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Research article

A class of impulsive vibration equation with fractional derivatives

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Abstract: In this paper, we study a class of second order impulsive vibration equation containing fractional derivatives. By using monotone iterative technique, some new results on the multiplicity for solutions of the equations under nonlinear boundary conditions are obtained, and the properties of the solutions are discussed. Finally, the practicability of our results is discussed through a concrete example.

Keywords: fractional derivative; impulsive vibration equation; Caputo derivative; nonlinear boundary conditions
Mathematics Subject Classification: 26A33, 34A08, 34A37, 34B15

1. Introduction

As we all know, the vibration equation is one of the important research topics in mechanics, physics and other disciplines. In the recent decades, researchers have been paying more and more attentions to the fractional differential equations due to its wide applications on mechanics, physical science, biological sciences and engineering disciplines, etc., see [3, 6, 7, 12, 13, 17–20, 25] and the references therein. The development of fractional differential equations provides some new theoretical bases for the study of vibration problems. In [22], the vibration equations with fractional derivatives are used to describe the vibration behavior of viscoelastic polymers and good results are obtained. The theoretical study of vibration equations with fractional derivative has also been widely concerned. These studies include the mechanical properties, dynamic characteristics of the system and the correlation functions with various influences for the vibration equations with fractional derivatives, and so on see [15, 16, 21, 22]. In addition, mutation often occurs in vibration system. These abrupt changes can be simulated by the impulsive vibration equation. And this simulation are effective in describing the behavior of real system. There have been a large number of references for the study of fractional impulsive differential equations, see [1, 4, 5, 9, 11, 23, 24, 26].

In this paper, we study a class of second order impulsive vibration equation containing fractional derivatives under the nonlinear boundary conditions

$$\begin{cases} x''(t) - \lambda^c D^{\alpha}_{0^+} x(t) = f(t, x(t), x'(t)),\ t \in J', \\ \Delta x(t)|_{t=t_k} = I_k(x(t_k)),\ k = 1,\ 2, \cdots,\ m, \\ \Delta x'(t)|_{t=t_k} = Q_k(x(t_k)),\ k = 1,\ 2, \cdots,\ m, \\ g_0(x(0), x(1)) = 0,\ g_1(x'(0), x'(1)) = 0, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $\lambda > 0$ and ${}^c D^{\alpha}_{0^+}$ is the Caputo derivative, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. Set $J = [0, 1]$, $J' = J \backslash \{t_1,\ t_2,\ \cdots,\ t_m\}$, $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1,\ 2,\ \cdots,\ m$. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$. $x(t_k^-)$, $x(t_k^+)$ denote the left limit and the right limit of $x(t)$ at $t = t_k$. $x'(t_k^-)$, $x'(t_k^+)$ denote the left limit and the right limit of $x'(t)$ at $t = t_k$. Let $x(t_k) = x(t_k^-)$. $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, I_k, $Q_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \cdots, m$. g_0, $g_1 \in C(\mathbb{R}^2, \mathbb{R})$ are given nonlinear functions. By using monotone iterative technique, some new results on multiplicity of boundary value problems are obtained, and the properties of the solutions are discussed. Finally, an example is given out to illustrate the applicability of our main results.

2. Preliminaries

In this section, we present some basic definitions and lemmas, which will be used to prove our main results.

Definition 2.1. (See [8], P67) *Let α, $\beta > 0$. The function $E_{\alpha,\beta}$ is defined by*

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\alpha + \beta)},$$

whenever the series converge is called the two parameters Mittag-Leffler function with parameters α and β.

Lemma 2.1. (See [8], P68) *Consider the two parameters Mittag-Leffler function $E_{\alpha,\beta}$ for some α, $\beta > 0$. The power series defining $E_{\alpha,\beta}$ is convergent for all $z \in \mathbb{C}$. In other words, $E_{\alpha,\beta}$ is an entire function.*

Lemma 2.2. *Let α, $\beta > 0$, $k = 0,\ 1,\ 2,\ \cdots$, $z \in \mathbb{R}$. Then*

$$E^{(k)}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j \Gamma(k + j + 1)}{\Gamma(j + 1)\Gamma(\alpha(k + j) + \beta)}.$$

Proof. By Definition 2.1 and Lemma 2.1, we get

$$\begin{aligned} E^{(k)}_{\alpha,\beta}(z) = \frac{d^k}{dz^k} E_{\alpha,\beta}(z) &= \sum_{j=0}^{\infty} \frac{d^k}{dz^k}\Big(\frac{z^j}{\Gamma(j\alpha + \beta)}\Big) = \sum_{j=k}^{\infty} \frac{\Gamma(j + 1) z^{j-k}}{\Gamma(j - k + 1)\Gamma(\alpha j + \beta)} \\ &= \sum_{j=0}^{\infty} \frac{z^j \Gamma(k + j + 1)}{\Gamma(j + 1)\Gamma(\alpha(k + j) + \beta)}. \end{aligned}$$

□

Lemma 2.3. (See [14], P314) *Let* $0 < \alpha < \beta,\ n-1 < \alpha \le n,\ l-1 < \beta \le l\ (n, l \in \mathbb{N},\ n \le l,\ \lambda \in \mathbb{R})$. *Then*

$$ {}^cD^{\beta}_{0^+}x(t) - \lambda {}^cD^{\alpha}_{0^+}x(t) = 0\ (t > 0) $$

has its linearly independent solutions given by

$$ x_j(t) = t^j E_{\beta-\alpha,j+1}(\lambda t^{\beta-\alpha}) - \lambda t^{\beta-\alpha+j} E_{\beta-\alpha,\beta-\alpha+j+1}(\lambda t^{\beta-\alpha})\ \ (j = 0, 1, \cdots, n-1), \tag{2.1} $$

$$ x_j(t) = t^j E_{\beta-\alpha,j+1}(\lambda t^{\beta-\alpha})\ (j = n, \cdots, l-1). \tag{2.2} $$

Lemma 2.4. (See [14], P324) *Let* $l-1 < \beta \le l\ (l \in \mathbb{N}),\ 0 < \alpha < \beta$ *be such that* $\beta - l + 1 \ge \alpha,\ \lambda \in \mathbb{R}$, *and* $h(t)$ *be a given real function defined on* $\mathbb{R}_+$. *The general solution to the nonhomogeneous linear differential equation*

$$ {}^cD^{\beta}_{0^+}x(t) - \lambda {}^cD^{\alpha}_{0^+}x(t) = h(t)\ (t > 0) $$

is given by

$$ x(t) = \int_0^t (t-s)^{\beta-1} E_{\beta-\alpha,\beta}(\lambda(t-s)^{\beta-\alpha})h(s)\mathrm{d}s + \sum_{j=0}^{l-1} c_j x_j(t), $$

where $x_j(t)$ *are given by (2.1) and (2.2),* c_j *are arbitrary real constants* $(j = 0,\ 1,\ \cdots,\ l-1)$.

Lemma 2.5. *Let* $L[0,1]$ *denote the space of Lebesgue integrable functions on [0,1],* $h \in L[0,1]$ *and* $0 < \alpha < 1$*, then the Cauchy problem of the second order vibration equation with fractional derivative*

$$ \begin{cases} x''(t) - \lambda {}^cD^{\alpha}_{0^+}x(t) = h(t),\ t \in J, \\ x(\xi) = x_0,\ x'(\xi) = x_1,\ \xi \in J \end{cases} $$

has a unique solution, which is given by

$$ \begin{aligned} x(t) = &\int_0^t (t-s)E_{2-\alpha,2}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s - \int_0^{\xi} (\xi-s)E_{2-\alpha,2}(\lambda(\xi-s)^{2-\alpha})h(s)\mathrm{d}s + x_0 \\ &+ \frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha}) - \xi E_{2-\alpha,2}(\lambda \xi^{2-\alpha})}{E_{2-\alpha,1}(\lambda \xi^{2-\alpha})}\Big(x_1 - \int_0^{\xi} E_{2-\alpha,1}(\lambda(\xi-s)^{2-\alpha})h(s)\mathrm{d}s\Big), \end{aligned} \tag{2.3} $$

and $x(t)$ *is derivable while its derivative is given by*

$$ x'(t) = \int_0^t E_{2-\alpha,1}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s + \frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{E_{2-\alpha,1}(\lambda \xi^{2-\alpha})}\Big(x_1 - \int_0^{\xi} E_{2-\alpha,1}(\lambda(\xi-s)^{2-\alpha})h(s)\mathrm{d}s\Big). \tag{2.4} $$

Proof. In view of Lemma 2.4, for $\beta = 2,\ 0 < \alpha < 1$, the general solution of the equation

$$ x''(t) - \lambda {}^cD^{\alpha}_{0^+}x(t) = h(t) $$

is given by

$$ x(t) = \int_0^t (t-s)E_{2-\alpha,2}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s + c_1 tE_{2-\alpha,2}(\lambda t^{2-\alpha}) + c_0, $$

where $c_0,\ c_1 \in \mathbb{R}$, and

$$x'(t) = \int_0^t E_{2-\alpha,1}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s + c_1 E_{2-\alpha,1}(\lambda t^{2-\alpha}).$$

Therefore,

$$\begin{aligned} x(\xi) &= \int_0^\xi (\xi - s)E_{2-\alpha,2}(\lambda(\xi-s)^{2-\alpha})h(s)\mathrm{d}s + c_1\xi E_{2-\alpha,2}(\lambda\xi^{2-\alpha}) + c_0 = x_0, \\ x'(\xi) &= \int_0^\xi E_{2-\alpha,1}(\lambda(\xi-s)^{2-\alpha})h(s)\mathrm{d}s + c_1 E_{2-\alpha,1}(\lambda\xi^{2-\alpha}) = x_1. \end{aligned}$$

We get

$$\begin{aligned} c_1 &= \frac{1}{E_{2-\alpha,1}(\lambda\xi^{2-\alpha})}\Big(x_1 - \int_0^\xi E_{2-\alpha,1}(\lambda(\xi-s)^{2-\alpha})h(s)\mathrm{d}s\Big), \\ c_0 &= x_0 - \int_0^\xi (\xi-s)E_{2-\alpha,2}(\lambda(\xi-s)^{2-\alpha})h(s)\mathrm{d}s - c_1\xi E_{2-\alpha,2}(\lambda\xi^{2-\alpha}). \end{aligned}$$

So

$$\begin{aligned} x(t) &= \int_0^t (t-s)E_{2-\alpha,2}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s - \int_0^\xi (\xi-s)E_{2-\alpha,2}(\lambda(\xi-s)^{2-\alpha})h(s)\mathrm{d}s + x_0 \\ &\quad + c_1\big(tE_{2-\alpha,2}(\lambda t^{2-\alpha}) - \xi E_{2-\alpha,2}(\lambda\xi^{2-\alpha})\big) \\ &= \int_0^t (t-s)E_{2-\alpha,2}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s - \int_0^\xi (\xi-s)E_{2-\alpha,2}(\lambda(\xi-s)^{2-\alpha})h(s)\mathrm{d}s + x_0 \\ &\quad + \frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha}) - \xi E_{2-\alpha,2}(\lambda\xi^{2-\alpha})}{E_{2-\alpha,1}(\lambda\xi^{2-\alpha})}\Big(x_1 - \int_0^\xi E_{2-\alpha,1}(\lambda(\xi-s)^{2-\alpha})h(s)\mathrm{d}s\Big), \end{aligned}$$

and

$$x'(t) = \int_0^t E_{2-\alpha,1}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s + \frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{E_{2-\alpha,1}(\lambda\xi^{2-\alpha})}\Big(x_1 - \int_0^\xi E_{2-\alpha,1}(\lambda(\xi-s)^{2-\alpha})h(s)\mathrm{d}s\Big).$$

The proof is completed. □

Let $PC^1(J) = \{x : J \to \mathbb{R} \mid x,\ x' \in C(J',\mathbb{R}),\ x(t_k^+),\ x(t_k^-),\ x'(t_k^+),\ x'(t_k^-)$ exist, and $x(t_k) = x(t_k^-),\ k = 1,2,\cdots,m\}$ and endowed with the normal $\|x\| = \max\{\sup_{t\in[0,1]}|x(t)|, \sup_{t\in[0,1]}|x'(t)|\}$. Then $PC^1(J)$ is a Banach space.

