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Research article

Some characterizations of non-null rectifying curves in dual Lorentzian 3-space $\mathbb{D}_1^3$

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Abstract: This paper gives several properties and characterization of non-null rectifying curves in dual Lorentzian 3-space $\mathbb{D}_1^3$. In considering a causal character of a dual curve we give some parameterization of rectifying dual curves, and a dual differential equation of third order is constructed for every non-null dual curve. Then several well-known characterizations of spherical, normal and rectifying dual curves are consequences of this differential equation.

Keywords: Serret-Frenet formulae; rectifying dual curve; dual helices
Mathematics Subject Classification: 53C50, 53C40

1. Introduction

Dual numbers were first introduced by W. Clifford after him E. Study used it as a tool for his research on the differential line geometry. He devoted special attention to the representation of oriented lines by dual unit vectors and defined the mapping that is known by his name. The set of all oriented lines in Euclidean 3-space $\mathbb{E}^3$ is in one-to- one correspondence with set of points of the dual unit sphere in the dual 3-space $\mathbb{D}^3$. More details on the necessary basic concepts about the dual elements and the one-to-one correspondence between ruled surfaces and one-parameter dual spherical motions can be found in [1–3]. If we take the Minkowski 3-space $\mathbb{E}_1^3$ instead of $\mathbb{E}^3$ the E. Study's map can be stated as follows: The timelike and spacelike dual unit vectors of hyperbolic and Lorentzian dual unit spheres $\mathbb{H}_+^2$ and $\mathbb{S}_1^2$ at the Lorentzian 3-space $\mathbb{D}_1^3$ are in one-to-one correspondence with the directed timelike and spacelike lines of the space of Lorentzian lines $\mathbb{E}_1^3$, respectively. Then a differentiable curve on $\mathbb{H}_+^2$ corresponds to a timelike ruled surface at $\mathbb{E}_1^3$. Similarly the timelike (resp. spacelike) curve on $\mathbb{S}_1^2$ corresponds to any spacelike (resp. timelike) ruled surface at $\mathbb{E}_1^3$ [4–6].

In recent years, with the development of the theory of relativity, geometers have extended some topics in classical differential geometry of Riemannian manifolds to that of Lorentzian manifolds. For instance, in the Euclidean 3-space $\mathbb{E}^3$, a curve whose tangent vector makes a constant angle with a fixed direction is called a helix (characterized by τ/κ is a non-constant linear function of the arc length parameter), spherical curves (characterized by $\left(\sigma\rho'\right)^2 + \rho^2 = \text{constant}$, with $\kappa\rho = 1$, $\sigma\tau = 1$), and rectifying curves (characterized by τ/κ is a non-constant linear function of the arc length parameter), where τ and κ stands for the torsion and curvature of the curve, respectively (see [7–11]). Due to its relationship with physical sciences in Minkowski space, the helices, spherical, normal and rectifying curves have been studied widely by [12–18]. In the course of time, the extensions of the results concerning the special curves in the Lorentzian 3-space $\mathbb{D}_1^3$ have been presented [19–24].

In this work, we revisit the non-null rectifying curves in Lorentzian 3-space $\mathbb{D}_1^3$ that they were characterized by [16, 20]. Especially, the fundamentals of this notion in Euclidean space are based on [7]. In this context, with a new perspective, we prove that the ratio of torsion and curvature of any non-null rectifying dual curve is a non-constant linear function of the pseudo dual arc-length parameter. Then, by considering the causal characters of a dual curve we give some parameterization of rectifying dual curves. Thereafter, we prove that the tangential height dual function of every non-null curve satisfies a third-order dual differential equation. Thus, several known properties of differentiable curves in the Euclidean 3-space are extended to the Lorentzian 3-space $\mathbb{D}_1^3$.

2. Preliminaries

In this section, we give a brief summary of the theory of dual numbers and dual Lorentzian vectors. (See for instance Refs. [1–6]). If x, and x^* are real numbers, the combination $X = x + \varepsilon x^*$ is called a dual number. Here ε is a dual unit subject to the rules $\varepsilon \neq 0$, $\varepsilon^2 = 0$, $\varepsilon.1 = 1.\varepsilon = \varepsilon$. The set of dual numbers, $\mathbb{D}$, forms a commutative ring having the numbers $\varepsilon x^*(x^* \in \mathbb{R})$ as divisors of zero, not a field. No number εx^* has inverse in the algebra. But the other lows of the algebra of dual numbers are the same as of the complex numbers. Then the set

$$\mathbb{D}^3 = \{\mathbf{X} := \mathbf{x} + \varepsilon\mathbf{x}^* = (X_1, X_2, X_3)\}, \tag{2.1}$$

together with the Lorentzian scalar product

$$< \mathbf{X}, \mathbf{Y} >= X_1Y_1 + X_2Y_2 - X_3Y_3, \tag{2.2}$$

forms the so called dual Lorentzian 3-space $\mathbb{D}_1^3$. This yields

$$\left.\begin{array}{l} \mathbf{F}_1 \times \mathbf{F}_2 = -\mathbf{F}_3,\ \mathbf{F}_1 \times \mathbf{F}_3 = -\mathbf{F}_2,\ \mathbf{F}_2 \times \mathbf{F}_3 = \mathbf{F}_1, \\ < \mathbf{F}_1, \mathbf{F}_1 > = < \mathbf{F}_2, \mathbf{F}_2 >= - < \mathbf{F}_3, \mathbf{F}_3 >=1, \end{array}\right\} \tag{2.3}$$

where $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, are the dual base at the origin point $\mathbf{O}\,(0, 0, 0)$ of the dual Lorentzian 3-space $\mathbb{D}_1^3$. Thereby a point $X = (X_1, X_2, X_3)^t$ has dual coordinates $X_i = (x_i + \varepsilon x_i^*) \in \mathbb{D}$. If $\mathbf{x} \neq \mathbf{0}$ the norm $\|\mathbf{X}\|$ of $\mathbf{X}$ is defined by

$$\|\mathbf{X}\| = \sqrt{|< \mathbf{X}, \mathbf{X} >|} = \|\mathbf{x}\|\,(1+\varepsilon\frac{< \mathbf{x}, \mathbf{x}^* >}{\|\mathbf{x}\|^2}), \tag{2.4}$$

then the vector $\mathbf{X}$ is called a spacelike dual unit vector if $< \mathbf{X}, \mathbf{X} >=1$ and a timelike dual unit vector if $< \mathbf{X}, \mathbf{X} >= -1$. It is clear that

$$\|\mathbf{X}\|^2 = \pm 1 \Longleftrightarrow \|\mathbf{x}\|^2 = \pm 1,\ < \mathbf{x}, \mathbf{x}^* >=0. \tag{2.5}$$

The six components x_i, $x_i^*(i = 1, 2, 3)$ of $\mathbf{x}$, and $\mathbf{x}^*$ are called the normed Plücker coordinates of the line.

Let $\widehat{\mathbf{m}}(\widehat{m}_1, \widehat{m}_2, \widehat{m}_3)$ be a fixed dual point in $\mathbb{D}_1^3$. Then

$$\widehat{\mathbb{S}}_1^2(\widehat{\mathbf{m}}) = \left\{\mathbf{X} \in \mathbb{D}_1^3 \mid (X_1 - \widehat{m}_1)^2 + (X_2 - \widehat{m}_2)^2 - (X_2 - \widehat{m}_3)^2 = 1\right\}, \tag{2.6}$$

is the Lorentzian dual unit sphere with the center $\widehat{\mathbf{m}}$ and

$$\widehat{\mathbb{H}}_+^2(\widehat{\mathbf{m}}) = \left\{\mathbf{X} \in \mathbb{D}_1^3 \mid (X_1 - \widehat{m}_1)^2 + (X_2 - \widehat{m}_2)^2 - (X_3 - \widehat{m}_3)^2 = -1\right\}, \tag{2.7}$$

is the hyperbolic dual unit sphere with the center $\widehat{\mathbf{m}}$. Therefore, the hyperbolic, and Lorentzian (de Sitter space) dual unit spheres with the center $\mathbf{O}$ are:

$$\widehat{\mathbb{H}}_+^2 = \left\{\mathbf{X} \in \mathbb{D}_1^3 \mid -X_1^2 + X_2^2 + X_3^2 = -1\right\}, \tag{2.8}$$

and

$$\widehat{\mathbb{S}}_1^2 = \left\{\mathbf{X} \in \mathbb{D}_1^3 \mid X_1^2 + X_2^2 - X_3^2 = 1\right\}, \tag{2.9}$$

respectively. Via this we have the following map (E. Study's map): The dual unit spheres are shaped as a pair of conjugate hyperboloids. The common asymptotic cone represents the set of null (lightlike) lines, the ring shaped hyperboloid represents the set of spacelike lines, and the oval shaped hyperboloid forms the set of timelike lines, opposite points of each hyperboloid represent the pair of opposite vectors on a line (see Figure 1).

