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Research article

On the entire solutions for several partial differential difference equations (systems) of Fermat type in $\mathbb{C}^2$

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Abstract: By utilizing the Nevanlinna theory of meromorphic functions in several complex variables, we will establish some theorems about the existence and the forms of entire solutions for several partial differential difference equations (systems) of Fermat type with two complex variables such as

$$f(z)^2 + \left[f(z+c) + \frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_2}\right]^2 = 1$$

and

$$\begin{cases} f_1(z)^2 + \left[f_2(z+c) + \dfrac{\partial f_1}{\partial z_1} + \dfrac{\partial f_1}{\partial z_2}\right]^2 = 1, \\ f_2(z)^2 + \left[f_1(z+c) + \dfrac{\partial f_2}{\partial z_1} + \dfrac{\partial f_2}{\partial z_2}\right]^2 = 1, \end{cases}$$

which are some extensions and generalizations of the previous theorems given by Xu and Cao [29,30], Xu, Liu and Li [28], and Liu, Yang [18–20]. Moreover, we give some examples to explain that our results are precise to some extent.

Keywords: Nevanlinna theory; entire function; partial differential difference equation system
Mathematics Subject Classification: 30D35, 35M30, 32W50, 39A45

1. Introduction

We first assume that the readers are familiar with the notations of the Nevanlinna theory such as $T(r, f), m(r, f), N(r, f)$ and so on, which can be found, for instant, in Hayman [12], Yang [31], Yi and Yang [32]. As is known to all, Nevanlinna theory is a powerful tool in analysing the properties of meromorphic functions including meromorphic functions in arbitrary plane regions, algebroid functions, functions of several variables, holomorphic curves, uniqueness theory of meromorphic function and complex differential equations, and so on. Moreover, it is also used in lots of areas of mathematics including potential theory, measure theory, differential geometry, topology and others.

Around 2006, Chiang and Feng [6], Halburd and Korhonen [8] established independently the difference analogues of Nevanlinna theory in $\mathbb{C}$, respectively. In the past two decades, many scholars paid a lot of attention to a large number of interesting topics on complex difference of meromorphic functions by making use of the difference analogue of Nevanlinna theory, and complex difference results are in the rapid development.

In all the previous articles, Korhonen, Halburd, Chen and his students investigated the properties of solutions for a series of linear difference equations, the difference Painlevé equations, the Riccati equations, the Pielou logistic equations, and obtained the conditions of the existence and estimations of growth order for the above difference equations [1,4,5,9,27,35]; Laine, Korhonen, Yang, Chen, etc. established the difference analogue of Clunie Lemma and Mohonko Lemma, and gave the existence theorems on solutions for some complex difference equations including difference Malmquist type and some nonlinear difference equations [16, 17, 21]; Korhonen, Zhang, Liu, Yang paid close attention to some complex differential difference equations [11, 18–20, 24, 34], in special, Liu and his co-authors [18–20] investigated some types of complex differential difference equations of Fermat type such as

$$f'(z)^2 + f(z+c)^2 = 1, \tag{1.1}$$

$$f'(z)^2 + [f(z+c) - f(z)]^2 = 1. \tag{1.2}$$

They proved that the transcendental entire solutions with finite order of equation (1.1) must satisfy $f(z) = \sin(z \pm Bi)$, where B is a constant and $c = 2k\pi$ or $c = (2k+1)\pi$, k is an integer, and the transcendental entire solutions with finite order of equation (1.2) must satisfy $f(z) = 12\sin(2z + Bi)$, where $c = (2k+1)\pi$, k is an integer, and B is a constant. In 2016, Gao [7] discussed the solutions for the system of complex differential-difference equations

$$\begin{cases} [f_1'(z)]^2 + f_2(z+c)^2 = 1, \\ [f_2'(z)]^2 + f_1(z+c)^2 = 1, \end{cases} \tag{1.3}$$

and extend the results given by Liu and Cao.

Theorem 1.1. (see [7, Theorem 1.1]). *Suppose that (f_1, f_2) is a pair of transcendental entire solutions with finite order for the system of differential-difference equations* (1.3). *Then (f_1, f_2) satisfies*

$$(f_1(z), f_2(z)) = (\sin(z - bi), \sin(z - b_1 i)),$$

or

$$(f_1(z), f_2(z)) = (\sin(z + bi), \sin(z + b_1 i)),$$

where b, b_1 are constants, and $c = k\pi$, k is a integer.

In 2012, Korhonen [14] firstly established the difference version of logarithmic derivative lemma (shortly, we may say logarithmic difference lemma) for meromorphic functions on $\mathbb{C}^m$ with hyper order strictly less than $\frac{2}{3}$, and then used it to consider a class of partial difference equations in the same paper. Later, Cao and Korhonen [2] improved the logarithmic difference lemma to the case where the hyper order is strictly less than one. In 2019, Cao and Xu [3] further improved the logarithmic difference lemma in several variables under the condition of minimal type $\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0$. As far as we know, however, there are very little of results on solutions of complex partial difference equations by using Nevanlinna theory. By making use of the Nevanlinna theory with several complex variables (see [2, 3, 14]), Xu and Cao [3, 29, 30], Lu and Li [22] investigated the existence of the solutions for some Fermat type partial differential-difference equations with several variables, and obtained the following theorem.

