AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(2): 1816–1821.
DOI:10.3934/math.2021109
Received: 04 October 2020
Accepted: 29 November 2020
Published: 30 November 2020

Research article

Derivation of some integrals in Gradshteyn and Ryzhik

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Abstract: In this work we present derivations of the formula listed in entry 4.113 in the sixth edition of Gradshteyn and Rhyzik's table of integrals. We evaluate two definite integrals of the form

$$\int_0^\infty \frac{e^{-iay}(-iy+\log(z))^k + e^{iay}(iy+\log(z))^k}{\cosh(by)}dy$$

and

$$\int_0^\infty \frac{e^{iay}(iy+\log(z))^k - e^{-iay}(-iy+\log(z))^k}{\sinh(by)}dy$$

in terms of the Lerch function where k, a, z and b are arbitrary complex numbers. The entries in the table(s) are obtained as special cases in the paper below.

Keywords: entries of Gradshteyn and Ryzhik; hyperbolic integrals; hypergeometric function; Lerch function
Mathematics Subject Classification: 01A55, 11M06, 11M35, 30-02, 30D10, 30D30, 30E20

1. Introduction

We will derive integrals as indicated in the abstract in terms of the Lerch function. The derivations follow the method used by us in [6]. A generalization of Cauchy's integral formula is given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw \tag{1.1}$$

where C will be defined below. This method involves using a form of Eq (1.1) then multiplying both sides by a function, then take a definite integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Eq (1.1) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

2. Definite integral of the contour integral

We use the method in [6]. Using a generalization of Cauchy's integral formula we multiply both sides by in the first term e^{iay} and replacing y by $iy + \log(z)$ and in the second by e^{-iay} and replacing y by $-iy + \log(z)$ then add these two equations, followed by multiplying both sides by $\frac{1}{2\cosh(by)}$ to get

$$\frac{e^{-iay}(-iy+\log(z))^k + e^{iay}(iy+\log(z))^k}{2\Gamma(k+1)\cosh(by)} = \frac{1}{2\pi i}\int_C z^w w^{-k-1}\operatorname{sech}(by)\cos((a+w)y)dw \tag{2.1}$$

where the logarithmic function is defined in Eq (4.2) in [2]. The variables a, b, k and z are general complex numbers. Define $\vec{a} = (\Re(a), \Im(a))$, $\vec{b} = (\Re(a), \Im(b)))$ and $\vec{w} = (\Re(w), \Im(w))$. Then the integral over the contour C has w replaced by $w + a$. We can define the cut and contour C as lying on the side of $\vec{w} + \vec{a}$ opposite to $\vec{b}$, coming from infinity initially parallel to $\vec{w} + \vec{a}$, around the origin with zero radius and back to infinity parallel to $\vec{w} + \vec{a}$. The contour lies on opposite sides of the cut. We then take the definite integral over $y \in [0, \infty)$ of both sides to get

$$\begin{aligned}
&\frac{1}{\Gamma(k+1)}\int_0^\infty \frac{e^{-iay}(-iy+\log(z))^k + e^{iay}(iy+\log(z))^k}{2\cosh(by)}dy \\
&= \frac{1}{4\pi i}\int_0^\infty\int_C z^w w^{-k-1}\operatorname{sech}(by)\cos((a+w)y)dwdy \\
&= \frac{1}{4\pi i}\int_C\left(\int_0^\infty \operatorname{sech}(by)\cos((a+w)y)dy\right)\frac{z^w dw}{w^{k+1}} \\
&= \frac{1}{4ib}\int_C z^w w^{-k-1}\operatorname{sech}\left(\frac{\pi(a+w)}{2b}\right)dw
\end{aligned} \tag{2.2}$$

from Eq (1.7.7.1) in [1]. In a similar manner we can derive the second integral formula by subtracting the two equations and multiplying by $\frac{1}{2i\sinh(by)}$ we get

$$\frac{e^{iay}(iy+\log(z))^k - e^{-iay}(-iy+\log(z))^k}{\Gamma(k+1)2i\sinh(by)} = \frac{1}{2\pi i}\int_C z^w w^{-k-1}\operatorname{csch}(by)\sin((a+w)y)dw \tag{2.3}$$

We then take the definite integral over $y \in [0, \infty)$ of both sides to get

$$\begin{aligned}
\frac{1}{k!}\int_0^\infty \frac{e^{iay}(iy+\log(z))^k - e^{-iay}(-iy+\log(z))^k}{2i\sinh(by)}dy &= \frac{1}{2\pi i}\int_0^\infty\int_C z^w w^{-k-1}\operatorname{csch}(by)\sin((a+w)y)dwdy \\
&= \frac{1}{2\pi i}\int_C\left(\int_0^\infty \operatorname{csch}(by)\sin((a+w)y)dy\right)\frac{z^w dw}{w^{k+1}} \\
&= \frac{1}{4bi}\int_C z^w w^{-k-1}\tanh\left(\frac{\pi(a+w)}{2b}\right)dw
\end{aligned} \tag{2.4}$$

from Eq (2.7.7.6) in [1].

