manifolds. Some natural properties of warped product were investigated in [11].

In 1981, Chen first used the idea of warped products for CR-submanifolds of Kaehler manifolds ([13, 15]). Basically, Chen proved the existence of CR-warped product submanifolds of the type $N_T \times_f N_\perp$ in the setting of Kaehler manifold, where N_T and $N_\perp$ are the holomorphic and totally real submanifolds. Further, Hasegawa and Mihai [19] extended the study of Chen for the contact CR-warped product submanifolds of the Sasakian manifolds. Mihai [22] obtained an estimate for the squared norm of the second fundamental form in terms of the warping function for the contact CR-warped product submanifolds in the frame of Sasakian space form. Since then, many authors have studied warped product submanifolds in the different settings of Riemannian manifolds and numerous existence results have been explored (see the survey article [17]).

In 1999, Chen [14] discovered a relationship between Ricci curvature and squared mean curvature vector for an arbitrary Riemannian manifold. On the line of Chen a series of articles have been appeared to formulate the relationship between Ricci curvature and squared mean curvature in the setting of some important structures on Riemannian manifolds (see [6, 12, 21–23, 29]). Recently Ali et al. [2] established a relationship between Ricci curvature and squared mean curvature for warped product submanifolds of a sphere and provide many physical applications.

In this paper our aim is to obtain a relationship between Ricci curvature and squared mean curvature for contact CR-warped product submanifolds in the setting of generalized Sasakian space form admitting a nearly Sasakian structure. Further, we provide some applications in terms of Hamiltonians and Euler-Lagrange equation. In the last we also worked out some applications of Obata's differential equation.

2. Some basic results

A $(2n+1)-$dimensional $C^\infty-$manifold $\bar{M}$ is said to have an almost contact structure if there exist on $\bar{M}$ a tensor field ϕ of the type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \oplus \xi, \;\; \phi\xi = 0, \;\; \eta \circ \phi = 0, \;\; \eta(\xi) = 1.$$

There always exists a Riemannian metric g on an almost contact metric manifold $\bar{M}$ satisfying the following conditions

$$\eta(X) = g(X, \xi), \;\; g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all $X, Y \in T\bar{M}$.

An almost contact metric manifold is said to be nearly Sasakian manifold, if

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = -2g(X, Y)\xi + \eta(Y)X + \eta(X)Y, \tag{2.1}$$

for all $X, Y \in T\bar{M}$.

Moreover, the structure vector field ξ is Killing vector field on a Riemannian manifold if it satisfies the following equation

$$\bar{\nabla}_X \xi = 0.$$

In [1] Alegre et al. introduced the notion of generalized Sasakian space form as that an almost contact metric manifold $(\bar{M}, \phi, \xi, \eta, g)$ whose curvature tensor $\bar{R}$ satisfies

$$\begin{aligned}\bar{R}(X,Y,Z,W) =& f_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\ &- f_2[g(\phi X,Z)g(\phi Y,W) - g(\phi X,W)g(\phi Y,Z)\\ &+ 2g(\phi X,Y)g(\phi Z,W)] - f_3[\eta(Z)\{\eta(Y)g(X,W)\\ &- \eta(X)g(Y,W)\} + \eta(W)\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}]\end{aligned} \tag{2.2}$$

for all vector fields X, Y, Z, W and certain differentiable functions f_1, f_2, f_3 on $\bar{M}$. A generalized Sasakian space form with functions f_1, f_2, f_3 is denoted by $\bar{M}(f_1, f_2, f_3)$. If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, then $\bar{M}(f_1, f_2, f_3)$ is a Sasakian space form $\bar{M}(c)$ [1]. If $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$, then $\bar{M}(f_1, f_2, f_3)$ is a Kenmotsu space form $\bar{M}(c)$ [1] and if $f_1 = f_2 = f_3 = \frac{c}{4}$, then $\bar{M}(f_1, f_2, f_3)$ is a cosymplectic space form $\bar{M}(c)$ [1].

Now, we discuss some examples of space forms. Let $M(c)$ be a complex space form with the complex structure (J, g_M), consider the product manifold $N^{2n+1} = M(c) \times R$ and define the following tensors on N^{2n+1}

$$\phi = J \circ d\pi,\ \ \xi = \frac{\partial}{\partial t},\ \ \eta = dt,\ \text{and}\ g_N = g_M + d(t) \otimes dt,$$

where $\pi : M(c) \times R \to M(c)$ is the projection map and t is the standard coordinate function on the real axis. Then $(N^{2n+1}, \phi, \xi, \eta, g_N)$ is a cosymplectic space form with constant $\phi-$sectional curvature equal to c ([1, 7]).

The Hyperbolic space $H(-1)$ of constant sectional curvature -1 is an example of Kenmotsu space form.

Nearly Sasakian structure was introduced on the 5-dimensional sphere $S^5(2)$ of constant sectional curvature 2 as totally umbilical hypersurface of nearly Kaehler 6-sphere S^6, which is not a Sasakian structure [8].

One of the present author Ibrahim Al-Dayel [18] proved that there exists two other structures which are Sasakian as well as nearly cosymplectic structure. More precisely,

Theorem 2.1. *[18] There are two structures $(\phi_i, \xi, \eta, g), i = 1, 2$ related to the nearly Sasakian structure (ϕ, ξ, η, g) on the 5-sphere $S^5(2)$ such that $S^5(2)(\phi_1, \xi, \eta, g)$ is homothetic to Sasakian manifold and $S^5(2)(\phi_2, \xi, \eta, g)$ is a nearly cosymplectic manifold.*

Another example of generalized Sasakian space form admitting the nearly cosymplectic structure is totally geodesic hypersurface $S^5(\psi_1, \xi, \eta, g)$ of the nearly Kaehler 6-sphere (S^6, J, g) [9]. Further extending this study one of the present author Ibrahim Al-Dayel in [18] obtained two more structures ψ_2 and ψ_3 such that $S^5(\psi_2, \xi, \eta, g)$ and $S^5(\psi_3, \xi, \eta, g)$ are Sasakian and nearly cosymplectic manifolds respectively.

