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Research article

Multivalued weakly Picard operators via simulation functions with application to functional equations

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Abstract: The aim of this paper is to introduce the notion of Suzuki type multivalued contraction with simulation functions and then to set up some new fixed point and data dependence results for these type of contraction mappings. We produce an example to support our results. Moreover, we present an application to functional equation arising in dynamical system.

Keywords: multivalued Picard operator; data dependence; $\mathcal{Z}$-contraction; functional equation; dynamical system
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1. Introduction and preliminaries

We pronounce $(\mathcal{N}, \wp)$ a metric space, the classes $\mathcal{P}(\mathcal{N})$ and $\mathcal{CB}(\mathcal{N})$ of all nonempty, closed and bounded subsets of $\mathcal{N}$.

If $\mathcal{U}, \mathcal{V} \in \mathcal{CB}(\mathcal{N})$, review the functionals:

$\mathcal{D} : \mathcal{P}(\mathcal{N}) \times \mathcal{P}(\mathcal{N}) \to \mathbb{R}_+$, $\mathcal{D}(\mathcal{W}, \mathcal{V}) = inf\{\wp(\omega, \lambda) : \omega \in \mathcal{W},\ \lambda \in \mathcal{V}\}, \mathcal{W} \subset \mathcal{N}$;

$\mathcal{H} : \mathcal{P}(\mathcal{N}) \times \mathcal{P}(\mathcal{N}) \to \mathbb{R}_+, \mathcal{H}(\mathcal{U}, \mathcal{V}) := max\{\sup_{u\in\mathcal{U}} \inf_{v\in\mathcal{V}} \wp(u, v), \sup_{v\in\mathcal{V}} \inf_{u\in\mathcal{U}} \wp(u, v)\}$ - the Pompeiu- Hausdorff functional.

Lemma 1.1. *[12] Suppose $\mathcal{U} \subseteq \mathcal{N}$ and $\xi > 1$. Then for $\mu \in \mathcal{N}$ there is $v \in \mathcal{V}$ such that $\wp(\mu, v) \leq \xi\mathcal{D}(\mu, \mathcal{V})$.*

Definition 1.2. *[16] A function $Q : \mathcal{N} \to \mathcal{CB}(\mathcal{N})$ is called multi-valued weakly Picard operator (MWP operator) if for all $\mu \in \mathcal{N}$ and $\mu \in Q\mu$, there is a sequence (μ_m) in $\mathcal{N}$ such that (i) $\mu_0 = \mu, \mu_1 = \lambda$, (ii) $\mu_{n+1} \in Q\mu_m \; \forall \; m \geq 0$, and (iii) (μ_m) tends to the fixed point of Q.*

Utilizing the approach of Rus [16], Popescu [13] defined a multivalued operator and named it as (s, r)-contractive multivalued and siglevalued operators and showed that the underlying mapping is an MWP operator. Following Popescu [13], Kamran [9] extended (s, r)-contractive operators to weakly (s, r)-contractive operators and proved that the underlying mappings are still MWP operators.

A new generalization of Banach's theorem was given by Khojasteh *et al.* [11] by considering a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$, which follows the assertions given below:

(ζ_a) $\zeta(0, 0) = 0$,

(ζ_b) $\zeta(\alpha,\beta) < \beta - \alpha$ for all $\alpha,\beta > 0$,

(ζ_c) The sequences $\{\alpha_n\}, \{\beta_n\}$ in $(0, \infty)$ with $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n > 0$, then

$$\lim_{n\to\infty} sup \; \zeta(\alpha_n,\beta_n) < 0.$$

They named such functions as simulation function and the contractive map is named as $\mathcal{Z}$-contraction. Later on, Rold*á*n-L*ó*pez-de-Hierro *et al.* [15] replaced (ζ_c) with (ζ_c'),

(ζ_c') : the sequences $(\alpha_n), (\beta_n)$ in $(0, \infty)$ with $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n > 0$ and $\alpha_n < \beta_n$, then

$$\lim_{n\to\infty} sup\zeta(\alpha_n,\beta_n) < 0.$$

The lemma stated below is essential:

Lemma 1.3. *[14] Suppose a sequence (μ_n) in $\mathcal{N}$ with*

$$\lim_{n\to\infty} \wp(\mu_n, \mu_{n+1}) = 0. \tag{1.1}$$

If the sequence is not a Cauchy, then $\exists \; \varepsilon > 0$ and sequences $\mu_{m(j)}$ and $\mu_{n(j)}$ such that $n(j) > m(j) > j$ and the sequences

$$\wp(\mu_{m(j)}, \mu_{n(j)}), \wp(\mu_{m(j)}, \mu_{n(j)+1}), \wp(\mu_{m(j)-1}, \mu_{n(j)}),$$

$$\wp(\mu_{m(j)-1}, \mu_{n(j)+1}), \wp(\mu_{m(j)+1}, \mu_{n(j)+1})$$

converges to ε^+ when $k \to \infty$

The purpose of this paper is to introduce the notion of Suzuki type multivalued $\mathcal{Z}_{(\phi,\lambda)}$-contraction and to prove the fixed point and data dependence results for such contraction mapping. We give an example to show the validity of our results. Moreover, we present an application to functional equation arising in dynamical system to show the usability of our results.

2. Fixed point results

We now give the definition of Suzuki type multivalued $\mathcal{Z}_{(\phi,\lambda)}$-contraction and obtain some fixed point results.

We start this section with the following notion:

Denote Φ the set consisting of all strictly increasing functions $\varphi : [0,1) \to (0,1]$. We define

$$\varphi(s) = \begin{cases} \frac{1}{s+1} & if\ 0 \leq s < \frac{1}{2} \\ 1-s & if\ \frac{1}{2} \leq s < 1. \end{cases}$$

Definition 2.1. *A mapping $Q : \mathcal{N} \to C\mathcal{B}(\mathcal{N})$ is called Suzuki type multivalued $\mathcal{Z}_{(\varphi,\lambda)}$-contraction with respect to ζ, if there is a function $\varphi \in \Phi$ such that*

$$\varphi(s) max\{\mathcal{D}(\mu, Q\mu), \frac{1}{2}\mathcal{D}(\lambda, Q\mu)\} \leq \wp(\mu, \lambda) \tag{2.1}$$

implies

$$\zeta(\mathcal{H}(Q\mu, Q\lambda), \rho\mathcal{R}(\mu, \lambda)) \geq 0 \tag{2.2}$$

where $\rho \in [0, 1)$, and

$$\mathcal{R}(\mu, \lambda) = max\{\wp(\mu, \lambda), \mathcal{D}(\mu, Q\mu), \mathcal{D}(\lambda, Q\lambda), \frac{1}{2}\left(\mathcal{D}(\mu, Q\lambda) + \mathcal{D}(\lambda, Q\mu)\right)\}.$$