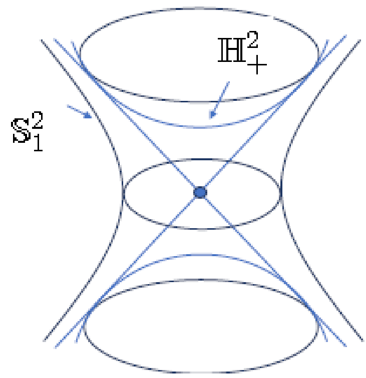

Figure 1. The dual hyperbolic and dual Lorentzian unit spheres.

Let $\psi(t) = (\psi_1(t), \psi_2(t), \psi_3(t))$, and $\psi^*(t) = \left(\psi_1^*(t), \psi_2^*(t), \psi_3^*(t)\right)$ be real valued curves in the Mnkowski 3-space $\mathbb{E}_1^3$. Then, a differentiable curve

$$\begin{aligned} \mathbf{\Psi} &: \quad t \in \mathbb{R} \to \mathbb{D}_1^3 \\ \mathbf{\Psi} &= \quad (\psi_1(t), \psi_2(t), \psi_3(t)) + \varepsilon\left(\psi_1^*(t), \psi_2^*(t), \psi_3^*(t)\right), \ t \in \mathbb{R}, \end{aligned}$$

represents a curve in the dual Lorentzian 3-space $\mathbb{D}_1^3$ and is called a dual space curve. The dual arc length of $\mathbf{\Psi}(t)$ from t_0 to t is defined by

$$\widehat{s} = s + \varepsilon s^* = \int_{t_0}^{t} \left\|\frac{d\mathbf{\Psi}}{dt}\right\| dt = \int_{t_0}^{t} \left\|\frac{d\psi}{dt}\right\| dt + \varepsilon \int_{t_0}^{t} < \mathbf{t}, \frac{d\psi^*}{dt} > dt,$$

where $\mathbf{t}$ is a unit tangent vector of $\psi(t)$. From now on, we will take the pseudo dual arc length $\widehat{s}$ of $\mathbf{\Psi}(t)$ as the parameter instead of $t \in \mathbb{R}$. Then $\mathbf{\Psi}(\widehat{s})$ is called a dual arc-length parameter curve. From now on, we shall often not write the dual parameter $\widehat{s}$ explicitly in our formulae.

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving dual Serret-Frenet frame along $\Psi(\widehat{s})$ in the dual space $\mathbb{D}^3_1$. Then $\mathbf{t} + \varepsilon\mathbf{t}^* = \mathbf{T}(\widehat{s})$, $\mathbf{n} + \varepsilon\mathbf{n}^* = \mathbf{N}(\widehat{s})$, and $\mathbf{b} + \varepsilon\mathbf{b}^* = \mathbf{B}(\widehat{s})$ are the dual unit tangent, dual unit principal normal, and dual unit binormal vectors of the curve at the point $\mathbf{\Psi}(\widehat{s})$. The dual arc-length derivative of the dual Serret-Frenet frame is governed by the relations:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \widehat{\kappa} & 0 \\ -\epsilon_0\epsilon_1\widehat{\kappa} & 0 & \widehat{\tau} \\ 0 & --\epsilon_1\epsilon_2\widehat{\tau} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \left(' = \frac{d}{d\widehat{s}}\right), \tag{2.10}$$

where $\epsilon_0 =< \mathbf{T}, \mathbf{T} >= \pm 1$, $\epsilon_1 = < \mathbf{N}, \mathbf{N} >= \pm 1$, $\epsilon_2 =< \mathbf{B}, \mathbf{B} > = \pm 1$, $\epsilon_0\epsilon_1\epsilon_2 = -1$, $\widehat{\kappa} = \kappa + \varepsilon\kappa^*$ is nowhere pure dual curvature, and $\widehat{\tau} = \tau + \varepsilon\tau^*$ is nowhere pure dual torsion. The above formulae are called the Serret-Frenet formulae of dual curve in $\mathbb{D}^3_1$. Here "prime" denotes the derivative with respect to the pseudo dual parameter $\widehat{s}$.

3. Characterizations of a non-null dual curve

In the next theorem, we prove that the ratio of torsion and curvature of any non-null dual rectifying curve is a non-constant linear function of the pseudo dual arc length parameter $\widehat{s}$. Also, we emphasize that this property is invariant with respect to the causal character of a dual curve and its rectifying dual plane [7, 16, 20].

Theorem 3.1. Let $\mathbf{\Psi} = \Psi(\widehat{s})$ be a non-null unit speed dual curve with spacelike or timelike rectifying dual plane and $\widehat{\kappa}$ is nowhere pure-dual. Then the following are equivalent:
(i) There is a dual point $\widehat{\mathbf{m}} \in \mathbb{D}^3_1$ such that every spacelike or timelike rectifying dual plane of $\mathbf{\Psi} = \Psi(\widehat{s})$ goes through $\widehat{\mathbf{m}}$.
(iii) $\widehat{\tau}/\widehat{\kappa}$ is a non-constant linear function $A_1\widehat{s} + A_2$, with $A_i = a_i + \varepsilon a_i^*$, where $a_i \neq 0$, and $i = 1, 2$.
(iii) There is a dual point $\widehat{\mathbf{m}}_0 \in \mathbb{D}^3_1$ such that $\left\|\widehat{\alpha}(\widehat{s}) - \widehat{\mathbf{m}}_0\right\|^2 = \left|\epsilon_0(\widehat{s} + C)^2 + \epsilon_2 D\right|$. The dual constants are related by

$$A_1 = \frac{\epsilon_0}{D},\ A_2 = \frac{\epsilon_0\widehat{C}}{D};\ \text{with } D = d + \varepsilon d^*,\ \text{where } d \neq 0.$$

And by the uniqueness of $\widehat{\mathbf{m}}$, $\widehat{\mathbf{m}}$ is equal to $\widehat{\mathbf{m}}_0$.

Proof. (i) Suppose that every spacelike or timelike rectifying dual plane of $\mathbf{\Psi} = \Psi(\widehat{s})$ goes through a fixed dual point $\widehat{\mathbf{m}} \in \mathbb{D}^3_1$. Then we have

$$< \mathbf{\Psi} - \widehat{\mathbf{m}}, \mathbf{N} >= 0.$$

By differentiating this equation and using the Serret-Frenet formulae, thus obtaining

$$< \mathbf{\Psi} - \widehat{\mathbf{m}}, -\epsilon_2\widehat{\kappa}\mathbf{T} + \widehat{\tau}\mathbf{B} >= 0. \tag{3.1}$$

From the last two equations, it follows that the rectifying dual plane is orthogonal to both $\mathbf{N}$ and $-\epsilon_2\widehat{\kappa}\mathbf{T} + \widehat{\tau}\mathbf{B}$. Hence, we can write

$$\mathbf{\Psi} - \widehat{\mathbf{m}} = \widehat{\eta}(\widehat{s})(\epsilon_0\widehat{\tau}\mathbf{T} - \widehat{\kappa}\mathbf{B}), \tag{3.2}$$

for a differentiable dual function $\widehat{\eta} = \widehat{\eta}(\widehat{s})$. By differentiation of Eq (3.1), we have that:

$$-\epsilon_1\widehat{\kappa} + < \mathbf{\Psi} - \widehat{\mathbf{m}}, \epsilon_2\widehat{\kappa}^{\prime}\mathbf{T} + \widehat{\tau}\mathbf{B} >= 0. \tag{3.3}$$

Combing the Eqs (3.2) and (3.3), implies that

$$\widehat{\eta} = \frac{\widehat{\kappa}\epsilon_0}{\widehat{\tau}^{\prime}\widehat{\kappa} - \widehat{\kappa}^{\prime}\widehat{\tau}}. \tag{3.4}$$

Substituting Eq (3.4) into Eq (3.2), we obtain

$$\mathbf{\Psi} - \widehat{\mathbf{m}} = \frac{\widehat{\kappa}\widehat{\tau}}{\widehat{\tau}^{\prime}\widehat{\kappa} - \widehat{\kappa}^{\prime}\widehat{\tau}}\mathbf{T} - \frac{\widehat{\kappa}^{2}\epsilon_0}{\widehat{\tau}^{\prime}\widehat{\kappa} - \widehat{\kappa}^{\prime}\widehat{\tau}}\mathbf{B}. \tag{3.5}$$

