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Research article

New identities involving Hardy sums $S_3(h,k)$ and general Kloosterman sums

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Abstract: The main purpose of this paper is to obtain some exact computational formulas or upper bounds for hybrid mean value involving Hardy sums $S_3(h,p)$ and general Kloosterman sums $K(r,l,\lambda;p)$. By applying the properties of Gauss sums and the mean value theorems of Dirichlet L-function, we derive some new identities. As the special cases, we also deduce some exact computational formulas for hybrid mean value involving $S_3(h,p)$ and classical Kloosterman sums $K(n,p)$.

Keywords: Hardy sums; general Kloosterman sums; hybrid mean value
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1. Introduction and main results

If h and k are integers with $k > 0$, the classical Dedekind sums $S(h,k)$ are defined as

$$S(h,k) = \sum_{a=1}^{k} \left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

The various properties of $S(h,k)$ were investigated by many authors, one of which is reciprocity theorem (see Tom M. Apostol [1] or L. Carlitz [2]). That is, for all positive integers h and k with $(h,k) = 1$, we have the identity

$$S(h,k) + S(k,h) = \frac{h^2 + k^2 + 1}{12hk} - \frac{1}{4}.$$

Conrey et al. [3] studied the mean value distribution of $S(h,k)$ and deduced the important asymptotic formula

$$\sum_{h=1}^{k}{}' |S(h,k)|^{2m} = f_m(k)\left(\frac{k}{12}\right)^{2m} + O\left(\left(k^{\frac{9}{5}} + k^{2m-1+\frac{1}{m+1}}\right)\log^3 k\right),$$

where $\sum_{h=1}^{k}{}'$ denotes the summation over all h such that $(h,k)=1$ and

$$\sum_{n=1}^{\infty}\frac{f_m(n)}{n^s} = 2\frac{\zeta^2(2m)}{\zeta(4m)}\cdot\frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\zeta(s).$$

Moreover, X. L. He and W. P. Zhang [4] gave an interesting asymptotic formula for the Dedekind sums with a weight of Hurwitz zeta-function as follows:

$$\sum_{h=1}^{k}{}'\zeta^2\left(\frac{1}{2},\frac{h}{k}\right)S^2(h,k) = \frac{k^3}{144}\zeta(3)\prod_{p|k}\left(1-\frac{1}{p^3}\right) + O\left(k^{\frac{5}{2}}\exp\left(\frac{3\log k}{\log\log k}\right)\right).$$

Other sums analogous to the Dedekind sums are the Hardy sums. Using the notation of Berndt and Goldberg [5], they defined

$$S_1(h,k) = \sum_{j=1}^{k-1}(-1)^{j+1+\left[\frac{hj}{k}\right]},$$

where h and k are integers with $k>0$.

In 2014, H. Zhang and W. P. Zhang [6] obtained some beautiful identities involving $S_1(h,k)$ in the forms of

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}K(m,p)K(n,p)S_1(2m\overline{n},p),$$

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,p)|^2\,|K(n,p)|^2\,S_1(2m\overline{n},p),$$

where $K(n,p)$ denotes the reduced form of the general Kloosterman sums attached to a Dirichlet character λ modulo k as

$$K(r,l,\lambda;k) = \sum_{a=1}^{k}{}'\lambda(a)e\left(\frac{ra+l\overline{a}}{k}\right),$$

where $e(x)=e^{2\pi ix}$, $\overline{a}$ denotes the solution of the congruence $x\cdot a\equiv 1 \bmod k$.

Recently, H. F. Zhang and T. P. Zhang [7] extended the results in [6] to a more general situation as

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}K(m,s,\lambda;p)\overline{K(n,t,\lambda;p)}S_1(2m\overline{n},p),$$

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_1(2m\overline{n},p),$$

where $\overline{K(n,t,\lambda;p)}$ denotes complex conjugate of $K(n,t,\lambda;p)$.

