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Research article

Fixed point theorems for weakly compatible mappings under implicit relations in quaternion valued G-metric spaces

Mohamed Gamal[1] **and Watcharaporn Cholamjiak**[2,*]

[1] Department of Mathematics, Faculty of Science, South Valley University, Luxor 85844, Egypt
[2] School of Science, University of Phayao, Phayao 56000, Thailand

* **Correspondence:** watcharaporn.ch@up.ac.th

Abstract: In this paper, we established some common fixed point theorems of four self-mappings in completed quaternion valued $G-$metric space. Moreover, we gave an example of completed quaternion valued $G-$metric space and example for supporting our main results. The results obtained in this paper extend and improve some recent results.

Keywords: Quaternion valued $G-$metric space; common fixed point; weakly compatible mapping, implicit relation; coincident point.
Mathematics Subject Classification: 47H10, 55H02.

1. Introduction

The fixed point theory plays a major role in mathematics and applied sciences, such as mathematical models, optimization, and economic theories. This is the reason that why the study of metric fixed point theory has been researched extensively in the past decades. Since 1963, many mathematicians tried to generalize the usual notation of metric space and extend some known metric space theorems in more general setting (see [2, 3, 5, 8–15, 21, 22]).

In 2005, Mustafa and Sims [20] introduced and study a new generalized metric spaces which is called G-metric spaces. They found a new fixed point of various mappings in new structure of these spaces. The G-metric spaces is defined by the following:

Definition 1 [20] Let X be a nonempty set and a function G be defined on the product set $X \times X \times X$ into the interval $[0, +\infty)$ satisfying the following properties:
(G_1) $G(a, b, c) = 0$ if and only if $a = b = c$;
(G_2) $G(a, a, b) > 0$ for all $a, b \in X$ with $a \neq b$;
(G_3) $G(a, a, b) \leq G(a, b, c)$ for all $a, b, c \in X$ with $c \neq b$;

(G_4) $G(a,b,c) = G(a,c,b) = G(b,a,c) = ...$ (symmetry);
(G_5) $G(a,b,c) \leq G(a,d,d) + G(d,b,c)$ for all $a,b,c,d \in X$ (rectangle inequality).
The function G is called $G-$metric on X and (X,G) is called $G-$metric space.

In 1843, Irish mathematician, Hamilton [18] gave the definition of quaternion as the quotient of two directed lines in a three-dimensional space or equivalently as the quotient of two vectors. The study of quaternion has a lot set of applications such as applying to mechanics in three-dimensional space and practical uses in applied mathematics in particular for calculations involving three-dimensional rotations. There are many features of quaternions are that number system that extends the complex numbers and multiplication of two quaternions is noncommutative.

We denote $\mathbf{H}$ for the skew field of quaternion and $q \in \mathbf{H}$ has the form $p = a + bi + cj + dk$ where $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $kj = -jk = -i$, $ki = -ik = j$ and the modulus of p, $|p| = \sqrt{a^2 + b^2 + c^2 + d^2}$ where a,b,c and d are real numbers, and i,j and k are the fundamental quaternion units. Thus a quaternion p may be viewed as a four-dimensional vector (a,b,c,d). By simple treating, quaternion can be written as simply quadruples of real numbers $[a,b,c,d]$, with addition and multiplication operations that are suitably defined. The components group into the imaginary part (b,c,d) , which we consider this part as a vector and the purely real part a which is called a scalar. Sometimes, we write a quaternion as $[V,a]$ with $V = (b,c,d)$.

$$[V,a] = [(b,c,d),a] = [a,b,c,d] = a + bi + cj + dk \quad \text{for all } \ a,b,c,d \in \mathbb{R}.$$

For more properties of quaternion analysis, see [7, 16, 17] and and the references therein.

In order to prove our results, we present some necessary basic notions and concepts in the following. Let $\mathbf{H}$ be the set of quaternion and $p_1, p_2 \in \mathbf{H}$. Define a partial order $\precsim$ on $\mathbf{H}$ as follows:
$p_1 \precsim p_2$ iff $\mathrm{Re}(p_1) \leq \mathrm{Re}(p_2)$ and $\mathrm{Im}_s(p_1) \leq \mathrm{Im}_s(p_2)$, $p_1, p_2 \in \mathbf{H}$, $s = i,j,k$ where $\mathrm{Im}_i = b$, $\mathrm{Im}_j = c$ and $\mathrm{Im}_k = d$.

