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Research article

Closure properties of generalized λ-Hadamard product for a class of meromorphic Janowski functions

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Abstract: In this paper, we introduce a class of meromorphic starlike function by subordination relationship and generalized λ-Hadamard product. We obtain the necessary and sufficient conditions and closure properties of the class. In addition, some new results of the class are given.

Keywords: meromorphic function; Janowski functions; Hadamard product; generalized λ-Hadamard product; closure property
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1. Introduction

Let $\Sigma(a, k)$ be the class of meromorphic functions f of the form

$$f(z) = \frac{a}{z} + \sum_{n=k}^{\infty} a_n z^n \quad (a > 0, k \geq 2, k \in \mathbb{N}), \tag{1.1}$$

which is analytic in the punctured open unit disk $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$.

In 2009, El-Ashwah [1] introduced the subclass $\Sigma S_k^*(a,\beta)$ of generalized meromorphically starlike of order β as follows,

$$\Sigma S_k^*(a,\beta) = \left\{ f(z) \in \Sigma(a, k) : -\text{Re}\frac{zf'(z)}{f(z)} > \beta, \beta \in [0, 1) \right\}.$$

An analytic function $g : \mathbb{U} = \{z : |z| < 1\} \to \mathbb{C}$ is subordinate to an analytic function $h : \mathbb{U} \to \mathbb{C}$, if there is a function ω satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$, such that $g(z) = h(\omega(z))(z \in \mathbb{U})$.

Note that $g(z) \prec h(z)$. Especially, if h is univalent in $\mathbb{U}$, then the following conclusion is true (see [2]):

$$g(z) \prec h(z) \Longleftrightarrow g(0) = h(0) \text{ and } g(\mathbb{U}) \subset h(\mathbb{U}).$$

Using the subordinate relationship, we introduce the following subclass of generalized meromorphic Janowski functions (see [3,5,6]).

Definition 1. Let $f(z) \in \Sigma(a,k), a > 0, A, B \in \mathbb{R}$, $|A| \leq 1, |B| \leq 1$ and $A \neq B$. The function $f \in \Sigma S_k^*(a, A, B)$ if and only if

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}^*).$$

It is obvious that

$$\Sigma S_k^*(a, A, B) \subset \Sigma S_k^*(a, \frac{1-A}{1-B}) \ (-1 \leq B < A \leq 1).$$

According to the subordination relationship, the function $f(z) \in \Sigma S_k^*(a, A, B)$ if and only if there exists an analytic function $\omega(z)$ in the open unit dick $\mathbb{U}$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$, such that

$$-\frac{zf'(z)}{f(z)} = \frac{1+Aw(z)}{1+Bw(z)} \quad (z \in \mathbb{U}^*),$$

which is equivalent to

$$\left|\frac{zf'(z)+f(z)}{Af(z)+Bzf'(z)}\right| < 1 \quad (z \in \mathbb{U}^*). \tag{1.2}$$

Let $T(a,k)$ be the subclass of $\Sigma(a,k)$, the function f belonging to $T(a,k)$ of the following form

$$f(z) = \frac{a}{z} - \sum_{n=k}^{\infty} |a_n| z^n. \tag{1.3}$$

Let

$$\overline{\Sigma S}_k^*(a, A, B) = T(a,k) \cap \Sigma S_k^*(a, A, B).$$

Next, we introduce the generalized λ-Hadamard product of the class $T(a,k)$.

Definition 2. Let $a > 0, p > 0, q > 0, \lambda \geq 0, k \in \mathbb{N}, k \geq 2, f_i(z) = \frac{a}{z} - \sum_{n=k}^{\infty} |a_{n,j}| z^n \in T(a,k)$ $(i = 1,2)$. The generalized λ-Hadamard product $(f_1 \widetilde{\Delta} f_2)_\lambda(p,q,a;z)$ of the function f_1 and f_2 is defined by

$$(f_1 \widetilde{\Delta} f_2)_\lambda(p,q,a;z) = (1-\lambda)(f_1 \Delta f_2)(p,q,a;z) - \lambda z (f_1 \Delta f_2)'(p,q,a;z), \tag{1.4}$$

where

$$(f_1 \Delta f_2)(p,q,a;z) = \frac{a^2}{z} - \sum_{n=k}^{\infty} |a_{n,1}|^p |a_{n,2}|^q z^n. \tag{1.5}$$