Let (M^n, g) be an $n-$dimensional Riemannian manifold isometrically immersed in a $m-$dimensional Riemannian manifold $\bar{M}$. Then the Gauss and Weingarten formulas are $\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$ and $\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$ respectively, for all $X, Y \in TM$ and $N \in T^{\perp}M$. Where ∇ is the induced Levi-Civita connection on M, N is a vector field normal to M, h is the second fundamental form of M, $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ and A_N is the shape operator of the second fundamental form. The second fundamental form h and the shape operator are associated by the following formula

$$g(h(X, Y), N) = g(A_N X, Y). \tag{2.3}$$

The equation of Gauss is given by

$$R(X,Y,Z,W) = \bar{R}(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \tag{2.4}$$

for all $X,Y,Z,W \in TM$. Where, $\bar{R}$ and R are the curvature tensors of $\bar{M}$ and M respectively.
For any $X \in TM$ and $N \in T^{\perp}M$, ϕX and ϕN can be decomposed as follows

$$\phi X = PX + FX$$

and

$$\phi N = tN + fN,$$

where PX (resp. tN) is the tangential and FX (resp. fN) is the normal component of ϕX (resp. ϕN).

For any orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space T_xM, the mean curvature vector $H(x)$ and its squared norm are defined as follows

$$H(x) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \quad \|H\|^2 = \frac{1}{n^2}\sum_{i,j=1}^{n} g(h(e_i, e_i), h(e_j, e_j)),$$

where n is the dimension of M. If $h = 0$ then the submanifold is said to be totally geodesic and minimal if $H = 0$. If $h(X,Y) = g(X,Y)H$ for all $X,Y \in TM$, then M is called totally umbilical.

The scalar curvature of $m-$ dimensional Riemannian manifold $\bar{M}$ is denoted by $\bar{\pi}(\bar{M})$ and is defined as

$$\bar{\pi}(\bar{M}) = \sum_{1\le p<q\le m} \bar{\kappa}_{pq},$$

where $\bar{\kappa}_{pq} = \bar{\kappa}(e_p \wedge e_q)$. Throughout this study, we shall use the equivalent version of the above equation, which is given by

$$2\bar{\pi}(\bar{M}) = \sum_{1\le p<q\le m} \bar{\kappa}_{pq}.$$

In a similar way, the scalar curvature $\bar{\pi}(L_x)$ of a $L-$plane is expressed as

$$\bar{\pi}(L_x) = \sum_{1\le p<q\le m} \bar{\kappa}_{pq}. \tag{2.5}$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_xM and if e_r belongs to the orthonormal basis $\{e_{n+1}, \ldots e_m\}$ of the normal space $T^{\perp}M$, then we have

$$h^r_{pq} = g(h(e_p, e_q), e_r) \tag{2.6}$$

and

$$\|h\|^2 = \sum_{p,q=1}^{n} g(h(e_p, e_q), h(e_p, e_q)).$$

Let κ_{pq} and $\bar{\kappa}_{pq}$ be the sectional curvatures of the plane sections generated by e_p and e_q at the point $x \in M^n$ and in the Riemannian space form $\bar{M}^m(c)$, respectively. Thus by Gauss equation, we have

$$\kappa_{pq} = \bar{\kappa}_{pq} + \sum_{r=n+1}^{m} (h^r_{pp}h^r_{qq} - (h^r_{pq})^2). \tag{2.7}$$

The global tensor field for orthonormal frame of vector field $\{e_1, \ldots, e_n\}$ on M^n is defined as

$$\bar{S}(X, Y) = \sum_{i=1}^{n} \{g(\bar{R}(e_i, X)Y, e_i)\},$$

for all $X, Y \in T_x M^n$. The above tensor is named Ricci tensor. When we fix a specific vector e_u from $\{e_1, \ldots, e_n\}$ on M^n, which is designated by χ. Therefore the Ricci curvature is defined by

$$Ric(\chi) = \sum_{\substack{p=1 \\ p \neq u}}^{n} \kappa(e_p \wedge e_u) \tag{2.8}$$

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively and let ψ be a positive differentiable function on N_1. If $\pi : N_1 \times N_2 \rightarrow N_1$ and $\eta : N_1 \times N_2 \rightarrow N_2$ are the projection maps given by $\pi(p, q) = p$ and $\eta(p, q) = q$ for every $(p, q) \in N_1 \times N_2$, then the *warped product manifold* is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_* X, \pi_* Y) + (\psi \circ \pi)^2 g_2(\eta_* X, \eta_* Y),$$

for all $X, Y \in TM$. The function ψ is called the *warping function* of the warped product manifold [11]. If the warping function is constant, then the warped product is trivial i.e., simply Riemannian product. Then from Lemma 7.3 of [11], we have

$$\nabla_X Z = \nabla_Z X = (\frac{X\psi}{\psi})Z$$

where ∇ is the Levi-Civita connection on M. For a warped product $M = N_1 \times_\psi N_2$ it is easy to observe that

$$\nabla_X Z = \nabla_Z X = (X ln\psi)Z$$

for $X \in TM_1$ and $Z \in TM_2$.
We denote $\nabla\psi$ the gradient of ψ and it is defined as

$$g(\nabla\psi, X) = X\psi, \tag{2.9}$$

for all $X \in TM$.

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthogonal basis of the tangent space TM of a $n-$dimensional Riemannian manifold M. Then (2.9) provides the following,

$$||\nabla\psi||^2 = \sum_{i=1}^{n} (e_i(\psi))^2.$$

The Laplacian of ψ is defined by

$$\Delta\psi = \sum_{i=1}^{n} \{(\nabla_{e_i} e_i)\psi - e_i e_i \psi\}.$$

The Hessian tensor for a differentiable function ψ is a symmetric covariant tensor of rank 2 and is defined as

$$\Delta\psi = -traceH^{\psi},$$

where H^{ψ} is the Hessian of ψ.

For the warped product submanifolds, we observed the well known result, which is described as follows [16]

$$\sum_{p=1}^{n_1}\sum_{q=1}^{n_2}\kappa(e_p \wedge e_q) = \frac{n_2\Delta\psi}{\psi} = n_2(\Delta ln\psi - \|\nabla ln\psi\|^2). \tag{2.10}$$

For a compact orientable Riemannian manifold M with or without boundary and as a consequences of the integration theory of manifolds, we have

$$\int_M \Delta\psi dV = 0,$$

where ψ is a function on M and dV is the volume element of M.