Furthermore, one calculates

$$\widehat{\mathbf{m}}^{\prime} = \left(1 - \left(\frac{\widehat{\kappa}\widehat{\tau}}{\widehat{\tau}^{\prime}\widehat{\kappa} - \widehat{\kappa}^{\prime}\widehat{\tau}}\right)^{\prime}\right)\mathbf{T} + \left(\frac{\widehat{\kappa}^{2}\epsilon_0}{\widehat{\tau}^{\prime}\widehat{\kappa} - \widehat{\kappa}^{\prime}\widehat{\tau}}\right)\mathbf{B}.$$

Therefore, the coefficients vanishing identically if

$$1 - \left(\frac{\widehat{\kappa}\widehat{\tau}}{\widehat{\tau}^{\prime}\widehat{\kappa} - \widehat{\kappa}^{\prime}\widehat{\tau}}\right)^{\prime} = 0, \ \left(\frac{\widehat{\kappa}^{2}\epsilon_0}{\widehat{\tau}^{\prime}\widehat{\kappa} - \widehat{\kappa}^{\prime}\widehat{\tau}}\right)^{\prime} = 0,$$

whereby

$$\frac{\widehat{\kappa}\widehat{\tau}}{\widehat{\tau}^{\prime}\widehat{\kappa} - \widehat{\kappa}^{\prime}\widehat{\tau}} = \widehat{s} + C, \quad \frac{\widehat{\kappa}^{2}\epsilon_0}{\widehat{\tau}^{\prime}\widehat{\kappa} - \widehat{\kappa}^{\prime}\widehat{\tau}} = D, \ \ C = c + \varepsilon c^{*} \in \mathbb{D}. \tag{3.6}$$

Since $\widehat{\kappa}$ is nowhere pure-dual, then D is nowhere pure-dual too. From Eqs (3.5) and (3.6), we have

$$\left.\begin{array}{l} \mathbf{\Psi} - \widehat{\mathbf{m}} = (\widehat{s} + C)\mathbf{T} - D\mathbf{B}, \\ \frac{\widehat{\tau}}{\widehat{\kappa}} = A_1\widehat{s} + A_2; \ \ A_1 = \frac{\epsilon_0}{D}, \ A_2 = \epsilon_0\frac{C}{D}, \ D = d + \varepsilon d^{*} \in \mathbb{D}. \end{array}\right\} \tag{3.7}$$

Therefore, we can calculate that

$$\left\| \mathbf{\Psi} - \widehat{\mathbf{m}}_0 \right\|^2 = \left| \epsilon_0(\widehat{s} + C)^2 + \epsilon_2 D^2 \right|.$$

This shows (i) implies (ii), and (iii). If every spacelike or timelike rectifying dual plane goes through another dual point $\widehat{\mathbf{m}}_0$ then let $\widehat{\gamma}(\widehat{t})$ be the unit speed dual geodesic line through $\widehat{\mathbf{m}}_0$ and $\widehat{\mathbf{m}}$. Then for each $\widehat{t} \in \mathbb{D}$, there are dual constants $C(\widehat{t})$ and $D(\widehat{t}) \neq 0$ such that

$$\mathbf{\Psi} - \widehat{\gamma}(\widehat{t}) = (\widehat{s} + C(\widehat{t}))\mathbf{T} + D(\widehat{t})\mathbf{B}. \tag{3.8}$$

If dot denotes to derivation with respect to $\widehat{t}$, then from Eq (3.8) we have

$$-\dot{\widehat{\gamma}}(\widehat{t}) = \dot{C}(\widehat{t}))\mathbf{T} + \dot{D}(\widehat{t})\mathbf{B}.$$

Note that:

$$\frac{\widehat{\tau}}{\widehat{\kappa}} = A_1(\widehat{t})\widehat{s} + A_2(\widehat{t}); \ A_1(\widehat{t}) = \frac{\epsilon_0}{D(\widehat{t})}, \ A_2(\widehat{t}) = \epsilon_0\frac{C(\widehat{t})}{D(\widehat{t})}.$$

By differentiation with respect to $\widehat{t}$, we have

$$\dot{A}_1(\widehat{t})\widehat{s}+\dot{A}_2(\widehat{t}) = 0.$$

Thus, we have

$$\dot{A}_1(\widehat{t}) = \dot{A}_2(\widehat{t}) = 0 \Rightarrow \dot{\widehat{\gamma}}(\widehat{t}) = 0,\ \widehat{\mathbf{m}} = \widehat{\mathbf{m}}_0.$$

(ii) Suppose that $\frac{\widehat{\tau}}{\widehat{\kappa}} = A_1\widehat{s}+A_2;\ A_i \neq \varepsilon a_i^*$. If we let

$$\widehat{\mathbf{m}}=\mathbf{\Psi}-(\widehat{s}+\frac{A_2}{A_1})\mathbf{T}+\frac{\epsilon_0}{A_1}\mathbf{B},$$

then by the assumption, we get $\widehat{\mathbf{m}}' = \mathbf{0}$. Hence $\widehat{\mathbf{m}}$ is a fixed dual point in $\mathbb{D}_1^3$ and

$$\mathbf{\Psi}(\widehat{s}) - \widehat{\mathbf{m}}= (\widehat{s}+C)\mathbf{T}+D\mathbf{B},\ D = \frac{\epsilon_0}{A_1},\ C = \frac{A_2}{A_1}.$$

This shows (ii) implies (i), and (iii).

Now suppose that statement (iii) holds, then

$$< \mathbf{\Psi}(\widehat{s}) - \widehat{\mathbf{m}},\mathbf{T}>= \epsilon_0(\widehat{s}+C). \tag{3.9}$$

By differentiation of Eq (3.9) and using Serret-Frenet formulae, we have

$$\widehat{\kappa} < \mathbf{\Psi} - \widehat{\mathbf{m}}, \mathbf{N} > = 0;\ \widehat{\kappa} \neq 0 \Rightarrow < \mathbf{\Psi} - \widehat{\mathbf{m}}, \mathbf{N} > = 0,$$

which means that every rectifying dual plane of $\mathbf{\Psi}$ goes through a fixed dual point in $\mathbb{D}_1^3$. This shows (iii) implies (i). □

We end this section by determine some parameterization of a non-null rectifying dual curve in terms of its radial projection. Let us first assume that $\mathbf{\Psi} = \mathbf{\Psi}(\widehat{s})$ is a unit speed spacelike dual curve with a spacelike or timelike dual principal normal in $\mathbb{D}_1^3$. Then $\epsilon_0 = 1$, $\epsilon_1 = -\epsilon_2 = \epsilon = \pm 1$. For a fixed dual point $\widehat{\mathbf{m}}\in\mathbb{D}_1^3$ and by the proof of Theorem 3.1, let

$$\widehat{\beta}(\widehat{s}) = \frac{1}{R(\widehat{s})}(\mathbf{\Psi} - \widehat{\mathbf{m}});\quad R(\widehat{s})= \left\|\mathbf{\Psi} - \widehat{\mathbf{m}}\right\| = \sqrt{\left|(\widehat{s}+C)^2 - \epsilon D^2\right|}, \tag{3.10}$$

is the radial projection of $\mathbf{\Psi}(\widehat{s})$ into $\mathbb{S}_1^2(\widehat{\mathbf{m}})$ (resp. $\mathbb{H}_+^2(\widehat{\mathbf{m}})$). Then, we have the following:

$$\left.\begin{array}{l} \mathbf{T} = R'\widehat{\beta} + R\widehat{\beta}', \\ \widehat{\kappa}\mathbf{N} = R''\widehat{\beta} + 2R'\widehat{\beta}' + R\widehat{\beta}''. \end{array}\right\} \tag{3.11}$$

Theorem 3.2. Let $\mathbf{\Psi} = \mathbf{\Psi}(\widehat{s})$ be a unit speed non-null spacelike rectifying dual curve with a spacelike or timelike dual principal normal and $\widehat{\kappa}$ is nowhere pure-dual. If $\widehat{\mathbf{m}}\in\mathbb{D}_1^3$ is a fixed dual point, then the following statements hold:

(i) $\mathbf{\Psi} - \widehat{\mathbf{m}}$ is a spacelike position vector with a spacelike rectifying dual plane if and only if, up to a parametrization, $\mathbf{\Psi} - \widehat{\mathbf{m}}$ is given by

$$\mathbf{\Psi}(\Theta) - \widehat{\mathbf{m}} = \frac{D}{\cos\Theta}\widehat{\beta}(\Theta), \tag{3.12}$$

where $\widehat{\beta}(\Theta)$ is a unit speed spacelike dual curve lying fully in $\mathbb{S}_1^2(\widehat{\mathbf{m}})$, and

$$\widehat{\kappa}_\beta^2 = \frac{R^6}{D^4}\widehat{\kappa}^2 - 1.$$

(ii) $\mathbf{\Psi} - \widehat{\mathbf{m}}$ is a spacelike position vector with a timelike rectifying dual plane if and only if, up to a parametrization, $\mathbf{\Psi} - \widehat{\mathbf{m}}$ is given by

$$\mathbf{\Psi}(\Theta) - \widehat{\mathbf{m}} = \frac{D}{\sinh\Theta}\widehat{\beta}(\Theta) \tag{3.13}$$

where $\widehat{\beta}(\Theta)$ is a unit speed timelike dual curve lying fully in $\mathbb{S}_1^2(\widehat{\mathbf{m}})$ and

$$\widehat{\kappa}_\beta^2 = \frac{R^6}{D^4}\widehat{\kappa}^2 + 1.$$

(iii) $\mathbf{\Psi} - \widehat{\mathbf{m}}$ is a timelike position vector with a timelike rectifying dual plane if and only if, up to a parametrization, $\mathbf{\Psi} - \widehat{\mathbf{m}}$ is given by

$$\mathbf{\Psi}(\Theta) - \widehat{\mathbf{m}} = \frac{D}{\cosh\Theta}\widehat{\beta}(\Theta), \tag{3.14}$$

where $\widehat{\beta}(\Theta)$ is a unit speed spacelike dual curve lying fully in $\mathbb{H}_+^2(\widehat{\mathbf{m}})$, and

$$\widehat{\kappa}_\beta^2 = \frac{R^6}{D^4}\widehat{\kappa}^2 - 1.$$

Here $D = d + \varepsilon d^*$, where $d \neq 0$, and $\widehat{\kappa}_\beta$ denotes to the dual curvature of the dual curve $\widehat{\beta} = \widehat{\beta}(\Theta)$.

Proof. Assume that $\widehat{\beta}(S)$ is lying fully in $\mathbb{S}_1^2(\widehat{\mathbf{m}})$. Then, we can write

$$\left.\begin{array}{l} \widehat{\beta}(\widehat{s}) = \frac{1}{R(\widehat{s})}(\mathbf{\Psi} - \widehat{\mathbf{m}});\ R^2(\widehat{s}) = \left\|\mathbf{\Psi} - \widehat{\mathbf{m}}\right\|^2 = \left|(\widehat{s} + C)^2 - \epsilon D^2\right|, \\ <\widehat{\beta},\widehat{\beta}>= 1,\quad <\widehat{\beta},\widehat{\beta}^{\prime}>= 0. \end{array}\right\} \tag{3.15}$$

A straightforward calculations show that

$$\left\|\widehat{\beta}^{\prime}(\widehat{s})\right\|^2 = -\epsilon\frac{D^2}{R^4}, \tag{3.16}$$

which means that $\widehat{\beta}(\widehat{s})$ is a spacelike or timelike dual curve if $\epsilon = -1$ or $\epsilon = 1$, respectively.
(i) In the case of $\epsilon = -1$, the position vector $\mathbf{\Psi} - \widehat{\mathbf{m}}$ is lying fully in a spacelike rectifying dual plane. If $\Theta = \vartheta + \varepsilon\vartheta^*$ be the pseudo dual arc length parameter of the dual curve $\widehat{\beta}(\widehat{s})$, then we have

$$\Theta = \int_0^S \left\|\widehat{\beta}^{\prime}(\widehat{s})\right\| d\widehat{s} = \left(\frac{D}{(\widehat{s} + C)^2 + D^2}\right)d\widehat{s} = \tan^{-1}\left(\frac{\widehat{s} + C}{D}\right), \tag{3.17}$$

and therefore $\widehat{s} + C = D\tan\Theta$. Substituting this into Eq (3.15) with $\epsilon = -1$, we obtain

$$\mathbf{\Psi}(\Theta) - \widehat{\mathbf{m}} = \frac{D}{\cos\Theta}\widehat{\beta}(\Theta). \tag{3.18}$$

Conversely, assume that $\mathbf{\Psi}(\Theta)-\widehat{\mathbf{m}}$ is a dual curve defined by Eq (3.18), where $\widehat{\beta}(\Theta)$ is a unit speed spacelike dual curve lying on $\mathbb{S}_1^2(\widehat{\mathbf{m}})$; that is,

$$\left\|\widehat{\beta}(\widehat{s})\right\|^2=\left\|\widehat{\beta}'(\widehat{s})\right\|^2=1,\ \text{and} <\widehat{\beta},\widehat{\beta}'>=0.$$

Differentiating the Eq (3.18) with respect to Θ, we get

$$(\mathbf{\Psi}-\widehat{\mathbf{m}})'=\frac{D}{\cos^2\Theta}\left(\widehat{\beta}\sin\Theta+\widehat{\beta}'\cos\Theta\right). \tag{3.19}$$

Therefore, it follows that

$$<(\mathbf{\Psi}-\widehat{\mathbf{m}})',\mathbf{\Psi}-\widehat{\mathbf{m}}>=\frac{D^2\sin\Theta}{\cos^3\Theta},\ \left\|(\mathbf{\Psi}-\widehat{\mathbf{m}})'\right\|^2=\frac{D^2}{\cos^4\Theta}. \tag{3.20}$$

Let us write

$$\mathbf{\Psi}(\Theta)-\widehat{\mathbf{m}}=\widehat{\mu}(\Theta)(\mathbf{\Psi}(\Theta)-\widehat{\mathbf{m}})'+(\mathbf{\Psi}(\Theta)-\widehat{\mathbf{m}})^{\perp},$$

for dual function $\widehat{\mu}(\Theta)$, where$(\mathbf{\Psi}(\Theta)-\widehat{\mathbf{m}})^{\perp}$ is the normal component of the position vector $\mathbf{\Psi}(\Theta)-\widehat{\mathbf{m}}$. Then, in view of the last equations, we easily find that

$$\widehat{\mu}(\Theta)=\frac{<(\mathbf{\Psi}-\widehat{\mathbf{m}})',\mathbf{\Psi}-\widehat{\mathbf{m}}>}{\left\|(\mathbf{\Psi}-\widehat{\mathbf{m}})'\right\|^2}=\sin\Theta\cos\Theta\neq\text{constant}.$$

Therefore, we have

$$\left\|(\mathbf{\Psi}(\Theta)-\widehat{\mathbf{m}})^{\perp}\right\|^2=\left\|(\mathbf{\Psi}(\Theta)-\widehat{\mathbf{m}})\right\|^2-\frac{<(\mathbf{\Psi}-\widehat{\mathbf{m}})',\mathbf{\Psi}-\widehat{\mathbf{m}}>^2}{\left\|(\mathbf{\Psi}-\widehat{\mathbf{m}})'\right\|^2}=D^2=\text{constant},$$

which means that $\mathbf{\Psi}(\Theta)-\widehat{\mathbf{m}}$ is a rectifying dual curve.

We now calculate the curvature of $\widehat{\beta}(\widehat{s})$. By a direct calculation, we have the following:

$$\left.\begin{array}{l} <\widehat{\beta}(\widehat{s}),\widehat{\beta}''(\widehat{s})>=-\frac{D^2}{R4}, \\ <\widehat{\beta}'(\widehat{s}),\widehat{\beta}''(\widehat{s})>=-\frac{2D^2}{R5}R', \\ \left\|\widehat{\beta}''(\widehat{s})\right\|^2=\frac{4D^2R'^2}{R^6}-\frac{D^4}{R^8}\widehat{\kappa}_{\beta}^2. \end{array}\right\} \tag{3.21}$$

When Eq (3.21) is applied to Eq (3.11) we immediately find that:

$$\widehat{\kappa}^2:=\left\|\mathbf{\Psi}''(\widehat{s})\right\|^2=\frac{D^4}{R^6}(1+\widehat{\kappa}_{\beta}^2).$$

