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*Research article*

# On the stability of two functional equations for $(S, N)$-implications

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**Abstract:** The iterative functional equation $\alpha \to (\alpha \to \beta) = \alpha \to \beta$ and the law of importation $(\alpha \wedge \beta) \to \gamma = \alpha \to (\beta \to \gamma)$ are two tautologies in classical logic. In fuzzy logics, they are two important properties, and are respectively formulated as $I(\alpha,\beta) = I(\alpha, I(\alpha,\beta))$ and $I(T(\alpha,\beta),\gamma) = I(\alpha, I(\beta,\gamma))$ where $I$ is a fuzzy implication and $T$ is a $t$-norm. Over the past several years, solutions to these two functional equations involving different classes of fuzzy implications have been studied. However, there are no results about stability study of fuzzy functional equations involving fuzzy implication. This paper discusses fuzzy implications that do not strictly satisfying these equations, but approximately satisfy these equations. Then we establish the Hyers-Ulam stability of the iterative functional equation involving the $(S, N)$-implication, where the $(S, N)$-implication is a common class of fuzzy implications generated by a continuous $t$-conorm $S$ and a continuous fuzzy negation $N$. Furthermore, given a fixed $t$-norm (the minimum $t$-norm or the product $t$-norm) the Hyers-Ulam stability of the law of importation involving the $(S, N)$-implication is studied.

**Keywords:** fuzzy implications; functional equations; iterative Boolean-like law; law of importation; stability; $(S, N)$-implication
**Mathematics Subject Classification:** 03E72, 39B82

## 1. Introduction

Functional equations in fuzzy logic involving fuzzy implications are generalizations of the corresponding tautologies in classical logic with Boolean implications. They are general forms of properties of fuzzy implications, such as identity principle, laws of contraposition, exchange principle, the law of importation, and so on. These properties are very important in fuzzy reasoning [1] and image processing [2].

Investigations into functional equations have mainly focused on their solutions for various fuzzy implications. For example, the function $\alpha \to (\alpha \to \beta) = \alpha \to \beta$, called the derived iterative Boolean law, is a tautology in Boolean logic. It is formulated in fuzzy logic as

$$I(\alpha, I(\alpha,\beta)) = I(\alpha,\beta), \tag{1}$$

where $I$ is a fuzzy implication. Shi et al. [3] characterized the solutions of Eq (1) for several different types of fuzzy implications. Xie and Qin [4] studied the solutions of Eq (1) for D-operations. Massanet and Torrens [5] also studied the Eq (1) for D-operations. Xie et al. [6] also discussed the solutions of Eq (1) for several implications derived from uninorms.

The equation $(\alpha \wedge \beta) \to \gamma = \alpha \to (\beta \to \gamma)$, called the the *law of importation*, also is a tautology in Boolean logic. It is formulated in fuzzy logic as

$$I(T(\alpha,\beta), \gamma) = I(\alpha, I(\beta, \gamma)), \tag{2}$$

where $T$ is a $t$-norm. Jayaram [1] characterized the solutions of Eq (2) for R-, S-, QL-, $g$- and $f$-implications. Mas et al. [7] characterized the solutions of Eq (2) for several implications derived from smooth discrete $t$-norms and $t$-conorms. Mas and Monserrat [8] discussed the solutions of Eq (2) for some fuzzy implications derived from uninorms. Massanet et al [9–11] characterized the solutions of Eq (2) for several fuzzy implications with a fixed $t$-norm (or uninorm). Zhou [17] studied the law of importation for fuzzy implications generated by continuous multiplicative generators of $t$-norms.

However, stability problems of functional equations with fuzzy implications have received little attention in the literature to date. With the development of functional equations, it is natural to study the stability of these functional equations. There are many types of stability for various functional equations. In this paper, we study the Hyers-Ulam stability of above two functional equations.

