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Research article

On coupled impulsive fractional integro-differential equations with Riemann-Liouville derivatives

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Abstract: In this paper, we investigate the existence, uniqueness and stability of coupled impulsive fractional integro-differential equations with Riemann-Liouville derivatives. To prove the existence and uniqueness results for afore mentioned system, we use the techniques of Kransnoselskii's type fixed point theorem. Furthermore, different kinds of Ulam stabilities are discussed along with examples, to demonstrate the validity of main results.

Keywords: Riemann-Liouville fractional derivatives; fractional integro-differential equations; coupled system; impulses; Ulam stability
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1. Introduction

Fractional order differential equations are the generalizations of the classical integer order differential equations. The idea about the fractional order derivative was introduced at the end of the sixteenth century (1695) when Leibniz used the notation $\frac{d^n}{d\sigma^n}$ for n^{th} order derivative. By writing a letter to him, L'Hospital asked the question: what would be the result if $n = \frac{1}{2}$? Leibniz answered in such words, "An apparent Paradox, from which one day useful consequences will be drawn", and this question became the foundation of fractional calculus. Fractional calculus has become a speedily developing area and its applications can be found in diverse fields ranging from physical sciences, porous media, electrochemistry, economics, electromagnetics, medicine and engineering to biological sciences. Progressively, fractional differential equations play a very important role in fields such as thermodynamics, statistical physics, nonlinear oscillation of earthquakes, viscoelasticity, defence, optics, control, signal processing, electrical circuits, astronomy etc. There are some outstanding articles which provide the main theoretical tools for the qualitative analysis of this research field, and

at the same time, shows the interconnection as well as the distinction between integral models, classical and fractional differential equations, see [14, 16, 18, 19, 22, 25, 26, 28, 30]

Impulsive fractional differential equations are used to describe both physical, social sciences and many dynamical systems such as evolution processes pharmacotherapy. There are two types of impulsive fractional differential equations the first one is instantaneous impulsive fractional differential equations while the other one is non-instantaneous impulsive fractional differential equations. In last few decades, the theory of impulsive fractional differential equations are well utilized in medicine, mechanical engineering, ecology, biology and astronomy etc. There are some remarkable monographs [3, 6, 8, 15, 20, 23, 33, 34], considering fractional differential equations with impulses.

The most preferable research area in the field of fractional differential equations ($\mathcal{FDE}'s$), which received great attention from the researchers is the theory regarding the existence of solutions. Many researchers developed some interesting results about the existence of solutions of different boundary value problems (BVPs) using different fixed point theorems. For details we refer the reader to [2, 7, 9–11, 13, 27]. Most of the time, it is quite intricate to find the exact solutions of nonlinear differential equations, in such a situation different approximation techniques are introduced. The difference between exact and approximate solutions is nowadays dealt with using Hyers-Ulam ($\mathcal{HU}$) type stabilities, which were first introduced in 1940 by Ulam [29] and then answered by Hyers in the following year in the context of Banach spaces. Many researchers investigated $\mathcal{HU}$ type stabilities for different problems with different approaches [12, 17, 31, 35–37, 39, 40].

Zada and Dayyan [38], investigated the existence, uniqueness and Ulam's type stability for the implicit fractional differential equation with instantaneous impulses and Riemann-Liouville fractional integral boundary conditions having the following form

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}_{0,\sigma}\mathrm{u}(\sigma) - \phi_1(\sigma, \mathrm{u}(\sigma), {}^{c}\mathcal{D}^{\alpha}\mathrm{u}(\sigma)) = 0, \;\; \sigma \neq \sigma_j \in I, \;\; 0 < \alpha \leq 1, \\ \Delta\mathrm{u}(\sigma_j) - \mathcal{E}_j(\mathrm{u}(\sigma_j)) = 0, \;\; j = 1, 2, \ldots, q-1, \\ \eta_1\mathrm{u}(\sigma)|_{\sigma=0} + \xi_1\mathcal{I}^{\alpha}\mathrm{u}(\sigma)|_{\sigma=0} = \nu_1, \qquad \eta_2\mathrm{u}(\sigma)|_{\sigma=\mathbb{T}} + \xi_2\mathcal{I}^{\alpha}\mathrm{u}(\sigma)|_{\sigma=\mathbb{T}} = \nu_2, \end{cases}$$