Theorem 1.2. (see [29, Theorem 1.2]). *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then any transcendental entire solutions with finite order of the partial differential-difference equation*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1$$

has the form of $f(z_1, z_2) = \sin(Az_1 + B)$, where A is a constant on $\mathbb{C}$ satisfying $Ae^{iAc_1} = 1$, and B is a constant on $\mathbb{C}$; in the special case whenever $c_1 = 0$, we have $f(z_1, z_2) = \sin(z_1 + B)$.

In 2020, Xu, Liu and Li [28] further explored the existence and forms of several systems of complex partial differential difference equations and extend the previous results given by [29, 30].

Theorem 1.3. (see [28, Theorem 1.3]). *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then any pair of transcendental entire solutions with finite order for the system of Fermat type partial differential-difference equations*

$$\begin{cases} \left(\frac{\partial f_1(z_1, z_2)}{\partial z_1}\right)^2 + f_2(z_1 + c_1, z_2 + c_2)^2 = 1, \\ \left(\frac{\partial f_2(z_1, z_2)}{\partial z_1}\right)^2 + f_1(z_1 + c_1, z_2 + c_2)^2 = 1 \end{cases} \tag{1.4}$$

has the following form

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+B_1} + e^{-(L(z)+B_1)}}{2}, \frac{A_{21}e^{L(z)+B_1} + A_{22}e^{-(L(z)+B_1)}}{2}\right),$$

where $L(z) = a_1z_1 + a_2z_2$, B_1 is a constant in $\mathbb{C}$, and a_1, c, A_{21}, A_{22} satisfy one of the following cases

(i) $A_{21} = -i$, $A_{22} = i$, and $a_1 = i$, $L(c) = (2k + \frac{1}{2})\pi i$, or $a_1 = -i$, $L(c) = (2k - \frac{1}{2})\pi i$;

(ii) $A_{21} = i$, $A_{22} = -i$, and $a_1 = i$, $L(c) = (2k - \frac{1}{2})\pi i$, or $a_1 = -i$, $L(c) = (2k + \frac{1}{2})\pi i$;

(iii) $A_{21} = 1$, $A_{22} = 1$, and $a_1 = i$, $L(c) = 2k\pi i$, or $a_1 = -i$, $L(c) = (2k + 1)\pi i$;

(iv) $A_{21} = -1$, $A_{22} = -1$, and $a_1 = i$, $L(c) = (2k + 1)\pi i$, or $a_1 = -i$, $L(c) = 2k\pi i$.

The above results suggest the following questions.

Question 1.1. *What will happen about the solutions when $f(z)$, $f(z+c)$, $\frac{\partial f}{\partial z_1}$ and $\frac{\partial f}{\partial z_2}$ are included in the equations in Theorem B?*

Question 1.2. *What will happen about the solutions when the equations are turned into system under the hypothesis of Question 1.1?*

2. Results

In view of the above questions, this article is concerned with the description of the existence and the forms of entire solutions for several partial differential difference of Fermat type. The main tools are used in this paper are the Nevanlinna theory and difference Nevanlinna theory with several complex variables. Our main results are obtained generalized the previous theorems given by Xu and Cao, Liu, Cao and Cao [20,29]. Throughout this paper, for convenience, we assume that $z+w = (z_1+w_1, z_2+w_2)$ for any $z = (z_1, z_2), w = (w_1, w_2)$. The main results of this paper are stated below.

Theorem 2.1. *Let $c = (c_1, c_2) \in \mathbb{C}^2 - \{(0, 0)\}$. Then any transcendental entire solution $f(z_1, z_2)$ with finite order of the partial differential equation*

$$f(z)^2 + \left[f(z+c) + \frac{\partial f}{\partial z_1}\right]^2 = 1 \tag{2.1}$$

must be of the form

$$f(z_1, z_2) = \frac{e^{L(z)+b} + e^{-L(z)-b}}{2},$$

where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$, and $L(c) := \alpha_1 c_1 + \alpha_2 c_2$, α_1, $\alpha_2, b \in \mathbb{C}$ satisfy

(i) $L(c) = 2k\pi i + \frac{\pi}{2}i$ and $\alpha_1 = -2i$;

(ii) $L(c) = 2k\pi i + \frac{3\pi}{2}i$ and $\alpha_1 = 0$.

The following examples show that the forms of transcendental entire solutions with finite order for equation (2.1) are precise.

Example 2.1. *Let $\alpha_1 = -2i$, $\alpha_2 = \frac{3\pi}{2}$ and $b \in \mathbb{C}$, that is,*

$$f(z_1, z_2) = \frac{e^{2z_2+b} + e^{-2z_2-b}}{2}.$$

Then $f(z_1, z_2)$ satisfies equation (2.1) with $(c_1, c_2) = (c_1, \frac{3\pi i}{4})$, where $c_1 \in \mathbb{C}$.

Example 2.2. *Let $\alpha_1 = 0$, $\alpha_2 = 2$ and $b = 0$, that is,*

$$f(z_1, z_2) = \frac{e^{-2iz_1+\frac{3\pi}{2}} + e^{2iz_1-\frac{3\pi}{2}}}{2}.$$

Then $f(z_1, z_2)$ satisfies equation (2.1) with $(c_1, c_2) = (\frac{\pi}{2}, i)$.