3. Infinite sum of the contour integral

Again, using the method in [6], replacing y with $\pi(2p+1)/(2b)+\log(z)$ and multiplying both sides by $(-1)^p\left(\frac{\pi}{b}\right)e^{a\pi(2p+1)/(2b)}$ to yield

$$(-1)^p\left(\frac{\pi}{b}\right)e^{a(\pi(2p+1)/(2b))}\frac{(\pi(2p+1)/(2b)+\log(z))^k}{k!}=\frac{(-1)^p}{2\pi i}\left(\frac{\pi}{b}\right)\int_C\frac{e^{w\left(\frac{\pi(2p+1)}{2b}\right)+\frac{a\pi(2p+1)}{2b}}}{w^{k+1}}z^w dw \tag{3.1}$$

followed by taking the infinite sum of both sides of Eq (3.1) with respect to p over $[0,\infty)$ to get

$$\begin{aligned}\frac{b^{-k-1}e^{\frac{a\pi}{2b}}\pi^{k+1}}{k!}\Phi\left(-e^{\frac{a\pi}{b}},-k,\frac{1}{2}+\frac{b\log(z)}{\pi}\right)&=\frac{1}{2ib}\sum_{p=0}^{\infty}\int_C\frac{e^{w\left(\frac{\pi(2p+1)}{2b}\right)+\frac{a\pi(2p+1)}{2b}}}{(-1)^{-p}w^{k+1}}z^w dw\\&=\frac{1}{2ib}\int_C\sum_{p=0}^{\infty}\frac{e^{w\left(\frac{\pi(2p+1)}{2b}\right)+\frac{a\pi(2p+1)}{2b}}}{(-1)^{-p}w^{k+1}}z^w dw\\&=\frac{1}{4ib}\int_C z^w w^{-k-1}\operatorname{sech}\left(\frac{\pi(a+w)}{2b}\right)dw\end{aligned}\tag{3.2}$$

from (1.232.1) in [5] where $\Re\left((w+a)/b\right)<0$ for the convergence of the sum and if the $\Re(k)<0$ then the argument of the sum over p cannot be zero for some value of p. We use (9.550) in [5] where $\Phi(z,s,v)$ is the Lerch function which is a generalization of the Hurwitz Zeta and polylogarithm functions.

The Lerch function has a series representation given by

$$\Phi(z,s,v)=\sum_{n=0}^{\infty}(v+n)^{-s}z^n \tag{3.3}$$

where $|z|<1, v\neq 0,-1,..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt \tag{3.4}$$

where $\Re(v)>0$, or $|z|\leq 1, z\neq 1, \Re(s)>0$, or $z=1, \Re(s)>1$. Similarly, using the method in [6], replacing y with $2\pi(p+1)/(2b)+\log(z)$ and multiplying both sides by $(-1)^p\left(\frac{\pi}{b}\right)e^{2a\pi(p+1)/(2b)}$ to yield

$$(-1)^p\left(\frac{\pi}{b}\right)e^{\frac{2a\pi(p+1)}{2b}}\frac{\left(\frac{2\pi(p+1)}{2b}+\log(z)\right)^k}{k!}=\frac{(-1)^p}{2\pi i}\left(\frac{\pi}{b}\right)\int_C\frac{e^{w(2\pi(p+1)/(2b))+2a\pi(p+1)/(2b)}}{w^{k+1}}z^w dw \tag{3.5}$$

followed by taking the infinite sum of both sides of Eq (3.5) with respect to p over $[0,\infty)$ to get

$$\begin{aligned}\frac{e^{\frac{a\pi}{b}}\pi^{k+1}b^{-k-1}}{k!}\Phi\left(-e^{\frac{a\pi}{b}},-k,1+\frac{b\log(z)}{\pi}\right)&=\frac{1}{2bi}\sum_{p=0}^{\infty}\int_C\frac{e^{w(2\pi(p+1)/(2b))+2a\pi(p+1)/(2b)}}{(-1)^{-p}w^{k+1}}z^w dw\\&=\frac{1}{2bi}\int_C\sum_{p=0}^{\infty}\frac{e^{w(2\pi(p+1)/(2b))+2a\pi(p+1)/(2b)}}{(-1)^{-p}w^{k+1}}z^w dw\\&=\frac{1}{4bi}\int_C\left(z^w w^{-k-1}+z^w w^{-k-1}\tanh\left(\frac{\pi(a+w)}{2b}\right)\right)dw\end{aligned}\tag{3.6}$$

from Eq (1.232.1) in [5] where $\Re((a+w)/b)<0$.