3. Contact CR-warped product submanifolds

Suppose M be a $n-$dimensional submanifold isometrically immersed in an almost contact metric manifold $\bar{M}(g,\phi,\xi,\eta)$ such that the structure vector field ξ is tangent to M. The submanifold M is called contact CR-submanifold if it admits an invariant distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant such that $TM = D \oplus D^{\perp} \oplus \langle\xi\rangle$, where $\phi D \subseteq D$, $\phi D^{\perp} \subseteq T^{\perp}M$ and $\langle\xi\rangle$ is the 1-dimensional distribution spanned by ξ [24]. If μ is the invariant subspace of the normal bundle $T^{\perp}M$, then in the case of contact CR- submanifold, the normal bundle $T^{\perp}M$ can be decomposed as $T^{\perp}M = \mu \oplus \phi D^{\perp}$. A contact CR-submanifold is called contact CR-product submanifold if the distributions D and $D^{\perp}$ are parallel on M. Moreover, a contact CR-submanifold is said to be mixed totally geodesic if $h(D, D^{\perp}) = 0$. As a generalization of the product manifold submanifolds one can consider warped product submanifolds. In [19] Hasegawa and Mihai studied contact CR-warped product submanifolds in Sasakian manifolds basically, they proved the existence of the contact CR-warped product submanifolds of the type $N_T \times_f N_{\perp}$ such that the structure vector field ξ is tangent to N_T. After that Mihai [22] and Munteanu [26] investigated contact CR-warped product in Sasakian space forms and obtained an inequality for squared norm of second fundamental form and warping function. Moreover, Atceken [5] explored the existence of contact CR-warped product submanifolds in the expressions of some inequalities this study was extended to the setting of trans-Sasakian generalized Sasakian spaceforms by Sular and Ozgur [28]. More recently, Ishan and Khan [20] generalized contact CR-warped product submanifolds in the setting of generalized Sasakian manifolds admitting nearly Sasakian structure. Throughout, this study we consider $n-$dimensional contact CR-warped product submanifold $M^n = N_T^{n_1} \times_{\psi} N_{\perp}^{n_2}$, such that the structure vector field ξ is tangential to N_T, where n_1 and n_1 are the dimensions of the invariant and anti-invariant submanifold respectively.

Now, we start with some initial results.

Lemma 3.1. *Let $M = N_T^{n_1} \times_{\psi} N_{\perp}^{n_2}$ be a contact CR-warped product submanifold isometrically immersed in a nearly Sasakian manifold $\bar{M}$. Then*

(i) $g(h(X,Y),\phi Z) = 0$,
(ii) $g(h(\phi X,\phi X),N) = -g(h(X,X),N)$,

for any $X, Y \in TN_T$, $Z \in TN_\perp$ *and* $N \in \mu$.

Proof. By using Gauss and Weingarten formulae in Eq (2.1), we have

$$-A_{\phi Z}X - \nabla^{\perp}_{X}\phi Z - \phi\nabla_X Z - \phi h(X,Z) + \nabla_Z\phi X + h(\phi X,Z) \\ - \phi\nabla_Z X - \phi h(X,Z) = -\eta(X)\phi Z,$$

taking inner product with Y and using (2.3), we get the result that needed.
To show (ii), for any $X \in TN_T$ we have

$$\bar{\nabla}_X\phi X = (\bar{\nabla}_X\phi)X + \phi\bar{\nabla}_X X,$$

using Gauss formula and (2.1), we get

$$\nabla_X\phi X + h(\phi X,X) = -\eta(X)\phi X + \phi\nabla_X X + \phi h(X,X),$$

taking inner product with ϕN, above equation yields

$$g(h(\phi X,X),\phi N) = g(h(X,X),N), \tag{3.1}$$

replacing X by ϕX and using the fact that ξ is Killing vector field [9], the last equation gives

$$g(h(\phi X,X),JN) = -g(h(\phi X,\phi X),N). \tag{3.2}$$

From (3.1) and (3.2), we get the required result. □

By the Lemma 3.1 it is evident that the isometric immersion $N_T^{n_1} \times_\psi N_\perp^{n_2}$ into a nearly Sasakian manifold is $D-$ minimal. The $D-$ minimal property provides us a useful relationship between the contact CR-warped product submanifold $N_T \times_\psi N_\perp$ and the equation of Gauss.
Definition 3.1 The warped product $N_1 \times_\psi N_2$ isometrically immersed in a Riemannian manifold $\bar{M}$ is called N_i totally geodesic if the partial second fundamental form h_i vanishes identically. It is called N_i-minimal if the partial mean curvature vector H^i becomes zero for $i = 1, 2$.

Let $\{e_1, \ldots, e_\beta, e_{\beta+1} = \phi e_1 \ldots, \ldots e_{n_1-1} = \phi e_\beta, e_{n_1} = \xi, e_{n_1+1}, \ldots, e_n\}$ be a local orthonormal frame of vector fields on the contact CR-warped product submanifold $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ such that $\{\xi, e_1, \ldots, e_{n_1}\}$ are tangent to N_T and $\{e_{n_1+1}, \ldots e_n\}$ are tangent to $N_\perp$. Moreover, $\{e_1^* = \phi e_{n_1+1}, \ldots, e_n^* = \phi e_n, e_{n+1}^*, \ldots, e_m^*\}$ is a local orthonormal frame of the normal space $T^\perp M$.

From Lemma 3.1, one can observe

$$\sum_{r=n+1}^{m}\sum_{i,j=1}^{n_1} g(h(e_i,e_j),e_r) = 0.$$

Therefore, it knows that the trace of h due to N_T becomes zero. Hence in view of the Definition 3.1, we have the following important result.

Theorem 3.2. *Let $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a contact CR-warped product submanifold isometrically immersed in a nearly Sasakian manifold. Then M^n is $D-$ minimal.*

So, it is easy to conclude the following

$$\|H\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{m} (h^r_{n_1+1 n_1+1} + \cdots + h^r_{nn}),$$

where $\|H\|^2$ is the squared mean curvature.

4. Ricci curvature for contact CR-warped product submanifold

The present section deals to formulate the Ricci curvature in the expressions of mean curvature vector and warping function ψ.