If equation (2.1) contains both two partial differentials and difference, we obtain

Theorem 2.2. *Let $c = (c_1, c_2) \in \mathbb{C}^2 - \{(0,0)\}$ and $s = c_2z_1 - c_1z_2$. Then any transcendental entire solution $f(z_1, z_2)$ with finite order of the partial differential equation*

$$f(z)^2 + \left[f(z+c) + \frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_2}\right]^2 = 1 \tag{2.2}$$

must be of the form

$$f(z_1, z_2) = \frac{e^{L(z)+H(s)+b} + e^{-L(z)-H(s)-b}}{2},$$

where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$, $H(s)$ is a polynomial in s, and $L(c)$, α_1, $\alpha_2, b \in \mathbb{C}$ satisfy

(i) $c_1 = c_2 = k\pi + \frac{3}{4}\pi$, $\alpha_1 + \alpha_2 = -2i$ and $\deg_s H \geq 2$;

(ii) $c_1 \neq c_2$, $H(s) \equiv 0$, and $L(c) = 2k\pi i + \frac{3\pi}{2}i$, $\alpha_1 + \alpha_2 = 0$ or $L(c) = 2k\pi i + \frac{\pi}{2}i$, $\alpha_1 + \alpha_2 = -2i$.

We list the following examples to show that the forms of transcendental entire solutions with finite order for equation (2.2) are precise.

Example 2.3. *Let $L(z) = -3iz_1 + iz_2$, $H(s) = \frac{3\pi i}{4}(z_1 - z_2)^n$, $n \in \mathbb{N}_+$ and $b = 0$, that is,*

$$f(z_1, z_2) = \frac{e^{L(z)+H(s)} + e^{-L(z)-H(s)}}{2} = \cos[-3z_1 + z_2 + \frac{3\pi}{4}(z_1 - z_2)^n].$$

Then $f(z_1, z_2)$ satisfies equation (2.2) with $(c_1, c_2) = (\frac{3\pi}{4}, \frac{3\pi}{4})$.

Example 2.4. *Let $L(z) = -3iz_1 + iz_2$, $H(s) \equiv 0$, and $b = 0$, that is,*

$$f(z_1, z_2) = \frac{e^{L(z)} + e^{-L(z)}}{2} = \cos(-3z_1 + z_2).$$

Then $f(z_1, z_2)$ satisfies equation (2.2) with $(c_1, c_2) = (-\frac{\pi}{2}, \pi)$.

Example 2.5. *Let $L(z) = iz_1 - iz_2$, and $b = 0$, that is, $\alpha_1 + \alpha_2 = 0$ and*

$$f(z_1, z_2) = \frac{e^{L(z)} + e^{-L(z)}}{2} = \cos(z_1 - z_2).$$

Then $f(z_1, z_2)$ satisfies equation (2.2) with $(c_1, c_2) = (2\pi, \frac{\pi}{2})$.

When equation (2.1) is turned to system of functional equations, we have

Theorem 2.3. *Let $c = (c_1, c_2) \in \mathbb{C}^2 - \{(0,0)\}$. Then any pair of finite order transcendental entire solution (f_1, f_2) for the system of the partial differential equations*

$$\begin{cases} f_1(z)^2 + \left[f_2(z+c) + \dfrac{\partial f_1}{\partial z_1}\right]^2 = 1, \\ f_2(z)^2 + \left[f_1(z+c) + \dfrac{\partial f_2}{\partial z_1}\right]^2 = 1 \end{cases} \tag{2.3}$$

must be of the form

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+b} + e^{-L(z)-b}}{2}, \frac{A_1 e^{L(z)+b} + A_2 e^{-L(z)-b}}{2}\right),$$

where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$*, and* $\alpha_1, \alpha_2, b, A_1, A_2 \in \mathbb{C}$ *satisfy one of the following cases:*

(i) if $\alpha_1 = 0$*, then* $\alpha_2 c_2 = 2k\pi i + \frac{\pi}{2} i$*,* $A_1 = A_2 = -1$ *or* $\alpha_2 c_2 = 2k\pi i + \frac{3\pi}{2} i$*,* $A_1 = A_2 = 1$ *or* $\alpha_2 c_2 = 2k\pi i$*,* $A_1 = -i, A_2 = i$ *or* $\alpha_2 c_2 = (2k+1)\pi i$*,* $A_1 = i, A_2 = -i$*,* $k \in \mathbb{Z}$*;*

(ii) if $\alpha_1 = -2i$*, then* $\alpha_1 c_1 + \alpha_2 c_2 = 2k\pi i + \frac{\pi}{2} i$*,* $A_1 = A_2 = 1$ *or* $\alpha_1 c_1 + \alpha_2 c_2 = 2k\pi i + \frac{3\pi}{2} i$*,* $A_1 = A_2 = -1$*,* $k \in \mathbb{Z}$*.*

Some examples show the existence of transcendental entire solutions with finite order for system (2.3).