For $x \in PC^1(J)$, by the Lagrange mean value theorem, there exists $\xi_k \in [t_k - \varepsilon, t_k]$ such that

$$x(t_k) - x(t_k - \varepsilon) = x'(\xi_k)\varepsilon,$$

and

$$x'_-(t_k) = \lim_{\varepsilon\to 0^+}\frac{x(t_k) - x(t_k-\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0^+}\frac{x'(\xi_k)\varepsilon}{\varepsilon} = x'(t_k^-),\ k = 1,2,\cdots,m.$$

Thus, for $x \in PC^1(J)$, we denote

$$x'(t_k) = x'_-(t_k) = x'(t_k^-),\ k = 1,2,\cdots,m. \tag{2.5}$$

Let $P=\{x\in PC^1(J)|\ x(t)\geq 0,\ x'(t)\geq 0,\ t\in J\}$. It is obvious that $P\subset PC^1(J)$ is a normal solid cone. We denote $x\underline{\prec}y\in PC^1(J)$ if and only if $x(t)\leq y(t)$ and $x'(t)\leq y'(t)$ on $t\in[0,1]$, i.e. $y-x\in P$. We denote $x\prec y$ if $x\underline{\prec}y\in PC^1(J)$ and $x\neq y$, and $x\prec\prec y$ if $y-x\in \mathring{P}$.

Lemma 2.6. (See [10], P220, [2], P666) *Let E be a Banach space, and $P\subset E$ be a normal solid cone. Suppose that there exist $\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2\in E$ with $\alpha_1\prec\beta_1\prec\alpha_2\prec\beta_2$ and $A:[\alpha_1,\beta_2]\to E$ is a completely continuous strongly increasing operator such that*

$$\alpha_1\underline{\prec}A\alpha_1,\ A\beta_1\prec\beta_1,\ \alpha_2\prec A\alpha_2,\ A\beta_2\underline{\prec}\beta_2.$$

Then the operator A has at least three distinct fixed points $x_1,\ x_2,\ x_3$ on $[\alpha_1,\beta_2]$ such that

$$\alpha_1\underline{\prec}x_1\prec\prec\beta_1,\ \ \alpha_2\prec\prec x_2\underline{\prec}\beta_2,\ \alpha_2\underline{\nprec}x_3\underline{\nprec}\beta_1.$$

3. The solution of linear impulsive vibration equation

In this section, we obtain the solution of the linear impulsive vibration equation and discuss the properties of its kernel function.

Lemma 3.1. *For any $p_k,\ q_k\in\mathbb{R}\ (k=1,\ 2,\ \cdots,m)$, $m_i,\ n_i\in\mathbb{R}\ (i=1,\ 2)$ and $h\in L[0,1]$, the following boundary value problem of the second order impulsive vibration equation with fractional derivative*

$$\begin{cases} x''(t)-\lambda^cD^{\alpha}_{0^+}x(t)=h(t),\ t\in J',\\ \Delta x(t)|_{t=t_k}=p_k,\ k=1,\ 2,\cdots,\ m,\\ \Delta x'(t)|_{t=t_k}=q_k,\ k=1,\ 2,\cdots,\ m,\\ m_1x(0)+m_2x(1)=\gamma_0,\ n_1x'(0)+n_2x'(1)=\gamma_1\end{cases}\tag{3.1}$$

has a unique solution, which is given by

$$x(t)=\int_0^1G(t,s)h(s)\mathrm{d}s+\varphi(t)+\Big(\frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}-\frac{m_2E_{2-\alpha,2}(\lambda)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}\Big)\gamma_1+\frac{\gamma_0}{m_1+m_2},\ t\in J,\tag{3.2}$$

where

$$G(t,s)=\begin{cases}(t-s)E_{2-\alpha,2}(\lambda(t-s)^{2-\alpha})-\frac{n_2tE_{2-\alpha,2}(\lambda t^{2-\alpha})E_{2-\alpha,1}\left(\lambda(1-s)^{2-\alpha}\right)}{n_1+n_2E_{2-\alpha,1}(\lambda)}\\ \quad+\frac{m_2n_2E_{2-\alpha,2}(\lambda)E_{2-\alpha,1}\left(\lambda(1-s)^{2-\alpha}\right)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}-\frac{m_2(1-s)E_{2-\alpha,2}\left(\lambda(1-s)^{2-\alpha}\right)}{m_1+m_2},\ 0\leq s\leq t\leq 1,\\ \frac{m_2n_2E_{2-\alpha,2}(\lambda)E_{2-\alpha,1}\left(\lambda(1-s)^{2-\alpha}\right)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}-\frac{n_2tE_{2-\alpha,2}(\lambda t^{2-\alpha})E_{2-\alpha,1}\left(\lambda(1-s)^{2-\alpha}\right)}{n_1+n_2E_{2-\alpha,1}(\lambda)}\\ \quad-\frac{m_2(1-s)E_{2-\alpha,2}\left(\lambda(1-s)^{2-\alpha}\right)}{m_1+m_2},\qquad 0\leq t<s\leq 1,\end{cases}\tag{3.3}$$

$$\varphi(t)=\sum_{0<t_i<t}p_i-\frac{m_2}{m_1+m_2}\sum_{i=1}^{m}p_i+\sum_{0<t_i<t}\frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha})-t_iE_{2-\alpha,2}(\lambda t_i^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}q_i$$
$$+\sum_{i=1}^{m}\Big(\frac{m_2n_2E_{2-\alpha,2}(\lambda)E_{2-\alpha,1}(\lambda)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}-\frac{m_2(E_{2-\alpha,2}(\lambda)-t_iE_{2-\alpha,2}(\lambda t_i^{2-\alpha}))}{(m_1+m_2)E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}$$

$$-\frac{n_2E_{2-\alpha,1}(\lambda)tE_{2-\alpha,2}(\lambda t^{2-\alpha})}{(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}\Big)q_i,\ t\in J. \tag{3.4}$$

Furthermore,

$$x'(t)=\int_0^1 G_t'(t,s)h(s)\mathrm{d}s+\varphi'(t)+\frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}\gamma_1,\ t\in J, \tag{3.5}$$

$$G_t'(t,s)=\begin{cases} E_{2-\alpha,1}(\lambda(t-s)^{2-\alpha})-\frac{n_2E_{2-\alpha,1}(\lambda t^{2-\alpha})E_{2-\alpha,1}\left(\lambda(1-s)^{2-\alpha}\right)}{n_1+n_2E_{2-\alpha,1}(\lambda)},\ 0\le s\le t\le 1,\\ -\frac{n_2E_{2-\alpha,1}(\lambda t^{2-\alpha})E_{2-\alpha,1}\left(\lambda(1-s)^{2-\alpha}\right)}{n_1+n_2E_{2-\alpha,1}(\lambda)},\ 0\le t<s\le 1,\end{cases} \tag{3.6}$$

and

$$\varphi'(t)=\sum_{0<t_i<t}\frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}q_i-\sum_{i=1}^m\frac{n_2E_{2-\alpha,1}(\lambda)E_{2-\alpha,1}(\lambda t^{2-\alpha})}{(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}q_i,\ t\in J. \tag{3.7}$$

Proof. For $t\in[0,t_1]$, let ξ=0, $x(0)=c_0,\ x'(0)=c_1$, by Lemma 2.5, Cauchy problem

$$\begin{cases} x''(t)-\lambda^cD_{0^+}^{\alpha}x(t)=h(t),\\ x(0)=c_0,\ x'(0)=c_1\end{cases}$$

has a unique solution

$$x(t)=\int_0^t(t-s)E_{2-\alpha,2}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s+c_1tE_{2-\alpha,2}(\lambda t^{2-\alpha})+c_0,$$

and

$$x'(t)=\int_0^t E_{2-\alpha,1}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s+c_1E_{2-\alpha,1}(\lambda t^{2-\alpha}),\ c_0,\ c_1\in\mathbb{R}.$$

So

$$\begin{aligned} x(t_1^-)&=\int_0^{t_1}(t_1-s)E_{2-\alpha,2}(\lambda(t_1-s)^{2-\alpha})h(s)\mathrm{d}s+c_1t_1E_{2-\alpha,2}(\lambda t_1^{2-\alpha})+c_0,\\ x'(t_1^-)&=\int_0^{t_1}E_{2-\alpha,1}(\lambda(t_1-s)^{2-\alpha})h(s)\mathrm{d}s+c_1E_{2-\alpha,1}(\lambda t_1^{2-\alpha}).\end{aligned}$$

For $t\in(t_1,t_2]$, let $\xi=t_1,\ x(t_1^+)=x(t_1^-)+p_1,\ x'(t_1^+)=x'(t_1^-)+q_1$. By Lemma 2.5, we can obtain that

$$\begin{aligned} x(t)=&\int_0^t(t-s)E_{2-\alpha,2}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s-\int_0^{t_1}(t_1-s)E_{2-\alpha,2}(\lambda(t_1-s)^{2-\alpha})h(s)\mathrm{d}s+x(t_1^+)\\ &+\frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha})-t_1E_{2-\alpha,2}(\lambda t_1^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_1^{2-\alpha})}\Big(x'(t_1^+)-\int_0^{t_1}E_{2-\alpha,1}(\lambda(t_1-s)^{2-\alpha})h(s)\mathrm{d}s\Big)\\ =&\int_0^t(t-s)E_{2-\alpha,2}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s-\int_0^{t_1}(t_1-s)E_{2-\alpha,2}(\lambda(t_1-s)^{2-\alpha})h(s)\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+ x(t_1^-) + p_1 + \frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha}) - t_1 E_{2-\alpha,2}(\lambda t_1^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_1^{2-\alpha})}\Big(x'(t_1^-) + q_1 - \int_0^{t_1} E_{2-\alpha,1}(\lambda(t_1 - s)^{2-\alpha})h(s)\mathrm{d}s\Big)\\
=& \int_0^t (t-s)E_{2-\alpha,2}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s - \int_0^{t_1}(t_1 - s)E_{2-\alpha,2}(\lambda(t_1-s)^{2-\alpha})h(s)\mathrm{d}s + p_1\\
&+ \int_0^{t_1}(t_1 - s)E_{2-\alpha,2}(\lambda(t_1-s)^{2-\alpha})h(s)\mathrm{d}s + c_1 t_1 E_{2-\alpha,2}(\lambda t_1^{2-\alpha}) + c_0\\
&+ \frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha}) - t_1 E_{2-\alpha,2}(\lambda t_1^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_1^{2-\alpha})}\Big(\int_0^{t_1} E_{2-\alpha,1}(\lambda(t_1-s)^{2-\alpha})h(s)\mathrm{d}s + c_1 E_{2-\alpha,1}(\lambda t_1^{2-\alpha})\\
&+ q_1 - \int_0^{t_1} E_{2-\alpha,1}(\lambda(t_1-s)^{2-\alpha})h(s)\mathrm{d}s\Big)\\
=& \int_0^t (t-s)E_{2-\alpha,2}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s + tE_{2-\alpha,2}(\lambda t^{2-\alpha})c_1 + c_0 + p_1\\
&+ \frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha}) - t_1 E_{2-\alpha,2}(\lambda t_1^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_1^{2-\alpha})} q_1,
\end{aligned}
$$

and

$$
\begin{aligned}
x'(t) =& \int_0^t E_{2-\alpha,1}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s + \frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_1^{2-\alpha})}\Big(x'(t_1^-) + q_1\\
&- \int_0^{t_1} E_{2-\alpha,1}(\lambda(t_1-s)^{2-\alpha})h(s)\mathrm{d}s\Big)\\
=& \int_0^t E_{2-\alpha,1}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s + E_{2-\alpha,1}(\lambda t^{2-\alpha})c_1 + \frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_1^{2-\alpha})} q_1.
\end{aligned}
$$

For $t \in (t_k, t_{k+1}]$, let $\xi = t_k,\ x(t_k^+) = x(t_k^-) + p_k,\ x'(t_k^+) = x'(t_k^-) + q_k,\ k = 2, 3, \cdots, m$. In the same way, we have

$$
\begin{aligned}
x(t) =& \int_0^t (t-s)E_{2-\alpha,2}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s + tE_{2-\alpha,2}(\lambda t^{2-\alpha})c_1 + c_0 + \sum_{0<t_i<t} p_i\\
&+ \sum_{0<t_i<t} \frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha}) - t_i E_{2-\alpha,2}(\lambda t_i^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})} q_i,
\end{aligned}
$$

and

$$
x'(t) = \int_0^t E_{2-\alpha,1}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s + E_{2-\alpha,1}(\lambda t^{2-\alpha})c_1 + \sum_{0<t_i<t} \frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})} q_i.
$$