From (1.4) and (1.5), we get

$$(f_1 \widetilde{\Delta} f_2)_\lambda(p,q,a;z) = \frac{a^2}{z} - \sum_{n=k}^{\infty} [1-(n+1)\lambda] |a_{n,1}|^p |a_{n,2}|^q z^n.$$

Selecting different parameters a, λ, p and q, we can obtain the following three special convolutions:

(i) For $\lambda = 0$, $(f_1\widetilde{\Delta}f_2)_0(p, q, a; z)$ is the generalized Hadamard product:

$$(f_1\widetilde{\Delta}f_2)_0(p, q, a; z) =: (f_1\Delta f_2)(p, q, a; z) = \frac{a^2}{z} - \sum_{n=k}^{\infty} |a_{n,1}|^p |a_{n,2}|^q z^n.$$

In particular, for $a = 1$, the Hadamard product $(f_1\Delta f_2)(p, q, 1; z) = \frac{1}{z} - \sum_{n=k}^{\infty} |a_{n,1}|^p |a_{n,2}|^q z^n$ was studied by Janowski [3] and Tang et al. [6].

(ii) For $\lambda = 0$, $p = q = 1$, $(f_1\widetilde{\Delta}f_2)_0(1, 1, a; z)$ is the general Hadamard product:

$$(f_1\widetilde{\Delta}f_2)_0(1, 1, a; z) =: (f_1 * f_2)(a; z) = \frac{a^2}{z} - \sum_{n=k}^{\infty} |a_{n,1}||a_{n,2}|z^n.$$

In particular, for $a = 1$, $(f_1 * f_2)(1; z)$ is the well-known Hadamard product (see [1,4,7]):

$$(f_1 * f_2)(1; z) = (f_1 * f_2)(z) = \frac{1}{z} - \sum_{n=k}^{\infty} |a_{n,1}||a_{n,2}|z^n.$$

For $a = 1, k = 1, f_1(z) = z^{-1} + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^n, f_2(z) = \frac{a}{z} - \sum_{n=1}^{\infty} |a_{n,2}|z^n, (a)_n = a(a+1)\cdots(a+n-1) (n \in \mathbb{N})$, $(f_1 * f_2)(1; z) = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} |a_{n,2}|z^n$ was studied by Liu et al. [8] (see also [9]).

(iii) For $p = q = 1$, $(f_1\widetilde{\Delta}f_2)_\lambda(1, 1, a; z) = (f_1 \overline{*} f_2)_\lambda(z)$ is $\lambda-$Hadamard product:

$$(f_1 \overline{*} f_2)_\lambda(z) = (1 - \lambda)(f_1 * f_2) + \lambda z(f_1 * f_2)'(z) = \frac{a^2}{z} - \sum_{n=k}^{\infty} [1 - (n + 1)\lambda]|a_{n,1}||a_{n,2}|z^n.$$

In 1996, Choi et al. [4] studied the closure properties of the generalized λ-Hadamard product for univalent functions. In 2001, Liu et al. [8] investigated two new classes of meromorphically multivalent functions by using a linear operator. Based on this, in this paper, we mainly consider the closure properties of the generalized λ-Hadamard product for meromorphic functions. Firstly, we obtain the necessary and sufficient conditions of the class $\overline{\Sigma S}_k^*(a, A, B)$ by using subordination relationship. Secondly, we discuss the closure properties of the general Hadamard product, the generalized Hadamard product and λ-Hadamard product for functions of the class $\overline{\Sigma S}_k^*(a, A, B)$. Finally, we discuss the closure properties of the generalized λ-Hadamard product and generalize the known conclusions.

2. Perliminaries

Unless otherwise noted, the parameters a, k, A and B satisfy the condition: $a > 0, k \in \mathbb{N}, k \geq 2, A, B \in \mathbb{R}, |A| \leq 1, |B| \leq 1$ and $A \neq B$.

In order to establish our results, we need the following lemmas.

Lemma 1. Let f be of the form (1.1). If

$$\sum_{n=k}^{\infty} [(n + 1) + |A + nB|]|a_n| \leq a|A - B|, \tag{2.1}$$

then $f(z) \in \Sigma S_k^*(a, A, B)$.