Theorem 2.2. *Suppose that $Q : \mathcal{N} \to C\mathcal{B}(\mathcal{N})$ is Suzuki type multivalued $\mathcal{Z}_{(\varphi,\rho)}$-contraction, where $(\mathcal{N}, \wp)$ is a complete metric space. Then Q is an MWP operator.*

Proof. Choose $\mu_0 \in X$, $\mu_1 \in Q\mu_0$ and ρ_1 be real number such that $0 \leq \rho \leq s \leq \rho_1 < 1$. One can choose $\mu_2 \in Q\mu_1$ such that

$$\wp(\mu_1, \mu_2) \leq \frac{1}{\sqrt{\rho_1}} H(Q\mu_0, Q\mu_1).$$

Since

$$\begin{aligned} \varphi(s) max\{\mathcal{D}(\mu_0, Q\mu_0), \frac{1}{2}\mathcal{D}(\mu_1, Q\mu_0)\} &\leq \varphi(s)\mathcal{D}(\mu_0, Q\mu_0) \\ &\leq \wp(\mu_0, \mu_1), \end{aligned} \tag{2.3}$$

from (2.1) and (2.2), we have

$$0 \leq \zeta(\mathcal{H}(Q\mu_0, Q\mu_1), \rho\mathcal{R}(\mu_0, \mu_1)) \leq \rho\mathcal{R}(\mu_0, \mu_1) - \mathcal{H}(Q\mu_0, Q\mu_1)$$

this implies

$$\mathcal{H}(Q\mu_0, Q\mu_1) \leq \rho\mathcal{R}(\mu_0, \mu_1),$$

where

$$\begin{aligned} \mathcal{R}(\mu_0, \mu_1) &= max\{\wp(\mu_0, \mu_1), \mathcal{D}(\mu_0, Q\mu_0), \mathcal{D}(\mu_1, Q\mu_1), \frac{\mathcal{D}(\mu_0, Q\mu_1) + \mathcal{D}(\mu_1, Q\mu_0)}{2}\} \\ &= max\{\wp(\mu_0, \mu_1), \mathcal{D}(\mu_1, Q\mu_1), \frac{\mathcal{D}(\mu_0, Q\mu_1)}{2}\} \end{aligned}$$

$$\begin{aligned}&\leq max\{\wp(\mu_0,\mu_1),\wp(\mu_1,\mu_2),\frac{\wp(\mu_0,\mu_1)+\wp(\mu_1,\mu_2)}{2}\}\\&= max\{\wp(\mu_0,\mu_1),\wp(\mu_1,\mu_2)\}.\end{aligned}$$

If $\mathcal{R}(\mu_0,\mu_1)\leq\wp(\mu_1,\mu_2)$, then

$$\begin{aligned}\wp(\mu_1,\mu_2) &\leq \frac{1}{\sqrt{\rho_1}}\mathcal{H}(Q\mu_0,Q\mu_1)\\&\leq \frac{\rho}{\rho_1}\mathcal{R}(\mu_0,\mu_1)\\&\leq \sqrt{\rho_1}\wp(\mu_1,\mu_2).\end{aligned}$$

This implies $\wp(\mu_1,\mu_2)=0$ or $\mu_1=\mu_2\in Q\mu_1$, i.e. μ_1 is a fixed point. So, we consider the case when $\mathcal{R}(\mu_0,\mu_1)\leq\wp(\mu_0,\mu_1)$, this gives

$$\wp(\mu_1,\mu_2)\leq \sqrt{\rho_1}\wp(\mu_0,\mu_1)<\wp(\mu_0,\mu_1).$$

Hence we can recursively define a sequence (μ_n) such that $\mu_{n+1}\in Q\mu_n$ and

$$\wp(\mu_{n+1},\mu_{n+2})<\wp(\mu_n,\mu_{n+1})$$

which implies

$$\lim_{n\to\infty}\wp(\mu_n,\mu_{n+1})=0. \tag{2.4}$$

To prove that (μ_n) is Cauchy, contrary suppose that it is not, then by Lemma 1.3, there is $\epsilon>0$ and two subsequences $m(j)$, $n(j)$ with $n(j)>m(j)>j$ such that $\wp(\mu_{m(j)},\mu_{n(j)})\to\epsilon^+$, $\wp(\mu_{m(j)+1},\mu_{n(j)+1})\to\epsilon^+$ as $j\to\infty$ also

$$\lim_{j\to\infty}\rho\mathcal{R}(\mu_{m(j)},\mu_{n(j)})=\epsilon^+.$$

By (2.4), we have

$$\begin{aligned}&\varphi(s)max\{\mathcal{D}(\mu_{n(j)},Q\mu_{n(j)}),\frac{1}{2}\mathcal{D}(\mu_{m(j)},Q\mu_{n(j)})\} &(2.5)\\&\leq max\{\wp(\mu_{n(j)},\mu_{n(j)+1}),\frac{1}{2}\wp(\mu_{m(j)},\mu_{n(j)+1})\}\\&\leq max\{\wp(\mu_{m(j)},\mu_{n(j)})+\wp(\mu_{n(j)},\mu_{n(j)+1}),\wp(\mu_{m(j)},\mu_{n(j)+1})\}\\&= \wp(\mu_{m(j)},\mu_{n(j)})+\wp(\mu_{n(j)},\mu_{n(j)+1})\\&< \wp(\mu_{m(j)},\mu_{n(j)}), &(2.6)\end{aligned}$$

thus by (2.1), (2.2) and (ζ_b), we have

$$\begin{aligned}0 &\leq \zeta(\mathcal{H}(Q\mu_{m(j)},Q\mu_{n(j)}),\rho\mathcal{R}(\mu_{m(j)},\mu_{n(j)})) &(2.7)\\&\leq \rho\mathcal{R}(\mu_{m(j)},\mu_{n(j)})-\mathcal{H}(Q\mu_{m(j)},Q\mu_{n(j)}), &(2.8)\end{aligned}$$