Example 2.6. *Let* $L(z) = iz_2$ *and* $b = 0$*, that is,* $\alpha_1 = 0$*,* $\alpha_2 = i$ *and*

$$(f_1, f_2) = \left(\frac{e^{iz_2} + e^{-iz_2}}{2}, -\frac{e^{iz_2} + e^{-iz_2}}{2} \right) = (\cos z_2, -\cos z_2).$$

Then (f_1, f_2) *satisfies system* (2.3) *with* $(c_1, c_2) = (c_1, \frac{\pi}{2})$*, where* $c_1 \in \mathbb{C}$*.*

Example 2.7. *Let* $L(z) = iz_2$ *and* $b = 0$*, that is,* $\alpha_1 = 0$*,* $\alpha_2 = i$ *and*

$$(f_1, f_2) = \left(\frac{e^{iz_2} + e^{-iz_2}}{2}, \frac{-ie^{iz_2} + ie^{-iz_2}}{2} \right) = (\cos z_2, \sin z_2).$$

Then (f_1, f_2) *satisfies system* (2.3) *with* $(c_1, c_2) = (c_1, 2\pi)$*, where* $c_1 \in \mathbb{C}$*.*

Example 2.8. *Let* $L(z) = -2iz_1 + iz_2$ *and* $b = 0$*, that is,* $\alpha_1 = -2i$*,* $\alpha_2 = i$ *and*

$$\begin{aligned}(f_1, f_2) &= \left(\frac{e^{L(z)} + e^{-L(z)}}{2}, -\frac{e^{L(z)} + e^{-L(z)}}{2} \right) \\ &= (\cos(-2z_1 + z_2), -\cos(-2z_1 + z_2)).\end{aligned}$$

Then (f_1, f_2) *satisfies system* (2.3) *with* $(c_1, c_2) = (-\frac{\pi}{2}, \frac{\pi}{2})$*.*

Similar to the argument as in the proof of Theorem 2.3, we can get the following result easily.

Theorem 2.4. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$ *and* $c_1 \neq c_2$*. Then any pair of transcendental entire solution* (f_1, f_2) *with finite order for the system of the partial differential equations*

$$\begin{cases} f_1(z)^2 + \left[f_2(z+c) + \dfrac{\partial f_1}{\partial z_1} + \dfrac{\partial f_1}{\partial z_2} \right]^2 = 1, \\ f_2(z)^2 + \left[f_1(z+c) + \dfrac{\partial f_2}{\partial z_1} + \dfrac{\partial f_2}{\partial z_2} \right]^2 = 1 \end{cases} \tag{2.4}$$

must be of the form

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+b} + e^{-L(z)-b}}{2}, \frac{A_1 e^{L(z)+b} + A_2 e^{-L(z)-b}}{2} \right),$$

where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$*, and* $\alpha_1, \alpha_2, b, A_1, A_2 \in \mathbb{C}$ *satisfy one of the following cases:*

(i) if $\alpha_1 + \alpha_2 = 0$*, then* $\alpha_1 c_1 + \alpha_2 c_2 = 2k\pi i + \frac{\pi}{2} i$*,* $A_1 = A_2 = -1$ *or* $\alpha_1 c_1 + \alpha_2 c_2 = 2k\pi i + \frac{3\pi}{2} i$*,* $A_1 = A_2 = 1$ *or* $\alpha_1 c_1 + \alpha_2 c_2 = 2k\pi i$*,* $A_1 = -i, A_2 = i$ *or* $\alpha_1 c_1 + \alpha_2 c_2 = (2k+1)\pi i$*,* $A_1 = i, A_2 = -i$*,* $k \in \mathbb{Z}$*;*

(ii) if $\alpha_1 + \alpha_2 = -2i$*, then* $\alpha_1 c_1 + \alpha_2 c_2 = 2k\pi i + \frac{\pi}{2} i$*,* $A_1 = A_2 = 1$ *or* $\alpha_1 c_1 + \alpha_2 c_2 = 2k\pi i + \frac{3\pi}{2} i$*,* $A_1 = A_2 = -1$*,* $k \in \mathbb{Z}$*.*

The following example shows that the condition $c_1 \neq c_2$ in Theorem 2.4 can not be removed.

Example 2.9. *Let* $c_1 \neq 0$ *and*

$$(f_1, f_2) = \left(\frac{e^{g(z_1,z_2)} + e^{-g(z_1,z_2)}}{2}, \frac{-ie^{g(z_1,z_2)} + ie^{-g(z_1,z_2)}}{2} \right),$$

where $g(z_1, z_2) = z_1 - z_2 + \beta(z_1 - z_2)^n$, $\beta \neq 0$ *and* $n \in \mathbb{N}_+$. *Then* (f_1, f_2) *is a pair of transcendental entire solutions with finite order for system* (2.4) *with* $c = (c_1, c_1)$.

3. Proofs of Theorems 2.1 and 2.2

The following lemmas play the key roles in proving our results.

Lemma 3.1. *([25, 26]). For an entire function* F *on* $\mathbb{C}^n$, $F(0) \neq 0$ *and put* $\rho(n_F) = \rho < \infty$. *Then there exist a canonical function* f_F *and a function* $g_F \in \mathbb{C}^n$ *such that* $F(z) = f_F(z)e^{g_F(z)}$. *For the special case* $n = 1$, f_F *is the canonical product of Weierstrass.*

Remark 3.1. *Here, denote* $\rho(n_F)$ *to be the order of the counting function of zeros of* F.