4. Definite integral in terms of the Lerch function

Since the right hand-side of Eq (2.2) is equal to the right-hand side of (3.2), we can equate the left hand-sides and simplify to get

$$\int_0^\infty \frac{e^{-iay}(-iy+\log(z))^k + e^{iay}(iy+\log(z))^k}{\cosh(by)} dy = 2b^{-k-1} e^{\frac{a\pi}{2b}} \pi^{k+1} \Phi\left(-e^{\frac{a\pi}{b}}, -k, \frac{1}{2} + \frac{b\log(z)}{\pi}\right) \quad (4.1)$$

where a, b, k and z are general complex numbers.

To obtain the first contour integral in the last line of Eq (3.6) we use the Cauchy formula by replacing y by $\log(z)$ and multiplying both sides by $\frac{\pi}{2b}$ and simplifying we get

$$\frac{\pi \log^k(z)}{2bk!} = \frac{1}{4bi} \int_C z^w w^{-1-k} dw \quad (4.2)$$

We can write down an equivalent formula for the corresponding Lerch function for the second integral using Eqs (2.4), (3.6), and (4.2)

$$\int_0^\infty \frac{e^{iay}(iy+\log(z))^k - e^{-iay}(-iy+\log(z))^k}{\sinh(by)} dy = 2ie^{\frac{a\pi}{b}} \pi^{k+1} b^{-k-1} \Phi\left(-e^{\frac{a\pi}{b}}, -k, 1 + \frac{b\log(z)}{\pi}\right) - \frac{\pi i \log^k(z)}{b} \quad (4.3)$$

a, b, k, and z are general complex numbers.

5. Derivation of the integrals listed in Table 4.113 in [5]

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In this section we will derive definite integrals in terms of the Lerch function which simplify to the Hypergeometric function by Eq (1.11.10) in [3].

$$\Phi(z,1,v) = \sum_{n=0}^\infty \frac{z^n}{n+v} = v^{-1} {}_2F_1(1, v, 1+v; z) \quad (5.1)$$

For an extensive list of special cases of the hypergeometric function see [7]. These can be derived using the contiguous relations in section 15.2 in [2].

Here we form an equation by taking the difference between (4.3) and that equation where a is replaced by $-a$. Then set $k = -1$ and $z = e^\beta$ to get

$$\int_0^\infty \frac{\sin(ay)}{\sinh(by)} \frac{dy}{\beta^2+y^2} = \frac{1}{2\beta}\left(e^{\frac{a\pi}{b}} \Phi\left(-e^{\frac{a\pi}{b}}, 1, 1 + \frac{b\beta}{\pi}\right) - e^{-\frac{a\pi}{b}} \Phi\left(-e^{-\frac{a\pi}{b}}, 1, 1 + \frac{b\beta}{\pi}\right)\right) \quad (5.2)$$

Here we form an equation by taking the sum of (4.3) and that equation where a is replaced by $-a$. Then set $k = -1$ and $z = e^\beta$ to get

$$\int_0^\infty \frac{y\cos(ay)}{\sinh(by)} \frac{dy}{\beta^2+y^2} = \frac{1}{2}\left(\frac{\pi}{b\beta} - e^{-\frac{a\pi}{b}} \Phi\left(-e^{-\frac{a\pi}{b}}, 1, 1 + \frac{b\beta}{\pi}\right) - e^{\frac{a\pi}{b}} \Phi\left(-e^{\frac{a\pi}{b}}, 1, 1 + \frac{b\beta}{\pi}\right)\right) \quad (5.3)$$

We then set $k = -1$, $z = e^{\beta}$ by forming an equation using (4.1) by taking the difference between (4.1) and that equation where a is replaced by $-a$ to get

$$\int_0^\infty \frac{\cos(ay)}{\cosh(by)}\frac{dy}{\beta^2+y^2} = \frac{1}{2a}\left(e^{\frac{\beta\pi}{2b}}\Phi\left(-e^{\frac{\beta\pi}{b}},1,\frac{1}{2}+\frac{b\beta}{\pi}\right)+e^{-\frac{a\pi}{2b}}\Phi\left(-e^{-\frac{\beta\pi}{b}},1,\frac{1}{2}+\frac{b\beta}{\pi}\right)\right) \tag{5.4}$$

Here we form an equation using (4.1) by taking the sum between (4.1) and that equation where a is replaced by $-a$.

$$\int_0^\infty \frac{y\sin(ay)}{\cosh(by)}\frac{dy}{\beta^2+y^2} = \frac{1}{2}\left(e^{\frac{\beta\pi}{2b}}\Phi\left(-e^{\frac{\beta\pi}{b}},1,\frac{1}{2}+\frac{b\beta}{\pi}\right)-e^{-\frac{a\pi}{2b}}\Phi\left(-e^{-\frac{\beta\pi}{b}},1,\frac{1}{2}+\frac{b\beta}{\pi}\right)\right) \tag{5.5}$$