(ii) In the case of $\epsilon=1$, the position vector $\mathbf{\Psi}-\widehat{\mathbf{m}}$ is a spacelike position vector lying fully in a timelike rectifying dual plane. By Eqs (3.15) and (3.16), we have

$$\left\|\widehat{\beta}'(\widehat{s})\right\|=-\frac{D^2}{(\widehat{s}+C)^2-D^2},\ \left|\widehat{s}+C\right|>D, \tag{3.22}$$

which means that $\widehat{\beta}(\widehat{s})$ is a timelike dual curve. Hence, we easily get that

$$\Theta = \int_0^s \left(\frac{D}{D^2 - (\widehat{s} + C)^2} \right) d\widehat{s} = \coth^{-1} \left(\frac{\widehat{s} + C}{D} \right), \tag{3.23}$$

and therefore $\widehat{s} + C = -D \coth \Theta$. Substituting this into Eq (3.15) with $\epsilon = 1$, we obtain

$$\mathbf{\Psi}(\Theta) - \widehat{\mathbf{m}} = \frac{\widehat{d}}{\sinh \Theta} \widehat{\beta}(\Theta). \tag{3.24}$$

Conversely, let us assume that $\mathbf{\Psi}(\Theta)$ is a dual curve defined by Eq (3.24), where $\widehat{\beta}(\Theta)$ is a unit speed timelike dual curve lying on $\mathbb{S}_1^2(\widehat{\mathbf{m}})$; that is,

$$\left\| \widehat{\beta}(\widehat{s}) \right\|^2 = - \left\| \widehat{\beta}'(\widehat{s}) \right\|^2 = 1, \text{ and } < \widehat{\beta}, \widehat{\beta}' >= 0.$$

Differentiating the Eq (3.24) with respect to Θ, we get

$$(\mathbf{\Psi} - \widehat{\mathbf{m}})' = \frac{D}{\sinh^2 \Theta} \left(\widehat{\beta}' \sinh \Theta - \widehat{\beta} \cosh \Theta \right). \tag{3.25}$$

Therefore, it follows that

$$< (\mathbf{\Psi} - \widehat{\mathbf{m}})', \mathbf{\Psi} - \widehat{\mathbf{m}} >= -\frac{D^2 \cosh \Theta}{\sinh^3 \Theta}, \ \left\| (\mathbf{\Psi} - \widehat{\mathbf{m}})' \right\|^2 = -\frac{D^2}{\sinh^4 \Theta}. \tag{3.26}$$

Let us write

$$\mathbf{\Psi}(\widehat{s}_\beta) - \widehat{\mathbf{m}} = \widehat{\mu}(\Theta)(\mathbf{\Psi}(\Theta) - \widehat{\mathbf{m}})' + (\mathbf{\Psi}(\Theta) - \widehat{\mathbf{m}})^\perp,$$

for function $\widehat{\mu}(\Theta)$, where $(\mathbf{\Psi}(\Theta) - \widehat{\mathbf{m}})^\perp$ is the normal component of the position vector $\mathbf{\Psi}(\Theta) - \widehat{\mathbf{m}}$. Then, in view of the last equations, we easily find that

$$\widehat{\mu}(\Theta) = \frac{< (\mathbf{\Psi} - \widehat{\mathbf{m}})', \mathbf{\Psi} - \widehat{\mathbf{m}} >}{\left\| (\mathbf{\Psi} - \widehat{\mathbf{m}})' \right\|^2} = \sinh \Theta \cosh \Theta \neq \text{constant}.$$

Therefore, we have

$$\left\| (\mathbf{\Psi}(\Theta) - \widehat{\mathbf{m}})^\perp \right\|^2 = \left\| (\mathbf{\Psi}(\Theta) - \widehat{\mathbf{m}}) \right\|^2 - \frac{< (\mathbf{\Psi} - \widehat{\mathbf{m}})', \mathbf{\Psi} - \widehat{\mathbf{m}} >^2}{\left\| (\mathbf{\Psi} - \widehat{\mathbf{m}})' \right\|^2} = -D^2 = \text{constant},$$

which means that $\mathbf{\Psi}(\Theta) - \widehat{\mathbf{m}}$ is lying fully in a timelike rectifying dual plane. Additionally, an analogous arguments show that:

$$\left.\begin{array}{l} < \widehat{\beta}'(\widehat{s}), \widehat{\beta}''(\widehat{s}) >= \frac{2D^2 R}{R^5}, \\ < \widehat{\beta}(\widehat{s}), \widehat{\beta}''(\widehat{s}) >= \frac{D^2}{R^4}, \\ \left\| \widehat{\beta}''(\widehat{s}) \right\|^2 = -\frac{4D^2 R'}{R^6} + \frac{D^4}{R^8} \widehat{\kappa}_\beta^2, \end{array}\right\} \tag{3.27}$$

so that:

$$\widehat{\kappa}_\beta^2 = \frac{r^6}{\widehat{d}^4} \widehat{\kappa}^2 + 1.$$

(iii) The proof is analogous to the proofs of the statements (i) and (ii). □

If we repeat the above discussion for timelike curve, instead of spacelike curve with non-null principal normal, then $\epsilon_0 = -1$, $\epsilon_1 = \epsilon_2 = 1$. As a result, we have the following theorem which is analogous to the Theorem 3.2:

Theorem 3.3. Let $\mathbf{\Psi} = \mathbf{\Psi}(\widehat{s})$ is a unit speed timelike rectifying dual curve in $\mathbb{D}_1^3$. If $\widehat{\mathbf{m}} \in \mathbb{D}_1^3$ is a fixed point, then the following statements hold:
(i) $\mathbf{\Psi} - \widehat{\mathbf{m}}$ is a spacelike position vector lying fully in a timelike rectifying plane if and only if, up to a parametrization, $\mathbf{\Psi} - \widehat{\mathbf{m}}$ is given by

$$\mathbf{\Psi}(\Theta) - \widehat{\mathbf{m}} = \frac{D}{\sinh\Theta}\beta(\Theta), \tag{3.28}$$

where $\widehat{\beta}(\Theta)$ is a unit speed timelike curve lying fully in $\mathbb{S}_1^2(\widehat{\mathbf{m}})$, and there holds

$$\widehat{\kappa}_{\widehat{\beta}}^2 = 1 - \frac{R^6}{D^4}\widehat{\kappa}^2.$$

(ii) $\mathbf{\Psi} - \widehat{\mathbf{m}}$ is a timelike position vector lying fully in a timelike rectifying plane if and only if, up to a parametrization, $\mathbf{\Psi} - \widehat{\mathbf{m}}$ is given by

$$\mathbf{\Psi}(S_{\widehat{\beta}}) - \widehat{\mathbf{m}} = \frac{D}{\cosh\Theta}\widehat{\beta}(\Theta), \tag{3.29}$$

where $\widehat{\beta}(\Theta)$ is a unit speed spacelike curve lying fully in $\mathbb{H}_+^2(\widehat{\mathbf{m}})$, and there holds

$$\widehat{\kappa}_{\widehat{\beta}}^2 = 1 - \frac{R^6}{D^4}\widehat{\kappa}^2.$$

Here $D = d + \varepsilon d^*$, where $d \neq 0$, and $\widehat{\kappa}_\beta$ denote to the dual curvature of the dual curve $\widehat{\beta} = \widehat{\beta}(\Theta)$.

4. A differential equation for a non-null dual curve

In the next proposition, we derive a third-order dual differential equation satisfied for every non-null spherical, normal and rectifying dual curves in $\mathbb{D}_1^3$ with $\widehat{\kappa}$ is nowhere pure-dual. For this purpose, we define a smooth function on $\mathbf{\Psi}(\widehat{s})$ by

$$\widehat{h}(\widehat{s}) = <\mathbf{\Psi}, \mathbf{T}>, \tag{4.1}$$

we call $\widehat{h}(\widehat{s})$ the height dual tangential function (or, tangent directed dual distance functions).

Proposition 4.1. Let $\mathbf{\Psi} = \mathbf{\Psi}(\widehat{s})$ be a non-null unit speed dual curve in $\mathbb{D}_1^3$ with $\widehat{\kappa}$ is nowhere pure-dual. Then we have

$$\begin{aligned}
&\widehat{\rho}\widehat{\sigma}\widehat{h}''' + \left(2\widehat{\rho}'\widehat{\sigma} + \widehat{\rho}\widehat{\sigma}'\right)h'' + \left[\left(\widehat{\sigma}\widehat{\rho}'\right)' + \epsilon_1\left(\epsilon_0\frac{\widehat{\sigma}}{\widehat{\rho}} + \epsilon_2\frac{\widehat{\rho}}{\widehat{\sigma}}\right)\right]\widehat{h}' - \epsilon_2\left(\frac{\sigma}{\rho}\right)'\widehat{h} \\
=\ & \epsilon_0\left(\widehat{\sigma}\widehat{\rho}'\right)' - \frac{\widehat{\rho}}{\widehat{\sigma}},
\end{aligned} \tag{4.2}$$

where $\widehat{\rho}\widehat{\kappa} = 1$, $\widehat{\sigma}\widehat{\tau} = 1$.