In 1940, Ulam [12] asked the following stability question:

*Let $(K_1, \diamond)$ and $(K_2, \star)$ be two groups, $D(\cdot, \cdot)$ be a metric on $K_2$. For any given $\epsilon > 0$, whether there exist a $\delta(\epsilon) > 0$ such that if a function $h : K_1 \to K_2$ satisfies $D(h(\alpha \diamond \beta), h(\alpha) \star h(\beta)) < \delta(\epsilon)$ for all $\alpha,\beta \in K_1$, then there exists a group homomorphism $H : K_1 \to K_2$ with $D(h(\alpha), H(\alpha)) < \epsilon$ for all $\alpha \in K_1$?*

Clearly, the function $h$ is almost a homomorphism, and the mapping $H$ is a true homomorphism near $h$ with small error.

In 1941, Hyers [13] solved the stability problem of Ulam for the approximately additive mappings.

**Theorem 1.** *[13] Let $h : K_1 \to K_2$ be a function between two Banach spaces $K_1$ and $K_2$ such that*

$$|h(\alpha + \beta) - h(\alpha) - h(\beta)| \le \epsilon \tag{3}$$

*for some $\epsilon \ge 0$ and for all $\alpha,\beta \in K_1$. Then there exists a unique function $H : K_1 \to K_2$ satisfying*

$$|h(\alpha) - H(\alpha)| \le \epsilon \tag{4}$$
$$H(\alpha + \beta) = H(\alpha) + H(\beta) \tag{5}$$

*for any $\alpha,\beta \in K_1$.*

Since then, the stability problems of many different equations have been studied by many researchers. Furthermore, the stability of traditional functional equations such as additive mappings, quadratic equation, the cubic equation in fuzzy normed space and some fuzzy differential equations have been studied (cf. [15, 16, 18–22]). However, no fuzzy implication is involved in these equations.

How to study the Hyers-Ulam stability of fuzzy equation with fuzzy implication. For convenience, let us take the derived iterative Boolean law, i.e., Eq (1) for an example. Similar to Ulam's question, we consider the following problem:

*If the fuzzy implication I approximately satisfies Eq (1), i.e.,*

$$|I(\alpha, I(\alpha,\beta)) - I(\alpha,\beta)| \leq \epsilon, \tag{6}$$

*for some small nonnegative number* $\epsilon > 0$ *and for all* $\alpha,\beta \in [0,1]$*, does there exist a unique fuzzy implication J strictlysatisfying Eq (1) near I with a small error*?

In other words, when given a fuzzy implication $I$ satisfying inequality (6), we try to find a mapping $J : [0,1]^2 \to [0,1]$ satisfying

(i) $J$ is a $(S, N)$-implication;
(ii) $J(\alpha, J(\alpha,\beta)) = J(\alpha,\beta)$;
(iii) $|J(\alpha,\beta) - I(\alpha,\beta)| \leq \delta, \ \ \forall\alpha,\beta \in [0,1]$;
(iv) $J$ is the unique $(S, N)$-implication satisfying (ii)-(iii).

The error $\delta$ should be as small as possible. For instance, if $\delta = 1$, then all solutions of Eq (1) for $(S, N)$-implications are such mappings satisfying above (i)-(iii), which obviously does not make sense.

## 2. Preliminaries

**Definition 2.** *[23]. We say I:* $[0,1]^2 \to [0,1]$ *is a* ***fuzzy implication*** *if it is nonincreasing in the first variable and nondecreasing in the second one, and* $I(1,0) = 0$*,* $I(0,0) = I(0,1) = I(1,1) = 1$.

Here we denote $\mathbb{I}$ the set of all fuzzy implications.

**Definition 3.** *[24]. We say* $N : [0,1] \to [0,1]$ *is a* ***fuzzy negation*** *if it is decreasing and* $N(1) = 0$*,* $N(0) = 1$. *It is called*

*(i)* ***strict****, if it is a continuous and strictly decreasing mapping;*
*(ii)* ***strong****, if it is an involution mapping, i.e.,*

$$N(N(\alpha)) = \alpha$$

*for all* $\alpha \in [0,1]$.