therefore, we have

$$\wp(\mu_{m(j)+1},\mu_{n(j)+1}) \leq \mathcal{H}(Q\mu_{m(j)},Q\mu_{n(j)})$$

$$\begin{aligned} &\leq \rho\mathcal{R}(\mu_{m(j)}, \mu_{n(j)}) \\ &< \wp(\mu_{m(j)}, \mu_{n(j)}), \end{aligned}$$

hence

$$\lim_{j\to\infty} \mathcal{H}(Q\mu_{m(j)}, Q\mu_{n(j)}) = \epsilon^+,$$

so by (ζ_c) and (2.2), we have

$$0 \leq \lim_{j\to\infty} sup\zeta(\mathcal{H}(Q\mu_{m(j)}, Q\mu_{n(j)}), \rho\mathcal{R}(\mu_{m(j)}, \mu_{n(j)})) < 0,$$

this leads to a contradiction. Hence (μ_n) is a Cauchy sequence. Completeness of $\mathcal{N}$ implies that there is $u \in \mathcal{N}$ such that $\lim_{n\to\infty} \mu_n = u$. Moreover

$$\mathcal{D}(\mu_{n+1}, Qu) \leq \mathcal{H}(Q\mu_n, Qu).$$

Our claim is

$$\mathcal{D}(u, Q\mu) \leq \rho \max\{\wp(u,\mu), \mathcal{D}(\mu, Q\mu)\} \tag{2.9}$$

for all $\mu \neq u$. Since $\lim_{n\to\infty} \wp(u_n, u) = 0$, there exists an $n_0 \in \mathbb{N}$ such that $\wp(u, u_n) < \frac{1}{3}\wp(u,\mu)$ for all $n \geq n_0$. Note that $\varphi(s) \leq 1$ and

$$\begin{aligned} \varphi(s)\max\left\{\mathcal{D}(u_n, Qu_n), \frac{1}{2}\mathcal{D}(\mu, Qu_n)\right\} &\leq \max\left\{\mathcal{D}(u_n, Qu_n), \frac{1}{2}\mathcal{D}(\mu, Qu_n)\right\} \\ &\leq \max\{\wp(u_n, u_{n+1}), \frac{1}{2}\wp(\mu, u_{n+1})\} \\ &\leq \max\left\{\wp(u_n, u) + \wp(u, u_{n+1}), \frac{1}{2}\left(\wp(\mu, u) + \wp(u, u_{n+1})\right)\right\} \\ &\leq \max\left\{\frac{1}{3}\wp(u,\mu) + \frac{1}{3}\wp(u,\mu), \frac{1}{2}\left[\wp(\mu, u) + \frac{1}{3}\wp(u,\mu)\right]\right\} \\ &\leq \max\left\{\frac{2}{3}\wp(u,\mu), \frac{2}{3}\wp(\mu,u)\right\} = \frac{2}{3}\wp(\mu,u) \\ &= \left(\wp(u,\mu) - \frac{1}{3}\wp(u,\mu)\right) \leq (\wp(u,\mu) - \wp(u_n,u)) \\ &\leq \wp(u_n,\mu). \end{aligned}$$

Thus $\varphi(s)\max\left\{\mathcal{D}(u_n, Qu_n), \frac{1}{2}\mathcal{D}(\mu, Qu_n)\right\} \leq \wp(u_n,\mu)$ holds for all $n \geq n_0$. From (2.1) and (2.2), we have

$$\zeta(\mathcal{H}(Qu_n, Q\mu), \rho\mathcal{R}(u_{n+1}, \mu)) \geq 0,$$

this gives

$$\begin{aligned} \mathcal{D}(u_{n+1}, Q\mu) &\leq \mathcal{H}(Qu_n, Q\mu) \\ &\leq \rho\max\left\{\wp(u_n,\mu), \mathcal{D}(u_n, Qu_n), \mathcal{D}(\mu, Q\mu), \frac{\mathcal{D}(u_n, Q\mu) + \mathcal{D}(\mu, Qu_n)}{2}\right\} \\ &\leq \rho\max\left\{\wp(u_n,\mu), \wp(u_n, u_{n+1}), \mathcal{D}(\mu, Q\mu), \frac{\mathcal{D}(u_n, Q\mu) + \wp(\mu, u_{n+1})}{2}\right\}. \end{aligned} \tag{2.10}$$

Tending $n \to \infty$ to inequality (2.10), we arrived at

$$\mathcal{D}(u, Q\mu) \leq \rho \max\left\{\wp(u, \mu), \mathcal{D}(\mu, Q\mu), \frac{\mathcal{D}(u, Q\mu) + \wp(\mu, u)}{2}\right\}.$$

If

$$\max\left\{\wp(u, \mu), \mathcal{D}(\mu, Q\mu), \frac{\mathcal{D}(u, Q\mu) + \wp(\mu, u)}{2}\right\} = \frac{\mathcal{D}(u, Q\mu) + \wp(\mu, u)}{2},$$

then

$$\mathcal{D}(u, Q\mu) \leq \rho\left(\frac{\mathcal{D}(u, Q\mu) + \wp(\mu, u)}{2}\right)$$

and this implies that

$$\mathcal{D}(u, Q\mu) \leq \left(\frac{\rho}{2-\rho}\right)\wp(\mu, u) < \rho\wp(\mu, u) \leq \rho \max\{\wp(u, \mu), \mathcal{D}(\mu, Q\mu)\}.$$

Hence

$$\mathcal{D}(u, Q\mu) \leq \rho \max\{\wp(u, \mu), \mathcal{D}(\mu, Q\mu)\}$$

valid $\forall\ \mu \neq u$. To show that $u \in Qu$, we discus two cases:

(i) If $0 \leq r \leq s < \dfrac{1}{2}$. Contrary suppose that $u \notin Qu$. Since $Qu \neq \phi$, there exists $c \in Qu$ such that $c \neq u$ and

$$\wp(c, u) < \mathcal{D}(u, Qu) + \left(\frac{1}{\rho + 1} - 1\right)\mathcal{D}(u, Qu).$$