(Q_1) $\mathrm{Re}(p_1) = \mathrm{Re}(p_2)$ and $\mathrm{Im}_{s_1}(p_1) = \mathrm{Im}_{s_2} pz_2)$ where $s_1 = j,k$, $Im_i(p_1) < Im_i(p_2)$;
(Q_2) $Re(p_1) = R(p_2), Im_{s_2}(p_1) = Im_{s_2}(p_2)$ where $s_2 = i,k$, $Im_j(p_1) < Im_j(p_2)$;
(Q_3) $Re(p_1) = Re(p_2), Im_{s_3}(p_1) = Im_{s_3}(p_2)$ where $s_3 = i,j$, $Im_k(p_1) < Im_k(p_2)$;
(Q_4) $Re(p_1) = R(p_2)$, $Im_{s_1}(p_1) = Im_{s_1}(p_2)$, $Im_i(p_1) = Im_i(p_2)$;
(Q_5) $Re(p_1) = R(p_2)$, $Im_{s_2}(p_1) = Im_{s_2}(p_2)$, $Im_j(p_1) = Im_j(p_2)$;
(Q_6) $Re(p_1) = R(p_2)$, $Im_{s_3}(p_1) = Im_{s_3}(p_2)$, $Im_k(p_1) = Im_k(p_2)$;
(Q_7) $Re(p_1) = R(p_2)$, $Im_s(p_1) < Im_s(p_2)$;
(Q_8) $Re(p_1) < R(p_2)$, $Im_s(p_1) = Im_s(p_2)$;
(Q_9) $Re(p_1) < R(p_2), Im_{s_1}(p_1) = Im_{s_1}(p_2)$, where $s_1 = j,k$; $Im_i(p_1) < Im_i(p_2)$;
(Q_{10}) $Re(p_1) < R(p_2)$, $Im_{s_2}(p_1) = Im_{s_2}(p_2)$, $Im_j(p_1) < Im_j(p_2)$;
(Q_{11}) $Re(p_1) < R(p_2)$, $Im_{s_3}(p_1) = Im_{s_3}(p_2)$, $Im_k(p_1) < Im_k(p_2)$;
(Q_{12}) $Re(p_1) < R(p_2)$, $Im_{s_1}(p_1) < Im_{s_1}(p_2)$, $Im_i(p_1) = Im_i(p_2)$;
(Q_{13}) $Re(p_1) < R(p_2)$, $Im_{s_2}(p_1) < Im_{s_2}(p_2)$, $Im_j(p_1) = Im_j(p_2)$;
(Q_{14}) $Re(p_1) < R(p_2)$, $Im_{s_3}(p_1) < Im_{s_3}(p_2)$, $Im_k(p_1) = Im_k(p_2)$;
(Q_{15}) $Re(p_1) < R(p_2)$, $Im_s(p_1) < Im_s(p_2)$;
(Q_{16}) $Re(p_1) = R(p_2)$, $Im_s(p_1) = Im_s(p_2)$.

Particularly, we will write $p_1 \precnsim p_2$ if $p_1 \neq p_2$ and one from (Q_1) to (Q_{16}) is satisfied and we will write $p_1 \prec p_2$ if only (Q_{14}) is satisfied. It should be noted that

$$p_1 \preceq p_2 \text{ implies } |p_1| \leq |p_2| .$$

In 2014, Ahmed et al. [4] introduced the concept of quaternion metric spaces and established some fixed point theorems in quaternion setting. For presentation the common fixed point of four self-maps which satisfy a general contraction condition was shown in normal cone metric spaces. The quaternion valued metric function is defined as follow.

Definition 2 [4] Let X be a nonempty set and $g_H : X \times X \longrightarrow \mathbf{H}$ be a function satisfying the following properties:

(H_1) $g_{\mathbf{H}}(a,b) \succeq 0$ for all $a, b \in X$;
(H_2) $g_{\mathbf{H}}(a,b) = 0$ if and only if $a = b$;
(H_3) $g_{\mathbf{H}}(a,b) = g_{\mathbf{H}}(a,b)$ (symmetry);
(H_4) $g_{\mathbf{H}}(a,b) \preceq g_{\mathbf{H}}(a,c) + g_{\mathbf{H}}(c,b)$ for all $a, b, c \in X$ (triangle inequality).

The function $g_{\mathbf{H}}$ is called quaternion valued metric on X and $(X, g_{\mathbf{H}})$ is called quaternion valued metric space.

Motivated by Ahmed et al. work in [4], Adewale et al. [1] introduced the following concept of a quaternion valued G-metric spaces and gave some examples of this spaces.

Definition 3 [1] Let X be a nonempty set and let $G_{\mathbf{H}} : X \times X \times X \longrightarrow \mathbf{H}$ be a function satisfying the following conditions:

(QG_1) $G_{\mathbf{H}}(a,b,c) = 0_{\mathbf{H}}$ if $a = b = c$;
(QG_2) $0_{\mathbf{H}} \prec G_{\mathbf{H}}(a,a,b)$ for all $a, b \in X$ with $a \neq b$;
(QG_3) $G_{\mathbf{H}}(a,a,b) \preccurlyeq G_{\mathbf{H}}(a,b,c)$ for all $a, b, c \in X$ with $b \neq c$;
(QG_4) $G_{\mathbf{H}}(a,b,c) = G_{\mathbf{H}}(a,c,b) = G_{\mathbf{H}}(b,c,a) = ...$ (symmetry);
(QG_5) $G_{\mathbf{H}}(a,b,c) \preccurlyeq G_{\mathbf{H}}(a,d,d) + G_{\mathbf{H}}(d,b,c)$ for all $a, b, c, d \in X$ (rectangle inequality).

Therefore, the function $G_{\mathbf{H}}$ is called quaternion valued $G_{\mathbf{H}}$-metric on X and the pair (X,$G_{\mathbf{H}}$) is called quaternion valued $G-$metric space.