Hence,

$$
\begin{aligned}
x(1) =& \int_0^1 (1-s)E_{2-\alpha,2}(\lambda(1-s)^{2-\alpha})h(s)\mathrm{d}s + E_{2-\alpha,2}(\lambda)c_1 + c_0 + \sum_{i=1}^m p_i\\
&+ \sum_{i=1}^m \frac{E_{2-\alpha,2}(\lambda) - t_i E_{2-\alpha,2}(\lambda t_i^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})} q_i,
\end{aligned}
$$

$$x'(1)=\int_0^1 E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})h(s)\mathrm{d}s+E_{2-\alpha,1}(\lambda)c_1+\sum_{i=1}^{m}\frac{E_{2-\alpha,1}(\lambda)}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}q_i.$$

By the boundary conditions $m_1x(0)+m_2x(1)=\gamma_0,\ n_1x'(0)+n_2x'(1)=\gamma_1$, we can get that

$$\begin{cases} -(m_1+m_2)c_0-m_2E_{2-\alpha,2}(\lambda)c_1=m_2\int_0^1(1-s)E_{2-\alpha,2}(\lambda(1-s)^{2-\alpha})h(s)\mathrm{d}s+m_2\sum\limits_{i=1}^{m}p_i\\ \qquad\qquad +m_2\sum\limits_{i=1}^{m}\frac{E_{2-\alpha,2}(\lambda)-t_iE_{2-\alpha,2}(\lambda t_i^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}q_i-\gamma_0,\\ -(n_1+n_2E_{2-\alpha,1}(\lambda))c_1=n_2\int_0^1E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})h(s)\mathrm{d}s+\sum\limits_{i=1}^{m}\frac{n_2E_{2-\alpha,1}(\lambda)}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}q_i-\gamma_1.\end{cases}$$

So

$$\begin{aligned}c_0=&\int_0^1\Big(-\frac{m_2(1-s)E_{2-\alpha,2}(\lambda(1-s)^{2-\alpha})}{m_1+m_2}+\frac{m_2n_2E_{2-\alpha,2}(\lambda)E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}\Big)h(s)\mathrm{d}s\\&-\frac{m_2}{m_1+m_2}\sum_{i=1}^{m}p_i+\sum_{i=1}^{m}\Big(\frac{m_2n_2E_{2-\alpha,2}(\lambda)E_{2-\alpha,1}(\lambda)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}\\&-\frac{m_2(E_{2-\alpha,2}(\lambda)-t_iE_{2-\alpha,2}(\lambda t_i^{2-\alpha}))}{(m_1+m_2)E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}\Big)q_i-\frac{m_2E_{2-\alpha,2}(\lambda)\gamma_1}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}+\frac{\gamma_0}{m_1+m_2},\end{aligned}$$

$$c_1=-\frac{n_2}{n_1+n_2E_{2-\alpha,1}(\lambda)}\Big(\int_0^1E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})h(s)\mathrm{d}s+\sum_{i=1}^{m}\frac{E_{2-\alpha,1}(\lambda)}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}q_i\Big)+\frac{\gamma_1}{n_1+n_2E_{2-\alpha,1}(\lambda)}.$$

Therefore,

$$\begin{aligned}x(t)=&\int_0^t(t-s)E_{2-\alpha,2}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s+\int_0^1\Big(\frac{m_2n_2E_{2-\alpha,2}(\lambda)E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}\\&-\frac{n_2tE_{2-\alpha,2}(\lambda t^{2-\alpha})E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}-\frac{m_2(1-s)E_{2-\alpha,2}(\lambda(1-s)^{2-\alpha})}{m_1+m_2}\Big)h(s)\mathrm{d}s\\&+\sum_{0<t_i<t}p_i-\frac{m_2}{m_1+m_2}\sum_{i=1}^{m}p_i+\sum_{0<t_i<t}\frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha})-t_iE_{2-\alpha,2}(\lambda t_i^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}q_i\\&+\sum_{i=1}^{m}\Big(\frac{m_2n_2E_{2-\alpha,2}(\lambda)E_{2-\alpha,1}(\lambda)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}-\frac{m_2(E_{2-\alpha,2}(\lambda)-t_iE_{2-\alpha,2}(\lambda t_i^{2-\alpha}))}{(m_1+m_2)E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}\\&-\frac{n_2E_{2-\alpha,1}(\lambda)tE_{2-\alpha,2}(\lambda t^{2-\alpha})}{(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}\Big)q_i+\frac{\gamma_0}{m_1+m_2}\\&+\Big(\frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}-\frac{m_2E_{2-\alpha,2}(\lambda)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}\Big)\gamma_1\\=&\int_0^1G(t,s)h(s)\mathrm{d}s+\varphi(t)+\Big(\frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}-\frac{m_2E_{2-\alpha,2}(\lambda)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}\Big)\gamma_1+\frac{\gamma_0}{m_1+m_2},\ t\in[0,1].\end{aligned}$$

And

$$x'(t)=\int_0^tE_{2-\alpha,1}(\lambda(t-s)^{2-\alpha})h(s)\mathrm{d}s-\int_0^1\frac{n_2E_{2-\alpha,1}(\lambda t^{2-\alpha})E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}h(s)\mathrm{d}s$$

$$+\sum_{0<t_i<t}\frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}q_i-\sum_{i=1}^{m}\frac{n_2E_{2-\alpha,1}(\lambda)E_{2-\alpha,1}(\lambda t^{2-\alpha})}{(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}q_i+\frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}\gamma_1$$

$$=\int_0^1 G'_t(t,s)h(s)\mathrm{d}s+\varphi'(t)+\frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}\gamma_1,\ t\in[0,1].$$

Therefore, boundary value problem (3.1) has a unique solution $x = x(t)$ which is given by (3.2), and $G(t, s)$, $\varphi(t)$ are given by (3.3) and (3.4), respectively. Furthermore, $x'(t)$ is also established. □

For convenience, we give out the following hypothesis:

(H1) The constants $m_i,\ n_i \in \mathbb{R}(i = 1, 2)$ satisfy $m_2(m_1 + m_2) < 0$ and $n_2(n_1 + n_2E_{2-\alpha,1}(\lambda)) < 0$.

Lemma 3.2. *Suppose that (H1) holds. Then functions G and φ defined by* (3.3) *and* (3.4) *satisfy the following properties:*

(1) $G(t, s)$ is continuous for $t,\ s \in [0, 1]$.

(2) $G(t, s) > 0$ for $t, s \in [0, 1]$ and $\max_{t\in[0,1]} G(t, s) = G(1, s),\ \min_{t\in[0,1]} G(t, s) = G(0, s)$.

(3) $G'_t(t, s) > 0$ for $t, s \in [0, 1]$ and $\max_{t\in[0,1]} G'_t(t, s) = G'_t(1, s),\ \min_{t\in[0,1]} G'_t(t, s) = G'_t(0, s)$.

(4) If $p_k \ge 0,\ q_k \ge 0, k = 1,\ 2,\ \cdots,\ m$, then $\varphi(t) \ge 0,\ \varphi'(t) \ge 0$, for $t \in J_k$.

Proof. (1) By the definition of $G(t, s)$, $G \in C([0, 1] \times [0, 1])$ is obvious.

(2) By (H1), for $0 \le s \le t \le 1$,

$$G'_t(t,s)=\frac{\partial G(t,s)}{\partial t}=E_{2-\alpha,1}(\lambda(t-s)^{2-\alpha})-\frac{n_2E_{2-\alpha,1}(\lambda t^{2-\alpha})E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}>0.$$

Then $G(s, s) \le G(t, s) \le G(1, s)$, for $s \in [0, 1]$ and $t \in [s, 1]$. And

$$G(s,s)=\frac{m_2n_2E_{2-\alpha,2}(\lambda)E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}-\frac{n_2sE_{2-\alpha,2}(\lambda s^{2-\alpha})E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}$$
$$-\frac{m_2(1-s)E_{2-\alpha,2}(\lambda(1-s)^{2-\alpha})}{m_1+m_2}>0.$$

Hence, $G(t, s) > 0$ for $0 \le s \le t \le 1$ and $G(t, s) \le G(1, s)$ for $t \in [s, 1]$.

For $0 \le t < s \le 1$,

$$\frac{\partial G(t,s)}{\partial t}=-\frac{n_2E_{2-\alpha,1}(\lambda t^{2-\alpha})E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})}{n_1-n_2E_{2-\alpha,1}(\lambda)}>0,$$

we can get that $G(0, s) \le G(t, s) < G(s, s)$, for $s \in [0, 1]$ and $t \in [0, s)$. And

$$G(0,s)=\frac{m_2n_2E_{2-\alpha,2}(\lambda)E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}-\frac{m_2(1-s)E_{2-\alpha,2}(\lambda(1-s)^{2-\alpha})}{m_1+m_2}>0.$$

Hence, $G(t, s) > 0$ for $0 \le t < s \le 1$ and $G(t, s) < G(s, s)$ for $t \in [0, s)$.

Therefore, $G(t, s) > 0$ for any $t, s \in [0, 1]$. And $G(t, s)$ is monotone increasing with respect to $t \in [0, 1]$, so $\max_{t\in[0,1]} G(t, s) = G(1, s),\ \min_{t\in[0,1]} G(t, s) = G(0, s)$.

(3) Since

$$(E_{2-\alpha,1}(\lambda t^{2-\alpha}))'=\Big(\sum_{k=0}^{\infty}\frac{(\lambda t^{2-\alpha})^k}{\Gamma((2-\alpha)k+1)}\Big)'=\sum_{k=1}^{\infty}\frac{(2-\alpha)k\lambda^k t^{(2-\alpha)k-1}}{\Gamma((2-\alpha)k+1)}\ge 0,\ t\in[0,1].$$

By (H1), for $0 \leq s \leq t \leq 1$,

$$\frac{\partial^2 G(t,s)}{\partial t^2} = \left(E_{2-\alpha,1}(\lambda(t-s)^{2-\alpha})\right)' - \frac{n_2 E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})(E_{2-\alpha,1}(\lambda t^{2-\alpha}))'}{n_1 + n_2 E_{2-\alpha,1}(\lambda)} \geq 0.$$

Then $G_t'(s,s) \leq G_t'(t,s) \leq G_t'(1,s)$, for any $s \in [0,1],\ t \in [s,1]$. Because

$$G_t'(s,s) = 1 - \frac{n_2 E_{2-\alpha,1}(\lambda s^{2-\alpha}) E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})}{n_1 + n_2 E_{2-\alpha,1}(\lambda)} > 0,$$

we can get that $G_t'(t,s) > 0$ for $0 \leq s \leq t \leq 1$.

For $0 \leq t < s \leq 1$,

$$\frac{\partial G^2(t,s)}{\partial t^2} = -\frac{n_2 E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})(E_{2-\alpha,1}(\lambda t^{2-\alpha}))'}{n_1 - n_2 E_{2-\alpha,1}(\lambda)} \geq 0,$$

we can get that $G_t'(0,s) \leq G_t'(t,s) \leq G_t'(s,s)$ for any $s \in [0,1]$ and $t \in [0,s)$. Since

$$G_t'(0,s) = -\frac{n_2 E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})}{n_1 - n_2 E_{2-\alpha,1}(\lambda)} > 0,$$

we have $G_t'(t,s) > 0$ for $0 \leq t < s \leq 1$.

Therefore, $G_t'(t,s) > 0$ for any $t, s \in [0,1]$ and $\max\limits_{t\in[0,1]} G_t'(t,s) = G_t'(1,s),\ \min\limits_{t\in[0,1]} G_t'(t,s) = G_t'(0,s)$.

(4) If $p_k, q_k \geq 0$, $k = 1, 2, \cdots, m$, by (H1), (3.4) and (3.7), we can easily get that

$$\varphi(t) \geq 0,\quad \varphi'(t) \geq 0,\ t \in J_k.$$

The proof is completed. □

4. Existence of three solutions

In this section, we will establish the existence results of the solutions for the boundary value problem (1.1).