**Definition 4.** *[24]. We say* $T : [0,1]^2 \to [0,1]$ *is a* ***t-norm*** *if it is increasing in both variables, associative, commutative, and has 1 as its identity.*

This paper uses the minimum $t$-norm $T_M(\alpha,\beta) = \min(\alpha,\beta)$ and the product $t$-norm $T_P(\alpha,\beta) = \alpha \cdot \beta$.

The minimum $T_M$ is the strongest $t$-norm, i.e., $T_M(\alpha,\beta) \geq T(\alpha,\beta)$ for any $t$-norm $T$ and for all $\alpha,\beta \in [0,1]$.

**Definition 5.** *[24]. We say* $S : [0,1]^2 \to [0,1]$ *is a* ***t-conorm*** *if it is increasing in both variables, associative, commutative, and has 0 as its identity.*

This paper uses the maximum $t$-conorm $S_M(\alpha,\beta) = \max(\alpha,\beta)$ and the probabilistic sum $t$-conorm $S_P(\alpha,\beta) = \alpha + \beta - \alpha \cdot \beta$.

The maximum $S_M$ is the weakest $t$-conorm, i.e., $S_M(\alpha,\beta) \leq S(\alpha,\beta)$ for any $t$-conorm $S$ and for all $\alpha,\beta \in [0,1]$.

Let $N$ be a strong negation, a $t$-norm given by

$$T(\alpha,\beta) = N\Big(S(N(\alpha), N(\beta))\Big)$$

is said to be the *$N$-dual t-norm* of $S$, and, analogously, a $t$-conorm given by

$$S(\alpha,\beta) = N\Big(T(N(\alpha), N(\beta))\Big)$$

is said to be the *$N$-dual t-conorm* of $T$.

**Definition 6.** *[23]. We say $I$ is an $(S,N)$-**implication** if it is defined from a t-conorm $S$ and a negation $N$ in the way $I(\alpha,\beta) = S(N(\alpha),\beta)$, $\forall \alpha,\beta \in [0,1]$.*

**Lemma 7.** *If $N$ is a continuous fuzzy negation, then for any $\epsilon > 0$, $N_1(x) = ((1+\epsilon) \cdot N(x)) \wedge 1$ is also a continuous fuzzy negation.*

## 3. Stability of $I(\alpha, I(\alpha,\beta)) = I(\alpha,\beta)$

It is well-known that the solution of the iterative functional equation $I(\alpha,\beta) = I(\alpha, I(\alpha,\beta))$ for $(S,N)$-implications was solved by Shi et al [3] as follows.

**Theorem 8.** *[3] Let $I \in \mathbb{I}$ be an $(S,N)$-implication defined by a continuous negation $N$ and a t-conorm $S$, then $I$ satisfies Eq (1) if and only if $S(\alpha,\beta) = \max(\alpha,\beta)$ for any $\alpha,\beta \in [0,1]$.*

Now we solve the stability problem of the iterative functional equation $I(\alpha,\beta) = I(\alpha, I(\alpha,\beta))$ for $(S,N)$-implications.

**Theorem 9.** *Let $I \in \mathbb{I}$ be an $(S,N)$-implication defined by a continuous negation $N$ and a t-conorm $S$. If for some $\epsilon > 0$, $I$ satisfies inequality (6), then there exists an $(S,N)$-implication $J$ satisfying Eq (1) and*

$$|J(\alpha,\beta) - I(\alpha,\beta)| \leq \epsilon \tag{7}$$

*for all $\alpha,\beta \in [0,1]$.*

*Proof.* (i) Let $J(\alpha,\beta) = \max(N(\alpha),\beta)$, then $J$ is an S-implication satisfying Eq (1) (see above Theorem 8).