We found some gap in one of examples (Example 2, [1]), the domain of quaternion valued $G_{\mathbf{H}}$-metric is not no the product space $X \times X \times X$. So, we now give a new example of $G_{\mathbf{H}}$-metric as follow:

Example 4 Let $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ with

$$G_{\mathbf{H}}(a,b,c) = G_{\mathbf{H}}(b,c,a) = G_{\mathbf{H}}(a,c,b) = ...,$$

for all $a, b, c \in X$. $G_{\mathbf{H}} : X \times X \times X \longrightarrow \mathbf{H}$ is defined by

$$G_{\mathbf{H}}(q_1, q_2, q_3) = \triangle G_{\mathbf{H}} + \triangle G_{\mathbf{H}} i + \triangle G_{\mathbf{H}} j$$

where $\triangle G_{\mathbf{H}} = \sum_{i,j \in \{1,2,3\}} \left(\left|a_i - a_j\right| + \left|b_i - b_j\right| + \left|c_i - c_j\right| \right)$ and

$$q_1 = (a_1, b_1, c_1), \;\; q_2 = (a_2, b_2, c_2), \;\; q_3 = (a_3, b_3, c_3) \in X^3.$$

We see that $G_{\mathbf{H}}$ is quaternion valued $G-$metric on X but not $G-$metric on X.

We next recall some definition and basic results that will be used in our subsequent analysis.

Proposition 5 [19] Let $(X, G_{\mathbf{H}})$ be quaternion valued $G-$metric space. Then for all $a, b, c, d \in X$ the following properties hold:

(1) $G_{\mathbf{H}}(a,b,c) = 0$ implies $a = b = c$;
(2) $G_{\mathbf{H}}(a,b,c) \preccurlyeq G_{\mathbf{H}}(a,a,b) + G_{\mathbf{H}}(a,a,c)$;

(3) $G_{\mathbf{H}}(a,b,b) \preccurlyeq 2\ G_{\mathbf{H}}(b,a,a)$;
(4) $G_{\mathbf{H}}(a,b,c) \preccurlyeq G_{\mathbf{H}}(a,d,c) + G_{\mathbf{H}}(d,b,c)$;
(5) $G_{\mathbf{H}}(a,b,c) \preccurlyeq \frac{2}{3}\left(G_{\mathbf{H}}(a,b,d) + G_{\mathbf{H}}(a,d,c) + G_{\mathbf{H}}(d,b,c)\right)$;
(6) $G_{\mathbf{H}}(a,b,c) \preccurlyeq 2\left(G_{\mathbf{H}}(a,a,d) + G_{\mathbf{H}}(b,b,d) + G_{\mathbf{H}}(c,c,d)\right)$.

Definition 6 [6] Let K, L, M and N be four self-mappings of a nonempty set X. Then
(i) a point $u \in X$ is said to be a fixed point of K if $Ku = u$,
(ii) a point $u \in X$ is said to be a common fixed point of K and L if $Ku = Lu = u$,
(iii) a point $u \in X$ is said to be a coincidence point of K and N if $Ku = Nu$ and a point $t \in X$ such that $t = Ku = Nu$ is called a point of coincidence of K and N,
(iv) a point $t \in X$ is said to be a common point of coincidence of the pairs (K, N) and (L, M) if there exist $u, v \in X$ such that $Ku = Nu = t$ and $Lv = Mv = t$.

In this paper, we proved existence theorems of coincidence of the pairs (K, N) and (L, M) under some suitable conditions. Moreover, the weakly compatible condition of the pairs (K, N) and (L, M) was added for finding a common fixed point of mappings K, L, M and N. Finally, we presented examples and conditions which satisfy our main theorems.

2. Results

We begin with the following definition:

An implicit relation. Let F be the set of all complex valued lower semi-continuous functions $F : \mathbf{H}^6 \to \mathbf{H}$ satisfying the following conditions: for $u, u', v \in \mathbf{H}$
(F_1) F is non-increasing in the 5^{th} and 6^{th} variable,
(F_2) for $u, v \succsim 0_{\mathbf{H}}$, there exists $q \in [0,1)$ such that $|u| \leq q\,|v|$ if $F(u, v, u, v, 0_{\mathbf{H}}, u+v) \precsim 0_{\mathbf{H}}$,
(F_3) for $u, u' \succ 0_{\mathbf{H}}$, there exists $q \in [0,1)$ such that $|u| \leq q\left|u'\right|$ if $F(u, u, 0_{\mathbf{H}}, 0_{\mathbf{H}}, u', u) \precsim 0_{\mathbf{H}}$.

Now, we present our first main theorems in complete quaternion valued $G-$metric space.