where $I = [0, \mathbb{T}]$, and ${}^{c}\mathcal{D}^{\alpha}_{0,\sigma}$ is a generalization of classical Caputo derivative of order α with lower bound at 0, $\phi_1 : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Furthermore, $\mathrm{u}(\sigma_j^+)$ and $\mathrm{u}(\sigma_j^+)$ represent the right-sided and left-sided limits respectively at $\sigma = \sigma_j$ for $j = 1, 2, \ldots, q-1$.

Ali et al. [4], studied a coupled system for the existence and uniqueness of solution using Riemann-Liouville derivative

$$\begin{cases} \mathcal{D}^{\alpha}\mathrm{u}(\sigma) = \phi_1(\sigma, \mathrm{v}(\sigma), \mathcal{D}^{\alpha}\mathrm{u}(\sigma)), \;\; \mathcal{D}^{\beta}\mathrm{v}(\sigma) = \phi_2(\sigma, \mathrm{u}(\sigma), \mathcal{D}^{\beta}\mathrm{v}(\sigma)), \;\; \sigma \in \mathcal{J}, \\ \mathcal{D}^{\alpha-1}\mathrm{u}(0^+) = \beta_1\mathcal{D}^{\alpha-1}\mathrm{u}(\mathbb{T}^-), \;\; \mathcal{D}^{\alpha-1}\mathrm{u}(0^+) = \gamma_1\mathcal{D}^{\alpha-1}\mathrm{u}(\mathbb{T}^-), \\ \mathcal{D}^{\beta-1}\mathrm{v}(0^+) = \beta_2\mathcal{D}^{\beta-1}\mathrm{v}(\mathbb{T}^-), \;\; \mathcal{D}^{\beta-1}\mathrm{v}(0^+) = \gamma_2\mathcal{D}^{\beta-1}\mathrm{v}(\mathbb{T}^-), \end{cases}$$

where $\sigma \in \mathcal{J} = [0, \mathbb{T}]$, $\mathbb{T} > 0$, $\alpha, \beta \in (1, 2]$, and $\beta_1, \beta_2, \gamma_1, \gamma_2 \neq 1$. $\mathcal{D}^{\alpha}$, $\mathcal{D}^{\beta}$ are the Riemann-Liouville fractional derivatives and $\phi_1, \phi_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Wang et al. [32], presented stability of the following coupled system of implicit fractional integro-

differential equations having anti-periodic boundary conditions:

$$\begin{cases} {}^c\mathcal{D}^{\alpha}\mathrm{u}(\sigma) - \phi_1(\sigma, \mathrm{v}(\sigma), {}^c\mathcal{D}^{\alpha}\mathrm{u}(\sigma)) - \dfrac{1}{\Gamma(\gamma_1)}\displaystyle\int_0^{\sigma} (\sigma - \mathrm{s})^{\gamma_1 - 1} f(s, \mathrm{v}(s), {}^c\mathcal{D}^{\alpha}\mathrm{u}(s))ds = 0, \quad \forall\ \sigma \in \mathcal{J}, \\ {}^c\mathcal{D}^{\beta}\mathrm{v}(\sigma) - \phi_2(\sigma, \mathrm{u}(\sigma), {}^c\mathcal{D}^{\beta}\mathrm{v}(\sigma)) - \dfrac{1}{\Gamma(\gamma_2)}\displaystyle\int_0^{\sigma} (\sigma - \mathrm{s})^{\gamma_2 - 1} g(s, \mathrm{u}(s), {}^c\mathcal{D}^{\beta}\mathrm{v}(s))ds = 0, \quad \forall\ \sigma \in \mathcal{J}, \\ \mathrm{u}(\sigma)|_{\sigma=0} = -\mathrm{u}(\sigma)|_{\sigma=\mathbb{T}} = 0, \quad {}^c\mathcal{D}^{\mathrm{r}_1}\mathrm{u}(\sigma)|_{\sigma=0} = -{}^c\mathcal{D}^{\mathrm{r}_1}\mathrm{u}(\sigma)|_{\sigma=\mathbb{T}}, \\ \mathrm{v}(\sigma)|_{\sigma=0} = -\mathrm{v}(\sigma)|_{\sigma=\mathbb{T}} = 0, \quad {}^c\mathcal{D}^{\mathrm{r}_2}\mathrm{v}(\sigma)|_{\sigma=0} = -{}^c\mathcal{D}^{\mathrm{r}_2}\mathrm{v}(\sigma)|_{\sigma=\mathbb{T}}, \end{cases}$$