Lemma 3.2. *([23]). If* g *and* h *are entire functions on the complex plane* $\mathbb{C}$ *and* $g(h)$ *is an entire function of finite order, then there are only two possible cases: either*

(a) the internal function h *is a polynomial and the external function* g *is of finite order; or else*

(b) the internal function h *is not a polynomial but a function of finite order, and the external function* g *is of zero order.*

Lemma 3.3. *([13, Lemma 3.1]). Let* $f_j(\not\equiv 0)$, $j = 1, 2, 3$, *be meromorphic functions on* $\mathbb{C}^m$ *such that* f_1 *is not constant, and* $f_1 + f_2 + f_3 = 1$, *and such that*

$$\sum_{j=1}^{3} \left\{ N_2(r, \frac{1}{f_j}) + 2\overline{N}(r, f_j) \right\} < \lambda T(r, f_1) + O(\log^+ T(r, f_1)),$$

for all r *outside possibly a set with finite logarithmic measure, where* $\lambda < 1$ *is a positive number. Then either* $f_2 = 1$ *or* $f_3 = 1$, *where* $N_2(r, \frac{1}{f})$ *is the counting function of the zeros of* f *in* $|z| \leq r$, *where the simple zero is counted once, and the multiple zero is counted twice.*

3.1. The Proof of Theorem 2.1

Suppose that $f(z)$ is a transcendental entire solution with finite order of equation (2.1). Thus, we rewrite (2.1) as the following form

$$\left[f(z) + i\left(f(z + c) + \frac{\partial f}{\partial z_1} \right) \right] \left[f(z) - i\left(f(z + c) + \frac{\partial f}{\partial z_1} \right) \right] = 1, \tag{3.1}$$

which implies that both $f(z) + i\left(f(z + c) + \frac{\partial f}{\partial z_1} \right)$ and $f(z) - i\left(f(z + c) + \frac{\partial f}{\partial z_1} \right)$ have no poles and zeros. Thus, by Lemmas 3.1 and 3.2, there thus exists a polynomial $p(z)$ such that

$$f(z) + i\left(f(z + c) + \frac{\partial f}{\partial z_1} \right) = e^{p(z)},$$

$$f(z) - i\left(f(z+c) + \frac{\partial f}{\partial z_1}\right) = e^{-p(z)}.$$

This leads to

$$f(z) = \frac{e^{p(z)} + e^{-p(z)}}{2}, \tag{3.2}$$

$$f(z+c) + \frac{\partial f(z)}{\partial z_1} = \frac{e^{p(z)} - e^{-p(z)}}{2i}. \tag{3.3}$$

Substituting (3.2) into (3.3), it yields that

$$-\left(\frac{\partial p}{\partial z_1} + i\right)e^{p(z+c)+p(z)} + \left(\frac{\partial p}{\partial z_1} + i\right)e^{p(z+c)-p(z)} - e^{2p(z+c)} = 1. \tag{3.4}$$

Obviously, $\frac{\partial p}{\partial z_1} + i \not\equiv 0$. Otherwise, $e^{2p(z+c)} \equiv -1$, thus $p(z)$ is a constant. Then $f(z)$ is a constant, this a contradiction. By Lemma 3.3, it follows from (3.4) that

$$\left(\frac{\partial p}{\partial z_1} + i\right)e^{p(z+c)-p(z)} = 1. \tag{3.5}$$

Thus, in view of (3.4) and (3.5), we conclude that

$$-\left(\frac{\partial p}{\partial z_1} + i\right)e^{p(z)-p(z+c)} = 1. \tag{3.6}$$

Equations (3.5) and (3.6) mean that $e^{2[p(z+c)-p(z)]} = -1$. Thus, it yields that $p(z) = L(z) + H(s) + b$, where L is a linear function as the form $L(z) = \alpha_1 z_1 + \alpha_2 z_2$, $e^{2L(c)} = -1$, and $H(s)$ is a polynomial in $s := c_2 z_1 - c_1 z_2, \alpha_1, \alpha_2, b \in \mathbb{C}$.

On the other hand, it follows from (3.5) and (3.6) that $\left(\frac{\partial p}{\partial z_1} + i\right)^2 = -1$, that is, $\frac{\partial p}{\partial z_1} = 0$ or $\frac{\partial p}{\partial z_1} = -2i$. In view of (3.5) or (3.6), it yields that $e^{L(c)} = i$ and $\alpha_1 = -2i$ or $e^{L(c)} = -i$ and $\alpha_1 = 0$, that is, $L(c) = 2k\pi i + \frac{\pi}{2}i$ and $\alpha_1 = -2i$ or $L(c) = 2k\pi i + \frac{3\pi}{2}i$ and $\alpha_1 = 0$. Moreover, by combining with $p(z) = L(z) + H(s) + b$, we have that $H'c_2 = \eta$, where η is a constant. If $c_2 = 0$, then $H(s) = H(-c_1 z_2)$, and combining with $e^{2(p(z+c)-p(z))} = -1$, then it follows that $\deg_s H \leq 1$. If $c_2 \neq 0$, it is easy to get that $H(s)$ is a polynomial in s with $\deg_s H \leq 1$. Hence, we can conclude that $L(z)+H(s)+b$ is a linear form in z_1, z_2, w.l.o.g, we denote that the form is $\alpha_1 z_1 + \alpha_2 z_2 + b$, where $\alpha_1, \alpha_2, b \in \mathbb{C}$. Thus, by combining with (3.5) and (3.6), we have

$$f(z_1, z_2) = \frac{e^{L(z)+b} + e^{-L(z)-b}}{2},$$

where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$, and $L(c)$, α_1, $\alpha_2, b \in \mathbb{C}$ satisfy $L(c) = 2k\pi i + \frac{\pi}{2}i$ and $\alpha_1 = -2i$ or $L(c) = 2k\pi i + \frac{3\pi}{2}i$ and $\alpha_1 = 0$, $k \in \mathbb{Z}$.