Theorem 7 Let K, L, M and N be four self-mappings on a complete quaternion valued $G-$metric space $(X, G_{\mathbf{H}})$ such that $K(X) \subseteq M(X)$ and $L(X) \subseteq N(X)$. Assume that there exists $\phi_1, \phi_2 \in F$ such that for all $a, b \in X$, $a \neq b$,

$$\begin{cases} \phi_1\left(G_{\mathbf{H}}(La, La, Kb), G_{\mathbf{H}}(Ma, Ma, Nb), G_{\mathbf{H}}(Ma, La, La),\right. \\ \left.G_{\mathbf{H}}(Nb, Kb, Kb), G_{\mathbf{H}}(Ma, Kb, Kb), G_{\mathbf{H}}(Nb, La, La)\right) \precsim 0_{\mathbf{H}}, \\ \\ \phi_2\left(G_{\mathbf{H}}(Ka, Ka, Lb), G_{\mathbf{H}}(Na, Na, Mb), G_{\mathbf{H}}(Na, Ka, Ka),\right. \\ \left.G_{\mathbf{H}}(Mb, Lb, Lb), G_{\mathbf{H}}(Na, Lb, Lb), G_{\mathbf{H}}(Mb, Ka, Ka)\right) \precsim 0_{\mathbf{H}}. \end{cases} \tag{2.1}$$

If $M(X) \cup N(X)$ is complete subspace of X, then the pairs (K, N) and (L, M) have a unique common point of coincidence. Moreover, if the pairs (K, N) and (L, M) are weakly compatible, then the four mappings have a unique common fixed point.

Proof. Let x_0 be arbitrary point in X. Since $K(X) \subseteq M(X)$ and $L(X) \subseteq N(X)$, then we can define the sequence $\{a_n\}$ in X such that,

$$\begin{cases} b_{2n+1} = Ma_{2n+1} = Ka_{2n}, \\ b_{2n+2} = Na_{2n+2} = La_{2n+1}. \end{cases} \tag{2.2}$$

Since $\{b_n\} \subseteq M(X) \cup N(X)$. We then show that $\{b_n\}$ is a Cauchy sequence. By putting $a = a_{2n+1}$ and $b = a_{2n}$ in (ϕ_1), we have

$$\phi_1\,(G_{\mathbf{H}}(La_{2n+1}, La_{2n+1}, Ka_{2n}), G_{\mathbf{H}}(Ma_{2n+1}, Ma_{2n+1}, Na_{2n}), G_{\mathbf{H}}(Ma_{2n+1}, La_{2n+1}, La_{2n+1}) \\ , G_{\mathbf{H}}(Na_{2n}, Ka_{2n}, Ka_{2n}), G_{\mathbf{H}}(Ma_{2n+1}, Ka_{2n}, Ka_{2n}), G_{\mathbf{H}}(Na_{2n}, La_{2n+1}, La_{2n+1})) \precsim 0_{\mathbf{H}}.$$

This implies that

$$\phi_1\,(G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n}), G_{\mathbf{H}}(b_{2n+1}, b_{2n+2}, b_{2n+2}), \\ G_{\mathbf{H}}(b_{2n}, b_{2n+1}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n+1}), G_{\mathbf{H}}(b_{2n}, b_{2n+2}, b_{2n+2})) \precsim 0_{\mathbf{H}},$$

This means that

$$\phi_1\,(G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n}), G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), \\ G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n}), 0_{\mathbf{H}}, G_{\mathbf{H}}(b_{2n}, b_{2n+1}, b_{2n+1}) + G_{\mathbf{H}}(b_{2n+1}, b_{2n+2}, b_{2n+2})) \precsim 0_{\mathbf{H}}.$$

From (F_1) and (QG_5), we get

$$\phi_1\,(G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n}), G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), \\ G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n}), 0_{\mathbf{H}}, G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}) + G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n})) \precsim 0_{\mathbf{H}}.$$

From (F_2), we obtain

$$|G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1})| \leq q\ |G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n})|\,.$$

By a similar way, by putting $a = a_{2n+2}$ and $b = a_{2n+1}$ in (ϕ_2), we have successively

$$\phi_2\,(G_{\mathbf{H}}(Ka_{2n+2}, Ka_{2n+2}, La_{2n+1}), G_{\mathbf{H}}(Na_{2n+2}, Na_{2n+2}, Ma_{2n+1}), \\ G_{\mathbf{H}}(Na_{2n+2}, Ka_{2n+2}, Ka_{2n+2}), G_{\mathbf{H}}(Ma_{2n+1}, La_{2n+1}, La_{2n+1}), \\ G_{\mathbf{H}}(Na_{2n+2}, La_{2n+1}, La_{2n+1}), G_{\mathbf{H}}(Ma_{2n+1}, Ka_{2n+2}, Ka_{2n+2})) \precsim 0_{\mathbf{H}},$$

that is,

$$\phi_2\,(G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2}), G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+2}, b_{2n+3}, b_{2n+3}), \\ G_{\mathbf{H}}(b_{2n+1}, b_{2n+2}, b_{2n+2}), G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+2}), G_{\mathbf{H}}(b_{2n+1}, b_{2n+3}, b_{2n+3})) \precsim 0_{\mathbf{H}}.$$

This tends to

$$\phi_2\,(G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2}), G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2}), \\ G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), 0_{\mathbf{H}}, G_{\mathbf{H}}(b_{2n+1}, b_{2n+2}, b_{2n+2}) + G_{\mathbf{H}}(b_{2n+2}, b_{2n+3}, b_{2n+3})) \precsim 0_{\mathbf{H}}.$$

From (F_1) and (QG_5), we obtain

$$\phi_2\,(G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2}), G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2}), \\ G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), 0_{\mathbf{H}}, G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}) + G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2})) \precsim 0_{\mathbf{H}}.$$