Theorem 4.1. *Let $M = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a contact CR-warped product submanifold isometrically immersed in a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting nearly Sasakian structure. If for each orthogonal unit vector field $\chi \in T_xM$ orthogonal to ξ, either tangent to N_T or $N_\perp$. Then we have*

(1) The Ricci curvature satisfy the following inequality

(i) If χ is tangent to $N_T^{n_1}$, then

$$Ric(\chi) \leq \frac{1}{4} n^2 \|H\|^2 - \frac{n_2 \Delta\psi}{\psi} + (n + n_1 n_2 - 1) f_1 + \frac{3 f_2}{2} - (n_2 + 1) f_3. \tag{4.1}$$

(ii) χ is tangent to $N_\perp^{n_2}$, then

$$Ric(\chi) \leq \frac{1}{4} n^2 \|H\|^2 - \frac{n_2 \Delta\psi}{\psi} + (n + n_1 n_2 - 1) f_1 - (n_2 + 1) f_3. \tag{4.2}$$

(2) If $H(x) = 0$ for each point $x \in M^n$. Then there is a unit vector field X which satisfies the equality case of (1) if and only if M^n is mixed totally geodesic and χ lies in the relative null space N_x at x.

(3) For the equality case we have

(a) The equality case of (4.1) holds identically for all unit vector fields tangent to N_T at each $x \in M^n$ if and only if M^n is mixed totally geodesic and $D-$totally geodesic contact CR-warped product submanifold in $\bar{M}^m(f_1, f_2, f_3)$.

(b) The equality case of (4.2) holds identically for all unit vector fields tangent to $N_\perp$ at each $x \in M^n$ if and only if M is mixed totally geodesic and either M^n is $D^\perp$- totally geodesic contact CR-warped product or M^n is a $D^\perp$ totally umbilical in $\bar{M}^m(f_1, f_2, f_3)$ with dim $D^\perp = 2$.

(c) The equality case of (1) holds identically for all unit tangent vectors to M^n at each $x \in M^n$ if and only if either M^n is totally geodesic submanifold or M^n is a mixed totally geodesic totally umbilical and $D-$ totally geodesic submanifold with dim $N_\perp = 2$,

where n_1 and n_2 are the dimensions of N_T and $N_\perp$ respectively.

Proof. Suppose that $M = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a contact CR-warped product submanifold of a generalized Sasakian space form. From Gauss equation, we have

$$n^2\|H\|^2 = 2\pi(M^n) + \|h\|^2 - 2\bar{\pi}(M^n). \tag{4.3}$$

Let $\{e_1, \ldots, e_{n_1}, e_{n_1+1}, \ldots, e_n\}$ be a set of orthonormal vector fields on M^n such that the frame $\{e_1, \ldots, e_{n_1}\}$ is tangent to N_T and $\{e_{n_1+1}, \ldots, e_n\}$ is tangent to $N_\perp$. So, the unit tangent vector $\chi = e_A \in \{e_1, \ldots, e_n\}$ can be expanded (4.3) as follows

$$n^2\|H\|^2 = 2\pi(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}\{(h_{11}^r + \cdots + h_{nn}^r - h_{AA}^r)^2 + (h_{AA}^r)^2\}$$
$$- \sum_{r=n+1}^{m}\sum_{1\le p\neq q\le n} h_{pp}^r h_{qq}^r - 2\bar{\pi}(M^n).$$

□

The above expression can be expanded as

$$n^2\|H\|^2 = 2\pi(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}\{(h_{11}^r + \cdots + h_{nn}^r)^2$$
$$+ (2h_{AA}^r - (h_{11}^r + \cdots + h_{nn}^r))^2\} + 2\sum_{r=n+1}^{m}\sum_{1\le p<q\le n}(h_{pq}^r)^2$$
$$- 2\sum_{r=n+1}^{m}\sum_{1\le p<q\le n} h_{pp}^r h_{qq}^r - 2\bar{\pi}(M^n).$$

In view of the Lemma 3.1, the preceding expression takes the form

$$\begin{aligned} n^2\|H\|^2 = &\sum_{r=n+1}^{m}\{(h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r)^2 + +(2h_{AA}^r - (h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r))^2\} \\ &+ 2\pi(M^n) + \sum_{r=n+1}^{m}\sum_{1\le p<q\le n}(h_{pq}^r)^2 - \sum_{r=n+1}^{m}\sum_{1\le p<q\le n} h_{pp}^r h_{qq}^r + \sum_{r=n+1}^{m}\sum_{\substack{a=1\\ a\neq A}}(h_{aA}^r)^2 \\ &+ \sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\neq A}}(h_{pq}^r)^2 - \sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\neq A}} h_{pp}^r h_{qq}^r - 2\bar{\pi}(M^n). \end{aligned} \tag{4.4}$$

By Eq (2.7), we have

$$\begin{aligned} &\sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\neq A}}(h_{pq}^r)^2 - \sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\neq A}} h_{pp}^r h_{qq}^r \\ &\qquad = \sum_{\substack{1\le p<q\le n\\ p,q\neq A}} \bar{\kappa}_{p,q} - \sum_{\substack{1\le p<q\le n\\ p,q\neq A}} \kappa_{p,q} \end{aligned} \tag{4.5}$$

Substituting the values of Eq (4.5) in (4.4), we discover

$$
\begin{aligned}
n^2\|H\|^2 =& 2\pi(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{AA} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2 \\
&+ \sum_{r=n+1}^{m}\sum_{1\le p<q\le n}(h^r_{pq})^2 - \sum_{r=n+1}^{m}\sum_{1\le p<q\le n} h^r_{pp}h^r_{qq} - 2\bar{\pi}(M^n) \\
&+ \sum_{r=n+1}^{m}\sum_{\substack{a=1\\ a\neq A}}(h^r_{aA})^2 + \sum_{\substack{1\le p<q\le n\\ p,q\neq A}}\bar{\kappa}_{p,q} - \sum_{\substack{1\le p<q\le n\\ p,q\neq A}}\kappa_{p,q}.
\end{aligned} \tag{4.6}
$$