Therefore, this completes the proof of Theorem 2.1.

3.2. The Proof of Theorem 2.2

Suppose that $f(z)$ is a transcendental entire solution with finite order of equation (2.2). Similar to the argument as in the proof of Theorem 2.1, there exists a nonconstant polynomial $p(z)$ such that (3.2) and

$$\left(\frac{\partial p}{\partial z_1} + \frac{\partial p}{\partial z_2} + i\right)e^{p(z+c)-p(z)} = 1, \tag{3.7}$$

$$-\left(\frac{\partial p}{\partial z_1}+\frac{\partial p}{\partial z_2}+i\right)e^{p(z)-p(z+c)}=1. \tag{3.8}$$

In view of (3.7) and (3.8), we have $e^{2(p(z+c)-p(z))}=-1$. Thus, it yields that $p(z)=L(z)+H(s)+b$, where L is a linear function as the form $L(z)=\alpha_1 z_1+\alpha_2 z_2$, $H(s)$ is a polynomial in s, and $\alpha_1,\alpha_2,b\in\mathbb{C}$, and $e^{2L(c)}=-1$.

On the other hand, from (3.7) and (3.8), we have $\left(\frac{\partial p}{\partial z_1}+\frac{\partial p}{\partial z_2}+i\right)^2=-1$. Noting that $p(z)=L(z)+H(s)+b$, it follow that

$$[\alpha_1+\alpha_2+H'(c_2-c_1)+i]^2=-1. \tag{3.9}$$

If $c_1=c_2$, then $H'(c_2-c_1)\equiv 0$. This shows that $H(s)$ can be any degree polynomial in s. In fact, we can assume that $\deg_s H\geq 2$. If $\deg_s H\leq 1$, then $L(z)+H(s)+b$ is still a linear function of the form $a_1z_1+a_2z_2$, this means that $H(s)\equiv 0$. In view of (3.9), it yields that $\alpha_1+\alpha_2=0$ or $\alpha_1+\alpha_2=-2i$. If $\alpha_1+\alpha_2=0$, that is, $\alpha_1=-\alpha_2$, combining with $c_1=c_2$, then it leads to $L(c)=\alpha_1c_1-\alpha_1c_1=0$, that is, $e^{2L(c)}=1$, this is a contradiction with $e^{2L(c)}=-1$. If $\alpha_1+\alpha_2=-2i$, then $L(c)=(\alpha_1+\alpha_2)c_1=-2ic_1$. By combining with $e^{L(c)}=-i$, we have $c_1=c_2=k\pi+\frac{3}{4}\pi$.

If $c_1\neq c_2$, then H' must be a constant, that is, $\deg_s H\leq 1$. Hence, we can conclude that $L(z)+H(s)+b$ is a linear form in z_1,z_2. Thus, this means that $H(s)\equiv 0$ and $H'(c_2-c_1)\equiv 0$. Hence, we have $p(z)=\alpha_1z_1+\alpha_2z_2+b$, where $\alpha_1,\alpha_2,b\in\mathbb{C}$. So, in view of (3.9), we also have $\alpha_1+\alpha_2=0$ or $\alpha_1+\alpha_2=-2i$. Further, In view of (3.7) or (3.8), we conclude that $L(c)=2k\pi i+\frac{\pi}{2}i$ and $\alpha_1+\alpha_2=-2i$ or $L(c)=2k\pi i+\frac{3\pi}{2}i$ and $\alpha_1+\alpha_2=0$, $k\in\mathbb{Z}$.

Therefore, this completes the proof of Theorem 2.2.

4. The Proof of Theorem 2.3

Suppose that $(f_1(z),f_2(z))$ is a pair of transcendental entire solutions with finite order of (2.3). By using the same argument as in the proof of Theorem 2.1, there exist two polynomials $p(z),q(z)$ in $\mathbb{C}^2$ such that

$$\begin{cases} f_1(z)=\dfrac{e^{p(z)}+e^{-p(z)}}{2},\\ f_2(z+c)+\dfrac{\partial f_1}{\partial z_1}=\dfrac{e^{p(z)}-e^{-p(z)}}{2i},\\ f_2(z)=\dfrac{e^{q(z)}+e^{-q(z)}}{2},\\ f_1(z+c)+\dfrac{\partial f_2}{\partial z_1}=\dfrac{e^{q(z)}-e^{-q(z)}}{2i}. \end{cases} \tag{4.1}$$

In view of (4.1), it follows that

$$-(i+\frac{\partial p}{\partial z_1})e^{p(z)+q(z+c)}+(i+\frac{\partial p}{\partial z_1})e^{-p(z)+q(z+c)}-e^{2q(z+c)}\equiv 1, \tag{4.2}$$

and

$$-(i+\frac{\partial q}{\partial z_1})e^{q(z)+p(z+c)}+(i+\frac{\partial q}{\partial z_1})e^{-q(z)+p(z+c)}-e^{2p(z+c)}\equiv 1. \tag{4.3}$$

Thus, by applying Lemma 3.3, we can deduce from (4.2) and (4.3) that

$$(i+\frac{\partial p}{\partial z_1})e^{-p(z)+q(z+c)}=1,\quad or\quad -(i+\frac{\partial p}{\partial z_1})e^{p(z)+q(z+c)}=1$$

and

$$(i+\frac{\partial q}{\partial z_1})e^{-q(z)+p(z+c)}=1,\quad or\quad -(i+\frac{\partial q}{\partial z_1})e^{q(z)+p(z+c)}=1.$$

Now, we will discuss four cases below.