(ii) Now we prove the Inequality (7). Let $\beta = 0$, then $I(\alpha,\beta) = I(\alpha,0) = S(N(\alpha),0) = N(\alpha)$ and $I(\alpha, I(\alpha,0)) = S(N(\alpha),N(\alpha))$. So we have

$$|S(N(\alpha),N(\alpha)) - N(\alpha)| \leq \epsilon, \quad \forall \alpha \in [0,1]. \tag{8}$$

For any $\alpha,\beta \in [0,1]$, if $N(\alpha) \leq N(\beta)$, then

$$\begin{aligned} S(N(\alpha),N(\beta)) &\leq S(N(\beta),N(\beta)) \\ &\leq N(\beta) + \epsilon \text{ (Using Eq.(8))} \end{aligned}$$

$$= \max(N(\alpha), N(\beta)) + \epsilon.$$

If $N(\beta) \leq N(\alpha)$, then

$$\begin{aligned} S(N(\alpha), N(\beta)) &\leq S(N(\alpha), N(\alpha)) \\ &\leq N(\alpha) + \epsilon \ \text{ (Using Eq.(8))} \\ &= \max(N(\alpha), N(\beta)) + \epsilon. \end{aligned}$$

Thus we have

$$S(N(\alpha), N(\beta)) \leq \max(N(\alpha), N(\beta)) + \epsilon$$

for any $\alpha,\beta \in [0, 1]$. Moreover,

$$S(N(\alpha), N(\beta)) \geq \max(N(\alpha), N(\beta)).$$

Then

$$|\max(N(\alpha), N(\beta)) - S(N(\alpha), N(\beta))| \leq \epsilon, \tag{9}$$

for all $\alpha,\beta \in [0, 1]$. Since $N$ is a continuous negation, the range of $N$ is $[0, 1]$. Then the above equation could be rewritten as

$$|\max(N(\alpha), \gamma) - S(N(\alpha), \gamma)| \leq \epsilon, \quad \forall \alpha, \gamma \in [0, 1]. \tag{10}$$

Thus $|J(\alpha, \gamma)) - I(\alpha, \gamma)| \leq \epsilon$ for any $\alpha, \gamma \in [0, 1]$.

(iii) However, such $(S, N)$-implication is not unique. Let $N_1(\alpha) = ((1 + \epsilon) \cdot N(\alpha)) \wedge 1$ for any $\alpha \in [0, 1]$, then $N_1(\alpha)$ is still a continuous negation by Lemma 7. Obviously, $J_1(\alpha,\beta) = \max(N_1(\alpha),\beta)$ satisfies Eq (1). Moreover, we have $J_1(\alpha,\beta) \geq J(\alpha,\beta)$ from $N_1(\alpha) \geq N(\alpha)$, and

$$\begin{aligned} J_1(\alpha,\beta) - J(\alpha,\beta) &= \max(N_1(\alpha),\beta) - \max(N(\alpha),\beta) \\ &\leq N_1(\alpha) - N(\alpha) \\ &= ((1 + \epsilon) \cdot N(\alpha)) \wedge 1 - N(\alpha) \\ &\leq (1 + \epsilon) \cdot N(\alpha) - N(\alpha) \\ &= \epsilon \cdot N(\alpha) \\ &\leq \epsilon. \end{aligned}$$

Then $0 \leq J_1(\alpha,\beta) - J(\alpha,\beta) \leq \epsilon$ for any $\alpha,\beta \in [0, 1]$. Combined with $0 \leq I(\alpha,\beta) - J(\alpha,\beta) \leq \epsilon$ for any $\alpha,\beta \in [0, 1]$, we obtain $0 \leq J_1(\alpha,\beta) - I(\alpha,\beta) \leq \epsilon$ for any $\alpha,\beta \in [0, 1]$. Thus $|J_1(\alpha,\beta)) - I(\alpha,\beta)| \leq \epsilon$ for any $\alpha,\beta \in [0, 1]$. □