Thus

$$(\rho + 1)\,\wp(c, u) < \mathcal{D}(u, Qu). \tag{2.11}$$

As $\varphi(s)\max\left\{\mathcal{D}(u, Qu), \frac{1}{2}\mathcal{D}(c, Qu)\right\} \leq \max\{\wp(u, c), \wp(c, c)\} \leq \wp(u, c)$, so by (2.1) and (2.2), we have

$$\zeta(\mathcal{H}(Qu, Qc), \rho\mathcal{R}(u, c)) \geq 0,$$

this gives

$$\begin{aligned}
\mathcal{H}(Qu, Qc) &\leq \rho \max\left\{\wp(u, c), \mathcal{D}(u, Qu), \mathcal{D}(c, Qc), \frac{\mathcal{D}(u, Qc) + \mathcal{D}(c, Qu)}{2}\right\} \\
&\leq \rho \max\left\{\wp(u, c), \mathcal{D}(c, Qc), \frac{[\wp(u, c) + \mathcal{D}(c, Qc)]}{2}\right\} \\
&= \rho \max\{\wp(u, c), \mathcal{D}(c, Qc)\}.
\end{aligned}$$

Thus

$$\mathcal{H}(Qu, Qc) \leq \rho \max\{\wp(u, c), \mathcal{D}(c, Qc)\}. \tag{2.12}$$

Since $\mathcal{D}(c, Qc) \leq \mathcal{H}(Qu, Qc)$, therefore, we have

$$\mathcal{H}(Qu, Qc) \leq \rho \max\{\wp(u, c), \mathcal{H}(Qu, Qc)\},$$

hence

$$\mathcal{H}(Qu, Qc) \leq \rho\wp(u, c). \tag{2.13}$$

Since $\rho < 1$, therefore

$$\mathcal{D}(c, Qc) \leq \wp(u, c). \tag{2.14}$$

By (2.9) and (2.14), we have

$$\mathcal{D}(u, Qc) \leq \rho \max\{\wp(u, c), \mathcal{D}(c, Qc)\} < \rho \max\{\wp(u, c), \wp(u, c)\} < \rho\wp(u, c) < \wp(u, c). \tag{2.15}$$

Using (2.9), (2.12), (2.13) and (2.15), we have

$$\begin{aligned} \mathcal{D}(u, Qu) &\leq \mathcal{D}(u, Qc) + \mathcal{H}(Qu, Qc) < \wp(u, c) + \rho\wp(u, c) \\ &= (1 + \rho)\,\wp(u, c) < \mathcal{D}(u, Qu), \end{aligned}$$

a contradiction. Hence $u \in Qu$.

(ii) If $\frac{1}{2} \leq \rho \leq s < 1$, then we first show that

$$\mathcal{H}(Q\mu, Qu) \leq \rho\mathcal{R}(\mu, u) \tag{2.16}$$

for all $\mu \in \mathcal{N}$ with $\mu \neq u$.For each $n \in \mathbb{N}$, there exists $\lambda_n \in Q\mu$ such that $\wp(u, \lambda_n) < \mathcal{D}(u, Q\mu) + \frac{1}{n}\wp(\mu, u)$. So we have

$$\begin{aligned} \max\left\{\mathcal{D}(\mu, Q\mu), \frac{1}{2}\mathcal{D}(u, Q\mu)\right\} &\leq \max\left\{\wp(\mu, \lambda_n), \frac{1}{2}\wp(u, \lambda_n)\right\} \\ &\leq \max\left\{(\wp(\mu, u) + \wp(u, \lambda_n)), \wp(u, \lambda_n)\right\} \\ &= (\wp(\mu, u) + \wp(u, \lambda_n)) \\ &< \left(\wp(\mu, u) + \mathcal{D}(u, Q\mu) + \frac{1}{n}\wp(\mu, u)\right). \end{aligned}$$

Now by (2.9), we have

$$\max\left\{\mathcal{D}(\mu, Q\mu), \frac{1}{2}\mathcal{D}(u, Q\mu)\right\} < \left(\wp(\mu, u) + \rho \max\{\wp(u, \mu), \mathcal{D}(\mu, Q\mu)\} + \frac{1}{n}\wp(\mu, u)\right). \tag{2.17}$$

If $\max\{\wp(u, \mu), \mathcal{D}(\mu, Q\mu)\} = \wp(\mu, u)$, then from (2.17), we obtain that

$$\begin{aligned} \max\left\{\mathcal{D}(\mu, Q\mu), \frac{1}{2}\mathcal{D}(u, Q\mu)\right\} &< \left(\wp(\mu, u) + \rho\wp(u, \mu) + \frac{1}{n}\wp(\mu, u)\right) \\ &= \left(1 + \rho + \frac{1}{n}\right)\wp(\mu, u) \\ &< (1 + s + \frac{1}{n})\wp(\mu, u), \end{aligned}$$

which gives

$$\begin{aligned} \varphi(s) \max\left\{\mathcal{D}(\mu, Q\mu), \frac{1}{2}\mathcal{D}(u, Q\mu)\right\} &= (1 - s) \max\left\{\mathcal{D}(\mu, Q\mu), \frac{1}{2}\mathcal{D}(u, Q\mu)\right\} \\ &\leq \left(\frac{1}{2}\right) \max\left\{\mathcal{D}(\mu, Q\mu), \frac{1}{2}\mathcal{D}(u, Q\mu)\right\} \end{aligned}$$

$$
\begin{aligned}
&\le \left(\frac{1}{1+s}\right)\max\left\{\mathcal{D}(\mu,Q\mu),\frac{1}{2}\mathcal{D}(u,Q\mu)\right\}\\
&< \left(\frac{1}{1+s}\right)(1+s+\frac{1}{n})\wp(\mu,u),\\
&\le \left(\frac{1+s}{1+s}+\frac{1}{(1+s)n}\right)\wp(\mu,u)\\
&\le \left(1+\frac{1}{(1+s)n}\right)\wp(\mu,u).
\end{aligned}
$$

Tending $n\to\infty$, we have

$$\varphi(s)\max\left\{\mathcal{D}(\mu,Q\mu),\frac{1}{2}\mathcal{D}(u,Q\mu)\right\}\le\wp(\mu,u).$$