Actually there are six forms of Hardy sums (see Berndt [8] and Goldberg [9]). A natural question is whether we can obtain similar results by replacing $S_1(h,k)$ with other forms of Hardy sums. Due to some technical reasons, for most of other forms of Hardy sums, the answer is no! Thanks to the important relationships among Hardy sums and Dedekind sums built by R. Sitaramachandrarao [10], we are lucky to find the only one $S_3(h,p)$ to replace, with

$$S_3(h,k)=\sum_{j=1}^{k}(-1)^j\left(\left(\frac{hj}{k}\right)\right).$$

Our starting point relies heavily on the following in [10] as:

Proposition 1. Let k be an odd positive integer, h be an integer with $(h,k)=1$. Then

$$S_3(h,k)=2S(h,k)-4S(2h,k).$$

Then applying the properties of Gauss sums and the mean square value of Dirichlet L-functions, we have

Theorem 1. Let p be an odd prime. Then for any Dirichlet character λ mod p and any integer s, t with $(s,p)=(t,p)=1$, we have

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}K(m,s,\lambda;p)\overline{K(n,t,\lambda;p)}S_3(m\overline{n},p)=\begin{cases}\dfrac{p-1}{2}, & \text{if } \overline{\lambda}\chi=\chi_0;\\ \dfrac{p(p-1)}{2}, & \text{if } \overline{\lambda}\chi\neq\chi_0,\end{cases}$$

where χ is an odd Dirichlet character modulo p and χ_0 is the principal character modulo p.

Theorem 2. Let p be an odd prime with $p\equiv 1 \bmod 4$. Then for any Dirichlet character λ mod p and any integer s, t with $(s,p)=(t,p)=1$, we have

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)$$

$$\begin{cases}=\dfrac{p^2(p-1)}{2}, & \text{if } \overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi\neq\chi_0;\\ =\dfrac{p(p-1)}{2}, & \text{if } \overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi=\chi_0;\\ \leq p^{\frac{9}{2}}+\dfrac{1}{2}p^4-4p^{\frac{7}{2}}-p^3+5p^{\frac{5}{2}}+p^2-2p^{\frac{3}{2}}-\dfrac{1}{2}p, & \text{if } \overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi=\chi_0;\\ \leq p^5-3p^4+3p^3-\dfrac{1}{2}p^2-\dfrac{1}{2}p, & \text{if } \overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi\neq\chi_0.\end{cases}$$

Theorem 3. Let p be an odd prime with $p\equiv 3 \bmod 8$. Then for any Dirichlet character λ mod p and any integer s, t with $(s,p)=(t,p)=1$, we have

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)$$

$$\begin{cases} = \dfrac{p^2(p-1)}{2} - 6p^2h_p^2, & \text{if } \overline{\lambda\chi} \neq \chi_0,\ \overline{\lambda}\chi \neq \chi_0; \\ = \dfrac{p(p-1)}{2} - 6ph_p^2, & \text{if } \overline{\lambda\chi} \neq \chi_0,\ \overline{\lambda}\chi = \chi_0; \\ \leq p^{\frac{9}{2}} + \dfrac{1}{2}p^4 - 4p^{\frac{7}{2}} - p^3 + 5p^{\frac{5}{2}} + p^2 - 2p^{\frac{3}{2}} - \dfrac{1}{2}p + 6p^2h_p^2, & \text{if } \overline{\lambda\chi} = \chi_0,\ \overline{\lambda}\chi = \chi_0; \\ \leq p^5 - 3p^4 + 3p^3 - \dfrac{1}{2}p^2 - \dfrac{1}{2}p + 6p^3h_p^2, & \text{if } \overline{\lambda\chi} = \chi_0,\ \overline{\lambda}\chi \neq \chi_0, \end{cases}$$

where h_p denotes the class number of the quadratic field $Q\left(\sqrt{-p}\right)$.

Theorem 4. Let p be an odd prime with $p \equiv 7 \bmod 8$. Then for any Dirichlet character λ mod p and any integer s, t with $(s,p) = (t,p) = 1$, we have

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} |K(m,s,\lambda;p)|^2 |K(n,t,\lambda;p)|^2 S_3(m\overline{n},p)$$

$$\begin{cases} = \dfrac{p^2(p-1)}{2} + 2p^2h_p^2, & \text{if } \overline{\lambda\chi} \neq \chi_0,\ \overline{\lambda}\chi \neq \chi_0; \\ = \dfrac{p(p-1)}{2} + 2ph_p^2, & \text{if } \overline{\lambda\chi} \neq \chi_0,\ \overline{\lambda}\chi = \chi_0; \\ \leq p^{\frac{9}{2}} + \dfrac{1}{2}p^4 - 4p^{\frac{7}{2}} - p^3 + 5p^{\frac{5}{2}} + p^2 - 2p^{\frac{3}{2}} - \dfrac{1}{2}p + 2p^2h_p^2, & \text{if } \overline{\lambda\chi} = \chi_0,\ \overline{\lambda}\chi = \chi_0; \\ \leq p^5 - 3p^4 + 3p^3 - \dfrac{1}{2}p^2 - \dfrac{1}{2}p + 2p^3h_p^2, & \text{if } \overline{\lambda\chi} = \chi_0,\ \overline{\lambda}\chi \neq \chi_0. \end{cases}$$

Taking $\lambda = \lambda_0$, $s = t = 1$ in Theorems 1–4, we immediately deduce the following results.