Proof. Let $|z| = 1$. By (2.1), we obtain

$$\begin{aligned}
&|zf'(z) + f(z)| - |Af(z) + Bzf'(z)| \\
&\leq |\sum_{n=k}^{\infty}(n+1)|a_n|z^n| - |\frac{a}{z}(A-B)| + |\sum_{n=k}^{\infty}(A+nB)|a_n|z^n| \\
&\leq \sum_{n=k}^{\infty}(n+1)|a_n| - a|A-B| + \sum_{n=k}^{\infty}|A+nB||a_n| \\
&= \sum_{n=k}^{\infty}[(n+1) + |A+nB|]|a_n| - a|A-B| \\
&\leq 0.
\end{aligned}$$

Using the principle of maximum modulus, we get

$$|zf'(z) + f(z)| - |Af(z) + Bzf'(z)| < 0 \ (z \in \mathbb{U}^*).$$

Then we conclude that $f(z) \in \Sigma S_k^*(a, A, B)$. Therefore, we complete the proof of Lemma 1.

Lemma 2. Let the function f be of the form (1.3).

(i) If $0 \leq B < A \leq 1$. Then the function $f(z) \in \overline{\Sigma S}_k^*(a, A, B)$ if and only if

$$\sum_{n=k}^{\infty}[(n+1) + (A+nB)]|a_n| \leq a(A-B). \tag{2.2}$$

(ii) If $-1 \leq A < B \leq 0$. Then the function $f(z) \in \overline{\Sigma S}_k^*(a, A, B)$ if and only if

$$\sum_{n=k}^{\infty}[(n+1) - (A+nB)]|a_n| \leq a(B-A). \tag{2.3}$$

Proof. It appears from (1.3) that $\overline{\Sigma S}_k^*(a, A, B) \subset \Sigma S_k^*(a, A, B)$. In view of Lemma 1, we just need prove the necessity part of Lemma 2. Let $f(z) \in \overline{\Sigma S}_k^*(a, A, B)$. By (1.2), we have

$$\left|\frac{zf'(z) + f(z)}{Af(z) + Bzf'(z)}\right| = \left|\frac{\sum_{n=k}^{\infty}(n+1)|a_n|z^{n-1}}{a(A-B) - \sum_{n=k}^{\infty}(A+nB)|a_n|z^{n-1}}\right| < 1.$$

Since $|\text{Re}z| \leq |z|$ holds true for all $z \in \mathbb{C}$. If $0 \leq B < A \leq 1$, then

$$\text{Re}\left\{\frac{\sum_{n=k}^{\infty}(1+n)|a_n|z^{n-1}}{a(A-B) - \sum_{n=k}^{\infty}(A+nB)|a_n|z^{n-1}}\right\} < 1. \tag{2.4}$$

Let $z = r \to 1^-$. From (2.4), we can get (2.2). Using the same method, we can obtain the result of (ii). The estimates of (i) and (ii) are sharp for the function f given by

$$f(z) = \frac{a}{z} - \frac{a|A-B|}{(k+1) + |A+kB|}z^k, \ k \geq 2. \tag{2.5}$$

So, we complete the proof of Lemma 2.

3. Main results

First of all, we discuss the closure properties of the general Hadamard product of the class $\overline{\Sigma S}_k^*(a, A, B)$.

Theorem 1. Let $f_i(z) = \frac{a}{z} - \sum_{n=k}^{\infty} |a_n| z^n \in \overline{\Sigma S}_k^*(a, A, B)$ $(i = 1, 2)$. If A and B satisfy the condition

$$0 \leq B < A \leq 1 \quad \text{or} \quad -1 \leq A < B \leq 0,$$

then $\frac{1}{a}(f_1 * f_2)(a; z) \in \overline{\Sigma S}_k^*(a, A, B)$.

Proof. Let $0 \leq B < A \leq 1$. If $f_i(z) \in \overline{\Sigma S}_k^*(a, A, B)$ $(i = 1, 2)$, from (2.2), we obtain

$$\sum_{n=k}^{\infty} [(n + 1) + (A + nB)]|a_{n,1}| \leq a(A - B) \tag{3.1}$$

and

$$\sum_{n=k}^{\infty} [(n + 1) + (A + nB)]|a_{n,2}| \leq a(A - B). \tag{3.2}$$

To obtain $\frac{1}{a}(f_1 * f_2)(a; z) \in \overline{\Sigma S}_k^*(a, A, B)$, we just prove

$$\sum_{n=k}^{\infty} \frac{1}{a}[(n + 1) + (A + nB)]|a_{n,1}||a_{n,2}| \leq a(A - B). \tag{3.3}$$

By using Cauchy-Schwarz inequality, from (3.1) and (3.2), we have

$$\sum_{n=k}^{\infty} [(n + 1) + (A + nB)] \sqrt{|a_{n,1}||a_{n,2}|} \leq a(A - B). \tag{3.4}$$

According to (3.3) and (3.4), we only need to prove

$$\sqrt{|a_{n,1}||a_{n,2}|} \leq a.$$

From (3.4) and the condition $0 \leq B < A \leq 1$, we have

$$\sqrt{|a_{n,1}||a_{n,2}|} \leq \frac{a(A - B)}{(n + 1) + (A + nB)} \leq a.$$

Therefore, we conclude that $\frac{1}{a}(f_1 * f_2)(a; z) \in \overline{\Sigma S}_k^*(a, A, B)$.