By (2.1) with $\lambda=u$, (2.16) satisfied. If $\max\{\wp(u,\mu),\mathcal{D}(\mu,Q\mu)\}=\mathcal{D}(\mu,Q\mu)$, then (2.9) gives

$$
\begin{aligned}
\mathcal{D}(\mu,Q\mu) &\le \wp(\mu,u)+\mathcal{D}(u,Q\mu)\\
&\le \wp(\mu,u)+\rho\mathcal{D}(\mu,Q\mu)
\end{aligned}
$$

and hence $\mathcal{D}(\mu,Q\mu)\le\left(\frac{1}{1-\rho}\right)\wp(\mu,u)$. By (2.17), we obtain that

$$
\begin{aligned}
\max\left\{\mathcal{D}(\mu,Q\mu),\frac{1}{2}\mathcal{D}(u,Q\mu)\right\} &\le \left(\wp(\mu,u)+\rho\mathcal{D}(\mu,Q\mu)+\frac{1}{n}\wp(\mu,u)\right)\\
&\le \left(\wp(\mu,u)+\left(\frac{\rho}{1-\rho}\right)\wp(\mu,u)+\frac{1}{n}\wp(\mu,u)\right)\\
&= \left(\frac{1}{1-\rho}\right)\wp(\mu,u)+\frac{1}{n}\wp(\mu,u).
\end{aligned}
$$

As $\frac{1}{2}\le\rho<1$ and $\rho\le s$, so

$$
\begin{aligned}
\varphi(s)\max\left\{\mathcal{D}(\mu,Q\mu),\frac{1}{2}\mathcal{D}(u,Q\mu)\right\} &= (1-s)\max\left\{\mathcal{D}(\mu,Q\mu),\frac{1}{2}\mathcal{D}(u,Q\mu)\right\}\\
&\le (1-\rho)\max\left\{\mathcal{D}(\mu,Q\mu),\frac{1}{2}\mathcal{D}(u,Q\mu)\right\}\\
&\le (1-\rho)\left(\left(\frac{1}{1-\rho}\right)\wp(\mu,u)+\frac{1}{n}\wp(\mu,u)\right)\\
&\le \wp(u,\mu)+\left(\frac{1-\rho}{n}\right)\wp(u,\mu)\\
&\le \wp(u,\mu)+\left(\frac{1-\rho}{n}\right)\wp(u,\mu),
\end{aligned}
$$

tending $n\to\infty$ gives

$$\varphi(s)\max\left\{\mathcal{D}(\mu,Q\mu),\frac{1}{2}\mathcal{D}(u,Q\mu)\right\}\le\wp(\mu,u)$$

and thus (2.16) satisfied. Using $\mu = u_n$ in (2.16), we have

$$\begin{aligned} \mathcal{D}(u_{n+1}, Qu) &\leq \mathcal{H}(Qu_n, Qu) \\ &\leq \rho \max\left\{\wp(u_n, u), \mathcal{D}(u_n, Qu_n), \mathcal{D}(u, Qu), \frac{\mathcal{D}(u_n, Qu) + \mathcal{D}(u, Qu_n)}{2}\right\} \\ &\leq \rho \max\left\{\wp(u_n, u),\ \wp(u_n, u_{n+1}), \mathcal{D}(u, Qu), \frac{\mathcal{D}(u_n, Qu) + \wp(u, u_{n+1})}{2}\right\}. \end{aligned} \tag{2.18}$$

Tending $n \to \infty$ to inequality (2.18), we arrived at

$$\mathcal{D}(u, Qu) \leq \rho \mathcal{D}(u, Qu)$$

which further gives that $\mathcal{D}(u, Qu) = 0$, that is, $u \in Qu$. □

For the case of single-valued Suzuki type $\mathcal{Z}_{(\varphi,\rho)}$-contraction, we have the following result:

Theorem 2.3. *Assume that $Q : \mathcal{N} \to \mathcal{N}$ be Suzuki type $\mathcal{Z}_{(\varphi,\rho)}$-contraction on a complete metric space $(\mathcal{N}, \wp)$, then $\mathcal{N}$ is an MWP operator and has a unique fixed point.*

Proof. By Theorem 2.2, Q is an MWP operator. Let $\mu \neq \lambda$ be two fixed points of Q, then

$$\varphi(s) max\{\wp(\mu, Q\mu), \frac{1}{2}\wp(\lambda, Q\mu)\} = \varphi(s)\frac{1}{2}\wp(\mu, \lambda) < \wp(\mu, \lambda)$$

this implies

$$\begin{aligned} 0 &\leq \zeta(\wp(Q\mu, Q\lambda), \rho\mathcal{R}(\mu, \lambda)) \\ &= \zeta(\wp(\mu, \lambda), \rho\wp(\mu, \lambda)) \\ &\leq \rho\wp(\mu, \lambda) - \wp(\mu, \lambda) \end{aligned}$$

this gives

$$\wp(\mu, \lambda) \leq \rho\wp(\mu, \lambda),$$

which is not possible, hence $\wp(\mu, \lambda) = 0$. Thus Q possesses a unique fixed point. □

Example 2.1. *Let $\mathcal{N} = \{0, 1, 2, 3\}$ and $\wp(\mu, \lambda) = |\mu - \lambda|$. Define $Q : \mathcal{N} \to \mathcal{CB}(\mathcal{N})$ by*

$$Q\mu = \begin{cases} \{0, 2\} & if\, \mu \neq 3, \\ \{1\} & if\, \mu = 3. \end{cases}$$

Now for $\mu, \lambda \in \mathcal{N}$ with $\mu \neq \lambda$, we have

(μ,λ)	$\wp(\mu,\lambda)$	$\mathcal{D}(\mu,\mathcal{Q}\mu)$	$\mathcal{D}(\lambda,\mathcal{Q}\mu)$	$\mathcal{H}(\mathcal{Q}\mu,\mathcal{Q}\lambda)$	$R(\mu,\lambda)$
(0, 1)	1	0	1	0	1
(0, 2)	2	0	0	0	2
(0, 3)	3	0	1	1	3
(1, 0)	1	1	0	0	1
(1, 2)	1	1	0	0	1
(1, 3)	2	1	1	1	2
(2, 0)	2	0	0	0	2
(2, 1)	1	0	1	0	1
(2, 3)	1	0	1	1	2
(3, 0)	3	2	1	1	3
(3, 1)	2	2	0	1	2
(3, 2)	1	2	1	1	2