Corollary 1. Let p be an odd prime. Then we have the identity

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} K(m,p)K(n,p)S_3(m\overline{n},p) = \frac{p(p-1)}{2}.$$

Corollary 2. Let p be an odd prime. Then we have

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} |K(m,p)|^2 |K(n,p)|^2 S_3(m\overline{n},p) = \begin{cases} \dfrac{p^2(p-1)}{2}, & \text{if } p \equiv 1 \bmod 4; \\ \dfrac{p^2(p-1)}{2} - 6p^2h_p^2, & \text{if } p \equiv 3 \bmod 8; \\ \dfrac{p^2(p-1)}{2} + 2p^2h_p^2, & \text{if } p \equiv 7 \bmod 8. \end{cases}$$

2. Some Lemmas

To prove the Theorems, we need the following Lemmas.

Lemma 1. Let $k > 2$ be an integer. Then for any integer a with $(a,k) = 1$, we have the identity

$$S(a,k) = \frac{1}{\pi^2 k}\sum_{d|k}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}}\chi(a)\left|L(1,\chi)\right|^2,$$

where $L(1,\chi)$ denotes the Dirichlet L-function corresponding to Dirichlet character χ mod d.

Proof. See Lemma 2 of [11].

Lemma 2. Let p be an odd prime, s be any integer with $(s,p)=1$. Then for any non-principal character χ mod p and any Dirichlet character λ mod p, we have

$$\left|\sum_{m=1}^{p-1}\chi(m)K(m,s,\lambda;p)\right| = \begin{cases} p^{\frac{1}{2}}, & \text{if } \overline{\lambda}\chi=\chi_0;\\ p, & \text{if } \overline{\lambda}\chi\neq\chi_0. \end{cases}$$

Proof. See Lemma 2 of reference [7].

Lemma 3. Let p be an odd prime, s be any integer with $(s,p)=1$. Then for any non-principal character χ mod p and any Dirichlet character λ mod p, we have

$$\left|\sum_{m=1}^{p-1}\chi(m)\left|K(m,s,\lambda;p)\right|^2\right| = \begin{cases} p\left|\tau\left(\overline{\chi}^2\right)\right|, & \text{if } \overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi\neq\chi_0;\\ p^{\frac{1}{2}}\left|\tau\left(\overline{\chi}^2\right)\right|, & \text{if } \overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi=\chi_0;\\ p\left|\tau\left(\overline{\chi}^2\right)+(p-1)\right|, & \text{if } \overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi=\chi_0;\\ p\left|-\tau\left(\overline{\chi}^2\right)\tau\left(\overline{\lambda}\chi\right)+(p-1)\right|, & \text{if } \overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi\neq\chi_0, \end{cases}$$

where $\tau(\chi)=\sum_{a=1}^{p}\chi(a)e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums.

Proof. See Lemma 1 of reference [7].

Lemma 4. Let p be an odd prime, then we have

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2 = \frac{\pi^2}{12}\cdot\frac{(p-1)^2(p-2)}{p^2},$$

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\cdot|L(1,\chi)|^2 = \frac{\pi^2}{24}\cdot\frac{(p-1)^2(p-5)}{p^2}.$$

Proof. See Lemma 5 of reference [6].

3. Proof of Theorems

Now we come to prove our Theorems.

Firstly, we prove Theorem 1. Applying Proposition 1 and Lemma 1, we obtain

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}K(m,s,\lambda;p)\overline{K(n,t,\lambda;p)}S_3(m\overline{n},p)\\
&=\frac{2p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\sum_{m=1}^{p-1}\chi(m)K(m,s,\lambda;p)\cdot\sum_{n=1}^{p-1}\chi(\overline{n})\overline{K(n,t,\lambda;p)}\cdot|L(1,\chi)|^2\\
&\quad-\frac{4p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\sum_{m=1}^{p-1}\chi(m)K(m,s,\lambda;p)\cdot\sum_{n=1}^{p-1}\chi(\overline{n})\overline{K(n,t,\lambda;p)}\cdot|L(1,\chi)|^2
\end{aligned}$$

$$= \frac{2p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left| \sum_{m=1}^{p-1} \chi(m) K(m,s,\lambda;p) \right|^2 |L(1,\chi)|^2$$
$$- \frac{4p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \left| \sum_{m=1}^{p-1} \chi(m) K(m,s,\lambda;p) \right|^2 |L(1,\chi)|^2 .$$