**Remark 10.** *We give two $(S, N)$-implications which satisfy Eqs (1) and (7). Uniqueness theorem does not hold, this is different from Hyers-Ulam stability results of traditional equations in [15, 16, 18].*

**Corollary 11.** *Let $I \in \mathbb{I}$ be an $(S, N)$-implication defined by a continuous negation $N$ and a t-conorm $S$, if $I$ satisfies Eq (1), then for any $\epsilon > 0$, there exists an $(S, N)$-implication $J$ such that*

*(i)* $J(\alpha, J(\alpha,\beta)) = J(\alpha,\beta)$;
*(ii)* $J \neq I$ *and* $|J(\alpha,\beta) - I(\alpha,\beta)| \leq \epsilon, \ \forall \alpha,\beta \in [0, 1]$.

Interestingly, if an $(S, N)$-implication $I$ satisfies Eq (1), then there exists another $(S, N)$-implication $J$ which is $\epsilon$−near $I$ with small error and satisfies Eq (1).

## 4. Stability of $I(T(\alpha,\beta),\gamma) = I(\alpha, I(\beta,\gamma))$

In this section, we consider the Hyers-Ulam stability of the equation $I(T(\alpha,\beta),\gamma) = I(\alpha, I(\beta,\gamma))$ for $(S, N)$-implications with a fixed $t$-norm.

### *4.1. Stability of* $I(T_M(\alpha,\beta),\gamma) = I(\alpha, I(\beta,\gamma))$

First, we consider the following problem for the case of the minimum $t$-norm, i.e.,

$$I(T_M(\alpha,\beta),\gamma) = I(\alpha, I(\beta,\gamma)). \tag{11}$$

Similar to Ulam's question, we have the following problem:
Given a fuzzy implication $I$ satisfying inequality

$$|I(T_M(\alpha,\beta),\gamma) - I(\alpha, I(\beta,\gamma))| \leq \epsilon \tag{12}$$

for all $\alpha,\beta,\gamma \in [0, 1]$, we try to find a mapping $J : [0, 1]^2 \to [0, 1]$ satisfying

(i) $J$ is a $(S, N)$-implication;
(ii) $J(T_M(\alpha,\beta),\gamma) = J(\alpha, J(\beta,\gamma))$;
(iii) $|J(\alpha,\beta) - I(\alpha,\beta)| \leq \delta, \quad \forall \alpha,\beta \in [0, 1]$;
(iv) $J$ is the unique $(S, N)$-implication satisfying (ii)-(iii).

The error $\delta$ is a positive real number. It should be as small as possible.

**Theorem 12.** *[1]. Let $I \in \mathbb{I}$ be an $(S, N)$-implication defined by a strong negation $N$ and a t-conorm $S$, then it satisfies Eq. (11) with a t-norm $T$ if and only if $T$ is the $N$-dual of $S$.*

**Theorem 13.** *Let $I \in \mathbb{I}$ be an $(S, N)$-implication defined by a strong negation $N$ and a t-conorm $S$. If for some $\epsilon > 0$, $I$ satisfies inequality (12), then there is an $(S, N)$-implication $J$ satisfying Eq. (11) and*

$$|J(\alpha,\beta)) - I(\alpha,\beta)| \leq \epsilon, \tag{13}$$

*for all $\alpha,\beta \in [0, 1]$.*

*Proof.* (i) Let $J(\alpha,\beta) = \max(N(\alpha),\beta)$, then $J$ is an $(S, N)$-implication satisfying Eq 11 (see Theorem 12).