Case 1.

$$\begin{cases}(i+\frac{\partial p}{\partial z_1})e^{-p(z)+q(z+c)}=1,\\ (i+\frac{\partial q}{\partial z_1})e^{-q(z)+p(z+c)}=1.\end{cases}\tag{4.4}$$

Thus, it follows that $-p(z)+q(z+c)=\eta_1$ and $-q(z)+p(z+c)=\eta_2$, that is, $p(z+2c)-p(z)=\eta_2+\eta_1$ and $q(z+2c)-q(z)=\eta_2+\eta_1$, here and below η_1,η_2 are constants in $\mathbb{C}$, which each occurrence can be inconsistent. So, we can deduce that $p(z)=L(z)+H(s)+B_1$ and $q(z)=L(z)+H(s)+B_2$, where $L(z)$ is a linear function of the form $L(z)=\alpha_1z_1+\alpha_2z_2$, $H(s)$ is a polynomial in $s:=c_1z_2-c_2z_1$, $\alpha_1,\alpha_2,B_1,B_2$ are constants in $\mathbb{C}$. Noting that $(i+\frac{\partial p}{\partial z_1})$ and $(i+\frac{\partial q}{\partial z_1})$ are constants, it follows that $p(z)=L(z)+B_1$ and $q(z)=L(z)+B_2$. Thus, by combining with (4.2)-(4.4), it yields that

$$\begin{cases}(i+\alpha_1)e^{L(c)+B_2-B_1}=1,\\ (i+\alpha_1)e^{L(c)+B_1-B_2}=1,\\ -(i+\alpha_1)e^{-L(c)+B_1-B_2}=1,\\ -(i+\alpha_1)e^{-L(c)+B_2-B_1}=1,\end{cases}\tag{4.5}$$

where $L(c):=\alpha_1c_1+\alpha_2c_2$, this thus leads to

$$(i+\alpha_1)^2=-1,\quad e^{2L(c)}=-1,\quad e^{2(B_1-B_2)}=1.\tag{4.6}$$

Then we have $\alpha_1=0$ or $\alpha_1=-2i$.

Assume that $\alpha_1=0$. If $e^{L(c)}=i$, that is, $L(c)=\alpha_2c_2=2k\pi i+\frac{\pi}{2}i$, then it follows from (4.5) that $e^{B_2-B_1}=-1$. Noting that system (4.1), we have

$$f_1(z)=\frac{e^{p(z)}+e^{-p(z)}}{2}=\frac{e^{L(z)+B_1}+e^{-L(z)-B_1}}{2},$$

and

$$\begin{aligned}f_2(z)=\frac{e^{L(z)+B_2}+e^{-L(z)-B_2}}{2}&=\frac{e^{L(z)+B_1+B_2-B_1}+e^{-L(z)-B_1+B_1-B_2}}{2}\\ &=-\frac{e^{L(z)+B_1}+e^{-L(z)-B_1}}{2}=-f_1(z).\end{aligned}$$

If $e^{L(c)}=-i$, that is, $L(c)=\alpha_2c_2=2k\pi i+\frac{3\pi}{2}i$, then it follows from (4.5) that $e^{B_2-B_1}=1$. Thus, it yields from (4.1) that

$$f_2(z)=\frac{e^{L(z)+B_2}+e^{-L(z)-B_2}}{2}=\frac{e^{L(z)+B_1+B_2-B_1}+e^{-L(z)-B_1+B_1-B_2}}{2}$$

$$= \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2} = f_1(z).$$

Assume that $\alpha_1 = -2i$. If $e^{L(c)} = i$, that is, $L(c) = -2ic_1 + \alpha_2 c_2 = 2k\pi i + \frac{\pi}{2}i$, then it follows from (4.5) that $e^{B_2-B_1} = 1$. Noting that system (4.1), we have

$$\begin{aligned} f_2(z) &= \frac{e^{L(z)+B_2} + e^{-L(z)-B_2}}{2} = \frac{e^{L(z)+B_1+B_2-B_1} + e^{-L(z)-B_1+B_1-B_2}}{2} \\ &= \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2} = f_1(z). \end{aligned}$$

If $e^{L(c)} = -i$, that is, $L(c) = -2ic_1 + \alpha_2 c_2 = 2k\pi i + \frac{3\pi}{2}i$, then it follows from (4.5) that $e^{B_2-B_1} = -1$. Thus, it yields from (4.1) that

$$\begin{aligned} f_2(z) &= \frac{e^{L(z)+B_2} + e^{-L(z)-B_2}}{2} = \frac{e^{L(z)+B_1+B_2-B_1} + e^{-L(z)-B_1+B_1-B_2}}{2} \\ &= -\frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2} = -f_1(z). \end{aligned}$$

Case 2.

$$\begin{cases} (i + \dfrac{\partial p}{\partial z_1})e^{-p(z)+q(z+c)} = 1, \\ -(i + \dfrac{\partial q}{\partial z_1})e^{q(z)+p(z+c)} = 1. \end{cases}$$

Thus, it follows that $-p(z) + q(z+c) = \eta_1$ and $q(z) + p(z+c) = \eta_2$, that is, $q(z+2c) + q(z) = \eta_1 - \eta_2$. Noting that $q(z)$ is a nonconstant polynomial in $\mathbb{C}^2$, we can get a contradiction.