For any $u \in PC^1(J)$, we consider the following boundary value problem

$$\begin{cases} x''(t) - \lambda\, {}^cD_{0^+}^{\alpha} x(t) = f(t, u(t), u'(t)),\ t \in J', \\ \Delta x(t)|_{t=t_k} = I_k(u(t_k)),\ k = 1,\ 2, \cdots,\ m, \\ \Delta x'(t)|_{t=t_k} = Q_k(u(t_k)),\ k = 1,\ 2, \cdots,\ m, \\ m_1 x(0) + m_2 x(1) = g_0(u(0), u(1)) + m_1 u(0) + m_2 u(1) := \gamma_{u,0}, \\ n_1 x'(0) + n_2 x'(1) = g_1(u'(0), u'(1)) + n_1 u'(0) + n_2 u'(1) := \gamma_{u,1}. \end{cases} \tag{4.1}$$

By Lemma 3.1, we can get that boundary value (4.1) is equivalent to the following integral equation

$$\begin{aligned} x(t) = &\int_0^1 G(t,s) f(s, u(s), u'(s)) \mathrm{d}s + \varphi_u(t) \\ &+ \Big(\frac{t E_{2-\alpha,2}(\lambda t^{2-\alpha})}{n_1 + n_2 E_{2-\alpha,1}(\lambda)} - \frac{m_2 E_{2-\alpha,2}(\lambda)}{(m_1 + m_2)(n_1 + n_2 E_{2-\alpha,1}(\lambda))}\Big)\gamma_{u,1} + \frac{\gamma_{u,0}}{m_1 + m_2},\ t \in J, \end{aligned}$$

and

$$x'(t) = \int_0^1 G'_t(t,s)f(s,u(s),u'(s))\mathrm{d}s + \varphi'_u(t) + \frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{n_1 + n_2E_{2-\alpha,1}(\lambda)}\gamma_{u,1}, \ t \in J,$$

where

$$\begin{aligned}\varphi_u(t) = &\sum_{0<t_i<t} I_i(u(t_i)) - \frac{m_2}{m_1+m_2}\sum_{i=1}^{m} I_i(u(t_i)) + \sum_{0<t_i<t}\frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha}) - t_iE_{2-\alpha,2}(\lambda t_i^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}Q_i(u(t_i))\\ &+ \sum_{i=1}^{m}\Big(\frac{m_2n_2E_{2-\alpha,2}(\lambda)E_{2-\alpha,1}(\lambda)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda t_i^{2-\alpha})} - \frac{m_2(E_{2-\alpha,2}(\lambda) - t_iE_{2-\alpha,2}(\lambda t_i^{2-\alpha}))}{(m_1+m_2)E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}\\ &- \frac{n_2E_{2-\alpha,1}(\lambda)tE_{2-\alpha,2}(\lambda t^{2-\alpha})}{(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}\Big)Q_i(u(t_i)), \ t \in J \end{aligned}\tag{4.2}$$

and

$$\varphi'_u(t) = \sum_{0<t_i<t}\frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}Q_i(u(t_i)) - \sum_{i=1}^{m}\frac{n_2E_{2-\alpha,1}(\lambda)E_{2-\alpha,1}(\lambda t^{2-\alpha})}{(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}Q_i(u(t_i)), \ t \in J. \tag{4.3}$$

We define an operator $T : PC^1(J) \to PC^1(J)$ by

$$\begin{aligned}Tu(t) = &\int_0^1 G(t,s)f(s,u(s),u'(s))\mathrm{d}s + \varphi_u(t)\\ &+ \Big(\frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)} - \frac{m_2E_{2-\alpha,2}(\lambda)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}\Big)\gamma_{u,1} + \frac{\gamma_{u,0}}{m_1+m_2}, \ t \in J.\end{aligned}$$

By Lemma 3.1 and (3.5),

$$(Tu)'(t) = \int_0^1 G'_t(t,s)f(s,u(s),u'(s))\mathrm{d}s + \varphi'_u(t) + \frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}\gamma_{u,1}.$$

We can easily get that the following Lemma 4.1 holds.

Lemma 4.1. *The function $x = x(t)$ is the solution of boundary value problem (1.1) if and only if x is a fixed point of the operator T in $PC^1(J)$.*

Lemma 4.2. *If (H1) holds, then $T : PC^1(J) \to PC^1(J)$ is completely continuous.*

Proof. Step 1: T is a continuous operator.

Suppose that $\{u_n\} \subset PC^1(J)$ and there exists $u \in PC^1(J)$ such that $\|u_n - u\| \to 0 (n \to \infty)$. Then there exists a constant $M > 0$ such that $\|u_n\| \le M, \ \|u\| \le M$.

Since $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, I_k, $Q_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \cdots, m$, g_0, $g_1 \in C(\mathbb{R}^2, \mathbb{R})$ and $\gamma_{u_n,0}$, $\gamma_{u_n,1}$, $\gamma_{u,0}$, $\gamma_{u,1}, \in \mathbb{R}$, then

$$\lim_{n\to\infty} |f(t,u_n(t),u'_n(t)) - f(t,u(t),u'(t))| = 0, \ t \in J,$$

$$\lim_{n\to\infty} |I_k(u_n(t_k)) - I_k(u(t_k))| = 0, \ \lim_{n\to\infty} |Q_k(u_n(t_k)) - Q_k(u(t_k))| = 0, \ k = 1, 2, \cdots, m,$$

$$\lim_{n\to\infty}|\gamma_{u_n,0}-\gamma_{u,0}|=0,\ \lim_{n\to\infty}|\gamma_{u_n,1}-\gamma_{u,1}|=0.$$

By (H1), Lemma 3.2 and Lebesgue dominated convergence theorem, for any $t\in J$, we have

$$\begin{aligned}
&|Tu_n(t)-Tu(t)|\\
=&\Big|\int_0^1 G(t,s)\Big(f(s,u_n(s),u_n'(s))-f(s,u(s),u'(s))\Big)\,\mathrm{d}s+(\varphi_{u_n}(t)-\varphi_u(t))\\
&+\Big(\frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}-\frac{m_2E_{2-\alpha,2}(\lambda)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}\Big)(\gamma_{u_n,1}-\gamma_{u,1})+\frac{\gamma_{u_n,0}-\gamma_{u,0}}{m_1+m_2}\Big|\\
\le&\int_0^1 G(1,s)\Big|f(s,u_n(s),u_n'(s))-f(s,u(s),u'(s))\Big|\,\mathrm{d}s\\
&+\frac{|m_1+m_2|+|m_2|}{|m_1+m_2|}\sum_{i=1}^m\Big|I_i(u_n(t_i))-I_i(u(t_i))\Big|\\
&+\frac{(|m_1|+2|m_2|)(|n_1|+2|n_2|E_{2-\alpha,1}(\lambda))E_{2-\alpha,2}(\lambda)}{|m_1+m_2||n_1+n_2E_{2-\alpha,1}(\lambda)|}\sum_{i=1}^m\Big|Q_i(u_n(t_i))-Q_i(u(t_i))\Big|\\
&+\frac{(|m_1|+2|m_2|)E_{2-\alpha,2}(\lambda)}{|m_1+m_2||n_1+n_2E_{2-\alpha,1}(\lambda)|}|\gamma_{u_n,1}-\gamma_{u,1}|+\frac{|\gamma_{u_n,0}-\gamma_{u,0}|}{|m_1+m_2|}\to 0\ (n\to\infty),
\end{aligned}$$

and

$$\begin{aligned}
&|(Tu_n)'(t)-(Tu)'(t)|\\
=&\Big|\int_0^1 G_t'(t,s)\Big(f(s,u_n(s),u_n'(s))-f(s,u(s),u'(s))\Big)\mathrm{d}s+(\varphi_{u_n}'(t)-\varphi_u'(t))\\
&+\frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}(\gamma_{u_n,1}-\gamma_{u,1})\Big|\\
\le&\int_0^1 G_t'(1,s)\Big|f(s,u_n(s),u_n'(s))-f(s,u(s),u'(s))\Big|\mathrm{d}s\\
&+\sum_{i=1}^m\frac{(|n_1|+2|n_2|E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda)}{|n_1+n_2E_{2-\alpha,1}(\lambda)|}\Big|Q_i(u_n(t_i))-Q_i(u(t_i))\Big|\\
&+\frac{E_{2-\alpha,1}(\lambda)}{|n_1+n_2E_{2-\alpha,1}(\lambda)|}|\gamma_{u_n,1}-\gamma_{u,1}|\to 0\ \ (n\to\infty).
\end{aligned}$$

So $\|Tu_n-Tu\|\to 0$ as $n\to\infty$, which means T is continuous.

Step 2: T is relatively compact.

Let $\Omega\subset PC^1(J)$ be a bounded set. By the continuity of the functions $f,\ I_k,\ Q_k\ (k=1,\ 2,\ \cdots,m)$, g_0 and g_1, there exists a constant $L>0$, for any $u\in\Omega$ and $t\in J$,

$$|f(t,u(t),u'(t))|\le L,\quad |\gamma_{u,0}|\le L,\ |\gamma_{u,1}|\le L,$$
$$|I_k(u(t))|\le L,\ |Q_k(u(t))|\le L,\ k=1,\ 2,\ \cdots,\ m.$$

Then

$$|\varphi_u(t)|\le\frac{m(|m_1|+2|m_2|)\Big(|n_1|(1+E_{2-\alpha,2}(\lambda))+|n_2|E_{2-\alpha,1}(\lambda)(1+2E_{2-\alpha,2}(\lambda))\Big)}{|m_1+m_2||n_1+n_2E_{2-\alpha,1}(\lambda)|}L,$$

$$|\varphi_u'(t)| \le \frac{m(|n_1| + 2|n_2|E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda)}{|n_1 + n_2E_{2-\alpha,1}(\lambda)|}L.$$

By Lemma 3.2, for any $t \in J$,

$$\begin{aligned}|Tu(t)| =&\Big|\int_0^1 G(t,s)f(s,u(s),u'(s))\mathrm{d}s + \varphi_u(t) + \frac{\gamma_{u,0}}{m_1+m_2}\\ &+\Big(\frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)} - \frac{m_2E_{2-\alpha,2}(\lambda)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}\Big)\gamma_{u,1}\Big|\\ \le&\Big(\int_0^1 G(1,s)\mathrm{d}s + \frac{(|m_1|+2|m_2|)E_{2-\alpha,2}(\lambda)+|n_1|+|n_2|E_{2-\alpha,1}(\lambda)}{|m_1+m_2||n_1+n_2E_{2-\alpha,1}(\lambda)|}\\ &+\frac{m(|m_1|+2|m_2|)\Big(|n_1|(1+E_{2-\alpha,2}(\lambda))+|n_2|E_{2-\alpha,1}(\lambda)(1+2E_{2-\alpha,2}(\lambda))\Big)}{|m_1+m_2||n_1+n_2E_{2-\alpha,1}(\lambda)|}\Big)L,\end{aligned}$$

and

$$\begin{aligned}|(Tu)'(t)| =&\Big|\int_0^1 G_t'(t,s)f(s,u(s),u'(s))\mathrm{d}s + \varphi_u'(t) + \frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}\gamma_{u,1}\Big|\\ \le&\Big(\int_0^1 G_t'(1,s)\mathrm{d}s + \frac{m(|n_1|+2|n_2|E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda)+E_{2-\alpha,1}(\lambda)}{|n_1+n_2E_{2-\alpha,1}(\lambda)|}\Big)L.\end{aligned}$$

Therefore, the operator $T(\Omega)$ is uniformly bounded.

Because $G(t,s)$ is continuous on $[0,1]\times[0,1]$, then it is uniformly continuous on $[0,1]\times[0,1]$. Thus, for any $\varepsilon > 0$, there exists a constant $\delta_1 > 0$ such that for any $s \in [0,1]$, $t_1',\ t_2' \in J_k,\ k = 0,\ 1,\ \cdots,\ m$, whenever $|t_1' - t_2'| < \delta_1$, we can get that

$$|G(t_1',s) - G(t_2',s)| < \frac{\varepsilon}{2L}.$$

Denote

$$G_0(t,s) = -\frac{n_2tE_{2-\alpha,2}(\lambda t^{2-\alpha})E_{2-\alpha,1}(\lambda(1-s)^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}.$$

By (3.6),

$$G_t'(t,s) = \begin{cases} E_{2-\alpha,1}(\lambda(t-s)^{2-\alpha}) + G_0(t,s),\ 0\le s\le t\le 1,\\ G_0(t,s),\ 0\le t< s\le 1.\end{cases}$$

Similarly, for the $\varepsilon > 0$, there exists a constant $\delta_2 > 0$ such that

$$|G_0(t_1',s) - G_0(t_2',s)| < \frac{\varepsilon}{3L}$$

whenever $|t_1' - t_2'| < \delta_2$ and $t_1',\ t_2' \in J_k, k = 0,\ 1,\ \cdots,\ m$.