So $max\{\mathcal{D}(\mu,\mathcal{Q}\mu):\mu\in\mathcal{N}\}=2$, $max\{\mathcal{D}(\lambda,\mathcal{Q}\mu):\mu,\lambda\in\mathcal{N}\}=1$ *and* $min\{\wp(\mu,\lambda):\mu,\lambda\in\mathcal{N}\}=1$. *Now for* $s=0.6,\varphi(s)=0.4$ *and*

$$\varphi(s)max\{\mathcal{D}(\mu,\mathcal{Q}\mu),\frac{1}{2}\mathcal{D}(\lambda,\mathcal{Q}\mu)\}=(0.4)(2)<1=min\{\wp(\mu,\lambda);\mu,\lambda\in\mathcal{N}\}$$

holds true for all $\mu,\lambda\in\mathcal{N}$. *Let* $\zeta(\alpha,\beta)=\frac{4}{5}\beta-\alpha$, *then for* $\rho=0.7$, *we have*

$$\begin{aligned}
\zeta(\mathcal{H}(\mathcal{Q}0,\mathcal{Q}1),\rho\mathcal{R}(0,1)) &= \zeta(0,0.7)=(0.8)(0.7)-0>0\\
\zeta(\mathcal{H}(\mathcal{Q}0,\mathcal{Q}2),\rho\mathcal{R}(0,2)) &= \zeta(0,1.4)=(0.8)(1.4)-0>0\\
\zeta(\mathcal{H}(\mathcal{Q}0,\mathcal{Q}3),\rho\mathcal{R}(0,3)) &= \zeta(1,2.1)=(0.8)(2.1)-1>0\\
\zeta(\mathcal{H}(\mathcal{Q}1,\mathcal{Q}2),\rho\mathcal{R}(1,2)) &= \zeta(0,0.7)=(0.8)(0.7)-0>0\\
\zeta(\mathcal{H}(\mathcal{Q}1,\mathcal{Q}3),\rho\mathcal{R}(1,3)) &= \zeta(1,1.4)=(0.8)(1.4)-1>0\\
\zeta(\mathcal{H}(\mathcal{Q}2,\mathcal{Q}3),\rho\mathcal{R}(2,3)) &= \zeta(1,1.4)=(0.8)(1.4)-1>0.
\end{aligned}$$

Hence $\mathcal{Q}$ *is* $\mathcal{Z}_{(\varphi,0.7)}$*-contraction. Thus by Theorem 2.2,* $\mathcal{Q}$ *is an MWP operator and* $Fix(\mathcal{Q})=\{0,2\}$.

3. Data dependence

We now prove a data dependence result.

Theorem 3.1. *Ssppose that* $\mathcal{Q}_1$, $\mathcal{Q}_2$ *are two multivalued operators on a metric space* $(\mathcal{N},\wp)$, *if*

1. $\mathcal{Q}_i$ *is* $\mathcal{Z}_{(\varphi,\rho_i)}$*-contraction for* $i=1,2$,
2. *there exist a real number* $\rho>0$ *such that* $\mathcal{H}(\mathcal{Q}_1\mu,\mathcal{Q}_2\mu)\leq\rho$, $\forall\ \mu\in\mathcal{N}$.

Then

1. $Fix(\mathcal{Q}_i)\in\mathcal{CL}(\mathcal{N})$, *for* $i=1,2$,
2. $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *are multivalued operators and*

$$\mathcal{H}(Fix(\mathcal{Q}_1),Fix(\mathcal{Q}_2))\leq\frac{\rho}{1-max\{\rho_1,\rho_2\}}.$$

Proof. Theorem 2.2 gives that $Fix(Q_i) \neq \phi$ for $i = 1, 2$. To prove that $Fix(Q)$ is closed, assume that $(\mu_n) \subset Fix(Q)$ such that $\mu_n \to t$ as $n \to \infty$, we have

$$\begin{aligned} \varphi(s) max\{\mathcal{D}(\mu_n, Q\mu_n), \frac{1}{2}\mathcal{D}(\omega, Q\mu_n)\} &= \varphi(s)\frac{1}{2}\mathcal{D}(\omega, Q\mu_n) \\ &\leq \frac{1}{2}\mathcal{D}(\omega, Q\mu_n) \\ &< \wp(\omega, \mu_n) \end{aligned}$$

this implies

$$\begin{aligned} 0 &\leq \zeta(\mathcal{H}(Q\mu_n, Q\omega), \rho\mathcal{R}(\mu_n, \omega)) \\ &\leq \rho\mathcal{R}(\mu_n, \omega) - \mathcal{H}(Q\mu_n, Q\omega) \end{aligned}$$

this gives

$$\mathcal{H}(Q\mu_n, Q\omega) \leq \rho\mathcal{R}(\mu_n, \omega).$$

Now

$$\begin{aligned} \mathcal{D}(\omega, Q\omega) &\leq \wp(\omega, \mu_n) + \mathcal{D}(\mu_n, Q\omega) \\ &\leq \wp(\omega, \mu_n) + \mathcal{H}(Q\mu_n, Q\omega) \\ &\leq \wp(\omega, \mu_n) + \rho\mathcal{R}(\mu_n, \omega) \\ &\leq \wp(\omega, \mu_n) + \rho max\{\wp(\omega, \mu_n), \mathcal{D}(\mu_n, Q\mu_n), \mathcal{D}(\omega, Q\omega), \frac{\mathcal{D}(\omega, Q\mu_n) + \mathcal{D}(\mu_n, Q\omega)}{2}\}, \end{aligned}$$

letting $n \to \infty$, we have $\mathcal{D}(\omega, Q\omega) = 0$, so $\omega \in Fix(Q)$ and hence $Fix(Q)$ is closed.
(b) From Theorem 2.2, it follows from (i) that Q_1 and Q_2 are MWP operators. Further, let μ_0 be fixed point of Q_1, then there exist $\mu_1 \in Q_2\mu_0$ and a real number $q > 1$ such that

$$\wp(\mu_0, \mu_1) \leq q\mathcal{H}(Q_1\mu_0, Q_2\mu_0),$$

since $\mu_1 \in Q_2\mu_0$, we can choose $\mu_2 \in Q_2\mu_1$ such that

$$\wp(\mu_1, \mu_2) \leq q\mathcal{H}(Q_2\mu_0, Q_2\mu_1)$$

also

$$\varphi(s) max\{\mathcal{D}(\mu_0, Q_2\mu_0), \frac{1}{2}\mathcal{D}(\mu_1, Q_2\mu_0)\} \leq \varphi(s)\wp(\mu_0, \mu_1) \leq \wp(\mu_0, \mu_1)$$

this implies

$$\begin{aligned} 0 &\leq \zeta(\mathcal{H}(Q_2\mu_0, Q_2\mu_1), \rho_2\mathcal{R}(\mu_0, \mu_1)) \\ &\leq \rho_2\mathcal{R}(\mu_0, \mu_1) - \mathcal{H}(Q_2\mu_0, Q_2\mu_1). \end{aligned}$$