From (F_2), we have

$$|G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2})| \leq q\ |G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1})|\,.$$

Consequently,

$$|G_{\mathbf{H}}(b_{n+1}, b_{n+1}, b_n)| \leq q\ |G_{\mathbf{H}}(b_n, b_n, b_{n-1})| \leq ... \leq q^n\ |G_{\mathbf{H}}(b_1, b_1, b_0)|\,.$$

Also, for all $n > m$, we get

$$\begin{aligned}|G_{\mathbf{H}}(b_n, b_n, b_m)| &\leq |G_{\mathbf{H}}(b_{m+1}, b_{m+1}, b_m)| + |G_{\mathbf{H}}(b_{m+2}, b_{m+2}, b_{m+1})| + ... + |G_{\mathbf{H}}(b_n, b_n, b_{n-1})|\\ &\leq (q^m + q^{m+1} + ... + q^{n-1})\,|G_{\mathbf{H}}(b_1, b_1, b_0)|\\ &\leq \frac{q^m}{1-q}\,|G_{\mathbf{H}}(b_1, b_1, b_0)| \longrightarrow 0 \quad \text{as} \quad m \longrightarrow +\infty.\end{aligned}$$

This means that $\{b_n\}$ is a Cauchy sequence in X. Since $(X, G_{\mathbf{H}})$ is complete, then there exists $u \in X$ such that $b_n \longrightarrow u$ as $n \longrightarrow +\infty$. Then from Eq. (2.2), we obtain

$$\lim_{n\longrightarrow+\infty} Ka_{2n} = \lim_{n\longrightarrow+\infty} Ma_{2n+1} = \lim_{n\longrightarrow+\infty} Na_{2n+2} = \lim_{n\longrightarrow+\infty} La_{2n+1} = u. \tag{2.3}$$

Since $K(X) \subseteq M(X)$, if $u \in M(X)$, then there exists $v \in X$ such that

$$M\,v = u. \tag{2.4}$$

We will show that $Lv = Mv$. By putting $a = v$ and $b = a_{2n}$ in (ϕ_1), we have

$$\begin{aligned}&\phi_1\,(G_{\mathbf{H}}(Lv, Lv, Ka_{2n}), G_{\mathbf{H}}(Mv, Mv, Na_{2n}), G_{\mathbf{H}}(Mv, Lv, Lv), G_{\mathbf{H}}(Na_{2n}, Ka_{2n}, Ka_{2n})\\ &\qquad\qquad\qquad , G_{\mathbf{H}}(Mv, Ka_{2n}, Ka_{2n}), G_{\mathbf{H}}(Na_{2n}, Lv, Lv)) \precsim 0_{\mathbf{H}}.\end{aligned}$$

Putting $n \longrightarrow +\infty$ and using Eqs. (2.3) and (2.4), we have

$$\phi_1\,(G_{\mathbf{H}}(Lv, Lv, u), G_{\mathbf{H}}(u, u, u), G_{\mathbf{H}}(u, Lv, Lv), G_{\mathbf{H}}(u, u, u), G_{\mathbf{H}}(u, u, u), G_{\mathbf{H}}(u, Lv, Lv)) \precsim 0_{\mathbf{H}}.$$

This tends to

$$\phi_1\,(G_{\mathbf{H}}(Lv, Lv, u), 0_{\mathbf{H}}, G_{\mathbf{H}}(Lv, Lv, u), 0_{\mathbf{H}}, 0_{\mathbf{H}}, G_{\mathbf{H}}(Lv, Lv, u)) \precsim 0_{\mathbf{H}}.$$

From (F_2), we obtain $G_{\mathbf{H}}(Lv, Lv, u) = 0_{\mathbf{H}}$ which tends to $Lv = u$. Hence

$$Lv = Mv = u. \tag{2.5}$$

Then, u is a point of coincidence of the pair (L, M).

Since $L(X) \subseteq N(X)$, there exists $w \in X$ such that

$$Nw = u. \tag{2.6}$$

We will show that $Kw = Nw$. By putting $a = w$ and $b = a_{2n+1}$ in (ϕ_2), we have

$$\begin{aligned}&\phi_2\,(G_{\mathbf{H}}(Kw, Kw, La_{2n+1}), G_{\mathbf{H}}(Nw, Nw, Ma_{2n+1}), G_{\mathbf{H}}(Nw, Kw, Kw),\\ &\quad G_{\mathbf{H}}(Ma_{2n+1}, La_{2n+1}, La_{2n+1}), G_{\mathbf{H}}(Nw, La_{2n+1}, La_{2n+1}), G_{\mathbf{H}}(Ma_{2n+1}, Kw, Kw)) \precsim 0_{\mathbf{H}}.\end{aligned}$$

Taking $n \longrightarrow +\infty$ and using Eqs. (2.3) and (2.6), we get

$$\phi_2\,(G_{\mathbf{H}}(Kw, Kw, u), 0_{\mathbf{H}}, G_{\mathbf{H}}(Kw, Kw, u), 0_{\mathbf{H}}, 0_{\mathbf{H}}, G_{\mathbf{H}}(Kw, Kw, u)) \precsim 0_{\mathbf{H}}.$$

From (F_2), we get $G_{\mathbf{H}}(Kw, Kw, u) = 0_{\mathbf{H}}$ which implies $Kw = u$. Hence

$$Kw = Nw = u. \tag{2.7}$$

Then, u is a point of coincidence of the pair (K, N).