For $-1 \leq A < B \leq 0$, using the same method above, we get $\frac{1}{a}(f_1 * f_2)(a; z) \in \overline{\Sigma S}_k^*(a, A, B)$. Therefore, we complete the proof of Theorem 1.

Next, we prove the closure properties of the generalized Hadamard product.

Theorem 2. Let $f_i(z) = \frac{a}{z} - \sum_{n=k}^{\infty} |a_n| z^n \in \overline{\Sigma S}_k^*(a, A, B)$ $(i = 1, 2)$ and $p > 1$. If anyone of the following conditions holds true:

(i) For $0 \le B < A \le 1$, $\hat{B}$ and a satisfy

$$\begin{cases} 0 \le \hat{B} \le \frac{aA[(k+1)+(A+kB)]-(A-B)(k+1+A)}{a[(k+1)+(A+kB)]+(A-B)k}, & 0 < a < 1, \\ \hat{B} \le \frac{aA(1+B)-(A-B)}{a(1+B)+(A-B)}, & a > 1. \end{cases}$$

(ii) For $-1 \le A < B \le 0$, $-1 \le A < \hat{B} \le 0$ and a satisfies $a > 0$. Then $\frac{1}{a}(f_1 \Delta f_2)\left(\frac{1}{p}, \frac{p-1}{p}; a, z\right) \in \overline{\Sigma S}^*_k(a, A, \hat{B})$.

Proof. (i) For $0 \le B < A \le 1$, by Lemma 2, we can get

$$\sum_{n=k}^{\infty} \frac{(n+1)+(A+nB)}{a(A-B)}|a_{n,i}| \le 1 \ (i = 1, 2).$$

We deduce that

$$\left\{\sum_{n=k}^{\infty} \frac{(n+1)+(A+nB)}{a(A-B)}|a_{n,1}|\right\}^{\frac{1}{p}} \le 1 \tag{3.5}$$

and

$$\left\{\sum_{n=k}^{\infty} \frac{(n+1)+(A+nB)}{a(A-B)}|a_{n,2}|\right\}^{\frac{p-1}{p}} \le 1. \tag{3.6}$$

According to (1.5), we have

$$(f_1 \Delta f_2)(\frac{1}{p}, \frac{p-1}{p}) = \frac{a^2}{z} - \sum_{n=k}^{\infty} |a_{n,1}|^{\frac{1}{p}} |a_{n,2}|^{\frac{p-1}{p}} z^n.$$

To prove $\frac{1}{a}(f_1 * f_2)(\frac{1}{p}, \frac{p-1}{p}, a; z) \in \overline{\Sigma S}^*_k(a, A, \hat{B})$, we only need to show

$$\sum_{n=k}^{\infty} \frac{(n+1)+(A+n\hat{B})}{a^2(A-\hat{B})}|a_{n,1}|^{\frac{1}{p}} |a_{n,2}|^{\frac{p-1}{p}} \le 1. \tag{3.7}$$

Applying Hölder inequality, from (3.5) and (3.6), we obtain

$$\sum_{n=k}^{\infty} \frac{(n+1)+(A+nB)}{a(A-B)}|a_{n,1}|^{\frac{1}{p}} |a_{n,2}|^{\frac{p-1}{p}} \le 1.$$

Thus, (3.7) holds if

$$\frac{(n+1)+(A+n\hat{B})}{a^2(A-\hat{B})} \le \frac{(n+1)+(A+nB)}{a(A-B)} \quad (\forall n \in \mathbb{N}, n \ge k),$$

that is,

$$\hat{B}[\frac{a[(n+1)+(A+nB)]}{A-B} + n] \le \frac{aA[(n+1)+(A+nB)]}{A-B} - (n+1+A). \tag{3.8}$$

Let

$$F_1(n) = \frac{a[(n+1)+(A+nB)]}{A-B} + n$$

and

$$G_1(n) = \frac{aA[(n+1)+(A+nB)]}{A-B} - (n+1+A).$$

It is easy to see $F_1(n) > 0$ holds true for $n \geq k$. To obtain (3.8), we only need to prove

$$\hat{B} \leq \frac{G_1(n)}{F_1(n)} = \frac{aA[(n+1)+(A+nB)] - (A-B)(n+1+A)}{a[(n+1)+(A+nB)] + (A-B)n}.$$