Hence

$$\mathcal{H}(Q_2\mu_0, Q_2\mu_1) \leq \rho_2\mathcal{R}(\mu_0, \mu_1) \leq \rho_2\wp(\mu_0, \mu_1).$$

(Choosing $\mathcal{R}(\mu_0, \mu_1) \leq \wp(\mu_1, \mu_2)$ gives fixed point). Hence

$$\wp(\mu_1, \mu_2) \leq q\mathcal{H}(Q_2\mu_0, Q_2\mu_1) \leq q\rho_2\wp(\mu_0, \mu_1).$$

Therefore, for operator Q_2, we get an iterative sequence of successive approximations satisfying $\mu_{n+1} \in Q_2\mu_n$ such that

$$\wp(\mu_n, \mu_{n+1}) \leq (q\rho_2)^n\wp(\mu_0, \mu_1)$$

for all $n \geq 0$. If there exists a positive integer N such that for $n \geq N$ and $l \geq 1$,

$$\begin{aligned} \wp(\mu_{n+l}, \mu_n) &\leq \wp(\mu_n, \mu_{n+1}) + \wp(\mu_{n+1}, \mu_{n+2}) + ... + \wp(\mu_{n+l-1}, \mu_{n+l}) \\ &\leq \frac{(q\rho_2)^n}{1 - q\rho_2}\wp(\mu_0, \mu_1). \end{aligned} \tag{3.1}$$

Choosing $1 < q < min\{\frac{1}{\rho_1}, \frac{1}{\rho_2}\}$ and taking limit as $n \to \infty$, we have (μ_n) is a Cauchy sequence in $\mathcal{N}$. Let $u \in \mathcal{N}$ be the limit point of (μ_n). We show that $u \in Q_2$. Otherwise, let N be the positive integer such that $\mathcal{D}(u, Q_2\mu_n) > 0 = \wp(u, \mu_n)$, $\forall\ n \geq N$. Then $\wp(\mu, \mu_{n+1}) > \wp(u, \mu_n)$ for all $n \geq N$, a contradiction. Hence there exist a subsequence $\mu_{n(j)}$ such that for all $j \in \mathbb{N}$, we have

$$\begin{aligned} \varphi(s)max\{\mathcal{D}(\mu_{n(j)}, Q_2\mu_{n(j)}), \frac{1}{2}\mathcal{D}(u, Q_2\mu_{n(j)})\} &\leq max\{\wp(\mu_{n(j)}, \mu_{n(j)+1}), \mathcal{D}(u, Q_2\mu_{n(j)})\} \\ &\leq \{\wp(\mu_{n(j)}, \mu_{n(j)+1}), \wp(u, \mu_{n(j)+1})\} \\ &\leq \{\wp(\mu_{n(j)}, \mu_{n(j)+1}), \wp(u, \mu_{n(j)}) + \wp(\mu_{n(j)}, \mu_{n(j)+1})\} \\ &= \wp(u, \mu_{n(j)}) + \wp(\mu_{n(j)}, \mu_{n(j)+1}) \\ &= \wp(u, \mu_{n(j)}) \end{aligned}$$

(since $\wp(\mu_{n(j)}, \mu_{n(j)+1}) \to 0$ as $j \to \infty$). Above inequality gives

$$0 \leq \zeta(\mathcal{H}(Q_2\mu_{n(j)}, Q_2u), \rho\mathcal{R}(\mu_{n(j)}, u))$$

implies

$$\mathcal{H}(Q_2\mu_{n(j)}, Q_2u) \leq \rho\mathcal{R}(\mu_{n(j)}, u).$$

Thus

$$\mathcal{D}(u, Q_2u) = \lim_{j\to\infty} \mathcal{D}(\mu_{n(j)+1}, Q_2u) \leq \lim_{j\to\infty} \mathcal{H}(Q_2\mu_{n(j)}, Q_2u) \leq \lim_{j\to\infty} \rho_2\wp(\mu_{n(j)}, u) = 0.$$

So $u \in Fix(Q_2)$. In (2.1), letting $l \to \infty$, we get

$$\wp(\mu_n, u) \leq \frac{(q\rho_2)^n}{1 - q\rho_2}\wp(\mu_0, \mu_1)$$

for all $n \in \mathbb{N}$. Then

$$\wp(\mu_0, u) \leq \frac{1}{1 - q\rho_2}\wp(\mu_0, \mu_1) \leq \frac{q\rho}{1 - q\rho_2}.$$

On the same lines, we have that for each $u_0 \in Fix(Q_2)$, there exists $\mu \in Fix(Q_1)$ such that

$$\wp(u_0, \mu) \leq \frac{1}{1 - q\rho_2}\wp(u_0, u_1) \leq \frac{q\rho}{1 - q\rho_2}.$$

Hence

$$\mathcal{H}(Fix(Q_1), Fix(Q_2)) \leq \frac{q\rho}{1 - max\{q\rho_1, q\rho_2\}}.$$

Taking limit as $q \searrow 1$, we get the desired result. □

4. An application

A variety of fixed point theorems have been used to explore the existence and uniqueness of solution of functional equations arising in dynamic programming. Utilizing Theorem 2.3, we explore the existence and uniqueness of solution for a nonlinear functional equation. From now on, $W_1(\mathbb{R})$ and $W_2(\mathbb{R})$ are assumed to be Banach spaces and $U_1 \subset W_1$, $U_2 \subset W_2$. Class of all bounded real-valued functions on U_1 is denoted by $\mathcal{B}(U_1)$. The metric space pair $\mathcal{B}(U_1, \wp)$ where

$$d_{\mathcal{B}}(f, g) = \sup_{\mu \in U_1} |f(\mu) - g(\mu)|, \quad f, g \in U_1,$$

is complete. Viewing U_1 and U_2 as the state and decision spaces respectively, the problem of dynamic programming reduces to the problem of solving the functional equation:

$$p(\mu) = \sup_{\lambda \in U_2} F(\mu, \lambda, p(\eta(\mu, \lambda))),$$

where $\eta : U_1 \times U_2 \to U_1$ is the transformation of the process and $p(\mu)$ represents the optimal return function with initial functional equation:

$$p(\mu) = \sup_{\lambda \in U_2}\{v(\mu, \lambda) + G(\mu, \lambda, p(\eta(\mu, \lambda)))\}, \ \mu \in U_1, \tag{4.1}$$

where $v : U_1 \times U_2 \to \mathbb{R}$ and $G : U_1 \times U_2 \times \mathbb{R} \to \mathbb{R}$ are bounded functions. Let $Q : \mathcal{B}(U_1) \to \mathcal{B}(U_1)$ be defined by:

$$Q(f(\mu)) = \sup_{\lambda \in U_2}\{v(\mu, \lambda) + G(\mu, \lambda, p(\eta(\mu, \lambda)))\}, \ \mu \in U_1, f \in \mathcal{B}(U_1).$$