Then from Lemma 2 and Lemma 4, if $\overline{\lambda}\chi = \chi_0$, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} K(m,s,\lambda;p)\overline{K(n,t,\lambda;p)} S_3(m\overline{n},p) \\
&= \frac{2p^2}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 - \frac{4p^2}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2)\,|L(1,\chi)|^2 \\
&= \frac{2p^2}{\pi^2(p-1)} \cdot \frac{\pi^2}{12} \frac{(p-1)^2(p-2)}{p^2} - \frac{4p^2}{\pi^2(p-1)} \cdot \frac{\pi^2}{24} \frac{(p-1)^2(p-5)}{p^2} \\
&= \frac{p-1}{2}.
\end{aligned}$$

While if $\overline{\lambda}\chi \neq \chi_0$, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} K(m,s,\lambda;p)\overline{K(n,t,\lambda;p)} S_3(m\overline{n},p) \\
&= \frac{2p^3}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 - \frac{4p^3}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2)\,|L(1,\chi)|^2 \\
&= \frac{2p^3}{\pi^2(p-1)} \cdot \frac{\pi^2}{12} \cdot \frac{(p-1)^2(p-2)}{p^2} - \frac{4p^3}{\pi^2(p-1)} \cdot \frac{\pi^2}{24} \cdot \frac{(p-1)^2(p-5)}{p^2} \\
&= \frac{p(p-1)}{2}.
\end{aligned}$$

This completes the proof of Theorem 1.

Then we prove Theorem 2. From Proposition 1 and Lemma 1, we obtain

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} |K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\, S_3(m\overline{n},p) \\
&= \frac{2p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left| \sum_{m=1}^{p-1} \chi(m)\,|K(m,s,\lambda;p)|^2 \right|^2 |L(1,\chi)|^2 \\
&\quad - \frac{4p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \left| \sum_{m=1}^{p-1} \chi(m)\,|K(m,s,\lambda;p)|^2 \right|^2 |L(1,\chi)|^2 .
\end{aligned}$$

Since $p \equiv 1 \bmod 4$, and notice that $\left|\tau\left(\overline{\chi}^2\right)\right| = \sqrt{p}$. From Lemma 3 and Lemma 4, if $\overline{\lambda\chi} \neq \chi_0$, $\overline{\lambda}\chi \neq \chi_0$, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)\\
&=\frac{2p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2-\frac{4p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\,|L(1,\chi)|^2\\
&=\frac{2p^4}{\pi^2(p-1)}\cdot\frac{\pi^2}{12}\cdot\frac{(p-1)^2(p-2)}{p^2}-\frac{4p^4}{\pi^2(p-1)}\cdot\frac{\pi^2}{24}\cdot\frac{(p-1)^2(p-5)}{p^2}\\
&=\frac{p^2(p-1)}{2}.
\end{aligned}$$

Similarly, if $\overline{\lambda\chi} \neq \chi_0,\ \overline{\lambda}\chi = \chi_0$, we have

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)=\frac{p(p-1)}{2}.$$

If $\overline{\lambda\chi} = \chi_0,\ \overline{\lambda}\chi = \chi_0$, we can obtain

$$\begin{aligned}
&\left|\sum_{m=1}^{p-1}\chi(m)\,|K(m,s,\lambda;p)|^2\right|^2\\
&=p^2\left[\left(Re\,\tau\left(\overline{\chi}^2\right)+(p-1)\right)^2+\left(Im\,\tau\left(\overline{\chi}^2\right)\right)^2\right]\\
&=p^2\left[p+2(p-1)Re\,\tau\left(\overline{\chi}^2\right)+(p-1)^2\right].
\end{aligned}$$