Since, $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$, then from (2.5), the scalar curvature of M^n can be defined as follows

$$
\begin{aligned}
\pi(M^n) &= \sum_{1\le p<q\le n}\kappa(e_p \wedge e_q) \\
&= \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}\kappa(e_i \wedge e_j) + \sum_{1\le i<k\le n_1}\kappa(e_i \wedge e_k) + \sum_{n_1+1\le l<o\le n}\kappa(e_l \wedge e_o)
\end{aligned} \tag{4.7}
$$

Using (2.5) and (2.10), we derive

$$
\pi(M^n) = \frac{n_2\Delta\psi}{\psi} + \pi(N_T^{n_1}) + \pi(N_\perp^{n_2}) \tag{4.8}
$$

Utilizing (4.8) together with (2.2) in (4.6), we have

$$
\begin{aligned}
\frac{1}{2}n^2\|H\|^2 =& \frac{n_2\Delta\psi}{\psi} + \sum_{\substack{1\le p<q\le n\\ p,q\neq A}}\bar{\kappa}_{p,q} + \bar{\pi}(N_T^{n_1}) + \bar{\pi}(N_\perp^{n_2}) \\
&+ \sum_{r=n+1}^{m}\{\sum_{1\le p<q\le n}(h^r_{pq})^2 - \sum_{\substack{1\le p<q\le n\\ p,q\neq A}} h^r_{pp}h^r_{qq}\} \\
&+ \sum_{r=n+1}^{m}\sum_{\substack{a=1\\ a\neq A}}(h^r_{aA})^2 + \sum_{r=n+1}^{m}\sum_{1\le i\neq j\le n_1}(h^r_{ii}h^r_{jj} - (h^r_{ij})^2) \\
&+ \sum_{r=n+1}^{m}\sum_{n_1+1\le s\neq t\le n}(h^r_{ss}h^r_{tt} - (h^r_{st})^2) \\
&+ \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{AA} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2 \\
&- \{f_1(n(n-1)) + f_2(3(n_1-1)) - f_3(2(n-1))\}.
\end{aligned} \tag{4.9}
$$

Considering $\chi = e_a$, we got two options: χ may be tangent to the submanifold $N_T^{n_1}$ or to the fibre $N_\perp^{n_2}$.

Option 1: If e_a is tangent to $N_T^{n_1}$, then fix a unit tangent vector from $\{e_1, \ldots, e_{n_1}\}$ suppose $\chi = e_a = e_1$, then from (4.9) and (2.8), we find

$$\begin{aligned} Ric(\chi) \leq & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} - \frac{1}{2}\sum_{r=n+1}^{m}(2h_{11}^r - (h_{n_1+1n_1+1}^r + \ldots h_{nn}^r))^2 \\ & - \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n_1}(h_{pq}^r)^2 + \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(h_{ij}^r)^2 - \sum_{1\leq i<j\leq n_1}h_{ii}^r h_{jj}^r] \\ & + \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n}(h_{st}^r)^2 + \sum_{r=n+1}^{m}[\sum_{n_1+1\leq s<t\leq n}(h_{ij}^r)^2 - \sum_{n_1+1\leq s<t\leq n}h_{ss}^r h_{tt}^r] \\ & + \sum_{r=n+1}^{m}\sum_{2\leq p<q\leq n}h_{pp}^r h_{qq}^r + f_1(n(n-1)) + f_2(3(n_1-1)) - f_3(2(n-1)) \\ & - \sum_{2\leq p<q\leq n}\bar{\kappa}_{p,q} - \bar{\pi}(N_T^{n_1}) - \bar{\pi}(N_\perp^{n_2}). \end{aligned} \tag{4.10}$$

From (2.2), (2.5) and (2.6), we have

$$\sum_{2\leq p<q\leq n}\bar{\kappa}_{p,q} = \frac{f_1}{2}((n-1)(n-2)) + \frac{f_2}{2}(3(n_1-2)) - \frac{f_3}{2}(2(n-2)), \tag{4.11}$$

$$\bar{\pi}(N_T^{n_1}) = \frac{f_1}{2}((n_1(n_1-1)) + \frac{f_2}{2}(3(n_1-1)) - \frac{f_3}{2}(2(n_1-1)), \tag{4.12}$$

$$\bar{\pi}(N_T^{n_1}) = \frac{f_1}{2}((n_2(n_2-1)) \tag{4.13}$$

Using in (4.10), we have

$$\begin{aligned} Ric(\chi) \leq & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2+1)f_3 \\ & - \frac{1}{2}\sum_{r=n+1}^{m}(2h_{11}^r - (h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r))^2 \\ & - \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n}(h_{pq}^r)^2 + \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(h_{ij}^r)^2 + \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n}(h_{st}^r)^2] \\ & - \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}h_{ii}^r h_{jj}^r + \sum_{n_1+1\leq s<t\leq n}h_{ss}^r h_{tt}^r] \\ & + \sum_{r=n+1}^{m}\sum_{2\leq p<q\leq n}h_{pp}^r h_{qq}^r. \end{aligned} \tag{4.14}$$

Further, the seventh and eighth terms on right hand side of (4.14) can be written as

$$\begin{aligned} \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(h_{ij}^r)^2 + \sum_{n_1+1\leq s<t\leq n}(h_{st}^r)^2] - \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n}(h_{pq}^r)^2 \\ = -\sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h_{pq}^r)^2. \end{aligned}$$

Likewise, we get

$$\sum_{r=n+1}^{m}[\sum_{1\le i<j\le n_1} h_{ii}^r h_{jj}^r + \sum_{n_1+1\le s\neq t\le n} h_{ss}^r h_{tt}^r - \sum_{2\le p<q\le n} h_{pp}^r h_{qq}^r]$$
$$= \sum_{r=n+1}^{m}[\sum_{p=2}^{n_1}\sum_{q=n_1+1}^{n} h_{pp}^r h_{qq}^r - \sum_{j=2}^{n_1} h_{11}^r h_{jj}^r].$$

Utilizing above two values in (4.14), we get

$$\begin{aligned} Ric(\chi) \le& \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2+1)f_3 \\ &- \frac{1}{2}\sum_{r=n+1}^{m}(2h_{11}^r - (h_{n_1+1n_1+1}^r + \dots h_{nn}^r))^2 \\ &- \sum_{r=n+1}^{m}[\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h_{pq}^r)^2 + \sum_{b=2}^{n_1} h_{11}^r h_{bb}^r - \sum_{p=2}^{n_1}\sum_{q=n_1+1}^{n} h_{pp}^r h_{qq}^r]. \end{aligned} \tag{4.15}$$