By the uniformly continuity of functions $tE_{2-\alpha,2}(\lambda t^{2-\alpha})$ and $E_{2-\alpha,1}(\lambda t^{2-\alpha})$ on $t \in [0,1]$, we can show that for the $\varepsilon > 0$, there exists a constant $\delta_3 > 0$ such that

$$|t_1'E_{2-\alpha,2}(\lambda {t_1'}^{2-\alpha}) - t_2'E_{2-\alpha,2}(\lambda {t_2'}^{2-\alpha})| < \frac{|n_1+n_2E_{2-\alpha,1}(\lambda)|\varepsilon}{2\Big(m(|n_1|+2|n_2|E_{2-\alpha,1}(\lambda))+1\Big)L},$$

and

$$|E_{2-\alpha,1}(\lambda {t'}_1^{2-\alpha}) - E_{2-\alpha,1}(\lambda {t'}_2^{2-\alpha})| < \frac{|n_1 + n_2 E_{2-\alpha,1}(\lambda)|\varepsilon}{3\Big(m(|n_1| + 2|n_2|E_{2-\alpha,1}(\lambda)) + 1\Big)L},$$

whenever $|t'_1 - t'_2| < \delta_3$ and $t'_1,\ t'_2 \in J_k, k = 0,\ 1,\ \cdots,\ m$. Hence, by (4.2) and (4.3), we have

$$\begin{aligned}
|\varphi_u(t'_1) - \varphi_u(t'_2)| \leq & \sum_{i=1}^{m} \Big|t'_1 E_{2-\alpha,2}(\lambda {t'}_1^{2-\alpha}) - t'_2 E_{2-\alpha,2}(\lambda {t'}_2^{2-\alpha})\Big||Q_i(u(t_i))| \\
& + \sum_{i=1}^{m} \frac{|n_2|E_{2-\alpha,1}(\lambda)|t'_1 E_{2-\alpha,2}(\lambda {t'}_1^{2-\alpha}) - t'_2 E_{2-\alpha,2}(\lambda {t'}_2^{2-\alpha})|}{|n_1 + n_2 E_{2-\alpha,1}(\lambda)|}|Q_i(u(t_i))| \\
\leq & \frac{mL(|n_1| + 2|n_2|E_{2-\alpha,1}(\lambda))}{|n_1 + n_2 E_{2-\alpha,1}(\lambda)|}\Big|t'_1 E_{2-\alpha,2}(\lambda {t'}_1^{2-\alpha}) - t'_2 E_{2-\alpha,2}(\lambda {t'}_2^{2-\alpha})\Big|,
\end{aligned}$$

and

$$\begin{aligned}
|\varphi'_u(t'_1) - \varphi'_u(t'_2)| \leq & \sum_{i=1}^{m} \Big|E_{2-\alpha,1}(\lambda {t'}_1^{2-\alpha}) - E_{2-\alpha,1}(\lambda {t'}_2^{2-\alpha})\Big||Q_i(u(t_i))| \\
& + \sum_{i=1}^{m} \frac{|n_2|E_{2-\alpha,1}(\lambda)|E_{2-\alpha,1}(\lambda {t'}_1^{2-\alpha}) - E_{2-\alpha,1}(\lambda {t'}_2^{2-\alpha})|}{|n_1 + n_2 E_{2-\alpha,1}(\lambda)|}|Q_i(u(t_i))| \\
\leq & \frac{mL(|n_1| + 2|n_2|E_{2-\alpha,1}(\lambda))}{|n_1 + n_2 E_{2-\alpha,1}(\lambda)|}\Big|E_{2-\alpha,1}(\lambda {t'}_1^{2-\alpha}) - E_{2-\alpha,1}(\lambda {t'}_2^{2-\alpha})\Big|.
\end{aligned}$$

By the uniformly continuity of $E_{2-\alpha,1}(\lambda(t - s)^{2-\alpha})$ on $D = \{(t, s)|\ 0 \leq s \leq t \leq 1\}$, for the $\varepsilon > 0$, there exists a constant $0 < \delta_4 < \frac{\varepsilon}{6LE_{2-\alpha,1}(\lambda)}$, for $(t'_1, s),\ (t'_2, s) \in D$, $|t'_1 - t'_2| < \delta_4$ and $t'_1,\ t'_2 \in J_k, k = 0,\ 1,\ \cdots,\ m$, we have

$$|E_{2-\alpha,1}(\lambda(t'_1 - s)^{2-\alpha}) - E_{2-\alpha,1}(\lambda(t'_2 - s)^{2-\alpha})| < \frac{\varepsilon}{6L}.$$

Then

$$\begin{aligned}
& \Big|\int_0^{t'_1} E_{2-\alpha,1}(\lambda(t'_1 - s)^{2-\alpha})f(s, u(s), u'(s))\mathrm{d}s - \int_0^{t'_2} E_{2-\alpha,1}(\lambda(t'_2 - s)^{2-\alpha})f(s, u(s), u'(s))\mathrm{d}s\Big| \\
\leq & \Big|\int_0^{t'_1} E_{2-\alpha,1}(\lambda(t'_1 - s)^{2-\alpha})f(s, u(s), u'(s))\mathrm{d}s - \int_0^{t'_1} E_{2-\alpha,1}(\lambda(t'_2 - s)^{2-\alpha})f(s, u(s), u'(s))\mathrm{d}s \\
& - \int_{t'_1}^{t'_2} E_{2-\alpha,1}(\lambda(t'_2 - s)^{2-\alpha})f(s, u(s), u'(s))\mathrm{d}s\Big| \\
\leq & L\int_0^{t'_1} \Big|(E_{2-\alpha,1}(\lambda(t'_1 - s)^{2-\alpha}) - E_{2-\alpha,1}(\lambda(t'_2 - s)^{2-\alpha}))\Big|\mathrm{d}s + L\Big|\int_{t'_1}^{t'_2} E_{2-\alpha,1}(\lambda(t'_2 - s)^{2-\alpha})\mathrm{d}s\Big| \\
\leq & L\Big|(E_{2-\alpha,1}(\lambda(t'_1 - s)^{2-\alpha}) - E_{2-\alpha,1}(\lambda(t'_2 - s)^{2-\alpha}))\Big| + L|E_{2-\alpha,1}(\lambda)||t'_2 - t'_1| \\
< & \frac{\varepsilon}{3}.
\end{aligned}$$

We take $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$. Therefore, for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for $t'_1,\ t'_2 \in J_k, k = 0,\ 1,\ \cdots,\ m$ whenever $|t'_1 - t'_2| < \delta$ and any $u \in \Omega$, we can get that

$$|Tu(t'_1) - Tu(t'_2)|$$

$$
\begin{aligned}
&\leq \int_0^1 \left|G(t_1', s) - G(t_2', s)\right|\left|f(s, u(s), u'(s))\right| \mathrm{d}s + \left|\varphi_u(t_1') - \varphi_u(t_2')\right| \\
&\quad + \frac{\left|t_1' E_{2-\alpha,2}(\lambda t'^{2-\alpha}_1) - t_2' E_{2-\alpha,2}(\lambda t'^{2-\alpha}_2)\right|}{|n_1 + n_2 E_{2-\alpha,1}(\lambda)|}\gamma_{u,1} \\
&\leq L \int_0^1 \left|G(t_1', s) - G(t_2', s)\right| \mathrm{d}s + \left|\varphi_u(t_1') - \varphi_u(t_2')\right| \\
&\quad + \frac{L}{|n_1 + n_2 E_{2-\alpha,1}(\lambda)|}\left|t_1' E_{2-\alpha,2}(\lambda {t_1'}^{2-\alpha}) - t_2' E_{2-\alpha,2}(\lambda {t_2'}^{2-\alpha})\right| \\
&< \frac{\varepsilon}{2} + \frac{\left(m(|n_1| + 2|n_2|E_{2-\alpha,1}(\lambda)) + 1\right)L}{|n_1 + n_2 E_{2-\alpha}(\lambda)|}\left|t_1' E_{2-\alpha,2}(\lambda {t_1'}^{2-\alpha}) - t_2' E_{2-\alpha,2}(\lambda {t_2'}^{2-\alpha})\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}
$$

and

$$
\begin{aligned}
&|(Tu)'(t_1') - (Tu)'(t_2')| \\
=&\Big| \int_0^{t_1'} E_{2-\alpha,1}(\lambda(t_1' - s)^{2-\alpha}) f(s, u(s), u'(s)) \mathrm{d}s - \int_0^{t_2'} E_{2-\alpha,1}(\lambda(t_2' - s)^{2-\alpha}) f(s, u(s), u'(s)) \mathrm{d}s \\
&+ \int_0^1 (G_0(t_1', s) - G_0(t_1', s)) f(s, u(s), u'(s)) \mathrm{d}s + \varphi_u'(t_1') - \varphi_u'(t_2') \\
&+ \frac{\gamma_{u,1}}{n_1 + n_2 E_{2-\alpha,1}(\lambda)} (E_{2-\alpha,1}(\lambda t'^{2-\alpha}_1) - E_{2-\alpha,1}(\lambda {t_2'}^{2-\alpha}))\Big| \\
\leq&\Big| \int_0^{t_1'} E_{2-\alpha,1}(\lambda(t_1' - s)^{2-\alpha}) f(s, u(s), u'(s)) \mathrm{d}s - \int_0^{t_2'} E_{2-\alpha,1}(\lambda(t_2' - s)^{2-\alpha}) f(s, u(s), u'(s)) \mathrm{d}s\Big| \\
&+ L\left|G_0(t_1', s) - G_0(t_2', s)\right| + \frac{\left(m(|n_1| + 2|n_2|E_{2-\alpha,1}(\lambda)) + 1\right)L}{|n_1 + n_2 E_{2-\alpha}(\lambda)|}\left|E_{2-\alpha,1}(\lambda {t_1'}^{2-\alpha}) - E_{2-\alpha,1}(\lambda {t_2'}^{2-\alpha})\right| \\
<&\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

Thus, the operator $T(\Omega)$ is equicontinuous on every interval J_k.
According to the Arzela-Ascoli theorem, $T(\Omega)$ is relatively compact.
Therefore, $T : PC^1(J) \to PC^1(J)$ is completely continuous.

□

In the following, we give out some hypotheses.

(H2) $f(t, x_1, y_1) \leq f(t, x_2, y_2)$ if $t \in J$, $x_1 \leq x_2$, $y_1 \leq y_2 \in \mathbb{R}$. Furthermore, $f(t, x_1, y_1) < f(t, x_2, y_2)$ if $x_1 < x_2$ and $y_1 \leq y_2$.

And for $x_1 \leq x_2 \in \mathbb{R}$, $I_k(x_1) \leq I_k(x_2),\ Q_k(x_1) \leq Q_k(x_2),\ k = 1, 2, \cdots, m$.

(H3.1) If $m_1 + m_2 > 0,\ m_2 < 0,\ n_1 + n_2 E_{2-\alpha,1}(\lambda) > 0,\ n_2 < 0$, then for $y_1 \leq y_2, z_1 \leq z_2 \in \mathbb{R}$,

$$
g_0(z_2, y_2) - g_0(z_1, y_1) \geq -m_1(z_2 - z_1) - m_2(y_2 - y_1),
$$

and

$$
g_1(z_2, y_2) - g_1(z_1, y_1) \geq -n_1(z_2 - z_1) - n_2(y_2 - y_1).
$$

(H3.2) If $m_1+m_2<0,\ m_2>0,\ n_1+n_2E_{2-\alpha,1}(\lambda)>0,\ n_2<0$, then for $y_1\le y_2,\ z_1\le z_2\in\mathbb{R}$,

$$g_0(z_2,y_2)-g_0(z_1,y_1)\le -m_1(z_2-z_1)-m_2(y_2-y_1),$$

and

$$g_1(z_2,y_2)-g_1(z_1,y_1)\ge -n_1(z_2-z_1)-n_2(y_2-y_1).$$

(H3.3) If $m_1+m_2>0,\ m_2<0,\ n_1+n_2E_{2-\alpha,1}(\lambda)<0,\ n_2>0$, then for $y_1\le y_2,\ z_1\le z_2\in\mathbb{R}$,

$$g_0(z_2,y_2)-g_0(z_1,y_1)\ge -m_1(z_2-z_1)-m_2(y_2-y_1),$$

and

$$g_1(z_2,y_2)-g_1(z_1,y_1)\le -n_1(z_2-z_1)-n_2(y_2-y_1).$$

(H3.4) If $m_1+m_2<0,\ m_2>0,\ n_1+n_2E_{2-\alpha,1}(\lambda)<0,\ n_2>0$, then for $y_1\le y_2,\ z_1\le z_2\in\mathbb{R}$,

$$g_0(z_2,y_2)-g_0(z_1,y_1)\le -m_1(z_2-z_1)-m_2(y_2-y_1),$$

and

$$g_1(z_2,y_2)-g_1(z_1,y_1)\le -n_1(z_2-z_1)-n_2(y_2-y_1).$$

Remark. It is easy to see that if one of (H3.1)–(H3.4) holds, then (H1) holds.