Proof. Assume that $\mathbf{\Psi} = \mathbf{\Psi}(\widehat{s})$ be a non-null unit speed dual curve in $\mathbb{D}_1^3$ with $\widehat{\kappa}$ is nowhere pure-dual. From Eqs (2.10) and (4.1), we have

$$\widehat{\rho}\left(\widehat{h}' - \epsilon_0\right) = < \mathbf{\Psi}, \mathbf{N} >, \tag{4.3}$$

which, again on differentiating gives

$$\widehat{\rho\sigma h}'' + \widehat{\sigma\rho}'\widehat{h}' - \epsilon_0\widehat{\sigma\rho}' - \epsilon_2\frac{\widehat{\sigma}}{\widehat{\rho}}\widehat{h} = < \mathbf{\Psi}, \mathbf{B} > \tag{4.4}$$

After differentiating Eq (4.4) and applying Eq (4.3) we obtain

$$\widehat{\rho\sigma h}''' + \left(2\widehat{\rho}'\widehat{\sigma} + \widehat{\rho\sigma}'\right)h'' + \left[\left(\widehat{\sigma\rho}'\right)' + \epsilon_1\left(\epsilon_0\frac{\widehat{\sigma}}{\widehat{\rho}} + \epsilon_2\frac{\widehat{\rho}}{\widehat{\sigma}}\right)\right]\widehat{h}' - \epsilon_2\left(\frac{\sigma}{\rho}\right)'\widehat{h} = \epsilon_0\left(\widehat{\sigma\rho}'\right)' - \frac{\widehat{\rho}}{\widehat{\sigma}},$$

as claimed. □

Applications of Proposition 4.1 We now show that Proposition 4.1 implies easily several well-known characterizations of non-null spherical, normal and rectifying dual curves in $\mathbb{D}_1^3$ [20, 21].

Corollary 4.1. A non-null unit speed dual curve $\mathbf{\Psi}= \mathbf{\Psi}(\widehat{s})$ in $\mathbb{D}_1^3$ with $\widehat{\kappa}$ is nowhere pure-dual is lying fully on the dual unit sphere $\mathbb{S}_1^2$ (resp. $\mathbb{H}_+^2$) if and only if it satisfies

$$\epsilon_0\left(\sigma\rho'\right)^2 + \rho^2 = C, \quad \epsilon_0 \in \{1, -1\}, \tag{4.5}$$

for some dual constant $C = c + \varepsilon c^*$, where $c \neq 0$.

Proof. Let $\mathbf{\Psi}= \mathbf{\Psi}(\widehat{s})$ be a non-null unit speed dual curve in $\mathbb{S}_1^2$ (resp. $\mathbb{H}_+^2$). Then we have

$$< \mathbf{\Psi}, \mathbf{\Psi} >= \epsilon_0, \text{ and } \widehat{h}(\widehat{s}) =< \mathbf{\Psi}, \mathbf{T} >= 0. \tag{4.6}$$

Hence differential of Eq (4.2) reduces to

$$\epsilon_0\left(\widehat{\sigma\rho}'\right)' + \frac{\widehat{\rho}}{\widehat{\sigma}} = 0.$$

By multiplying $2\widehat{\sigma\rho}'$ to this equation and integrating it give

$$\epsilon_0\left(\widehat{\sigma\rho}'\right)^2 + \widehat{\rho}^2 = C, \tag{4.7}$$

where $C = c + \varepsilon c^*$ and $c \neq 0$ .Thus $\mathbf{\Psi}= \mathbf{\Psi}(S)$ is a dual spherical curve in $\mathbb{D}_1^3$ and $\widehat{\kappa}$ is nowhere pure-dual. The converse is trivial. □

Corollary 4.2. A non-null unit speed dual curve $\mathbf{\Psi}= \mathbf{\Psi}(\widehat{s})$ in $\mathbb{D}_1^3$ with $\widehat{\kappa}$ is nowhere pure-dual is a rectifying dual curve if and only if it satisfies

$$(\widehat{s} + C)\left(\frac{\widehat{\kappa}}{\widehat{\tau}}\right)' + \frac{\widehat{\kappa}}{\widehat{\tau}} = 0, \tag{4.8}$$

for some dual $C = c + \varepsilon c^*$, where $c \neq 0$.

Proof. Assume that $\mathbf{\Psi}=\mathbf{\Psi}(\widehat{s})$ is a non-null unit speed rectifying dual curve in $\mathbb{D}_1^3$ with $\widehat{\kappa}$ is nowhere pure-dual. Then, in view of Eq (3.7), we have

$$\widehat{h}(\widehat{s}) = \epsilon_0(\widehat{s}+C). \tag{4.9}$$

Hence Eq (4.2) reduces to

$$\frac{\widehat{\sigma}}{\widehat{\rho}} + (\widehat{s}+C)\left(\frac{\widehat{\sigma}}{\widehat{\rho}}\right)' = 0,$$

which implies condition (4.8).

Conversely, if Eq (4.8) holds, then by integrating Eq (4.8), we find $\widehat{\tau}\overline{C} = (\widehat{s}+C)\widehat{\kappa}$ for a dual constant $\overline{C}$, which implies that is a rectifying curve. □

Since $\mathbf{\Psi}(\widehat{s})$ is a linear combination of the Serret-Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, we put:

$$\mathbf{\Psi} = <\mathbf{\Psi}, \mathbf{T}> \mathbf{T} + <\mathbf{\Psi}, \mathbf{N}> \mathbf{N} + <\mathbf{\Psi}, \mathbf{B}> \mathbf{B}. \tag{4.10}$$

As a result the following corollary can be given:

Corollary 4.3. Let $\mathbf{\Psi}(\widehat{s})$ be a non-null unit speed dual curve in $\mathbb{D}_1^3$ with $\widehat{\kappa}$ is nowhere pure-dual. Then

$$\epsilon_1\,(<\mathbf{\Psi}, \mathbf{N}>)^2 + \epsilon_2\,(<\mathbf{\Psi}, \mathbf{B}>)^2 = A^2, \tag{4.11}$$

holds for a dual constant A if and only if $\mathbf{\Psi}(\widehat{s})$ is either dual spherical or a dual normal curve in $\mathbb{D}_1^3$.

Proof. From Eqs (4.10) and (4.11), we get:

$$\|\mathbf{\Psi}\|^2 = \epsilon_0 <\mathbf{\Psi}, \mathbf{T}>^2 + A^2. \tag{4.12}$$

After differentiating Eq (4.12) and using Eqs (4.1) and (4.3), we find

$$<\mathbf{\Psi}, \mathbf{T}> = \epsilon_0 \widehat{h}\widehat{h'} \Rightarrow \widehat{h} = \epsilon_0 \widehat{h}\widehat{h'} \Rightarrow \widehat{h}(\epsilon_0 \widehat{h'} - 1) = 0. \tag{4.13}$$

Thus we have either $\widehat{h} = 0$, or $\epsilon_0\widehat{h'} - 1 = 0$. So we have either $\widehat{h} = 0$, or $\widehat{h} = \epsilon_0\,(\widehat{s}+C)$ for a dual constant C. Hence, $\mathbf{\Psi}(\widehat{s})$ is either is a dual spherical or a dual normal curve. The converse is clear. □

5. Characterizations of dual helices

A characterization of the dual helices in $\mathbb{D}_1^3$ by its dual curvature and dual torsion is the same as that of the helices in $\mathbb{E}^3$. For this reason, we give the dual Minkowski version of the helices which has been defined in the Euclidean 3-space $\mathbb{E}^3$ as a curve for which $<\mathbf{T}, \mathbf{U}>$ is a nonzero constant, where $\mathbf{T}$ is the dual unit tangent vector of the curve $\mathbf{\Psi}=\mathbf{\Psi}(\widehat{s})$ [8–12].

5.1. Spacelike dual helix with a spacelike dual axis

Let $\mathbf{\Psi}= \mathbf{\Psi}(\widehat{s})$ be a regular unit speed spacelike dual curve with a spacelike or timelike principal normal in $\mathbb{D}_1^3$. Then for non zero dual constant a spacelike dual helix with a spacelike dual axis in $\mathbb{D}_1^3$ is defined by

$$< \mathbf{T}, \mathbf{U} >= A \text{ and } \quad < \mathbf{U}, \mathbf{U} >= 1,$$

where $A \in \mathbb{D}$ and $\mathbf{U}$ is the fixed dual unit vector along the axis of $\mathbf{\Psi}(\widehat{s})$.