Because $\frac{G_1(n)}{F_1(n)}$ is an increasing function with respect to n and the condition (i) holds true, so we have

$$\hat{B} \leq \frac{aA[(k+1)+(A+kB)] - (A-B)(k+1+A)}{a[(k+1)+(A+kB)] + (A-B)k} = \frac{G(k)}{F(k)} \leq \frac{G(n)}{F(n)}.$$

Therefore, we conclude that $\frac{1}{a}(f_1 \Delta f_2)\left(\frac{1}{p}, \frac{p-1}{p}; a, z\right) \in \overline{\Sigma S}_k^*(a, A, \hat{B})$.

(ii) It is similar to the proof of (i). For $-1 \leq A < B \leq 0$, we only need to show

$$\frac{(n+1)-(A+n\hat{B})}{a^2(\hat{B}-A)} \leq \frac{(n+1)-(A+nB)}{a(B-A)} \quad (\forall n \in \mathbb{N}, n \geq k),$$

that is,

$$\hat{B}\left[\frac{a[(n+1)-(A+nB)]}{B-A} + n\right] \geq \frac{aA[(n+1)-(A+nB)]}{B-A} + (n+1-A). \tag{3.9}$$

Let

$$F_2(n) = \frac{a[(n+1)-(A+nB)]}{B-A} + n$$

and

$$G_2(n) = \frac{aA[(n+1)-(A+nB)]}{B-A} + (n+1-A).$$

It is easy to see $F_2(n) > 0$ holds true for $n \geq k$. To make (3.9) establish, we only need to prove

$$\hat{B} \geq \frac{G_2(n)}{F_2(n)} = \frac{aA[(n+1)-(A+nB)] + (B-A)(n+1-A)}{a[(n+1)+(A+nB)] + (B-A)n}. \tag{3.10}$$

It is clear that $\frac{G_2(n)}{F_2(n)}$ is an decreasing function with respect to n for $0 < a < 1$ and

$$A > \frac{aA[(k+1)-(A+kB)] - (B-A)(k+1-A)}{a[(k+1)-(A+kB)] + (B-A)k},$$

then we have

$$\hat{B} \geq \frac{aA[(k+1)-(A+kB)] - (B-A)(k+1-A)}{a[(k+1)-(A+kB)] + (B-A)k} = \frac{G_2(k)}{F_2(k)} \geq \frac{G_2(n)}{F_2(n)}.$$

It can be concluded that $\frac{1}{a}(f_1 \Delta f_2)\left(\frac{1}{p}, \frac{p-1}{p}; a, z\right) \in \overline{\Sigma S}_k^*(a, A, \hat{B})$. Clearly, $\frac{G_2(n)}{F_2(n)}$ is an increasing function with respect to n for $a > 1$ and

$$\lim_{n \to +\infty} \frac{G_2(n)}{F_2(n)} = \frac{aA(1-B) + (B-A)}{a(1-B) + (B-A)}.$$

Since $A > \frac{aA(1-B)+(B-A)}{a(1-B)+(B-A)}$, we have

$$\hat{B} > A > \frac{aA(1-B)+(B-A)}{a(1-B)+(B-A)} = \lim_{n\to+\infty}\frac{G_2(n)}{F_2(n)} \geq \frac{G_2(n)}{F_2(n)}.$$

Thus, we conclude that $\frac{1}{a}(f_1\Delta f_2)\left(\frac{1}{p},\frac{p-1}{p};a,z\right) \in \overline{\Sigma S}_k^*(a,A,\hat{B})$. The proof of Theorem 2 is completed. Putting $A=1, B=0$ in Theorem 2, we obtain the following corollary.

Corollary 1. Let $f_i(z) = \frac{a}{z} - \sum_{n=k}^{\infty}|a_{n,i}|z^n \in \overline{\Sigma S}_k^*(a,1,0)$ $(i=1,2)$ and $p>1$. If $\hat{B}$ and a satisfy

$$\begin{cases} 0 \leq \hat{B} \leq \frac{(a-1)(k+2)}{a(k+2)+k}, & 0<a<1, \\ \hat{B} \leq \frac{a-1}{a+1}, & a>1. \end{cases}$$

Then $\frac{1}{a}(f_1 \vartriangle f_2)\left(\frac{1}{p},\frac{p-1}{p};a,z\right) \in \overline{\Sigma S}_k^*(a,A,\hat{B})$. Finally, we consider the closure properties of λ-Hadamard product $(f_1\overline{*}f_2)_\lambda(z)$ and the generalized λ-Hadamard product $(f_1\widetilde{\Delta}f_2)_\lambda(p.q,a;z)$ as follows.