where $1 < \alpha, \beta \leq 2$, $0 \leq \mathrm{r}_1, \mathrm{r}_2 \leq 2$, $\gamma_1, \gamma_2 > 0$, and $\mathcal{J} = [0, \mathbb{T}]$, $\mathbb{T} > 0$. $\phi_1, \phi_2, f, g : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Motivated by the above work, we focus our attention on the following coupled impulsive fractional integro-differential equations with Riemann-Liouville derivatives of the form:

$$\begin{cases} \begin{cases} \mathcal{D}^{\alpha}\mathrm{u}(\sigma) - \phi_1(\sigma, \mathcal{I}^{\alpha}\mathrm{u}(\sigma), \mathcal{I}^{\beta}\mathrm{v}(\sigma)) = 0, \quad \sigma \in \omega, \quad \sigma \neq \sigma_j, \quad j = 1, 2, \ldots, p, \\ \Delta\mathrm{u}(\sigma_j) - \mathcal{E}_j(\mathrm{u}(\sigma_j)) = 0, \qquad \Delta\mathrm{u}'(\sigma_j) - \mathcal{E}_j^*(\mathrm{u}(\sigma_j)) = 0, \quad j = 1, 2, \ldots, p, \\ \nu_1\mathcal{D}^{\alpha-2}\mathrm{u}(\sigma)|_{\sigma=0} = \mathrm{u}_1, \qquad \mu_1\mathrm{u}(\sigma)|_{\sigma=\mathbb{T}} + \nu_2\mathcal{I}^{\alpha-1}\mathrm{u}(\sigma)|_{\sigma=\mathbb{T}} = \mathrm{u}_2, \end{cases} \\ \begin{cases} \mathcal{D}^{\beta}\mathrm{v}(\sigma) - \phi_2(\sigma, \mathcal{I}^{\alpha}\mathrm{u}(\sigma), \mathcal{I}^{\beta}\mathrm{v}(\sigma)) = 0, \quad \sigma \in \omega, \quad \sigma \neq \sigma_k, \quad k = 1, 2, \ldots, q, \\ \Delta\mathrm{v}(\sigma_k) - \mathcal{E}_k(\mathrm{v}(\sigma_k)) = 0, \qquad \Delta\mathrm{v}'(\sigma_k) - \mathcal{E}_k^*(\mathrm{v}(\sigma_k)) = 0, \quad k = 1, 2, \ldots, q, \\ \nu_3\mathcal{D}^{\beta-2}\mathrm{v}(\sigma)|_{\sigma=0} = \mathrm{v}_1, \qquad \mu_2\mathrm{v}(\sigma)|_{\sigma=\mathbb{T}} + \nu_4\mathcal{I}^{\beta-1}\mathrm{v}(\sigma)|_{\sigma=\mathbb{T}} = \mathrm{v}_2, \end{cases} \end{cases} \tag{1.1}$$