Hence, $u \in X$ is a common point of coincidence for the four mappings.

To show the uniqueness of a point of coincidence, Let $u^* \neq u$ be another point of coincidence of the four mappings. Then, there exists v^*, w^* such that $Lv^* = Mv^* = u^*$ and $Kw^* = Nw^* = u^*$. Taking $a = v^*$ and $b = w$ in (ϕ_1), one can write

$$\phi_1\left(G_{\mathbf{H}}(Lv^*, Lv^*, Kw), G_{\mathbf{H}}(Mv^*, Mv^*, Nw), G_{\mathbf{H}}(Mv^*, Lv^*, Lv^*), G_{\mathbf{H}}(Nw, Kw, Kw),\right.$$
$$\left. G_{\mathbf{H}}(Mv^*, Kw, Kw), G_{\mathbf{H}}(Nw, Lv^*, Lv^*)\right) \precsim 0_{\mathbf{H}}.$$

This tends to

$$\phi_1\left(G_{\mathbf{H}}(u^*, u^*, u), G_{\mathbf{H}}(u^*, u^*, u), G_{\mathbf{H}}(u^*, u^*, u^*), G_{\mathbf{H}}(u, u, u), G_{\mathbf{H}}(u^*, u, u), G_{\mathbf{H}}(u, u^*, u^*)\right) \precsim 0_{\mathbf{H}},$$

that is,

$$\phi_1\left(G_{\mathbf{H}}(u^*, u^*, u), G_{\mathbf{H}}(u^*, u^*, u), 0_{\mathbf{H}}, 0_{\mathbf{H}}, G_{\mathbf{H}}(u, u, u^*), G_{\mathbf{H}}(u^*, u^*, u)\right) \precsim 0_{\mathbf{H}}.$$

From (F_3), we get

$$|G_{\mathbf{H}}(u^*, u^*, u)| \leq q_1 \left|G_{\mathbf{H}}(u, u, u^*)\right|. \tag{2.8}$$

Similarly, Taking $a = v$ and $b = w^*$ in (ϕ_2), we obtain

$$|G_{\mathbf{H}}(u, u, u^*)| \leq q_1 \left|G_{\mathbf{H}}(u^*, u^*, u)\right|. \tag{2.9}$$

From Eqs (2.8) and (2.9), we get

$$|G_{\mathbf{H}}(u^*, u^*, u)| (1 - q_1^2) \leq 0,$$

which implies that $|G_{\mathbf{H}}(u^*, u^*, u)| = 0$, i.e, $u^* = u$. Consequently, the pairs (K, N) and (L, M) have a unique common point of coincidence.

Using Eqs (2.5), (2.7) and weak compatibility of the pairs (K, N) and (L, M), we obtain that

$$KNw = NKw \ , \ LMv = MLv. \tag{2.10}$$

Then,

$$Ku = Nu \ , \ Lu = Mu, \tag{2.11}$$

This implies that u is a point of coincidence of the pairs (K, N) and (L, M).

Now, we show that u is a common fixed point of K, L, M and N. Taking $a = u$ and $b = v$ in (ϕ_1), we have

$$\phi_1\left(G_{\mathbf{H}}(Lu, Lu, Kv), G_{\mathbf{H}}(Mu, Mu, Nv), G_{\mathbf{H}}(Mu, Lu, Lu), G_{\mathbf{H}}(Nv, Kv, Kv),\right.$$
$$\left. G_{\mathbf{H}}(Mu, Kv, Kv), G_{\mathbf{H}}(Nv, Lu, Lu)\right) \precsim 0.$$

This leads us to

$$\phi_1\left(G_{\mathbf{H}}(Lu, Lu, u), G_{\mathbf{H}}(Lu, Lu, u), G_{\mathbf{H}}(Lu, Lu, Lu), G_{\mathbf{H}}(u, u, u), G_{\mathbf{H}}(Lu, u, u), G_{\mathbf{H}}(u, Lu, Lu)\right) \precsim 0_{\mathbf{H}},$$

that is,

$$\phi_1\left(G_{\mathbf{H}}(Lu, Lu, u), G_{\mathbf{H}}(Lu, Lu, u), 0_{\mathbf{H}}, 0_{\mathbf{H}}, G_{\mathbf{H}}(u, u, Lu), G_{\mathbf{H}}(Lu, Lu, u)\right) \precsim 0_{\mathbf{H}}.$$

From (F_3), we have

$$|G_{\mathbf{H}}(Lu, Lu, u)| \leq q_1 \, |G_{\mathbf{H}}(u, u, Lu)| \, . \tag{2.12}$$

Similarly, taking $a = v$ and $b = u$ in (ϕ_2), we have

$$|G_{\mathbf{H}}(u, u, Lu)| \leq q_1 \, |G_{\mathbf{H}}(Lu, Lu, u)| \, . \tag{2.13}$$

From Eqs (2.12) and (2.13), we obtain

$$|G_{\mathbf{H}}(Lu, Lu, u)| \, (1 - q_1^2) \leq 0,$$

which implies that $G_{\mathbf{H}}(Lu, Lu, u) = 0_{\mathbf{H}}$, i.e., $Tu = u$. Thus, $Lu = Mu = u$. Similarly, we can prove $Ku = Nu = u$. This implies that

$$Ku = Lu = Mu = Nu = u.$$

i.e., u is a common fixed point of K, L, M and N.