6. Table 4.113 in [5] expressed in terms of the Lerch function

$f(x)$	$\int_0^\infty f(x)dx$
$\frac{\sin(ay)}{\sinh(\pi y)}\frac{dy}{y^2+\beta^2}$	$\frac{1}{2\beta}\left(e^a\Phi\left(-e^a,1,1+\beta\right)-e^{-a}\Phi\left(-e^{-a},1,1+\beta\right)\right)$
$\frac{\sin(ay)}{\sinh(\pi y)}\frac{dy}{y^2+1}$	$\frac{1}{2}\left(e^a\Phi\left(-e^a,1,2\right)-e^{-a}\Phi\left(-e^{-a},1,2\right)\right)$
$\frac{\sin(ay)}{\sinh\left(\frac{\pi}{2}y\right)}\frac{dy}{y^2+1}$	$\frac{1}{2}\left(e^{2a}\Phi\left(-e^{2a},1,\frac{3}{2}\right)-e^{-2a}\Phi\left(-e^{-2a},1,\frac{3}{2}\right)\right)$
$\frac{\sin(ay)}{\sinh\left(\frac{\pi}{4}y\right)}\frac{dy}{y^2+1}$	$\frac{1}{2}\left(e^{4a}\Phi\left(-e^{4a},1,\frac{5}{4}\right)-e^{-4a}\Phi\left(-e^{-4a},1,\frac{5}{4}\right)\right)$
$\frac{y\sin(ay)}{\cosh\left(\frac{\pi}{4}y\right)}\frac{dy}{1+y^2}$	$\frac{1}{2}\left(e^{4a}\Phi\left(-e^{4a},1,\frac{3}{4}\right)-e^{-4a}\Phi\left(-e^{-4a},1,\frac{3}{4}\right)\right)$
$\frac{y\cos(ay)}{\sinh(\pi y)}\frac{dy}{y^2+1}$	$\frac{1}{2}\left(-e^{-a}\Phi\left(-e^{-a},1,2\right)-e^{a}\Phi\left(-e^{a},1,2\right)+1\right)$
$\frac{y\cos(ay)}{\sinh\left(\frac{\pi}{2}y\right)}\frac{dy}{y^2+1}$	$\frac{1}{2}\left(-e^{-2a}\Phi\left(-e^{-2a},1,\frac{3}{2}\right)-e^{2a}\Phi\left(-e^{2a},1,\frac{3}{2}\right)+2\right)$
$\frac{\cos(ay)}{\cosh(\pi y)}\frac{dy}{\beta^2+y^2}$	$\frac{1}{2\beta}\left(e^{a/2}\Phi\left(-e^{a},1,\beta+\frac{1}{2}\right)+e^{-a/2}\Phi\left(-e^{-a},1,\beta+\frac{1}{2}\right)\right)$
$\frac{\cos(ay)}{\cosh(\pi y)}\frac{dy}{1+y^2}$	$\frac{1}{2}\left(e^{-a/2}\Phi\left(-e^{-a},1,\frac{3}{2}\right)+e^{a/2}\Phi\left(-e^{a},1,\frac{3}{2}\right)\right)$
$\frac{\cos(ay)}{\cosh\left(\frac{\pi}{2}y\right)}\frac{dy}{1+y^2}$	$\frac{1}{2}\left(e^{-a}\Phi\left(-e^{-2a},1,1\right)+e^{a}\Phi\left(-e^{2a},1,1\right)\right)$
$\frac{\cos(ay)}{\cosh\left(\frac{\pi}{4}y\right)}\frac{dy}{1+y^2}$	$\frac{1}{2}\left(e^{-2a}\Phi\left(-e^{-4a},1,\frac{3}{4}\right)+e^{2a}\Phi\left(-e^{4a},1,\frac{3}{4}\right)\right)$

Table 4.113 in [5] is correct with the exception of entry (4.113.1).

Note that the second parameter of the Lerch function in every entry in the table is unity so we can write it in terms of the hypergeometric function using (5.1). Many of these can be evaluated in terms of simpler functions from the extensive tables in [7].

7. Conclusion

In comparing our results with Table 4.113 our formulae have a wider range of the parameters than are listed in the Gradshteyn and Ryzhik book [5] due to the use of the Lerch function in the derivation of these integrals. We also provided correct formula for an integral supplied by Erdélyi et al [4]. We will be looking at other integrals using this contour integral method for future work. The results presented were numerically verified for both real and imaginary values of the parameters in the integrals using Mathematica by Wolfram.

Acknowledgments

This paper is fully supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) Grant No. 504070.

Conflict of interest

The authors declare no conflict of interest in this paper.

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