Lemma 4.3. *For $m_i,\ n_i\in\mathbb{R}(i=1,\ 2)$, if one of the following conditions holds, then $x(t)\ge 0$ and $x'(t)\ge 0$ for $t\in J$.*

(1) $m_1+m_2>0,\ m_2<0,\ n_1+n_2E_{2-\alpha,1}(\lambda)>0,\ n_2<0$ and $x\in PC^1(J)$ satisfies

$$\begin{cases} x''(t)-\lambda{}^cD_{0^+}^{\alpha}x(t)\ge 0,\ t\in J',\\ \Delta x(t)|_{t=t_k}\ge 0,\ k=1,\ 2,\cdots,\ m,\\ \Delta x'(t)|_{t=t_k}\ge 0,\ k=1,\ 2,\cdots,\ m,\\ m_1x(0)+m_2x(1)\ge 0,\ n_1x'(0)+n_2x'(1)\ge 0.\end{cases}$$

(2) $m_1+m_2<0,\ m_2>0,\ n_1+n_2E_{2-\alpha,1}(\lambda)>0,\ n_2<0$ and $x\in PC^1(J)$ satisfies

$$\begin{cases} x''(t)-\lambda{}^cD_{0^+}^{\alpha}x(t)\ge 0,\ t\in J',\\ \Delta x(t)|_{t=t_k}\ge 0,\ k=1,\ 2,\cdots,\ m,\\ \Delta x'(t)|_{t=t_k}\ge 0,\ k=1,\ 2,\cdots,\ m,\\ m_1x(0)+m_2x(1)\le 0,\ n_1x'(0)+n_2x'(1)\ge 0.\end{cases}$$

(3) $m_1+m_2>0,\ m_2<0,\ n_1+n_2E_{2-\alpha,1}(\lambda)<0,\ n_2>0$ and $x\in PC^1(J)$ satisfies

$$\begin{cases} x''(t)-\lambda{}^cD_{0^+}^{\alpha}x(t)\ge 0,\ t\in J',\\ \Delta x(t)|_{t=t_k}\ge 0,\ k=1,\ 2,\cdots,\ m,\\ \Delta x'(t)|_{t=t_k}\ge 0,\ k=1,\ 2,\cdots,\ m,\\ m_1x(0)+m_2x(1)\ge 0,\ n_1x'(0)+n_2x'(1)\le 0.\end{cases}$$

(4) $m_1+m_2<0,\ m_2>0,\ n_1+n_2E_{2-\alpha,1}(\lambda)<0,\ n_2>0$ and $x\in PC^1(J)$ satisfies

$$\begin{cases} x''(t)-\lambda{}^cD_{0^+}^{\alpha}x(t)\ge 0,\ t\in J',\\ \Delta x(t)|_{t=t_k}\ge 0,\ k=1,\ 2,\cdots,\ m,\\ \Delta x'(t)|_{t=t_k}\ge 0,\ k=1,\ 2,\cdots,\ m,\\ m_1x(0)+m_2x(1)\le 0,\ n_1x'(0)+n_2x'(1)\le 0.\end{cases}$$

Proof. (1) Denote $x''(t) - \lambda^c D_{0^+}^\alpha x(t) = h(t),\ \Delta x(t)|_{t=t_k} = p_k,\ \Delta x'(t)|_{t=t_k} = q_k,\ k = 1,\ 2, \cdots, m,\ m_1 x(0) + m_2 x(1) = \gamma_0,\ n_1 x'(0) + n_2 x'(1) = \gamma_1$. Then $h(t) \geq 0,\ p_k \geq 0,\ \ q_k \geq 0,\ k = 1,\ 2,\ \cdots, m$.

By Lemma 3.1, the following boundary value problem

$$\begin{cases} x''(t) - \lambda^c D_{0^+}^\alpha x(t) = h(t),\ t \in J', \\ \Delta x(t)|_{t=t_k} = p_k,\ k = 1,\ 2, \cdots, m, \\ \Delta x'(t)|_{t=t_k} = q_k,\ k = 1,\ 2, \cdots, m, \\ m_1 x(0) + m_2 x(1) = \gamma_0,\ n_1 x'(0) + n_2 x'(1) = \gamma_1, \end{cases}$$

has a unique solution

$$x(t) = \int_0^t G(t,s)h(s)\mathrm{d}s + \varphi(t) + \Big(\frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha})}{n_1 + n_2 E_{2-\alpha,1}(\lambda)} - \frac{m_2 E_{2-\alpha,2}(\lambda)}{(m_1+m_2)(n_1 + n_2 E_{2-\alpha,1}(\lambda))}\Big)\gamma_1 + \frac{\gamma_0}{m_1+m_2},\ t \in J,$$

and

$$x'(t) = \int_0^1 G'_t(t,s)h(s)\mathrm{d}s + \varphi'(t) + \frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{n_1 + n_2 E_{2-\alpha,1}(\lambda)}\gamma_1,\ t \in J.$$

It's easy to see that $x(t) \geq 0$ and $x'(t) \geq 0$ for $t \in J$.

Similarly, (2)–(4) are easy to be proved. □

Lemma 4.4. *Suppose (H2) and one of (H3.1)-(H3.4) holds, then T is a strongly increasing operator.*

Proof. Here, we will only prove the conclusion when (H3.1) holds, and other situations are similar.

For any $u_1,\ u_2 \in PC^1(J)$, and $u_1 \prec u_2$ which implies that $u_1(t) \leq u_2(t),\ u_1'(t) \leq u_2'(t)$ and $u_1(t) \not\equiv u_2(t)$ for $t \in J$. By (H2), for any $t \in J$,

$$f(t, u_2(t), u_2'(t)) - f(t, u_1(t), u_1'(t)) \geq 0,$$
$$I_k(u_2(t_k)) - I_k(u_1(t_2)) \geq 0,\ Q_k(u_2(t_k)) - Q_k(u_2(t_k)) \geq 0.$$

Since $u_1(t) \not\equiv u_2(t)$, there exists an interval $[a,b] \subset J$ such that $u_1(t) < u_2(t)$ for $t \in [a,b]$. It follows from (H2)

$$f(t, u_2(t), u_2'(t)) - f(t, u_1(t), u_1'(t)) > 0, t \in [a,b]. \tag{4.4}$$

By (H3.1), we can get that

$$\begin{aligned} \gamma_{u_2,0} - \gamma_{u_1,0} =& g_0(u_2(0), u_2(1)) - g_0(u_1(0), u_1(1)) \\ &+ (m_1 u_2(0) + m_2 u_2(1)) - (m_1 u_1(0) + m_2 u_1(1)) \\ \geq& 0, \\ \gamma_{u_2,1} - \gamma_{u_1,1} =& g_1(u_2(0), u_2(1)) - g_1(u_1(0), u_1(1)) \\ &+ (n_1 u_2(0) + n_2 u_2(1)) - (n_1 u_1(0) n_2 u_1(1)) \\ \geq& 0. \end{aligned}$$

By (4.2), (4.3), (H3.1) and (H2), we can get that for $t \in J$,

$$\varphi_{u_2}(t) - \varphi_{u_1}(t)$$

$$
\begin{aligned}
&= \sum_{0<t_i<t} \Big(I_i(u_2(t_i)) - I_i(u_1(t_i))\Big) - \frac{m_2}{m_1+m_2}\sum_{i=1}^{m}\Big(I_i(u_2(t_i)) - I_i(u_1(t_i))\Big)\\
&\quad + \sum_{0<t_i<t} \frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha}) - t_iE_{2-\alpha,2}(\lambda t_i^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}\Big(Q_i(u_2(t_i)) - Q_i(u_1(t_i))\Big)\\
&\quad + \sum_{i=1}^{m}\Big(- \frac{m_2(E_{2-\alpha,2}(\lambda) - t_iE_{2-\alpha,1}(\lambda t_i^{2-\alpha}))}{(m_1+m_2)E_{2-\alpha,1}(\lambda t_i^{2-\alpha})} - \frac{n_2E_{2-\alpha,1}(\lambda)(tE_{2-\alpha,2}(\lambda t^{2-\alpha}))}{(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}\\
&\quad + \frac{m_2n_2E_{2-\alpha,2}(\lambda)E_{2-\alpha,1}(\lambda)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}\Big)\Big(Q_i(u_2(t_i)) - Q_i(u_1(t_i))\Big)\\
&\geq \sum_{0<t_i<t} \Big(I_i(u_2(t_i)) - I_i(u_1(t_i))\Big) - \frac{m_2}{m_1+m_2}\sum_{i=1}^{m}\Big(I_i(u_2(t_i)) - I_i(u_1(t_i))\Big)\\
&\quad + \sum_{i=1}^{m} \frac{m_2n_2E_{2-\alpha,2}(\lambda)E_{2-\alpha,1}(\lambda)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda)}\Big(Q_i(u_2(t_i)) - Q_i(u_1(t_i))\Big)\\
&\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\varphi'_{u_2}(t) - \varphi'_{u_1}(t)\\
&= \sum_{0<t_i<t} \frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}\Big(Q_i(u_2(t_i)) - Q_i(u_1(t_i))\Big)\\
&\quad - \sum_{i=1}^{m} \frac{n_2E_{2-\alpha,1}(\lambda)E_{2-\alpha,1}(\lambda t^{2-\alpha})}{(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda t_i^{2-\alpha})}\Big(Q_i(u_2(t_i)) - Q_i(u_1(t_i))\Big)\\
&\geq \sum_{0<t_i<t} \frac{1}{E_{2-\alpha,1}(\lambda)}\Big(Q_i(u_2(t_i)) - Q_i(u_1(t_i))\Big)\\
&\quad - \sum_{i=1}^{m} \frac{n_2E_{2-\alpha,1}(\lambda)}{(n_1+n_2E_{2-\alpha,1}(\lambda))E_{2-\alpha,1}(\lambda)}\Big(Q_i(u_2(t_i)) - Q_i(u_1(t_i))\Big)\\
&\geq 0.
\end{aligned}
$$

By Lemma 3.2, (H2) and (4.4), for any $u_1,\ u_2 \in PC^1(J)$ and $u_1 \prec u_2$, as $t \in J$, we have

$$
\begin{aligned}
Tu_2(t) - Tu_1(t) &= \int_0^1 G(t,s)\Big(f(s,u_2(s),u_2'(s)) - f(s,u_1(s),u_1'(s))\Big)\mathrm{d}s + (\varphi_{u_2}(t) - \varphi_{u_1}(t))\\
&\quad + \Big(\frac{tE_{2-\alpha,2}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)} - \frac{m_2E_{2-\alpha,2}(\lambda)}{(m_1+m_2)(n_1+n_2E_{2-\alpha,1}(\lambda))}\Big)(\gamma_{u_2,1} - \gamma_{u_1,1})\\
&\quad + \frac{\gamma_{u_2,0} - \gamma_{u_1,0}}{m_1+m_2}\\
&\geq \int_a^b G(0,s)\Big(f(s,u_2(s),u_2'(s)) - f(s,u_1(s),u_1'(s))\Big)\mathrm{d}s\\
&> 0,
\end{aligned}
$$

and

$$
\begin{aligned}
(Tu_2)'(t)-(Tu_1)'(t)=&\int_0^1 G_t'(t,s)\Big(f(s,u_2(s),u_2'(s))-f(s,u_1(s),u_1'(s))\Big)\mathrm{d}s+(\varphi_{u_2}'(t)-\varphi_{u_1}'(t))\\
&+\frac{E_{2-\alpha,1}(\lambda t^{2-\alpha})}{n_1+n_2E_{2-\alpha,1}(\lambda)}(\gamma_{u_2,1}-\gamma_{u_1,1})\\
\geq&\int_a^b G_t'(0,s)\Big(f(s,u_2(s),u_2'(s))-f(s,u_1(s),u_1'(s))\Big)\mathrm{d}s\\
>&0.
\end{aligned}
$$

Thus,

$$Tu_2-Tu_1\in\mathring{P}$$

which implies that T is a strongly increasing operator.