The uniqueness of the common fixed point of K, L, M and N is more easy consequence of the uniqueness of the common point of coincidence of the pairs (K, N) and (L, M). Also, the proof is similar in case $u \in N(X)$. This completes the proof.

Remark 8 The conclusion of Theorem 7 remains true if the completeness of $M(X) \cup N(X)$ is replaced by the completeness of one of the subspaces $K(X), L(X), M(X)$ or $N(X)$.

The following theorem is a new version of Theorem 7 under generalized contractive condition.

Theorem 9 Let K, L, M and N be four self-mappings on a complete quaternion valued G-metric space $(X, G_{\mathbf{H}})$ such that $K(X) \subseteq M(X)$ and $L(X) \subseteq N(X)$. Assume that there exists $\phi_1, \phi_2 \in F$ such that for all $a, b \in X$, $a \neq b$,

$$\begin{cases} \phi_1(G_{\mathbf{H}}(La, La, Kb), G_{\mathbf{H}}(Ma, Ma, Nb), G_{\mathbf{H}}(Ma, La, La), G_{\mathbf{H}}(Nb, Kb, Kb), \\ \quad G_{\mathbf{H}}(Ma, Kb, Kb), \dfrac{[G_{\mathbf{H}}(Nb, La, La)]^3 + [G_{\mathbf{H}}(Ma, Ma, Kb)]^3}{[G_{\mathbf{H}}(Nb, La, La)]^2 + G_{\mathbf{H}}(Ma, Ma, Kb)]^2}) \precsim 0_{\mathbf{H}}, \\ \phi_2(G_{\mathbf{H}}(Ka, Ka, Lb), G_{\mathbf{H}}(Na, Na, Mb), G_{\mathbf{H}}(Na, Ka, Ka), G_{\mathbf{H}}(Mb, Lb, Lb), \\ \quad G_{\mathbf{H}}(Na, Lb, Lb), \dfrac{[G_{\mathbf{H}}(Mb, Ka, Ka)]^3 + [G_{\mathbf{H}}(Na, Na, Lb)]^3}{[G_{\mathbf{H}}(Mb, Ka, Ka)]^2 + G_{\mathbf{H}}(Na, Na, Lb)]^2}) \precsim 0. \end{cases} \tag{2.14}$$

If $M(X) \cup N(X)$ is complete subspace of X, then the pairs (K, N) and (L, M) have a unique common point of coincidence. Moreover, if the pairs (K, N) and (L, M) are weakly compatible, then the four mappings have a unique common fixed point.

Proof. Let x_0 be arbitrary points in X. Since $K(X) \subseteq M(X)$ and $L(X) \subseteq N(X)$, then we can define the sequence $\{a_n\}$ in X as (2.2).

Since $\{b_n\} \subseteq M(X) \cup N(X)$. We then show that $\{b_n\}$ is a Cauchy sequence. Putting $a = a_{2n+1}$ and $b = a_{2n}$ in (ϕ_1), we obtain

$$\phi_1(G_{\mathbf{H}}(La_{2n+1}, La_{2n+1}, Ka_{2n}), G_{\mathbf{H}}(Ma_{2n+1}, Ma_{2n+1}, Na_{2n}), G_{\mathbf{H}}(Ma_{2n+1}, La_{2n+1}, La_{2n+1}),$$
$$G_{\mathbf{H}}(Na_{2n}, Ka_{2n}, Ka_{2n}), G_{\mathbf{H}}(Ma_{2n+1}, Ka_{2n}, Ka_{2n}),$$
$$\frac{[G_{\mathbf{H}}(Na_{2n}, La_{2n+1}, La_{2n+1})]^3 + [G_{\mathbf{H}}(Ma_{2n+1}, Ma_{2n+1}, Ka_{2n})]^3}{[G_{\mathbf{H}}(Na_{2n}, La_{2n+1}, La_{2n+1})]^2 + [G_{\mathbf{H}}(Ma_{2n+1}, Ma_{2n+1}, Ka_{2n})]^2}) \precsim 0_{\mathbf{H}}.$$

This leads us to

$$\phi_1(G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n}), G_{\mathbf{H}}(b_{2n+1}, b_{2n+2}, b_{2n+2}), G_{\mathbf{H}}(b_{2n}, b_{2n+1}, b_{2n+1}),$$
$$G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n+1}), \frac{[G_{\mathbf{H}}(b_{2n}, b_{2n+2}, b_{2n+2})]^3 + [G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n+1})]^3}{[G_{\mathbf{H}}(b_{2n}, b_{2n+2}, b_{2n+2})]^2 + [G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n+1})]^2}) \precsim 0_{\mathbf{H}},$$

which implies that

$$\phi_1\left(G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n}), G_{\mathbf{H}}(b_{2n+2}, y_{2n+2}, b_{2n+1}),\right.$$
$$\left.G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n}), 0_{\mathbf{H}}, G_{\mathbf{H}}(b_{2n}, b_{2n+2}, b_{2n+2})\right) \precsim 0.$$