(ii) Now we prove the Eq (13). Let $\gamma = 0$, then

$$\begin{aligned} I(T_M(\alpha,\beta),\gamma) &= I(T_M(\alpha,\beta), 0) \\ &= S(N(T_M(\alpha,\beta)), 0) \\ &= N(\alpha \wedge \beta) \\ &= N(\alpha) \vee N(\beta) \end{aligned}$$

and

$$\begin{aligned} I(\alpha, I(\beta,\gamma)) &= I(\alpha, I(\beta, 0)) \\ &= S(N(\alpha), N(\beta)). \end{aligned}$$

From $\left|I(T_M(\alpha,\beta),\gamma) - I(\alpha, I(\beta,\gamma))\right| \leq \epsilon$ and $S(N(\alpha), N(\beta)) \geq \max(N(\alpha), N(\beta))$, we have

$$N(\alpha) \vee N(\beta) \leq S(N(\alpha), N(\beta)) \leq N(\alpha) \vee N(\beta) + \epsilon.$$

Since $N$ is continuous, the range of $N$ still is $[0,1]$. Then the above equation could be rewritten as $N(\alpha) \vee \gamma \leq S(N(\alpha),\gamma) \leq N(\alpha) \vee \gamma + \epsilon$ for any $\alpha,\gamma \in [0,1]$. Thus $0 \leq I(\alpha,\gamma) - J(\alpha,\gamma) \leq \epsilon$, then we have $|I(\alpha,\gamma) - J(\alpha,\gamma)| \leq \epsilon$ for any $\alpha,\gamma \in [0,1]$. □

**Remark 14.** *If the following condition holds: For any strong negation $N$, there exists a new strong negation $N_1$ such that $0 \leq N_1(\alpha) - N(\alpha) \leq \epsilon$ for any $\alpha \in [0,1]$, then such $(S,N)$-implication in above theorem is not unique. Let $J_1(\alpha,\beta) = \max(N_1(\alpha),\beta)$, then we have, for any $\alpha,\beta \in [0,1]$*

$$\begin{aligned} 0 \leq J_1(\alpha,\beta) - J(\alpha,\beta) &= \max(N_1(\alpha),\beta) - \max(N(\alpha),\beta) \\ &\leq N_1(\alpha) - N(\alpha) \\ &\leq \epsilon. \end{aligned}$$

*And we have $-\epsilon \leq J(\alpha,\beta) - I(\alpha,\beta) \leq 0$ from $0 \leq I(\alpha,\beta) - J(\alpha,\beta) \leq \epsilon$. Then we obtain*

$$-\epsilon + 0 \leq J_1(\alpha,\beta) - J(\alpha,\beta) + J(\alpha,\beta) - I(\alpha,\beta) \leq 0 + \epsilon$$

*i.e., $-\epsilon \leq J_1(\alpha,\beta) - I(\alpha,\beta) \leq \epsilon$ for any $\alpha,\beta \in [0,1]$.*

*Then $J_1(\alpha,\beta) = \max(N_1(\alpha),\beta)$ is a new $(S,N)$-implication satisfying Eqs.(11) and (13).*

**Corollary 15.** *Let $I \in \mathbb{I}$ be an $(S,N)$-implication defined by a strong negation $N$ and a t-conorm $S$, if $I$ satisfies Eq. (11), then for any $\epsilon > 0$, there exists an $(S,N)$-implication $J$ such that*

*(i) $J(T_M(\alpha,\beta),\gamma) = J(\alpha, J(\beta,\gamma))$;*
*(ii) $J \neq I$ and $|J(\alpha,\beta) - I(\alpha,\beta)| \leq \epsilon, \quad \forall \alpha,\beta \in [0,1]$.*

Interestingly, if an $(S,N)$-implication $I$ satisfies Eq. (11), and there exists another $(S,N)$-implication $J$ which is $\epsilon-$near $I$ with small error and satisfies Eq. (11).

### 4.2. Stability of $I(T_P(\alpha,\beta),\gamma) = I(\alpha, I(\beta,\gamma))$

Second, we consider the following problem for the case of the product $t$-norm, i.e.,

$$I(T_P(\alpha,\beta),\gamma) = I(\alpha, I(\beta,\gamma)) \tag{14}$$

We give a Hyers-Ulam stability of the functional equation $I(T_P(\alpha,\beta),\gamma) = I(\alpha, I(\beta,\gamma))$ for $(S,N)$-implications.