Since $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ is $N_T^{n_1}$-minimal then we can observe the following

$$\sum_{r=n+1}^{m}\sum_{p=2}^{n_1}\sum_{q=n_1+1}^{n} h_{pp}^r h_{qq}^r = -\sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n} h_{11}^r h_{qq}^r \tag{4.16}$$

and

$$\sum_{r=n+1}^{m}\sum_{b=2}^{n_1} h_{11}^r h_{bb}^r = -\sum_{r=n+1}^{m}(h_{11}^r)^2. \tag{4.17}$$

Simultaneously, we can conclude

$$\begin{aligned} &\frac{1}{2}\sum_{r=n+1}^{m}(2h_{11}^r - (h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r))^2 + \sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n} h_{11}^r h_{qq}^r \\ &\qquad = 2\sum_{r=n+1}^{m}(h_{11}^r)^2 + \frac{1}{2}n^2\|H\|^2. \end{aligned} \tag{4.18}$$

Using (4.16) and (4.17) in (4.15), after the assessment of (4.18), we finally get

$$\begin{aligned} Ric(\chi) \le& \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2+1)f_3 \\ &- \frac{1}{4}\sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n}(h_{qq}^r)^2 - \sum_{r=n+1}^{m}\{(h_{11}^r)^2 - \sum_{q=n_1+1}^{n} h_{11}^r h_{qq}^r \\ &+ \frac{1}{4}(h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r)^2\}. \end{aligned}$$

Further, using the fact that $\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}) = n^2\|H\|^2$, we get

$$Ric(\chi) \leq \frac{1}{4}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2 + 1)f_3 - \frac{1}{4}\sum_{r=n+1}^{m}(2h^r_{11} - \sum_{q=n_1+1}^{n} h^r_{qq})^2.$$

From the above inequality, we can conclude the inequality (4.1).

Option 2: If e_a is tangent to $N^{n_2}_{\perp}$, then we select the unit vector from $\{e_{n_1+1}, \ldots, e_n\}$, suppose that the unit vector is e_n i.e., $\chi = e_n$. Then from (2.2), (2.5) and (2.6), we have

$$\sum_{1\leq p<q\leq n-1} \bar{\kappa}_{p,q} = \frac{f_1}{2}((n-1)(n-2)) + \frac{f_2}{2}(3(n_1-1)) - \frac{f_3}{2}(2(n-2)). \tag{4.19}$$

$$\bar{\pi}(N^{n_1}_T) = \frac{f_1}{2}(n_1(n_1-1)) + \frac{f_2}{2}(3(n_1-1)) - \frac{f_3}{2}(2(n-1)).$$

$$\bar{\pi}(N^{n_2}_{\perp}) = \frac{f_1}{2}(n_2(n_2-1)).$$

Now, in a similar way as in option 1 using (4.19), we have

$$\begin{aligned} Ric(\chi) \leq & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} - \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \ldots h^r_{nn}) - 2h^r_{nn})^2 \\ & - \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n_1}(h^r_{pq})^2 + \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(h^r_{ij})^2 - \sum_{1\leq i<j\leq n_1} h^r_{ii}h^r_{jj}] \\ & + \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n}(h^r_{st})^2 + \sum_{r=n+1}^{m}[\sum_{n_1+1\leq s<t\leq n}(h^r_{ij})^2 - \sum_{n_1+1\leq s<t\leq n} h^r_{ss}h^r_{tt}] \\ & + \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n-1} h^r_{pp}h^r_{qq} + (n + n_1n_2 - 1)f_1 - (n_2+1)f_3. \end{aligned} \tag{4.20}$$

Using similar steps of option *i*, the above inequality takes the form

$$\begin{aligned} Ric(\chi) \leq & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 - (n_2+1)f_3 \\ & - \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \ldots h^r_{nn}) - 2h^r_{nn})^2 \\ & - \sum_{r=n+1}^{m}[\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2 + \sum_{b=n_1+1}^{n-1} h^r_{nn}h^r_{bb} - \sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n-1} h^r_{pp}h^r_{qq}]. \end{aligned} \tag{4.21}$$

By the Lemma 3.1, one can observe that

$$\sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n-1} h^r_{pp}h^r_{qq} = 0. \tag{4.22}$$

Utilizing this in (4.21), we get

$$\begin{aligned}Ric(\chi) \leq&\frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 - (n_2 + 1)f_3\\&- \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \dots h^r_{nn}) - 2h^r_{nn})^2\\&- \sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2 - \sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n-1}h^r_{nn}h^r_{bb}.\end{aligned} \tag{4.23}$$

The last term of the above inequality can be written as

$$-\sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n-1}h^r_{nn}h^r_{bb} = -\sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n}h^r_{nn}h^r_{bb} + \sum_{r=n+1}^{m}(h^r_{nn})^2$$

Moreover, the fifth term on right hand side of (4.23) can be expanded as

$$\begin{aligned}-\frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}) - 2h^r_{nn})^2 =&\\&- \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn})^2\\&- 2\sum_{r=n+1}^{m}(h^r_{nn})^2 + \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}h^r_{nn}h^r_{jj}.\end{aligned}$$

Using last two values in (4.23), we have

$$\begin{aligned}Ric(\chi) \leq&\frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 - (n_2 + 1)f_3\\&- \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \dots h^r_{nn})^2 - 2\sum_{r=n+1}^{m}(h^r_{nn})^2\\&+ 2\sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}h^r_{nn}h^r_{jj} - \sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2\\&- \sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n}h^r_{nn}h^r_{bb} + \sum_{r=n+1}^{m}(h^r_{nn})^2,\end{aligned}$$

or equivalently

$$\begin{aligned}Ric(\chi) \leq&\frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 - (n_2 + 1)f_3\\&- \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \dots h^r_{nn})^2 - \sum_{r=n+1}^{m}(h^r_{nn})^2\\&+ \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}h^r_{nn}h^r_{jj} - \sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2\end{aligned}$$

On applying similar techniques as in the proof of option 1, we arrive at

$$
\begin{aligned}
Ric(\chi) \leq & \frac{1}{4}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 - (n_2 + 1)f_3 \\
& - \frac{1}{4}\sum_{r=n+1}^{m}(h^r_{nn} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2,
\end{aligned}
$$

which gives the inequality (4.2).