The proof is completed. □

Theorem 4.5. *If (H2) and (H3.1) hold, there exist $z_i,\ y_i\in PC^1(J),\ i=1,\ 2$, with $z_1\prec y_1\prec z_2\prec y_2$ such that*

$$
\begin{cases}
z_i''(t)-\lambda^cD_{0^+}^{\alpha}z_i(t)\leq f(t,z_i(t),z_i'(t)),\ t\in J',\\
\Delta z_i(t)|_{t=t_k}\leq I_k(z_i(t_k)),\ k=1,\ 2,\cdots,\ m,\\
\Delta z_i'(t)|_{t=t_k}\leq Q_k(z_i(t_k)),\ k=1,\ 2,\cdots,\ m,\\
g_0(z_i(0),z_i(1))\geq 0,\ g_1(z_i'(0),z_i'(1))\geq 0,
\end{cases}
\tag{4.5}
$$

and

$$
\begin{cases}
y_i''(t)-\lambda^cD_{0^+}^{\alpha}y_i(t)\geq f(t,y_i(t),y_i'(t)),\ t\in J',\\
\Delta y_i(t)|_{t=t_k}\geq I_k(y_i(t_k)),\ k=1,\ 2,\cdots,\ m,\\
\Delta y_i'(t)|_{t=t_k}\geq Q_k(y_i(t_k)),\ k=1,\ 2,\cdots,\ m,\\
g_0(y_i(0),y_i(1))\leq 0,\ g_1(y_i'(0),y_i'(1))\leq 0,
\end{cases}
\tag{4.6}
$$

where $i=1,2$, z_2 and y_1 are not the solutions of boundary value problem (1.1). Then boundary value problem (1.1) has at least three distinct solutions $x_1,\ x_2,\ x_3\in[z_1,\ y_2]$ and satisfies

$$z_1(t)\leq x_1(t)<y_1(t),\ z_2(t)<x_2(t)\leq y_2(t),\ z_2(t)\nleq x_2(t)\nleq y_1(t),\ t\in J.$$

Proof. By Lemma 4.2 and Lemma 4.4, we can get that $T:[z_1,y_2]\to PC^1(J)$ is a completely continuous strongly increasing operator.

By the definition of operator T, we can show that

$$
\begin{cases}
(Tz_1)''(t)-\lambda^cD_{0^+}^{\alpha}(Tz_1)(t)=f(t,z_1(t),z_1'(t)),t\in J',\\
\Delta(Tz_1)(t)|_{t=t_k}=I_k(z_1(t_k)),\ k=1,\ 2,\cdots,\ m,\\
\Delta(Tz_1)'(t)|_{t=t_k}=Q_k(z_1(t_k)),\ k=1,\ 2,\cdots,\ m,\\
m_1(Tz_1)(0)+m_2(Tz_1)(1)=g_0(z_1(0),z_1(1))+m_1z_1(0)+m_2z_1(1),\\
n_1(Tz_1)'(0)+n_2(Tz_1)'(1)=g_1(z_1'(0),z_1'(1))+n_1z_1'(0)+n_2z_1'(1).
\end{cases}
$$

Let $x=Tz_1-z_1$. By (4.5),

$$x''(t)-\lambda^cD_{0^+}^{\alpha}x(t)=((Tz_1)''(t)-z_1''(t))-\lambda({}^cD_{0^+}^{\alpha}(Tz_1)(t)-{}^cD_{0^+}^{\alpha}z_1(t))$$

$$
\begin{aligned}
&= ((Tz_1)''(t) - \lambda^c D_{0^+}^{\alpha}(Tz_1)(t)) - (z_1''(t) - \lambda^c D_{0^+}^{\alpha} z_1(t)) \\
&= f(t, z_1(t), z_1'(t)) - (z_1''(t) - \lambda^c D_{0^+}^{\alpha} z_1(t)) \\
&\geq 0, \ t \in J'.
\end{aligned}
$$

$$
\begin{aligned}
\Delta x(t)|_{t=t_k} &= \Delta T z_1(t)|_{t=t_k} - \Delta z_1(t)|_{t=t_k} = I_k(z_1(t_k)) - \Delta z_1(t)|_{t=t_k} \geq 0. \\
\Delta x'(t)|_{t=t_k} &= \Delta (T z_1)'(t)|_{t=t_k} - \Delta z_1'(t)|_{t=t_k} = Q_k(z_1(t_k)) - \Delta z_1'(t)|_{t=t_k} \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
m_1 x(0) + m_2 x(1) &= m_1((Tz_1)(0) - z_1(0)) + m_2((Tz_1)(1) - z_1(1)) \\
&= m_1((Tz_1)(0) + m_2(Tz_1)(1)) - (m_1 z_1(0) + m_2 z_1(1)) \\
&= g_0(z_1(0), z_1(1)) + (m_1 z_1(0) + m_2 z_1(1)) - (m_1 z_1(0) + m_2 z_1(1)) \\
&= g_0(z_1(0), z_1(1)) \\
&\geq 0.
\end{aligned}
$$

Similarly, we can get that

$$n_1 x'(0) + n_2 x'(1) \geq 0.$$

By (1) in Lemma 4.3, we have

$$x(t) = (Tz_1)(t) - z_1(t) \geq 0, \ x'(t) = (Tz_1)'(t) - z_1'(t) \geq 0, \ t \in J.$$

Therefore, $z_1 \preceq T z_1$.

It is similar that we can obtain $z_2 \preceq T z_2$. Because z_2 is not the solution of boundary value problem (1.1), then $T z_2 \neq z_2$. Thus, $z_2 \prec T z_2$.

By using the same method, we can easily get $T y_1 \prec y_1, \ T y_2 \preceq y_2$.

By using Lemma 2.6, we can get that the operator T has at least three distinct fixed points $x_1, \ x_2, \ x_3 \in [z_1, \ y_2]$, and satisfies

$$z_1 \preceq x_1 \prec\prec y_1, \ z_2 \prec\prec x_2 \preceq y_2, \ z_2 \npreceq x_3 \npreceq y_1.$$

Therefore, by Lemma 4.1, we can obtain that boundary value problem (1.1) has at least three distinct solutions $x_1, \ x_2, \ x_3 \in [z_1, \ y_2]$, and

$$z_1(t) \leq x_1(t) < y_1(t), \ z_2(t) < x_2(t) \leq y_2(t), \ z_2(t) \nleq x_2(t) \nleq y_1(t), \ t \in J.$$

The proof is completed. □

In the similar way, the following three theorems can be established.

Theorem 4.6. *If (H2) and (H3.2) hold, there exist* $z_i, \ y_i \in PC^1(J), \ i = 1, \ 2,$ *with* $z_1 \prec y_1 \prec z_2 \prec y_2$ *such that*

$$
\begin{cases}
z_i''(t) - \lambda^c D_{0^+}^{\alpha} z_i(t) \leq f(t, z_i(t), z_i'(t)), \ t \in J', \\
\Delta z_i(t)|_{t=t_k} \leq I_k(z_i(t_k)), \ k = 1, \ 2, \cdots, \ m, \\
\Delta z_i'(t)|_{t=t_k} \leq Q_k(z_i(t_k)), \ k = 1, \ 2, \cdots, \ m, \\
g_0(z_i(0), z_i(1)) \leq 0, \ g_1(z_i'(0), z_i'(1)) \geq 0,
\end{cases}
$$

and

$$\begin{cases} y_i''(t) - \lambda^c D_{0^+}^{\alpha} y_i(t) \geq f(t, y_i(t), y_i'(t)), \ t \in J', \\ \Delta y_i(t)|_{t=t_k} \geq I_k(y_i(t_k)), \ k = 1, \ 2, \cdots, \ m, \\ \Delta y_i'(t)|_{t=t_k} \geq Q_k(y_i(t_k)), \ k = 1, \ 2, \cdots, \ m, \\ g_0(y_i(0), y_i(1)) \geq 0, \ g_1(y_i'(0), y_i'(1)) \leq 0, \end{cases}$$

where $i = 1, 2$, z_2 and y_1 are not the solutions of boundary value problem (1.1). Then the boundary value problem (1.1) has at least three distinct solutions $x_1, \ x_2, \ x_3 \in [z_1, \ y_2]$ and satisfies

$$z_1(t) \leq x_1(t) < y_1(t), \ z_2(t) < x_2(t) \leq y_2(t), \ z_2(t) \nleq x_2(t) \nleq y_1(t), \ t \in J.$$

Theorem 4.7. *If (H2) and (H3.3) hold, there exist $z_i, \ y_i \in PC^1(J), \ i = 1, \ 2$, with $z_1 \prec y_1 \prec z_2 \prec y_2$ such that*

$$\begin{cases} z_i''(t) - \lambda^c D_{0^+}^{\alpha} z_i(t) \leq f(t, z_i(t), z_i'(t)), \ t \in J', \\ \Delta z_i(t)|_{t=t_k} \leq I_k(z_i(t_k)), \ k = 1, \ 2, \cdots, \ m, \\ \Delta z_i'(t)|_{t=t_k} \leq Q_k(z_i(t_k)), \ k = 1, \ 2, \cdots, \ m, \\ g_0(z_i(0), z_i(1)) \geq 0, \ g_1(z_i'(0), z_i'(1)) \leq 0, \end{cases}$$

and

$$\begin{cases} y_i''(t) - \lambda^c D_{0^+}^{\alpha} y_i(t) \geq f(t, y_i(t), y_i'(t)), \ t \in J', \\ \Delta y_i(t)|_{t=t_k} \geq I_k(y_i(t_k)), \ k = 1, \ 2, \cdots, \ m, \\ \Delta y_i'(t)|_{t=t_k} \geq Q_k(y_i(t_k)), \ k = 1, \ 2, \cdots, \ m, \\ g_0(y_i(0), y_i(1)) \leq 0, \ g_1(y_i'(0), y_i'(1)) \geq 0, \end{cases}$$

where $i = 1, 2$, z_2 and y_1 are not the solutions of boundary value problem (1.1). Then the boundary value problem (1.1) has at least three distinct solutions $x_1, \ x_2, \ x_3 \in [z_1, \ y_2]$ and satisfies

$$z_1(t) \leq x_1(t) < y_1(t), \ z_2(t) < x_2(t) \leq y_2(t), \ z_2(t) \nleq x_2(t) \nleq y_1(t), \ t \in J.$$

Theorem 4.8. *If (H2) and (H3.4) hold, there exist $z_i, \ y_i \in PC^1(J), \ i = 1, \ 2$, with $z_1 \prec y_1 \prec z_2 \prec y_2$ such that*

$$\begin{cases} z_i''(t) - \lambda^c D_{0^+}^{\alpha} z_i(t) \leq f(t, z_i(t), z_i'(t)), \ t \in J', \\ \Delta z_i(t)|_{t=t_k} \leq I_k(z_i(t_k)), \ k = 1, \ 2, \cdots, \ m, \\ \Delta z_i'(t)|_{t=t_k} \leq Q_k(z_i(t_k)), \ k = 1, \ 2, \cdots, \ m, \\ g_0(z_i(0), z_i(1)) \leq 0, \ g_1(z_i'(0), z_i'(1)) \leq 0, \end{cases}$$

and

$$\begin{cases} y_i''(t) - \lambda^c D_{0^+}^{\alpha} y_i(t) \geq f(t, y_i(t), y_i'(t)), \ t \in J', \\ \Delta y_i(t)|_{t=t_k} \geq I_k(y_i(t_k)), \ k = 1, \ 2, \cdots, \ m, \\ \Delta y_i'(t)|_{t=t_k} \geq Q_k(y_i(t_k)), \ k = 1, \ 2, \cdots, \ m, \\ g_0(y_i(0), y_i(1)) \geq 0, \ g_1(y_i'(0), y_i'(1)) \geq 0, \end{cases}$$

where $i = 1, 2$, z_2 and y_1 are not the solutions of boundary value problem (1.1). Then the boundary value problem (1.1) has at least three distinct solutions $x_1, \ x_2, \ x_3 \in [z_1, \ y_2]$ and satisfies

$$z_1(t) \leq x_1(t) < y_1(t), \ z_2(t) < x_2(t) \leq y_2(t), \ z_2(t) \nleq x_2(t) \nleq y_1(t), \ t \in J.$$

5. Discussion on applicability of the main results

In this section, we discuss the applicability of our main results.