Case 3.

$$\begin{cases} -(i + \dfrac{\partial p}{\partial z_1})e^{p(z)+q(z+c)} = 1, \\ (i + \dfrac{\partial q}{\partial z_1})e^{-q(z)+p(z+c)} = 1. \end{cases}$$

Thus, it follows that $p(z) + q(z+c) = \eta_1$ and $-q(z) + p(z+c) = \eta_2$, that is, $p(z+2c) + p(z) = \eta_2 + \eta_1$. Noting that $p(z)$ is a nonconstant polynomial in $\mathbb{C}^2$, we can get a contradiction.

Case 4.

$$\begin{cases} -(i + \dfrac{\partial p}{\partial z_1})e^{p(z)+q(z+c)} = 1, \\ -(i + \dfrac{\partial q}{\partial z_1})e^{q(z)+p(z+c)} = 1. \end{cases} \tag{4.7}$$

Thus, it follows that $p(z) + q(z+c) = \eta_1$ and $q(z) + p(z+c) = \eta_2$, that is, $p(z+2c) - p(z) = \eta_2 - \eta_1$ and $q(z+2c) - q(z) = \eta_1 - \eta_2$. Similar to the same argument as in Case 1, we get that $p(z) = L(z) + B_1$ and $q(z) = -L(z) + B_2$, where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$, $\alpha_1, \alpha_2, B_1, B_2$ are constants in $\mathbb{C}$. Thus, by combining with (4.2)-(4.3) and (4.7), it yields that

$$\begin{cases} (i + \alpha_1)e^{L(c)-B_1-B_2} = 1, \\ (i - \alpha_1)e^{-L(c)-B_1-B_2} = 1, \\ -(i + \alpha_1)e^{-L(c)+B_1+B_2} = 1, \\ -(i - \alpha_1)e^{L(c)+B_1+B_2} = 1, \end{cases} \tag{4.8}$$

which leads to $(i-\alpha_1)^2=(i+\alpha_1)^2=-1$, that is, $\alpha_1=0$. Then, it leads to

$$e^{2L(c)}=1,\quad e^{2(B_1+B_2)}=-1. \tag{4.9}$$

If $e^{L(c)}=1$, that is, $L(c)=2k\pi i$, then it follows from (4.8) that $e^{B_1+B_2}=i$. Noting that system (4.1), we have

$$f_1(z)=\frac{e^{p(z)}+e^{-p(z)}}{2}=\frac{e^{L(z)+B_1}+e^{-L(z)-B_1}}{2},$$

and

$$\begin{aligned}f_2(z)&=\frac{e^{-L(z)+B_2}+e^{L(z)-B_2}}{2}=\frac{e^{L(z)+B_1-B_1-B_2}+e^{-L(z)-B_1+B_1+B_2}}{2}\\&=\frac{-ie^{L(z)+B_1}+ie^{-L(z)-B_1}}{2}.\end{aligned}$$

If $e^{L(c)}=-1$, that is, $L(c)=(2k+1)\pi i$, then it follows from (4.8) that $e^{B_1+B_2}=-i$. Noting that system (4.1), we have

$$\begin{aligned}f_2(z)&=\frac{e^{-L(z)+B_2}+e^{L(z)-B_2}}{2}=\frac{e^{L(z)+B_1-B_1-B_2}+e^{-L(z)-B_1+B_1+B_2}}{2}\\&=\frac{ie^{L(z)+B_1}-ie^{-L(z)-B_1}}{2}.\end{aligned}$$

Therefore, this completes the proof of Theorem 2.3.

5. Remarks

By observing the argument as in Theorems 2.1 and 2.2, when $f(z)$ is a function with one complex variable, we can obtain the following conclusions easily.

Theorem 5.1. *Then the following complex differential equation*

$$f(z)^2+[f(z)+f'(z)]^2=1 \tag{5.1}$$

has no any transcendental entire solutions with finite order.

Theorem 5.2. *Let $c\in\mathbb{C}-\{0\}$. Suppose that the complex differential difference equation*

$$f(z)^2+[f(z+c)+f'(z)]^2=1 \tag{5.2}$$

admits a transcendental entire solution with finite order. Then f must be of the form

$$f(z)=\frac{e^{2iz+b}+e^{-2iz-b}}{2},$$

and $c=-\frac{\pi}{4}\pm k\pi$, $k\in\mathbb{Z}$, where $b\in\mathbb{C}$.

Theorem 5.3. *The following system of complex differential equations*

$$\begin{cases}f_1(z)^2+[f_2(z)+f_2'(z)]^2=1,\\ f_2(z)^2+[f_1(z)+f_1'(z)]^2=1\end{cases} \tag{5.3}$$

has no any transcendental entire solutions with finite order.

For further studying the solutions of (2.1)-(2.4), we raise the following question which can not be solved now.

Question 5.1. *How to describe the transcendental meromorphic solutions of* (2.1)–(2.4)*?*

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Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

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