Theorem 3. Let $f_i(z) = az - \sum_{n=k}^{\infty}|a_{n,i}|z^n \in \overline{\Sigma S}_k^*(a,A,B)$ $(i=1,2)$. If A, B and λ satisfy

$$\begin{cases} \frac{(A-B)-(k+1-A-nB)}{(A-B)(k+1)} < \lambda < \frac{1}{k+1}, & 0 \leq B < A \leq 1, \\ \lambda < \frac{1}{k+1}, & -1 \leq A < B \leq 0. \end{cases}$$

Then $\frac{1}{a}(f_1\overline{*}f_2)_\lambda(z) \in \overline{\Sigma S}_k^*(a,A,B)$.

Proof. Let $f_i(z) \in \overline{\Sigma S}_k^*(a,A,B)$ $(i=1,2)$. If $0 \leq B < A \leq 1$, from Lemma 2, we obtain

$$\sum_{n=k}^{\infty}((n+1)+(A+nB))|a_{n,1}| \leq a(A-B), \tag{3.11}$$

and

$$\sum_{n=k}^{\infty}((n+1)+(A+nB))|a_{n,2}| \leq a(A-B). \tag{3.12}$$

By Definition 2, we have

$$\frac{1}{a}(f_1\overline{*}f_2)_\lambda(z) = \frac{a}{z} - \sum_{n=k}^{\infty}\frac{1}{a}(1-(n+1)\lambda|a_{n,1}||a_{n,2}|z^n.$$

To show $\frac{1}{a}(f_1\overline{*}f_2)_\lambda(z) \in \overline{\Sigma S}_k^*(a,A,B)$, we just prove

$$\sum_{n=k}^{\infty}\frac{1}{a}(1-(n+1)\lambda)((n+1)+(A+nB))|a_{n,1}||a_{n,2}| \leq a(A-B). \tag{3.13}$$

Applying Cauchy-Schwarz inequality to (3.11) and (3.12), we get

$$\sum_{n=k}^{\infty}((n+1)+(A+nB))\sqrt{|a_{n,1}||a_{n,2}|} \leq a(A-B). \tag{3.14}$$

According to (3.13) and (3.14), we only need to prove

$$\sqrt{|a_{n,1}||a_{n,2}|} \leq \frac{a}{1-(n+1)\lambda}.$$

Since $\sqrt{|a_{n,1}||a_{n,2}|} \leq \frac{a(A-B)}{(n+1)+(A+nB)}$, we only need to prove

$$A - B \leq \frac{(n+1)+(A+nB)}{1-(n+1)\lambda}, n \geq k. \tag{3.15}$$

Let

$$M(n) = \frac{(n+1)+(A+nB)}{1-(n+1)\lambda}.$$

Clearly, $M(n)$ is an increasing function for $n \geq k$. This implies that (3.15) holds true if

$$\frac{k+1+A+nB}{1-(k+1)\lambda} \geq A - B.$$

For $\lambda < \frac{1}{k+1}$, the above formula can be expressed as $\lambda \geq \frac{(A-B)-(k+1-A-nB)}{(A-B)(k+1)}$.

To summarize, we conclude that $\frac{1}{a}(f_1 \overline{*} f_2)_\lambda(z) \in \overline{\Sigma S}^*_k(a, A, B)$ for $\frac{(A-B)-(k+1-A-nB)}{(A-B)(k+1)} < \lambda < \frac{1}{k+1}$. Using the same method, we can get $\frac{1}{a}(f_1 \overline{*} f_2)_\lambda(z) \in \overline{\Sigma S}^*_k(a, A, B)$ for

$$\lambda < \min\left\{\frac{(B-A)-(k+1-A-nB)}{(B-A)(k+1)}, \frac{1}{k+1}\right\} = \frac{1}{k+1}.$$

Thus, we complete the proof of Theorem 3.

Theorem 4. Let $f_i(z) = \frac{a}{z} - \sum_{n=k}^{\infty} |a_n| z^n \in \overline{\Sigma S}^*_k(a, A, B)$ $(i = 1, 2)$ and $p > 1$.