Next, we explore the equality cases of the inequality (4.1). First, we redefine the notion of the relative null space $\mathcal{N}_x$ of the submanifold M^n in the generalized Sasakian space form $\bar{M}^m(f_1, f_2, f_3)$ at any point $x \in M^n$, the relative null space was defined by B. Y. Chen [14], as follows

$$
\mathcal{N}_x = \{X \in T_xM^n : h(X, Y) = 0, \forall Y \in T_xM^n\}.
$$

For $A \in \{1, \ldots, n\}$ a unit vector field e_A tangent to M^n at x the equality sign of (4.1) holds identically iff

$$
(i)\ \sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0\ (ii)\ \sum_{b=1}^{n}\sum_{\substack{A=1 \\ b\neq A}}^{n} h^r_{bA} = 0\ (iii)\ 2h^r_{AA} = \sum_{q=n_1+1}^{n} h^r_{qq},
$$

such that $r \in \{n + 1, \ldots m\}$ the condition (i) indicates that M^n is mixed totally geodesic contact CR-warped product submanifold. Combining statements (ii) and (iii) with the fact that M^n is contact CR-warped product submanifold, we get that the unit vector field $\chi = e_A$ belongs to the relative null space $\mathcal{N}_x$. The converse is straightforward and statement (2) is proved.

For a contact CR-warped product submanifold, the equality satisfies in (4.1)if for all unit tangent vector belong to N_T at x iff

$$
(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0\ \ (ii)\sum_{b=1}^{n}\sum_{\substack{A=1 \\ b\neq A}}^{n_1} h^r_{bA} = 0\ \ (iii)\ 2h^r_{pp} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{4.24}
$$

where $p \in \{1, \ldots, n_1\}$ and $r \in \{n + 1, \ldots, m\}$. Since M^n is contact CR-warped product submanifold, the third condition says that $h^r_{pp} = 0, \ \ p \in \{1, \ldots, n_1\}$. Using this in the condition (ii), we shall say that M^n is $D-$totally geodesic contact CR-warped product submanifold in $\bar{M}^m(f_1, f_2, f_3)$ and mixed totally geodesicness inheres from the condition (i). Which demonstrates (a) in (3).

For a contact CR-warped product submanifold, the equality sign of (4.1) holds identically for all unit tangent vector fields tangent to $N_\perp$ at x if and only if

$$
(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0\ (ii)\sum_{b=1}^{n}\sum_{\substack{A=n_1+1 \\ b\neq A}}^{n} h^r_{bA} = 0\ (iii)\ 2h^r_{KK} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{4.25}
$$

such that $K \in \{n_1 + 1, \ldots, n\}$ and $r \in \{n + 1, \ldots, m\}$. From the condition ($iii$) two cases emerge, that is

$$
h^r_{KK} = 0,\ \ \forall K \in \{n_1 + 1, \ldots, n\}\ \text{ and }\ r \in \{n + 1, \ldots, m\}\ \text{ or }\ dim\, N_\perp = 2.
$$

If the first case of (4.25) is satisfied, then by virtue of condition (ii), it is easy to conclude that M^n is a $D^\perp-$ totally geodesic contact CR-warped product submanifold in $\bar{M}^m(c)$. This is the first case of part (b) of statement (3).

On the other hand, let M^n is not $D^\perp-$totally geodesic contact CR-warped product submanifold and dim $N^\perp = 2$. Then condition (ii) of (4.25) implies that M^n is $D^\perp-$ totally umbilical contact CR-warped product submanifold in $\bar{M}^m(f_1, f_2, f_3)$, it is case second of the present part. Thus part (b) of (3) is verified.

To prove (c) using parts (a) and (b) of (3), we combine (4.24) and (4.25). For the first case of this part, assume that $dimN_\perp \neq 2$. Since from parts (a) and (b) of (3) we conclude that M^n is $D-$totally geodesic and $D^\perp-$ totally geodesic submanifold in $\bar{M}^m(f_1, f_2, f_3)$. Therefore, M^n is a totally geodesic submanifold in $\bar{M}^m(c)$.

Another case, suppose that first case does not satisfies. Then parts (a) and (b) provide that M^n is mixed totally geodesic and $D-$ totally geodesic submanifold of $\bar{M}^m(f_1, f_2, f_3)$ with $dimN_\perp = 2$. From the condition (b) it is clear that M^n is $D^\perp-$totally umbilical contact CR-warped product submanifold and from (a) it is $D-$totally geodesic, which is part (c). This proves the theorem.

In view of (2.10), we have another version of the theorem 4.1 as follows.

Theorem 4.2. *Let $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a contact CR-warped product submanifold isometrically immersed in a generalized Sasakian space form $\bar{M}^m(f_1, f_2, f_3)$ admitting nearly Sasakian structure. Then for each orthogonal unit vector field $\chi \in T_xM$ orthogonal to ξ, either tangent to N_T or $N_\perp$ the Ricci curvature satisfies the following inequalities:*

(i) If χ is tangent to N_T, then

$$Ric(\chi) \leq \frac{1}{4}n^2\|H\|^2 - n_2\Delta ln\psi + n_2\|\nabla ln\psi\|^2 + (n + n_1n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2 + 1)f_3. \tag{4.26}$$

(ii) If χ is tangent to $N_\perp$, then

$$Ric(\chi) \leq \frac{1}{4}n^2\|H\|^2 - n_2\Delta ln\psi + n_2\|\nabla ln\psi\|^2 + (n + n_1n_2 - 1)f_1 - (n_2 + 1)f_3. \tag{4.27}$$

The equality cases are similar as in the theorem 4.1.