Theorem 5.1. Let $\mathbf{\Psi}= \mathbf{\Psi}(\widehat{s})$ be a unit speed non-null curve in $\mathbb{D}_1^3$ and $\widehat{\kappa}$ is nowhere pure-dual. Then the following statements hold:
(i) $\mathbf{\Psi}= \mathbf{\Psi}(\widehat{s})$ is a spacelike dual helix with a spacelike dual principal normal and a spacelike dual axis if and only if, with respect to a suitable pseudo dual arc length parameter $\widehat{s}$, the function $\widehat{h}(\widehat{s})$ satisfies

$$\left(\widehat{\rho}\widehat{h}'\right)' - \left(\frac{\widehat{\sigma}}{\widehat{\rho}} - \frac{\widehat{\rho}}{\widehat{\sigma}}\right)\widehat{\tau}\widehat{h} - \widehat{\rho}' + \frac{\widehat{s}\widehat{\rho}}{\widehat{\sigma}^2} = 0, \tag{5.1}$$

(ii) $\mathbf{\Psi}= \mathbf{\Psi}(\widehat{s})$ is a spacelike dual helix with a timelike dual principal normal and a spacelike dual axis if and only if, with respect to a suitable pseudo dual arc length parameter S, the function $\widehat{h}(\widehat{s})$ satisfies

$$\left(\widehat{\rho}\widehat{h}'\right)' - \left(\frac{\widehat{\sigma}}{\widehat{\rho}} + \frac{\widehat{\rho}}{\widehat{\sigma}}\right)\tau\widehat{h} - \widehat{\rho}' + \frac{\widehat{s}\widehat{\rho}}{\widehat{\sigma}^2} = 0. \tag{5.2}$$

Proof. (i) By the assumption we have

$$< \mathbf{U}, \mathbf{T} >= \cosh\Theta = \text{constant}, \ < \mathbf{U}, \mathbf{N} >=0, \text{ with } < \mathbf{U}, \mathbf{U} >=1, \tag{5.3}$$

for any given dual number $\Theta = \vartheta + \varepsilon\vartheta^*$, with $\vartheta \neq 0$. Thus we get

$$\left.\begin{array}{l} \mathbf{U} = \cosh\Theta\mathbf{T} + \sinh\Theta\mathbf{B}, \\ \widehat{\kappa}\cosh\Theta + \widehat{\tau}\sinh\Theta = 0. \end{array}\right\} \tag{5.4}$$

Since $< \mathbf{U}, \mathbf{\Psi} >' = \cosh\Theta$ holds, we have

$$< \mathbf{U}, \mathbf{\Psi} >= \widehat{s}\cosh\Theta + \overline{C}, \tag{5.5}$$

for some duall constant $\overline{C}$. Now, from Eqs (5.4) and (5.5), we find

$$< \mathbf{T}, \mathbf{\Psi} >= \widehat{s} + C + < \mathbf{\Psi}, \mathbf{B} > \frac{\widehat{\kappa}}{\widehat{\tau}}, \text{ with } C = \frac{\overline{C}}{\cosh\vartheta}. \tag{5.6}$$

Combining this equation with Eq (4.1) yields

$$\widehat{h} = \widehat{s} + C + < \mathbf{\Psi}, \mathbf{B} > \frac{\widehat{\kappa}}{\widehat{\tau}}. \tag{5.7}$$

By differentiating Eq (5.7) and using Eq (5.4), we get

$$\widehat{\rho}\left(\widehat{h}' - 1\right) - < \mathbf{\Psi}, \mathbf{N} >= 0, \tag{5.8}$$

which again by differentiation leads to

$$\widehat{\rho}\widehat{h}'' + \widehat{\rho}'\left(\widehat{h}' - 1\right) = -\widehat{\kappa} < \mathbf{\Psi}, \mathbf{T} > + \widehat{\tau} < \mathbf{\Psi}, \mathbf{B} > . \tag{5.9}$$

When Eqs (5.6) and (5.7) are applied to Eq (5.9) we immediately find that:

$$\left(\widehat{\rho}\widehat{h}'\right)' + \left(\frac{\widehat{\sigma}}{\widehat{\rho}} - \frac{\widehat{\rho}}{\widehat{\sigma}}\right)\widehat{\tau}\widehat{h} - \widehat{\rho}' + \frac{\widehat{\rho}}{\widehat{\sigma}^2}(\widehat{s} + C) = 0, \tag{5.10}$$

which, with respect to a suitable pseudo arc length parameter $\widehat{s}$, leads to Eq (5.1).

Conversely, assume that $\mathbf{\Psi}$ is a non-null unit speed spacelike dual helix with a spacelike dual principal normal that satisfies Eq (5.1). Then, by differentiating Eq (5.10) we derive

$$\begin{aligned} &\widehat{\rho}\widehat{\sigma}\widehat{h}''' + \left(2\widehat{\rho}'\widehat{\sigma} + \widehat{\rho}\widehat{\sigma}'\right)\widehat{h}'' + \left[\left(\widehat{\sigma}\widehat{\rho}'\right)' + \left(\frac{\widehat{\sigma}}{\widehat{\rho}} - \frac{\widehat{\rho}}{\widehat{\sigma}}\right)\right]\widehat{h}' + \left(\frac{\widehat{\sigma}}{\widehat{\rho}} + \frac{\widehat{\rho}}{\widehat{\sigma}}\right)'\widehat{h} \\ = \ &\left(\widehat{\sigma}\widehat{\rho}'\right)' - \left(\frac{\widehat{\rho}}{\widehat{\sigma}}\right)'(\widehat{s} + C) - \frac{\widehat{\rho}}{\widehat{\sigma}}. \end{aligned} \tag{5.11}$$

Comparing Eq (5.11) with Eq (4.2) in Proposition 4.1, gives

$$\left(\frac{\widehat{\rho}}{\widehat{\sigma}}\right)'\left(\widehat{s} + C - \widehat{h}\right) = 0. \tag{5.12}$$

If $\widehat{s} + C = \widehat{h}$ holds, then Eq (5.10) reduced to

$$\left(\frac{\widehat{\sigma}}{\widehat{\rho}} - \frac{\widehat{\rho}}{\widehat{\sigma}}\right)(\widehat{s} + C) + \frac{\widehat{\rho}}{\widehat{\sigma}}(\widehat{s} + C) = 0, \tag{5.13}$$

which implies $\widehat{\sigma} = 0$ which is impossible since $\mathbf{\Psi}$ is assumed to be a Serret-Frenet dual curve. Hence we get $\left(\frac{\widehat{\rho}}{\widehat{\sigma}}\right)' = 0$ from Eq (5.12), which implies that is $\mathbf{\Psi}$ a helix.

(ii) As stated in the above case, we have

$$< \mathbf{U}, \mathbf{T} >= \cos\Theta, \ < \mathbf{U}, \mathbf{N} >=0, \text{ with } < \mathbf{U}, \mathbf{U} >=1, \tag{5.14}$$

for a constant dual angle $\widehat{\vartheta} = \vartheta + \varepsilon\vartheta^*$, with $\vartheta \neq \pi/2$, and $\vartheta \neq 0$. Thus we get

$$\left.\begin{aligned} \mathbf{U} &= \cos\Theta\mathbf{T} + \sin\Theta\mathbf{B}, \\ \widehat{\kappa}\cos\Theta &+ \widehat{\tau}\sin\Theta = 0. \end{aligned}\right\} \tag{5.15}$$

Since $< \mathbf{U}, \mathbf{\Psi} >' = \cos\Theta$ holds, we have

$$< \mathbf{U}, \mathbf{\Psi} >=\widehat{s}\cos\Theta + \overline{C}, \tag{5.16}$$

for some dual constant $\overline{C}$. Form Eqs (5.15) and (5.16), we find

$$< \mathbf{T}, \mathbf{\Psi} >=\widehat{s} + C + < \mathbf{\Psi}, \mathbf{B} > \frac{\widehat{\kappa}}{\widehat{\tau}}, \text{ with } C = \frac{\overline{C}}{\cos\Theta} \in \mathbb{D}. \tag{5.17}$$

Combining this equation with Eq (3.10) yields

$$\widehat{h} = \widehat{s} + C + < \mathbf{\Psi}, \mathbf{B} > \frac{\widehat{\kappa}}{\widehat{\tau}}. \tag{5.18}$$

By differentiating Eq (5.18) and using Eq (5.15), we get

$$\widehat{\rho}\left(\widehat{h}' - 1\right) - < \mathbf{\Psi}, \mathbf{N} > = 0, \tag{5.19}$$

which again by differentiating leads to

$$\widehat{\rho}\widehat{h}'' + \widehat{\rho}'\left(\widehat{h}' - 1\right) = \widehat{\kappa} < \mathbf{\Psi}, \mathbf{T} > + \widehat{\tau} < \mathbf{\Psi}, \mathbf{B} > . \tag{5.20}$$

When Eqs (5.17) and (5.18) are applied to Eq (5.7) we immediately find that:

$$\left(\widehat{\rho}\widehat{h}'\right)' - \left(\frac{\widehat{\sigma}}{\widehat{\rho}} + \frac{\widehat{\rho}}{\widehat{\sigma}}\right)\widehat{\tau}\widehat{h} - \widehat{\rho}' + \frac{\widehat{\rho}}{\widehat{\sigma}^2}\left(\widehat{s} + C\right) = 0, \tag{5.21}$$

which, with respect to a suitable pseudo arc length dual parameter $\widehat{s}$, leads to Eq (5.2).