Theorem 4.1. *Assume that there exists $\rho > 0$ such that for every $(\mu, \lambda) \in U_1 \times U_2$, $f, g \in \mathcal{B}(U_1)$ and $t \in U_1$ the inequality*

$$\varphi(s) max\{|f(t) - Q(f(t))|, \frac{1}{2}|g(t) - Q(f(t))|\} \le |f(t) - g(t)| \tag{4.2}$$

implies

$$|G(\mu, \lambda, f(t)) - G(\mu, \lambda, g(t))| \quad \le \quad \frac{\rho \mathcal{R}(f(t), g(t))}{\rho \mathcal{R}(f(t), g(t)) + 1} \tag{4.3}$$

where

$$\begin{aligned} \mathcal{R}(f(t), g(t)) \quad = \quad & max\{|f(t) - g(t)|, |f(t) - Qf(t)|, |g(t) - Q(g(t))|, \\ & \frac{|f(t) - Q(g(t))| + |g(t) - Q(f(t))|}{2}\}. \end{aligned}$$

Then (4.1) *possesses a bounded solution.*

Proof. It is obvious that Q is self map on $\mathcal{B}(U_1)$. Suppose γ is an arbitrary positive real number and $f_1, f_2 \in \mathcal{B}(U_1)$. Take $\mu \in U_1$ and choose $\lambda_1, \lambda_2 \in U_2$ such that

$$Q(f_1(\mu)) \quad < \quad v(\mu, \lambda_1) + G(\mu, \lambda_1, f_1(\eta_1)) + \gamma \tag{4.4}$$

$$Q(f_2(\mu)) \quad < \quad g(\mu,\lambda_2) + G(\mu,\lambda_2, f_2(\eta_2)) + \gamma, \tag{4.5}$$

where $\eta_i = \eta_i(\mu,\lambda_i)$, $i = 1, 2$. By definition of Q, we get

$$\begin{aligned} Q(f_1(\mu)) &\geq v(\mu,\lambda_2) + G(\mu,\lambda_2, f_1(\eta_2)) \\ Q(f_2(\mu)) &\geq v(\mu,\lambda_1) + G(\mu,\lambda_1, f_2(\eta_1)). \end{aligned}$$

If the inequality (4.2) holds with $f = f_1, g = f_2$, then from (4.3), (4.4) and (4.6), we have

$$\begin{aligned} Q(f_1(\mu)) - Q(f_2(\mu)) &< G(\mu,\lambda_1, f_1(\eta_1)) - G(\mu,\lambda_1, f_2(\eta_1)) + \gamma \\ &\leq \frac{\rho\mathcal{R}(f_1(t), f_2(t))}{\rho\mathcal{R}(f_1(t), f_2(t)) + 1} + \gamma. \end{aligned} \tag{4.6}$$

Similarly, from (4.3), (4.5) and (4.6), we have

$$Q(f_2(\mu)) - Q(f_1(\mu)) < \frac{\rho\mathcal{R}(f_1(t), f_2(t))}{\rho\mathcal{R}(f_1(t), f_2(t)) + 1} + \gamma. \tag{4.7}$$

Hence, from (4.6) and (4.7), we obtain that

$$|Q(f_1(\mu)) - Q(f_2(\mu))| < \frac{\rho\mathcal{R}(f_1(t), f_2(t))}{\rho\mathcal{R}(f_1(t), f_2(t)) + 1} + \gamma,$$

since the inequality holds for any $\mu \in U_1$ and $\gamma > 0$, we get that

$$\varphi(s)max\{d_{\mathcal{B}}(f_1, Q(f_1)), \frac{1}{2}d_{\mathcal{B}}(f_2, Q(f_1))\} \leq d_{\mathcal{B}}(f_1, f_2)$$

implies

$$d_{\mathcal{B}}(Q(f_1), Q(f_2)) \leq \rho max\{d_{\mathcal{B}}(f_1, f_2), d_b(f_1, Qf_1), d_{\mathcal{B}}(f_2, Qf_2), \frac{d_{\mathcal{B}}(f_1, Qf_2) + d_{\mathcal{B}}(f_2, Qf_1)}{2}\},$$

so if we choose $\zeta(\alpha,\beta) = \frac{\beta}{\beta+1} - \alpha$, then we have

$$\zeta(d_{\mathcal{B}}(Qf_1, Qf_2), \rho\mathcal{R}(f_1, f_2)) \geq 0.$$

Hence, all the assertions of Theorem 2.3 are verified and therefore we get the conclusion. □

Example 4.1. *Let $W_1 = \mathbb{R} = W_2$, $U_1 = [1, 50]$, $U_2 = \mathbb{R}^+$. We define $\eta : U_1 \times U_2 \to U_1$, $v : U_1 \times U_2 \to \mathbb{R}$ and $G : U_1 \times U_2\mathbb{R} \to \mathbb{R}$ by*

$$\eta(\mu,\lambda) = \frac{50\mu + \lambda^6}{\mu^2 + \lambda^6},$$

$$v(\mu,\lambda) = \frac{sin(\mu + \lambda^2)}{7\lambda^2 + \lambda + 1}$$

and

$$A(\mu,\lambda,z) = z|sin(\mu^2 + \lambda^2).$$

Now for $s = 0.5$, $\varphi(s) = \frac{1}{2}$. Therefore for $\rho = 0.65$, it is clear that (4.2)-(4.3) *are satisfied. Hence from Theorem 2.3 that the functional equation* (4.1) *possesses a bounded solution.*

5. Conclusions

In this paper we introduced a new type of set valued contraction in Suzuki sense together with simulation functions and proved some new fixed point and data dependence theorems for such class of mappings. We gave an example to prove the validity of our results. At last, we presented an application to nonlinear functional equation arising in dynamical system to show the usability of obtained results.

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Conflict of interest

The authors declare no conflict of interest.

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