Remark 4.3. *In particular, it is straightforward to see that the example given in Proposition 3.3 of [26] satisfies the inequalities of Theorem 4.2.*

5. Application of Obata's differential equation

This section is based on the study of Obata [26]. Basically, Obata characterized a Riemannian manifolds by a specific ordinary differential equation and derived that an $n-$dimensional complete and connected Riemannian manifold (M^n, g) to be isometric to the n-dimensional sphere if and only if there exists a non-constant smooth function τ on M^n that is the solution of the differential equation $H^\tau = -c\tau g$, where H^τ is the Hessian of τ. Moreover, for the warped product submanifolds the Obata's differential equation is used in ([3,25]). Recently, Alodan et al. [4] applied Obata's work in the study of hypersurface of Sasakian manifold. Inspired by these studies, we obtain the following characterization.

Theorem 5.1. *Let $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a compact orientable contact CR-warped product submanifold isometrically immersed in a generalized Sasakian space form $M^m(f_1, f_2, f_3)$ admitting nearly Sasakian structure with positive Ricci curvature and satisfying one of the following relation:*

(i) $\chi \in TN_T$ orthogonal to ξ and

$$||Hess\tau||^2 = -\frac{3\lambda_1 n^2}{4n_1n_2}||H||^2 - \frac{3\lambda_1}{n_1n_2}[(n + n_1n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2 + 1)f_3] \tag{5.1}$$

(ii) $\chi \in TN_\perp$ and

$$||Hess\tau||^2 = -\frac{3\lambda_1 n^2}{4n_1n_2}||H||^2 - \frac{3\lambda_1}{n_1n_2}[(n + n_1n_2 - 1)f_1 - (n_2 + 1)f_3], \tag{5.2}$$

where $\lambda_1 > 0$ is an eigenvalue of the warping function $\tau = ln\psi$. Then the base manifold $N_T^{n_1}$ is isometric to the sphere $S^{n_1}(\frac{\lambda_1}{n_1})$ with constant sectional curvature $\frac{\lambda_1}{n_1}$.

Proof. Let $\chi \in TN_T$. Consider that $\tau = ln\psi$ and define the following relation as

$$||Hess\tau - t\tau I||^2 = ||Hess\tau||^2 + t^2\tau^2||I||^2 - 2t\tau g(Hess\tau, I). \tag{5.3}$$

But we know that $||I||^2 = trace(II^*) = p$, where p is a real number and

$$g(Hess(\tau), I^*) = trace(Hess\tau, I^*) = traceHess(\tau).$$

Then Eq (5.3) transform to

$$||Hess\tau - t\tau I||^2 = ||Hess\tau||^2 + pt^2\tau^2 - 2t\tau\Delta\tau.$$

Assuming λ_1 is an eigenvalue of the eigen function τ then $\Delta\tau = \lambda_1\tau$. Thus we get

$$||Hess\tau - t\tau I||^2 = ||Hess\tau||^2 + (pt^2 - 2t\lambda)\tau^2. \tag{5.4}$$

On the other hand, we obtain $\Delta\tau^2 = 2\tau\Delta\tau + ||\nabla\tau||^2$ or $\lambda_1\tau^2 = 2\lambda_1\tau^2 + ||\nabla\tau||^2$ which implies that $\tau^2 = -\frac{1}{\lambda_1}||\nabla\tau||^2$, using this in Eq (5.4), we have

$$||Hess\tau - t\tau I||^2 = ||Hess\tau||^2 + (2t - \frac{pt^2}{\lambda_1})||\nabla\tau||^2. \tag{5.5}$$

In particular $t = -\frac{\lambda_1}{n_1}$ on (5.5) and integrating with respect to dV

$$\int_{M^n} ||Hess\tau + \frac{\lambda_1}{n_1}\tau I||^2 dV = \int_{M^n} ||Hess\tau||^2 dV - \frac{3\lambda_1}{n_1}\int_{M^n} ||\nabla\tau||^2 dV. \tag{5.6}$$

Integrating the inequality (4.26) and using the fact $\int_{M^n} \Delta\phi dV = 0$, we have

$$\begin{aligned}\int_{M^n} Ric(\chi) dV \leq &\frac{n^2}{4}\int_{M^n} ||H||^2 dV + n_2\int_{M^n} ||\nabla\tau||^2 dV + \\ &+ [(n + n_1n_2 - 1)f_1 - \frac{3f_2}{2} - (n_2 + 1)f_3]Vol(M^n).\end{aligned} \tag{5.7}$$

From (5.6) and (5.7) we derive

$$\frac{1}{n_2}\int_{M^n} Ric(\chi)dV \leq \frac{n^2}{4n_2}\int_{M^n}\|H\|^2 dV - \frac{n_1}{3\lambda_1}\int_{M^n}\|Hess\tau + \frac{\lambda_1}{n_1}\tau I\|^2 dV + \frac{n_1}{3\lambda_1}\int_{M^n}\|Hess\tau\|^2 dV + \frac{1}{n_2}[(n + n_1 n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2+1)f_3]Vol(M^n).$$

According to assumption $Ric(\chi) \geq 0$, the above inequality gives

$$\int_{M^n}\|Hess\tau + \frac{\lambda_1}{n_1}\phi I\|^2 dV \leq \frac{3n^2\lambda_1}{4n_1 n_2}\int_{M^n}\|H\|^2 dV + \int_{M^n}\|Hess\tau\|^2 dV - \frac{3\lambda_1}{n_1 n_2}[(n + n_1 n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2+1)f_3]Vol(M^n).$$

From (5.1), we get

$$\int_{M^n}\|Hess\tau + \frac{\lambda_1}{n_1}\tau I\|^2 dV \leq 0.$$

But we know that

$$\int_{M^n}\|Hess\tau + \frac{\lambda_1}{n_1}\tau I\|^2 dV \geq 0.$$

Combining last two statements, we get

$$\int_{M^n}\|Hess\tau + \frac{\lambda_1}{n_1}\tau I\|^2 dV = 0 \Rightarrow Hess\tau = -\frac{\lambda_1}{n_1}\tau I. \tag{5.8}$$

Since the warping function $\tau = ln\psi$ is not constant function on M^n so equation (5.8) is Obata's differential equation [26] with constant $c = \frac{\lambda_1}{n_1} > 0$. As $\lambda_1 > 0$ and therefore the base submanifold $N_T^{n_1}$ is isometric to the sphere $S^{n_1}(\frac{\lambda_1}{n_1})$ with constant sectional curvature $\frac{\lambda_1}{n_1}$. This proves the theorem. □

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Conflict of interest

The authors declare that they have no Competing interests.

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