(1) For $0 \leq B < A \leq 1, 0 \leq \hat{B} < A$. If anyone of the following conditions holds true.

(i) $a < 1, \lambda < \min\left\{\frac{1}{2(n+1)}, \frac{a(1+B)+(A-B)}{(2n+1)(A-B)}, \frac{k(A-B)+a(k+1+A+kB)}{k(k+1)(A-B)}\right\}$,

(ii) $a > 1, \frac{1}{2(k+1)} < \lambda < \min\left\{\frac{a(1+B)+(A-B)}{(2n+1)(A-B)}, \frac{k(A-B)+a(k+1+A+kB)}{k(k+1)(A-B)}\right\}$,

(iii) $a < 1, \max\left\{\frac{a(1+B)+(A-B)}{(2k+1)(A-B)}, \frac{k(A-B)+a(k+1+A+kB)}{k(k+1)(A-B)}\right\} < \lambda < \frac{1}{2(n+1)}$,

(iv) $a > 1, \max\left\{\frac{1}{2(k+1)}, \frac{a(1+B)+(A-B)}{(2k+1)(A-B)}, \frac{k(A-B)+a(k+1+A+kB)}{k(k+1)(A-B)}\right\} < \lambda$.

Then the generalized λ-Hadamard product: $\frac{1}{a}(f_1 \widetilde{\Delta} f_2)_\lambda\left(\frac{1}{p}, \frac{p-1}{p}, a; z\right) \in \overline{\Sigma S}^*_k(a, A, B)$.

(2) For $-1 \leq A < B \leq 0$. If anyone of the following conditions holds true.

(v) $a < 1, \lambda < \min\left\{\frac{1}{2(n+1)}, \frac{(B-A)+a(1-B)}{(2k-1)(B-A)}, \frac{k(B-A)+a(k+1-A-kB)}{k(k+1)(B-A)}\right\}$,

$\max(A, \frac{aA[(k+1)-(A+kB)]+[1-(k+1)\lambda](k+1+A)(B-A)}{a[(k+1)-(A+kB)]-k[1-(k+1)\lambda](B-A)}) \leq \hat{B} \leq 0$,

(vi) $a > 1, \frac{1}{2(k+1)} < \lambda < \min\left\{\frac{(B-A)+a(1-B)}{(2k-1)(B-A)}, \frac{k(B-A)+a(k+1-A-kB)}{k(k+1)(B-A)}\right\}$,

$\max(A, \frac{aA[(k+1)-(A+kB)]+[1-(k+1)\lambda](k+1+A)(B-A)}{a[(k+1)-(A+kB)]-k[1-(k+1)\lambda](B-A)}) \leq \hat{B} \leq 0$,

(vii) $a < 1, \max\left\{\frac{(B-A)+a(1-B)}{(2k-1)(B-A)}, \frac{k(B-A)+a(k+1-A-kB)}{k(k+1)(B-A)}\right\} < \lambda < \frac{1}{2(n+1)}, A \leq \hat{B} < 0$,

(viii) $a > 1, \max\left\{\frac{1}{2(k+1)}, \frac{(B-A)+a(1-B)}{(2k-1)(B-A)}, \frac{k(B-A)+a(k+1-A-kB)}{k(k+1)(B-A)}\right\} < \lambda, A \leq \hat{B} < 0$.

Then the generalized λ-Hadamard product: $\frac{1}{a}(f_1\widetilde{\Delta}f_2)_\lambda\left(\frac{1}{p},\frac{p-1}{p},a;z\right)\in\overline{\Sigma S}^*_k(a,A,B)$.

Proof. Let $f_i(z)\in\overline{\Sigma S}^*_k(a,A,B)(i=1,2)$. The method for the proof of Theorem 4 is similar to that of Theorem 2. If $-1\le A<B\le 0$, we just need to prove

$$\frac{[1-(n+1)\lambda][(n+1)-(A+n\hat{B})]}{a^2(A-\hat{B})}\le\frac{(n+1)-(A-nB)}{a(B-A)},\tag{3.16}$$

that is

$$\hat{B}[\frac{a[(n+1)-(A+nB)]}{B-A}+n(1-(n+1)\lambda)]\le\frac{aA[(n+1)-(A+nB)]}{B-A}+[1-(n+1)\lambda](n+1-A).$$

Let

$$P(n)=\frac{a[(n+1)-(A+nB)]}{B-A}+n[1-(n+1)\lambda]$$

and

$$Q(n)=\frac{aA[(n+1)+(A+nB)]}{B-A}+[1-(n+1)\lambda](n+1-A).$$

The following discussion can be divided into two cases:

(a) It is easy to see that $P(n)>0$ if $\lambda<\frac{(B-A)+a(1-B)}{(B-A)(2n-1)}$. To prove that (3.16) holds true, we need only to prove the following inequality

$$\hat{B}\le\frac{Q(n)}{P(n)}=\frac{\frac{aA[(n+1)-(A+nB)]}{B-A}+[1-(n+1)\lambda](n+1-A)}{\frac{a[(n+1)-(A+nB)]}{B-A}+n[1-(n+1)\lambda]}.\tag{3.17}$$

Clearly, $\frac{Q(n)}{P(n)}$ is an increasing function with respect to n for $a<1,\lambda<\frac{1}{2(n+1)}$ or $a>1,\lambda>\frac{1}{2(k+1)}$. If $\lambda<\frac{k(B-A)+a(k+1-A-kB)}{k(k+1)(B-A)}$, then we have

$$\hat{B}\le\frac{aA[(k+1)-(A+kB)]+[1-(k+1)\lambda](k+1-A)(B-A)}{a[(k+1)-(A+kB)]+k[1-(k+1)\lambda](B-A)}=\frac{Q(k)}{P(k)}\le\frac{Q(n)}{P(n)}.$$

Therefore, $\frac{1}{a}(f_1\widetilde{\Delta}f_2)_\lambda\left(\frac{1}{p},\frac{p-1}{p},a,z\right)\in\overline{\Sigma S}^*_k(a,A,B)$ if the conditions (v) or (vi) are satisfied.

(b) It is easy to see that $P(n)<0$ if $\lambda>\frac{(B-A)+a(1-B)}{(B-A)(2n-1)}$ and $P(k)=\frac{a[(k+1)-(A+kB)]}{B-A}+k[1-(k+1)\lambda]>0$. To prove that (3.16) holds true, we need only to prove the following inequality

$$\hat{B}\le\frac{Q(n)}{P(n)}=\frac{\frac{aA[(n+1)-(A+nB)]}{A-B}-[1-(n+1)\lambda])(n+1-A)}{\frac{a[(n+1)-(A+nB)]}{B-A}+n[1-(n+1)\lambda]}.\tag{3.18}$$

Clearly, $\frac{Q(n)}{P(n)}$ is a decreasing function with respect to n for $a<1,\lambda<\frac{1}{2(n+1)}$ or $a>1,\lambda>\frac{1}{2(k+1)}$.

$$\hat{B}\le\lim_{n\to+\infty}\frac{Q(n)}{P(n)}\le\frac{Q(n)}{P(n)}.$$

Therefore, $\frac{1}{a}(f_1\widetilde{\Delta}f_2)_\lambda\left(\frac{1}{p},\frac{p-1}{p},a,z\right)\in\overline{\Sigma S}^*_k(a,A,B)$ if the conditions (vii) or (viii) are satisfied.

To summarize, we conclude that the conclusion (2) is established. Using the same method to that in the proof of (2), the conclusion (1) holds true. Thus, we complete the proof of Theorem 4.

Let $A = 1$ and $B = 0$ in Theorem 4, we have the following corollary.

Corollary 2. Let $f_i(z) \in \overline{\Sigma S}^*_k(a, 1, 0)$ $(i = 1, 2)$ and $p > 1$. If anyone of the following conditions is satisfied.

(i) $a < 1, \lambda < \min\left\{\frac{1}{2(n+1)}, \frac{a+1}{(2n+1)}, \frac{k+a(k+2)}{k(k+1)}\right\}$,

(ii) $a > 1, \frac{1}{2(k+1)} < \lambda < \min\left\{\frac{a+1}{(2n+1)}, \frac{k+a(k+2)}{k(k+1)}\right\}$,

(iii) $a < 1, \max\left\{\frac{a+1}{(2k+1)}, \frac{k+a(k+2)}{k(k+1)}\right\} < \lambda < \frac{1}{2(n+1)}$,

(iv) $a > 1, \max\left\{\frac{1}{2(k+1)}, \frac{a+1}{(2k+1)}, \frac{k+a(k+2)}{k(k+1)}\right\} < \lambda$.

Then the generalized λ-Hadamard product $(f_1 \widetilde{\Delta} f_2)_\lambda \left(\frac{1}{p}, \frac{p-1}{p}, a; z\right) \in \overline{\Sigma S}^*_k(a, 1 - 2\beta, \hat{B})$.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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