**Lemma 16.** *Let $N$ be a strong negation, $T_P$ be the product t-norm, if the function $S_{N,P} : [0,1]^2 \to [0,1]$ is defined as $S_{N,P}(\alpha,\beta) = N(T_P(N(\alpha), N(\beta)))$ for any $\alpha,\beta \in [0,1]$, then $S_{N,P}$ is a N-dual t-conorm of $T_P$.*

*Proof.* We need to prove that $S_{N,P}$ is $t$-conorm. Here we only give the proof of the fact that $S_{N,P}$ is associtative and has 0 as its identity. Since $T_P$ is associtative and $N$ is a strong negation, for any $\alpha,\beta,\gamma \in [0,1]$

$$S_{N,P}(\alpha, S_{N,P}(\beta,\gamma)) = N(T_P(N(\alpha), N(N(T_P(N(\beta), N(\gamma))))))$$

$$
\begin{aligned}
&= N(T_P(N(\alpha), T_P(N(\beta), N(\gamma)))) \\
&= N(T_P(T_P(N(\alpha), N(\beta)), N(\gamma))),
\end{aligned}
$$

and

$$
\begin{aligned}
S_{N,P}(S_{N,P}(\alpha,\beta),\gamma)) &= N(T_P(N(N(T_P(N(\alpha), N(\beta))), N(\gamma)))) \\
&= N(T_P(T_P(N(\alpha), N(\beta), N(\gamma)))).
\end{aligned}
$$

Then $S_{N,P}(\alpha, S_{N,P}(\beta,\gamma)) = S_{N,P}(S_{N,P}(\alpha,\beta),\gamma))$. Since $T_P$ has 1 as its identity and $N$ is a strong negation, we have $S_{N,P}(\alpha, 0) = N(T_P(N(\alpha), N(0))) = N(T_P(N(\alpha), 1)) = N(N(\alpha)) = \alpha$. □

**Theorem 17.** *Let $I \in \mathbb{I}$ be an $(S, N)$-implication defined by a strong negation $N$ and a t-conorm $S$. If for some $\epsilon > 0$, $I$ satisfies inequality*

$$|I(T_P(\alpha,\beta),\gamma) - I(\alpha, I(\beta,\gamma))| \leq \epsilon \tag{15}$$

*then there is an $(S, N)$-implication $J$ satisfying Eq.(14) and*

$$|J(\alpha,\beta)) - I(\alpha,\beta)| \leq \epsilon, \tag{16}$$

*for all $\alpha,\beta \in [0, 1]$.*

*Proof.* (i) Let $J(\alpha,\beta) = S_{N,P}(N(\alpha),\beta)$, then $J$ is an $(S, N)$-implication satisfying Eq. (14) (see Theorem 13).

(ii) Now we prove the Eq (16). Let $\gamma = 0$, then

$$
\begin{aligned}
I(T_P(\alpha,\beta),\gamma) &= I(T_P(\alpha,\beta), 0) \\
&= S_{N,P}(N(T_P(\alpha,\beta)), 0) \\
&= N(T_P(\alpha,\beta)) \qquad (S_{N,P} \text{ has 0 as its identity}) \\
&= S_{N,P}(N(\alpha), N(\beta)) \quad (S_{N,P} \text{ is N-dual of } T_P)
\end{aligned}
$$

and

$$
\begin{aligned}
I(\alpha, I(\beta,\gamma)) &= I(\alpha, I(\beta, 0)) \\
&= S(N(\alpha), S(N(\beta), 0)) \\
&= S(N(\alpha), N(\beta)) \qquad (S \text{ has 0 as its identity})
\end{aligned}
$$

From $|I(T_P(\alpha,\beta),\gamma) - I(\alpha, I(\beta,\gamma))| \leq \epsilon$, we have $|S_{N,P}(N(\alpha), N(\beta)) - S(N(\alpha), N(\beta))| \leq \epsilon$.