Conversely, assume that $\mathbf{\Psi}$ is a non-null unit speed spacelike dual helix with a timelike dual principal normal that satisfies Eq (5.2). Then, by differentiating Eq (5.21) we derive

$$\begin{aligned} &\widehat{\rho}\widehat{\sigma}\widehat{h}''' + \left(2\widehat{\rho}'\widehat{\sigma} + \widehat{\rho}\widehat{\sigma}'\right)\widehat{h}'' + \left[\left(\widehat{\sigma}\widehat{\rho}'\right)' - \left(\frac{\widehat{\sigma}}{\widehat{\rho}} + \frac{\widehat{\rho}}{\widehat{\sigma}}\right)\right]\widehat{h}' - \left(\frac{\widehat{\sigma}}{\widehat{\rho}} + \frac{\widehat{\rho}}{\widehat{\sigma}}\right)'\widehat{h} \\ = \; &\left(\widehat{\sigma}\widehat{\rho}'\right)' - \left(\frac{\widehat{\rho}}{\widehat{\sigma}}\right)'\left(\widehat{s} + C\right) - \frac{\widehat{\rho}}{\widehat{\sigma}}. \end{aligned} \tag{5.22}$$

Comparing Eq (5.22) with Eq (4.2) in Proposition 4.1, gives

$$\left(\frac{\widehat{\rho}}{\widehat{\sigma}}\right)'\left(\widehat{s} + C - \widehat{h}\right) = 0. \tag{5.23}$$

If $\widehat{s} + C = \widehat{h}$ holds, then Eq (5.2) reduces to

$$-\left(\frac{\widehat{\sigma}}{\widehat{\rho}} + \frac{\widehat{\rho}}{\widehat{\sigma}}\right)\left(\widehat{s} + C\right) + \frac{\widehat{\rho}}{\widehat{\sigma}}\left(S + C\right) = 0, \tag{5.24}$$

which implies $\widehat{\sigma} = 0$ which is impossible since $\mathbf{\Psi}$ is assumed to be a Serret-Frenet dual curve. Hence we get $\left(\frac{\widehat{\rho}}{\widehat{\sigma}}\right)' = 0$ from Eq (5.23), which implies that is $\mathbf{\Psi}$ a dual helix. □

5.2. Spacelike dual helix with a timelike dual axis

We now consider that $\mathbf{\Psi}(\widehat{s})$ be a regular unit speed spacelike dual curve with a spacelike dual principal normal and a timelike dual axis in $\mathbb{D}_1^3$. Then for a non-pure dual constant a spacelike dual helix with a timelike dual axis in $\mathbb{D}_1^3$ is defined by

$$< \mathbf{U}, \mathbf{T} > = A, \text{ and } < \mathbf{U}, \mathbf{U} > = -1, \tag{5.25}$$

where $A \in \mathbb{D}$, and $\mathbf{U}$ is the fixed dual unit vector along the axis of $\mathbf{\Psi}(\widehat{s})$.

Theorem 5.2. Suppose that $\mathbf{\Psi}(\widehat{s})$ is a unit speed spacelike dual curve with $\widehat{\kappa}$ is nowhere pure-dual. Then $\mathbf{\Psi}(\widehat{s})$ is a spacelike dual helix with a spacelike dual principal normal and a timelike dual axis if and only if, with respect to a suitable pseudo dual arc length parameter S, the dual function $\widehat{h(\widehat{s})}$ satisfies

$$\left(\widehat{\rho}\widehat{h}'\right)' + \left(\frac{\widehat{\sigma}}{\widehat{\rho}} - \frac{\widehat{\rho}}{\widehat{\sigma}}\right)\tau\widehat{h} - \widehat{\rho}' + \frac{s\widehat{\rho}}{\widehat{\sigma}^2} = 0. \tag{5.26}$$

Proof. By the assumption we have

$$< \mathbf{U}, \mathbf{T} >= \sinh\Theta = \text{ dual } const., \ < \mathbf{U}, \mathbf{N} >=0, \text{ with } < \mathbf{U}, \mathbf{U} >= -1, \tag{5.27}$$

for any given dual number $\Theta \geq 0$. Thus we get

$$\left.\begin{array}{l} \mathbf{U} = \sinh\Theta\mathbf{T} + \cosh\Theta\mathbf{B}, \\ \widehat{\kappa}\sinh\Theta + \widehat{\tau}\cosh\Theta = 0. \end{array}\right\} \tag{5.28}$$

Since $< \mathbf{U}, \mathbf{\Psi} >' = \sinh\Theta$ holds, we have

$$< \mathbf{U}, \mathbf{\Psi} >=\widehat{s}\sinh\Theta + \overline{C}, \tag{5.29}$$

for some dual constant $\overline{C}$. Form Eqs (5.28) and (5.29), we find

$$< \mathbf{T}, \mathbf{\Psi} >=\widehat{s} + C- < \mathbf{\Psi}, \mathbf{B} >\frac{\widehat{\kappa}}{\widehat{\tau}}, \text{ with } C = \frac{\overline{C}}{\sinh\Theta}. \tag{5.30}$$

Combining this equation with Eq (4.1) yields

$$\widehat{h}=\widehat{s} + C- < \mathbf{\Psi}, \mathbf{B} >\frac{\widehat{\kappa}}{\widehat{\tau}}. \tag{5.31}$$

By differentiating Eq (5.31) and using Eq (5.29), we get

$$\widehat{\rho}\left(\widehat{h}' - 1\right) - < \mathbf{\Psi}, \mathbf{N} >= 0, \tag{5.32}$$

which again by differentiation leads to

$$\widehat{\rho}\widehat{h}'' + \widehat{\rho}'\left(\widehat{h}' - 1\right) = -\widehat{\kappa} < \mathbf{T}, \mathbf{\Psi} > + \widehat{\tau} < \mathbf{\Psi}, \mathbf{B} > . \tag{5.33}$$

When Eqs (5.30) and (5.31) are applied to Eq (5.33) we immediately find that:

$$\left(\widehat{\rho}\widehat{h}'\right)' + \left(\frac{\widehat{\sigma}}{\widehat{\rho}} - \frac{\widehat{\rho}}{\widehat{\sigma}}\right)\widehat{\tau}\widehat{h} - \widehat{\rho}' + \frac{\widehat{\rho}}{\widehat{\sigma}^2}(\widehat{s} + C) = 0, \tag{5.34}$$

which, with respect to a suitable pseudo dual arc length parameter $\widehat{s}$, leads to Eq (5.26).

Conversely, assume that $\mathbf{\Psi}$ is a non-null unit speed spacelike dual helix with a timelike dual principal normal and a timelike dual axis that satisfies Eq (5.26). Then, by differentiating Eq (5.34) we derive

$$\begin{aligned} &\widehat{\rho}\widehat{\sigma}\widehat{h}''' + \left(2\widehat{\rho}'\widehat{\sigma} + \widehat{\rho}\widehat{\sigma}'\right)\widehat{h}'' + \left[\left(\widehat{\sigma}\widehat{\rho}'\right)' + \left(\frac{\widehat{\sigma}}{\widehat{\rho}} - \frac{\widehat{\rho}}{\widehat{\sigma}}\right)\right]h' + \left(\frac{\widehat{\sigma}}{\widehat{\rho}} - \frac{\widehat{\rho}}{\widehat{\sigma}}\right)'\widehat{h} \\ = \ & -\left(\widehat{\sigma}\widehat{\rho}'\right)'(\widehat{s} + C) - \frac{\widehat{\rho}}{\widehat{\sigma}}. \end{aligned} \tag{5.35}$$

Comparing Eq (5.35) with Eq (4.2) in Proposition 4.1, the result is clear. □

Conflict of interest

The author declares no conflict of interest in this paper.

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