So we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)\\
&=\frac{2p^3}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left[p+2(p-1)Re\,\tau\left(\overline{\chi}^2\right)+(p-1)^2\right]|L(1,\chi)|^2\\
&\quad-\frac{4p^3}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\left[p+2(p-1)Re\,\tau\left(\overline{\chi}^2\right)+(p-1)^2\right]|L(1,\chi)|^2\\
&=\frac{2p^3[p+(p-1)^2]}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2+\frac{4p^3}{\pi^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}Re\left(\tau\left(\overline{\chi}^2\right)\right)|L(1,\chi)|^2\\
&\quad-\frac{4p^3[p+(p-1)^2]}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\,|L(1,\chi)|^2-\frac{8p^3}{\pi^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)Re\left(\tau\left(\overline{\chi}^2\right)\right)|L(1,\chi)|^2\\
&=\frac{p(p-1)(p^2-p+1)}{2}+\frac{4p^3}{\pi^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}Re\left(\tau\left(\overline{\chi}^2\right)\right)|L(1,\chi)|^2
\end{aligned}$$

$$-\frac{8p^3}{\pi^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)Re\left(\tau\left(\overline{\chi}^2\right)\right)|L(1,\chi)|^2.$$

Noting that $x \le |x|$ holds for any real number x, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)\\
&\le\left|\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)\right|\\
&\le\frac{p(p-1)(p^2-p+1)}{2}+\frac{4p^{\frac{7}{2}}}{\pi^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2+\frac{8p^{\frac{7}{2}}}{\pi^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2\\
&=\frac{p(p-1)(p^2-p+1)}{2}+\frac{4p^{\frac{7}{2}}}{\pi^2}\cdot\frac{\pi^2}{12}\cdot\frac{(p-1)^2(p-2)}{p^2}+\frac{8p^{\frac{7}{2}}}{\pi^2}\cdot\frac{\pi^2}{12}\cdot\frac{(p-1)^2(p-2)}{p^2}\\
&=p^{\frac{9}{2}}+\frac{1}{2}p^4-4p^{\frac{7}{2}}-p^3+5p^{\frac{5}{2}}+p^2-2p^{\frac{3}{2}}-\frac{1}{2}p.
\end{aligned}$$

Similarly, if $\overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi\neq\chi_0$, we have

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)\le p^5-3p^4+3p^3-\frac{1}{2}p^2-\frac{1}{2}p.$$

This completes the proof of Theorem 2.

Next we turn to prove Theorem 3. Since $p\equiv 3 \bmod 4$, note that $\left(\frac{-1}{p}\right)=\chi_2(-1)=-1$, $L(1,\chi_2)=\frac{\pi\cdot h_p}{\sqrt{p}}$, and $\tau\left(\overline{\chi}_2^2\right)=-1$. From Lemma 3 and Lemma 4, if $\overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi\neq\chi_0$, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)\\
&=\frac{2p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2-\frac{2p^4}{\pi^2(p-1)}|L(1,\chi_2)|^2+\frac{2p^3}{\pi^2(p-1)}|L(1,\chi_2)|^2\\
&\quad-\frac{4p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\,|L(1,\chi)|^2+\frac{4p^4}{\pi^2(p-1)}\chi_2(2)\,|L(1,\chi_2)|^2-\frac{4p^3}{\pi^2(p-1)}\chi_2(2)\,|L(1,\chi_2)|^2\\
&=\frac{p^2(p-1)}{2}-\frac{2p^3}{\pi^2}|L(1,\chi_2)|^2+\frac{4p^3}{\pi^2}\chi_2(2)\,|L(1,\chi_2)|^2\\
&=\frac{p^2(p-1)}{2}-2p^2h_p^2+4p^2h_p^2\left(\frac{2}{p}\right).
\end{aligned}$$

Similarly, if $\overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi=\chi_0$, we have

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)=\frac{p(p-1)}{2}-2ph_p^2+4ph_p^2\left(\frac{2}{p}\right).$$

If $\overline{\lambda\chi} = \chi_0,\ \overline{\lambda}\chi = \chi_0$, we can obtain

$$p^2\left|\tau\left(\overline{\chi}_2^2\right) + (p-1)\right|^2 = p^2\left[1 + 2(p-1)Re\,\tau\left(\overline{\chi}_2^2\right) + (p-1)^2\right].$$