where $1 < \alpha, \beta \leq 2$, $\phi_1, \phi_2 : [0, \mathbb{T}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ being continuous functions and

$$\Delta\mathrm{u}(\sigma_j) = \mathrm{u}(\sigma_j^+) - \mathrm{u}(\sigma_j^-), \qquad \Delta\mathrm{u}'(\sigma_j) = \mathrm{u}'(\sigma_j^+) - \mathrm{u}'(\sigma_j^-)$$

$$\Delta\mathrm{v}(\sigma_k) = \mathrm{v}(\sigma_k^+) - \mathrm{v}(\sigma_k^-), \qquad \Delta\mathrm{v}'(\sigma_k) = \mathrm{v}'(\sigma_k^+) - \mathrm{v}'(\sigma_k^-),$$

where $\mathrm{u}(\sigma_j^+), \mathrm{v}(\sigma_k^+)$ and $\mathrm{u}(\sigma_j^-), \mathrm{v}(\sigma_k^-)$ are the right limits and left limits respectively, $\mathcal{E}_j, \mathcal{E}_j^*, \mathcal{E}_k, \mathcal{E}_k^* : \mathbb{R} \to \mathbb{R}$ are continuous functions, and $\mathcal{D}^{\alpha}, \mathcal{I}^{\alpha}$ are the α-order Riemann-Liouville fractional derivative and integral operators respectively.

The remaining article is arranged as follows: In Section 2, we present some basic definitions, theorems, and lemmas that will be used in our main results. In Section 3, we use suitable cases for the existence and uniqueness of solution for the proposed system (1.1) using Kransnoselskii's type fixed point theorem. In Section 4, we discuss different kinds of stabilities in the sense of Ulam under certain conditions. In Section 5, an example is given to support the main results.

2. Auxiliary results

In this section, we present some basics notations, definitions, and results that are used in the whole article.

Let $\mathbb{T} > 0$, $\omega = [0, \mathbb{T}]$. The Banach space of all continuous functions from ω into $\mathbb{R}$ is denoted by $C(\omega, \mathbb{R})$ with the norm

$$\|\mathrm{u}\| = \sup\{|\mathrm{u}(\sigma)| : \sigma \in \omega\}$$

and the product of these spaces is also a Banach space with the norm

$$\|(\mathfrak{u}, \mathfrak{v})\| = \|\mathfrak{u}\| + \|\mathfrak{v}\|.$$

The piecewise continuous functions with $1 < \alpha, \beta \leq 2$ are denoted as follows:

$$\vartheta_1 = \mathcal{PC}_{2-\alpha}(\omega, \mathbb{R}^+) = \{\mathfrak{u} : \omega \to \mathbb{R}^+, \mathfrak{u}(\sigma_j^+), \mathfrak{u}(\sigma_j^-) \text{ and } \Delta\mathfrak{u}'(\sigma_j^+), \mathfrak{u}'(\sigma_j^-) \text{ exist for } j = 1, 2, \ldots, p\},$$

$$\vartheta_2 = \mathcal{PC}_{2-\beta}(\omega, \mathbb{R}^+) = \{\mathfrak{v} : \omega \to \mathbb{R}^+, \mathfrak{v}(\sigma_k^+), \mathfrak{v}(\sigma_k^-) \text{ and } \Delta\mathfrak{v}'(\sigma_k^+), \mathfrak{v}'(\sigma_k^-) \text{ exist for } k = 1, 2, \ldots, q\},$$

with the norms

$$\|\mathfrak{u}\|_{\vartheta_1} = \sup\{|\sigma^{2-\alpha}\mathfrak{u}(\sigma)| : \sigma \in \omega\},$$

$$\|\mathfrak{v}\|_{\vartheta_2} = \sup\{|\sigma^{2-\beta}\mathfrak{v}(\sigma)| : \sigma \in \omega\},$$

respectively. Their product $\vartheta = \vartheta_1 \times \vartheta_2$ is also a Banach space with the norm $\|(\mathfrak{u}, \mathfrak{v})\|_\vartheta = \|\mathfrak{u}\|_{\vartheta_1} + \|\mathfrak{v}\|_{\vartheta_2}$.