Since $N$ is continuous, the range of $N$ still is $[0, 1]$. Then the above equation could be rewritten as $|S_{N,P}(N(\alpha),\gamma) - S(N(\alpha),\gamma)| \leq \epsilon$ for any $\alpha,\gamma \in [0, 1]$. Thus $|I(\alpha,\gamma) - J(\alpha,\gamma))| \leq \epsilon$ for any $\alpha,\gamma \in [0, 1]$. □

**Remark 18.** *If the following condition holds: For any strong negation $N$, there exists a new strong negation $N_1$ such that $0 \leq N_1(\alpha) - N(\alpha) \leq \epsilon$. Let $J_1(\alpha,\beta) = S_P(N_1(\alpha),\beta)$, then we have*

$$J_1(\alpha,\beta) - J(\alpha,\beta) = S_P(N_1(\alpha),\beta) - S_P(N(\alpha),\beta)$$

$$\begin{aligned} &\leq N_1(\alpha)\beta - N(\alpha)\beta \\ &= (N_1(\alpha) - N(\alpha))\beta \\ &\leq \epsilon \cdot \beta \\ &\leq \epsilon. \end{aligned}$$

*Combined with* $|I(\alpha,\beta) - J(\alpha,\beta)| \leq \epsilon$ *for any* $\alpha,\beta \in [0,1]$*, we obtain*

$$\begin{aligned} & |J_1(\alpha,\beta) - I(\alpha,\beta)| \\ = \;& |J_1(\alpha,\beta) - J(\alpha,\beta) + J(\alpha,\beta) - I(\alpha,\beta)| \\ \leq \;& |J_1(\alpha,\beta) - J(\alpha,\beta)| + |J(\alpha,\beta) - I(\alpha,\beta)| \\ \leq \;& 2\epsilon \end{aligned}$$

*for any* $\alpha,\beta \in [0,1]$*. Thus* $J_1(\alpha,\beta)$ *is an* $(S,N)$*-implication satisfying Eq. (14) and* $|J_1(\alpha,\beta) - I(\alpha,\beta)| \leq 2\epsilon$*.*

**Corollary 19.** *Let* $I \in \mathbb{I}$ *be an* $(S,N)$*-implication defined by a strong negation* $N$ *and a t-conorm* $S$*, if* $I$ *satisfies Eq (15), then for any* $\epsilon > 0$*, there exists an* $(S,N)$*-implication* $J$ *such that*

*(i)* $J(T_P(\alpha,\beta),\gamma) = J(\alpha, J(\beta,\gamma))$*;*
*(ii)* $J \neq I$ *and* $|J(\alpha,\beta) - I(\alpha,\beta)| \leq 2\epsilon, \quad \forall \alpha,\beta \in [0,1]$.

Interestingly, if an $(S,N)$-implication $I$ satisfies Eq.(14), then there exists another $(S,N)$-implication $J$ which is $2\epsilon-$near $I$ with small error and satisfies Eq. (14).

## 5. Conclusions

In this paper, we establish the Hyers-Ulam stability of $I(\alpha,\beta) = I(\alpha, I(\alpha,\beta))$, $I(T_M(\alpha,\beta),\gamma) = I(\alpha, I(\beta,\gamma))$ and $I(T_P(\alpha,\beta),\gamma) = I(\alpha, I(\beta,\gamma))$ for $(S,N)$-implications with $N$ a continuous (or strong) fuzzy negation. Interestingly, if the law of importation (or the iterative functional equation) holds for an $(S,N)$-implication $I$, then there exists another $(S,N)$-implication $J$ which is near $I$ with small error and satisfies the law of importation (or the iterative functional equation).

There are many other equations with fuzzy implications. A more detailed discussion of stability problem of these equations will be both necessary and interesting.

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## Conflict of interest

The authors declare no conflict of interest.

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