So we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)\\
&= \frac{2p^3}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left[p + 2(p-1)Re\,\tau\left(\overline{\chi}^2\right) + (p-1)^2\right]|L(1,\chi)|^2\\
&\quad -\frac{2p^3}{\pi^2(p-1)}\left[p + 2(p-1)Re\,\tau\left(\overline{\chi}_2^2\right) + (p-1)^2\right]|L(1,\chi_2)|^2\\
&\quad +\frac{2p^3}{\pi^2(p-1)}\left[1 + 2(p-1)Re\,\tau\left(\overline{\chi}_2^2\right) + (p-1)^2\right]|L(1,\chi_2)|^2\\
&\quad -\frac{4p^3}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\left[p + 2(p-1)Re\,\tau\left(\overline{\chi}^2\right) + (p-1)^2\right]|L(1,\chi)|^2\\
&\quad +\frac{4p^3}{\pi^2(p-1)}\left[p + 2(p-1)Re\,\tau\left(\overline{\chi}_2^2\right) + (p-1)^2\right]\chi_2(2)\,|L(1,\chi_2)|^2\\
&\quad -\frac{4p^3}{\pi^2(p-1)}\left[1 + 2(p-1)Re\,\tau\left(\overline{\chi}_2^2\right) + (p-1)^2\right]\chi_2(2)\,|L(1,\chi_2)|^2\\
&= \frac{p(p-1)(p^2-p+1)}{2} + \frac{4p^3}{\pi^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}Re\,\left(\tau\left(\overline{\chi}^2\right)\right)|L(1,\chi)|^2\\
&\quad -\frac{8p^3}{\pi^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)Re\,\left(\tau\left(\overline{\chi}^2\right)\right)|L(1,\chi)|^2 - 2p^2h_p^2 + 4p^2h_p^2\left(\frac{2}{p}\right).
\end{aligned}$$

Similarly, if $\overline{\lambda\chi} = \chi_0,\ \overline{\lambda}\chi \neq \chi_0$, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2|K(n,t,\lambda;p)|^2S_3(m\overline{n},p)\\
&= \frac{p(p-1)(2p^2-2p+1)}{2} - \frac{4p^3}{\pi^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}Re\,\left(\tau(\overline{\chi}^2)\tau(\overline{\lambda}\chi)\right)|L(1,\chi)|^2\\
&\quad +\frac{8p^3}{\pi^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)Re\,\left(\tau(\overline{\chi}^2)\tau(\overline{\lambda}\chi)\right)|L(1,\chi)|^2 - 2p^3h_p^2 + 4p^3h_p^2\left(\frac{2}{p}\right).
\end{aligned}$$

Combining the fact that

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1, & \text{if } p \equiv \pm 1 \bmod 8;\\ -1, & \text{if } p \equiv \pm 3 \bmod 8,\end{cases}$$

we deduce that if $p \equiv 3 \bmod 8$, then

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)$$
$$=\begin{cases}\dfrac{p^2(p-1)}{2}-6p^2h_p^2, & \text{if } \overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi\neq\chi_0;\\ \dfrac{p(p-1)}{2}-6ph_p^2, & \text{if } \overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi=\chi_0.\end{cases}$$

If $\overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi=\chi_0$, we have

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)$$
$$=\frac{p(p-1)(p^2-p+1)}{2}+\frac{4p^3}{\pi^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}Re\left(\tau\left(\overline{\chi}^2\right)\right)|L(1,\chi)|^2$$
$$-\frac{8p^3}{\pi^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)Re\left(\tau\left(\overline{\chi}^2\right)\right)|L(1,\chi)|^2-6p^2h_p^2.$$

Then

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2\,|K(n,t,\lambda;p)|^2\,S_3(m\overline{n},p)$$
$$\leq\frac{p(p-1)(p^2-p+1)}{2}+\frac{4p^3}{\pi^2}\left|\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}Re\left(\tau\left(\overline{\chi}^2\right)\right)|L(1,\chi)|^2\right|$$
$$+\frac{8p^3}{\pi^2}\left|\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)Re\left(\tau\left(\overline{\chi}^2\right)\right)|L(1,\chi)|^2\right|+6p^2h_p^2$$
$$=p^{\frac{9}{2}}+\frac{1}{2}p^4-4p^{\frac{7}{2}}-p^3+5p^{\frac{5}{2}}+p^2-2p^{\frac{3}{2}}-\frac{1}{2}p+6p^2h_p^2.$$

Similarly, if $\overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi\neq\chi_0$, we have

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,s,\lambda;p)|^2|K(n,t,\lambda;p)|^2S_3(m\overline{n},p)\leq p^5-3p^4+3p^3-\frac{1}{2}p^2-\frac{1}{2}p+6p^3h_p^2.$$

This completes the proof of Theorem 3.

Theorem 4 can be derived by the same method. This completes the proof of our Theorems.

4. Conclusions

In this paper, we obtain some exact computational formulas or upper bounds for hybrid mean value involving Hardy sums and Kloosterman sums (both classical Kloosterman sums and general

Kloosterman sums) by applying the properties of Gauss sums and the mean value of Dirichlet L-function. But in some cases, unluckily, it is difficult to get the exact formula. So how to get the exact formula in all cases remains open.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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