Definition 2.1. *[1] The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $\mathfrak{u} : \mathbb{R}^+ \to \mathbb{R}$ is defined as*

$$\mathcal{I}^\alpha \mathfrak{u}(\sigma) = \frac{1}{\Gamma(\alpha)} \int_0^\sigma (\sigma - \pi)^{\alpha-1} \mathfrak{u}(\pi) d\pi,$$

where $\Gamma(\cdot)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-\sigma}\sigma^{\alpha-1}d\sigma, \ \ \alpha > 0.$

Definition 2.2. *For a function $\mathfrak{u} : \mathbb{R}^+ \to \mathbb{R}$, the Riemann-Liouville derivative of fractional order $\alpha > 0$, $p = [\alpha] + 1$, is defined as*

$$\mathcal{D}^\alpha \mathfrak{u}(\sigma) = \frac{1}{\Gamma(p - \alpha)} \left(\frac{d}{d\sigma}\right)^p \int_0^\sigma (\sigma - \pi)^{p-\alpha-1} \mathfrak{u}(\pi) d\pi,$$

provided that integral on the right side exists. $[\alpha]$ denotes the integer part of the real number α. For more properties, the reader may refer to [1].

Lemma 2.1. *[1] Let $\mathfrak{u}$ be any function, and let $\alpha > 0$, then the Riemann-Liouville fractional derivative for the Homogeneous differential equation*

$$\mathcal{D}^\alpha \mathfrak{u}(\sigma) = 0, \ \ \alpha > 0,$$

has a solution

$$\mathfrak{u}(\sigma) = c_1\sigma^{\alpha-1} + c_2\sigma^{\alpha-2} + \cdots + c_{p-1}\sigma^{\alpha-p-1} + c_p\sigma^{\alpha-p},$$

and for non-homogeneous differential equation

$$\mathcal{D}^\alpha \mathfrak{u}(\sigma) = \phi_1(\sigma), \ \ \alpha > 0,$$

has a solution

$$\mathcal{I}^\alpha \mathcal{D}^\alpha \mathfrak{u}(\sigma) = \mathcal{I}^\alpha \phi_1(\sigma) + c_1\sigma^{\alpha-1} + c_2\sigma^{\alpha-2} + \cdots + c_{p-1}\sigma^{\alpha-p-1} + c_p\sigma^{\alpha-p},$$

where $p = [\alpha] + 1$ and $c_i, i = 1, 2, \ldots, p$, are real constants.

Theorem 2.1. *(Altman [5]) Let $\Lambda \neq 0$ be a convex and closed subset of Banach space ϑ. Consider two operators $\mathfrak{I}_1, \mathfrak{I}_2$ such that*

(1) $\mathfrak{I}_1(\mathfrak{u}, \mathfrak{v}) + \mathfrak{I}_2(\mathfrak{u}, \mathfrak{v}) \in \Lambda$;
(2) $\mathfrak{I}_1$ is a contractive operator;
(3) $\mathfrak{I}_2$ is a compact and continuous operator.

Then there exists $(\mathfrak{u}, \mathfrak{v}) \in \Lambda$ such that $\mathfrak{I}_1(\mathfrak{u}, \mathfrak{v}) + \mathfrak{I}_2(\mathfrak{u}, \mathfrak{v}) = (\mathfrak{u}, \mathfrak{v}) \in \vartheta$.

2.1. Ulam's Stabilities and Remarks

The following definitions and remarks are taken from [21, 24].