From (F_1), (QG_4) and (QG_5) we have

$$\phi_1\left(G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n}), G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n})\right.$$
$$\left., 0_{\mathbf{H}}, G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}) + G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n})\right) \precsim 0_{\mathbf{H}}.$$

From (F_2), we get

$$|G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1})| \leq q\ |G_{\mathbf{H}}(b_{2n+1}, b_{2n+1}, b_{2n})|\,.$$

Similarly, taking $a = a_{2n+2}$ and $b = a_{2n+1}$ in (ϕ_2), we get successively

$$\phi_2(G_{\mathbf{H}}(Ka_{2n+2}, Ka_{2n+2}, La_{2n+1}), G_{\mathbf{H}}(Na_{2n+2}, Na_{2n+2}, Ma_{2n+1}), G_{\mathbf{H}}(Na_{2n+2}, Ka_{2n+2}, Ka_{2n+2}),$$
$$G_{\mathbf{H}}(Ma_{2n+1}, La_{2n+1}, La_{2n+1}), G_{\mathbf{H}}(Na_{2n+2}, La_{2n+1}, La_{2n+1}),$$
$$\frac{[G_{\mathbf{H}}(Ma_{2n+1}, Ka_{2n+2}, Ka_{2n+2})]^3 + [G_{\mathbf{H}}(Na_{2n+2}, Na_{2n+2}, La_{2n+1})]^3}{[G_{\mathbf{H}}(Ma_{2n+1}, Ka_{2n+2}, Ka_{2n+2})]^2 + [G_{\mathbf{H}}(Na_{2n+2}, Na_{2n+2}, La_{2n+1})]^2}) \precsim 0_{\mathbf{H}}.$$

This implies that

$$\phi_2(G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2}), G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+2}, b_{2n+3}, b_{2n+3}), G_{\mathbf{H}}(b_{2n+1}, b_{2n+2}, b_{2n+2}),$$
$$G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+2}), \frac{[G_{\mathbf{H}}(b_{2n+1}, b_{2n+3}, b_{2n+3})]^3 + [G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+2})]^3}{[G_{\mathbf{H}}(b_{2n+1}, b_{2n+3}, b_{2n+3})]^2 + [G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+2})]^2}) \precsim 0_{\mathbf{H}},$$

that is,

$$\phi_2\left(G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2}), G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2}),\right.$$
$$\left.G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), 0_{\mathbf{H}}, G_{\mathbf{H}}(b_{2n+1}, b_{2n+2}, b_{2n+2}) + G_{\mathbf{H}}(b_{2n+2}, b_{2n+3}, b_{2n+3})\right) \precsim 0_{\mathbf{H}}.$$

From (F_1), (QG_4) and (QG_5), one can write

$$\phi_2\left(G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2}), G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2}),\right.$$
$$\left.G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}), 0_{\mathbf{H}}, G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1}) + G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2})\right) \precsim 0_{\mathbf{H}}.$$

From (F_2), we obtain

$$|G_{\mathbf{H}}(b_{2n+3}, b_{2n+3}, b_{2n+2})| \leq q \; |G_{\mathbf{H}}(b_{2n+2}, b_{2n+2}, b_{2n+1})| \,.$$

By a similar way, (step by step) of the proof of Theorem 7, one can complete the proof.

Example 10 Let $X = \{\frac{1}{n} \; : \; n \in \mathbb{N}\}$ and $G_{\mathbf{H}} : X \times X \times X \longrightarrow \mathbf{H}$ as defined in Example 4. Let $K, L, M, N : X^3 \longrightarrow X^3$ be defined by

$$Kx = \frac{1}{2}\,x, \quad Lx = \frac{1}{4}\,x, \quad Mx = x, \quad Nx = \frac{1}{2}\,x.$$

for all $x = (x_1, x_2, x_3) \in X^3$.

It is easy to see that $K(X) \subseteq M(X)$ and $L(X) \subseteq N(X)$. So there exist $\phi_1, \phi_2 : \mathbf{H}^6 \to \mathbf{H}$ such that

$$\phi_1(a, b, c, d, e, f) = (e + f) - (a + b + c + d) = \phi_2(a, b, c, d, e, f),$$

for all $a, b, c, d, e, f \in \mathbf{H}$. We see that $\phi_1, \phi_2 \in F$ and satisfy the following conditions (2.1) and (2.14). Moreover, We see that $(0, 0, 0)$ is unique common coincidence point of (K, N) and (L, M), and also unique common fixed point of mappings K, L, M and N. So, Theorem 7 and Theorem 9 are supported by this example.

3. Conclusions

In this work, we prove existence theorems of unique common points for the pairs (K, N) and (L, M), and unique common fixed points for four mappings K, N, L, M are presented in Theorem 7 and Theorem 9. The Example 10 is shown for supporting our main theorems (Theorem 7 and Theorem 9).

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Conflict of interest

The authors declare no conflict of interest.

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