Consider the following the boundary value problem

$$\begin{cases} x''(t) - \frac{1}{100}{}^cD_{0^+}^{\frac{1}{2}}x(t) = \frac{2t^3}{\pi}\arctan(x(t) + x'(t)), \ t \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \\ \Delta x(t)|_{t=\frac{1}{2}} = \frac{1}{2}x(\frac{1}{2}), \ \Delta x'(t)|_{t=\frac{1}{2}} = \frac{1}{209}x(\frac{1}{2}), \\ -\cos(2\pi x(0)) + 0.02x(1) - 0.52 = 0, \\ -6\pi\cos(\pi x'(0)) + \pi x'(1) - 6\pi = 0, \end{cases} \tag{5.1}$$

where $\alpha = \frac{1}{2}, \ \lambda = \frac{1}{100}, \ f(t, x, y) = \frac{2t^3}{\pi}\arctan(x + y), \ I_1(x) = \frac{1}{2}x, \ Q_1(x) = \frac{1}{209}x.$

$$g_0(x, y) = -\cos(2\pi x) + 0.02y - 0.52, \ g_1(x, y) = -6\pi\cos(\pi x) + \pi y - 6\pi.$$

Obviously, $f \in C([0, 1] \times \mathbb{R}^2, \mathbb{R}), \ I_1, \ Q_1 \in C(\mathbb{R}, \mathbb{R}), \ g_0, \ g_1 \in C(\mathbb{R}^2, \mathbb{R})$ are nonlinear functions. And $f, \ I_1, \ Q_1$ satisfy (H2).

Let $m_1 = 2\pi, \ m_2 = -0.02, \ n_1 = 6\pi, \ n_2 = -\pi$. Since

$$\begin{aligned} g_0(x_2, y_2) - g_0(x_1, y_1) &= -\cos(2\pi x_2) + \cos(2\pi x_1) + 0.02(y_2 - y_1) \\ &\geq -2\pi(x_2 - x_1) + 0.02(y_2 - y_1), \\ g_1(x_2, y_2) - g_1(x_1, y_1) &= -6\pi\cos(\pi x_2) + 6\pi\cos(\pi x_1) + \pi(y_2 - y_1) \\ &\geq -6\pi(x_2 - x_1) + \pi(y_2 - y_1), \end{aligned}$$

then $g_0, \ g_1$ satisfy (H3.1).

For $t \in [0, 1]$, we take

$$z_1(t) = \begin{cases} \frac{3}{2} + t, \ t \in [0, \ \frac{1}{2}], \\ 2 + t, \ t \in (\frac{1}{2}, \ 1], \end{cases} \quad y_1(t) = \begin{cases} 4 + \frac{3}{2}t + \frac{1}{8}t^2 + \frac{1}{4}t^4, \ t \in [0, \ \frac{1}{2}], \\ 9 + \frac{3}{2}t + \frac{1}{6}t^2 + \frac{1}{4}t^4, \ t \in (\frac{1}{2}, \ 1], \end{cases}$$

$$z_2(t) = \begin{cases} \frac{23}{2} + 3t, \ t \in [0, \ \frac{1}{2}], \\ 12 + 3t, \ t \in (\frac{1}{2}, \ 1], \end{cases} \quad y_2(t) = \begin{cases} 16 + \frac{7}{2}t + \frac{2}{5}t^2 + \frac{1}{6}t^4, \ t \in [0, \ \frac{1}{2}], \\ 33 + \frac{7}{2}t + \frac{3}{4}t^2 + \frac{1}{6}t^4, \ t \in (\frac{1}{2}, \ 1]. \end{cases}$$

We can easily get that

$$z_1''(t) - \frac{1}{100}{}^cD_{0^+}^{\frac{1}{2}}z_1(t) - \frac{2t^3}{\pi}\arctan(z_1(t) + z_1'(t)) = \begin{cases} -\frac{\sqrt{t}}{50\sqrt{\pi}} - \frac{2t^3\arctan(\frac{5}{2}+t)}{\pi} < 0, \ t \in (0, \frac{1}{2}), \\ -\frac{\sqrt{t}}{50\sqrt{\pi}} - \frac{2t^3\arctan(3+t)}{\pi} < 0, \ t \in (\frac{1}{2}, 1), \end{cases}$$

$$\begin{aligned} & y_1''(t) - \frac{1}{100}{}^cD_{0^+}^{\frac{1}{2}}y_1(t) - \frac{2t^3}{\pi}\arctan(y_1(t) + y_1'(t)) \\ = & \begin{cases} \frac{1}{4} + 3t^2 - \frac{\sqrt{t}(315+35t+96t^3)}{10500\sqrt{\pi}} - \frac{2t^3\arctan(\frac{11}{2}+\frac{7t}{4}+\frac{t^2}{8}+t^3+\frac{t^4}{4})}{\pi} > 0, \quad t \in [0, \frac{1}{2}], \\ \frac{84000\sqrt{\pi}\sqrt{t}(1+9t^2)-7560t-840t^2-2304t^4-35\sqrt{2}\left(\sqrt{t(2t-1)}+4\sqrt{t^3(2t-1)}\right)}{252000\sqrt{\pi}\sqrt{t}} \\ \quad -\frac{2t^3\arctan(\frac{21}{2}+\frac{11t}{6}+\frac{t^2}{6}+t^3+\frac{t^4}{4})}{\pi} > 0, \quad t \in (\frac{1}{2}, 1], \end{cases} \end{aligned}$$

$$z_2''(t)-\frac{1}{100}{}^cD_{0^+}^{\frac{1}{2}}z_2(t)-\frac{2t^3}{\pi}\arctan(z_2(t)+z_2'(t))=\begin{cases}-\frac{3\sqrt{t}}{50\sqrt{\pi}}-\frac{2t^3\arctan(\frac{29}{2}+3t)}{\pi}<0,\ t\in(0,\frac{1}{2}),\\ -\frac{3\sqrt{t}}{50\sqrt{\pi}}-\frac{2t^3\arctan(15+3t)}{\pi}<0,\ t\in(\frac{1}{2},1),\end{cases}$$

and

$$\begin{aligned}&y_2''(t)-\frac{1}{100}{}^cD_{0^+}^{\frac{1}{2}}y_2(t)-\frac{2t^3}{\pi}\arctan(y_2(t)+y_2'(t))\\ =&\begin{cases}\frac{4}{5}+2t^2-\frac{\sqrt{t}(735+112t+64t^3)}{10500\sqrt{\pi}}-\frac{2t^3\arctan(\frac{39}{2}+\frac{43t}{10}+\frac{2t^2}{5}+\frac{2t^3}{3}+\frac{t^4}{6})}{\pi}>0,\quad t\in[0,\frac{1}{2}],\\ \frac{21000\sqrt{\pi}\sqrt{t}(3+4t^2)-2940t-448t^2-256t^4-49\sqrt{2}\left(\sqrt{t(2t-1)}+4\sqrt{t^3(2t-1)}\right)}{42000\sqrt{\pi}\sqrt{t}}\\ \quad-\frac{2t^3\arctan(\frac{73}{2}+5t+\frac{3t^2}{4}+\frac{2t^3}{3}+\frac{t^4}{6})}{\pi}>0,\quad t\in(\frac{1}{2},1].\end{cases}\end{aligned}$$

So

$$\begin{cases}z_1''(t)-\frac{1}{100}{}^cD_{0^+}^{\frac{1}{2}}z_1(t)<\frac{2t^3}{\pi}\arctan(z_1(t)+z_1'(t)),\ t\in(0,\frac{1}{2})\cup(\frac{1}{2},1),\\ \Delta z_1(t)|_{t=\frac{1}{2}}=z_1(\frac{1}{2}^+)-z_1(\frac{1}{2}^-)=\frac{1}{2}<\frac{1}{2}z_1(\frac{1}{2})=1,\\ \Delta z_1'(t)|_{t=\frac{1}{2}}=z_1'(\frac{1}{2}^+)-z_1'(\frac{1}{2}^-)=0<\frac{1}{209}z_1(\frac{1}{2})=\frac{2}{209},\\ -\cos(2\pi z_1(0))+0.02z_1(1)-0.52>0,\\ -6\pi\cos(\pi z_1'(0))+\pi z_1'(1)-6\pi>0,\end{cases}$$

$$\begin{cases}y_1''(t)-\frac{1}{100}{}^cD_{0^+}^{\frac{1}{2}}y_1(t)>\frac{2t^3}{\pi}\arctan(y_1(t)+y_1'(t)),\ t\in(0,\frac{1}{2})\cup(\frac{1}{2},1),\\ \Delta y_1(t)|_{t=\frac{1}{2}}=y_1(\frac{1}{2}^+)-y_1(\frac{1}{2}^-)=\frac{481}{96}>\frac{1}{2}y_1(\frac{1}{2})=\frac{307}{128},\\ \Delta y_1'(t)|_{t=\frac{1}{2}}=y_1'(\frac{1}{2}^+)-y_1'(\frac{1}{2}^-)=\frac{1}{24}>\frac{1}{209}y_1(\frac{1}{2})=\frac{307}{13376},\\ -\cos(2\pi y_1(0))+0.02y_1(1)-0.52<0,\\ -6\pi\cos(\pi y_1'(0))+\pi y_1'(1)-6\pi<0,\end{cases}$$

$$\begin{cases}z_2''(t)-\frac{1}{100}{}^cD_{0^+}^{\frac{1}{2}}z_2(t)<\frac{2t^3}{\pi}\arctan(z_2(t)+z_2'(t)),\ t\in(0,\frac{1}{2})\cup(\frac{1}{2},1),\\ \Delta z_2(t)|_{t=\frac{1}{2}}=z_2(\frac{1}{2}^+)-z_2(\frac{1}{2}^-)=\frac{1}{2}<\frac{1}{2}z_2(\frac{1}{2})=\frac{13}{2},\\ \Delta z_2'(t)|_{t=\frac{1}{2}}=z_2'(\frac{1}{2}^+)-z_2'(\frac{1}{2}^-)=0<\frac{1}{209}z_2(\frac{1}{2})=\frac{13}{209},\\ -\cos(2\pi z_2(0))+0.02z_2(1)-0.52>0,\\ -6\pi\cos(\pi z_2'(0))+\pi z_2'(1)-6\pi>0\end{cases}$$

and

$$\begin{cases}y_2''(t)-\frac{1}{100}{}^cD_{0^+}^{\frac{1}{2}}y_2(t)>\frac{2t^3}{\pi}\arctan(y_2(t)+y_2'(t)),\ t\in(0,\frac{1}{2})\cup(\frac{1}{2},1),\\ \Delta y_2(t)|_{t=\frac{1}{2}}=y_2(\frac{1}{2}^+)-y_2(\frac{1}{2}^-)=\frac{1367}{80}>\frac{1}{2}y_2(\frac{1}{2})=\frac{8573}{960},\\ \Delta y_2'(t)|_{t=\frac{1}{2}}=y_2'(\frac{1}{2}^+)-y_2'(\frac{1}{2}^-)=\frac{7}{20}>\frac{1}{209}y_2(\frac{1}{2})=\frac{8573}{100320},\\ -\cos(2\pi y_2(0))+0.02y_2(1)-0.52<0,\\ -6\pi\cos(\pi y_2'(0))+\pi y_2'(1)-6\pi<0.\end{cases}$$

We can easily get that $z_1(t),\ z_2(t)$ satisfy (4.5) and $y_1(t),\ y_2(t)$ satisfy (4.6) with $z_1\prec y_1\prec z_2\prec y_2$.

The conditions of Theorem 4.5 are all satisfied. So by Theorem 4.5, the boundary value problem (5.1) has at least three distinct solutions $x_1,\ x_2,\ x_3\in[z_1,y_2]$, and moreover,

$$z_1(t)\le x_1(t)<y_1(t),\ z_2(t)<x_2(t)\le y_2(t),\ z_2(t)\nleq x_3(t)\nleq y_1(t),\ t\in[0,1].$$

6. Conclusion

In this work, we investigate the existence of solutions for a class of second order impulsive vibration equation with fractional derivatives. Some sufficient conditions for existence of the multiplicity solutions are established by applying monotone iterative technique. Finally, a concrete example is given to illustrate the wide range of potential applications of our main results.

Further extensions of this paper are to study the motion state of the vibrator in the system described by boundary value problem (1.1) and the existence of solutions to the boundary value problems with other boundary conditions. Moreover, fractional differential equation models such as the rheological model of the fractional derivative and singular systems model of fractional differential equations have real world applications. So, we also can consider using the boundary value problem of impulsive differential equations to simulate the abrupt changes in the systems described by these models.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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