Definition 2.3. *The given system* (1.1) *is* $\mathcal{HU}$ *stable if there exists* $\mathcal{N}_{\alpha,\beta} = \max\{\mathcal{N}_\alpha, \mathcal{N}_\beta\} > 0$ *such that, for* $\kappa = \max\{\kappa_\alpha, \kappa_\beta\} > 0$ *and for every solution* $(\xi, \zeta) \in \vartheta$ *of the inequality*

$$\begin{cases} \begin{cases} |\mathcal{D}^\alpha \xi(\sigma) - \phi_1(\sigma, \mathcal{I}^\alpha \xi(\sigma), \mathcal{I}^\beta \zeta(\sigma))| \le \kappa_\alpha, \quad \sigma \in \omega, \\ |\Delta\xi(\sigma_j) - \mathcal{E}_j(\xi(\sigma_j))| \le \kappa_\alpha, \quad j = 1, 2, \ldots, p, \\ |\Delta\xi'(\sigma_j) - \mathcal{E}_j^*(\xi(\sigma_j))| \le \kappa_\alpha, \quad j = 1, 2, \ldots, p, \end{cases} \\ \begin{cases} |\mathcal{D}^\beta \zeta(\sigma) - \phi_2(\sigma, \mathcal{I}^\alpha \xi(\sigma), \mathcal{I}^\beta \zeta(\sigma))| \le \kappa_\beta, \quad \sigma \in \omega, \\ |\Delta\zeta(\sigma_k) - \mathcal{E}_k(\zeta(\sigma_k))| \le \kappa_\beta, \quad k = 1, 2, \ldots, q, \\ |\Delta\zeta'(\sigma_k) - \mathcal{E}_k^*(\zeta(\sigma_k))| \le \kappa_\beta, \quad k = 1, 2, \ldots, q, \end{cases} \end{cases} \tag{2.1}$$

there exists a solution $(u, v) \in \vartheta$ *with*

$$\|(u, v) - (\xi, \zeta)\|_\vartheta \le \mathcal{N}_{\alpha,\beta}\kappa, \quad \sigma \in \omega.$$

Definition 2.4. *The given system* (1.1) *is generalized* $\mathcal{HU}$ *stable if there exists* $\mathcal{N}' \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\mathcal{N}'(0) = 0$ *such that, for any approximate solution* $(\xi, \zeta) \in \vartheta$ *of inequality* (2.1)*, there exists a solution* $(u, v) \in \vartheta$ *of* (1.1) *satisfying*

$$\|(u, v) - (\xi, \zeta)\|_\vartheta \le \mathcal{N}'(\kappa), \quad \sigma \in \omega.$$

Definition 2.5. *The given system* (1.1) *is* $\mathcal{HUR}$ *stable with respect to* $\psi_{\alpha,\beta} = \max\{\psi_\alpha, \psi_\beta\}$ *with* $\psi_{\alpha,\beta} \in C(\omega, \mathbb{R})$ *if there exists a constant* $\mathcal{N}_{\psi_\alpha,\psi_\beta} = \max\{\mathcal{N}_{\psi_\alpha}, \mathcal{N}_{\psi_\beta}\} > 0$ *such that, for any* $\kappa = \max\{\kappa_\alpha, \kappa_\beta\} > 0$ *and for any approximate solution* $(\xi, \zeta) \in \vartheta$ *of the inequality*

$$\begin{cases} \begin{cases} |\mathcal{D}^\alpha \xi(\sigma) - \phi_1(\sigma, \mathcal{I}^\alpha \xi(\sigma), \mathcal{I}^\beta \zeta(\sigma))| \le \psi_\alpha(\sigma)\kappa_\alpha, \quad \sigma \in \omega, \\ |\Delta\xi(\sigma_j) - \mathcal{E}_j(\xi(\sigma_j))| \le \psi_\alpha(\sigma)\kappa_\alpha, \quad j = 1, 2, \ldots, p, \\ |\Delta\xi'(\sigma_j) - \mathcal{E}_j^*(\xi(\sigma_j))| \le \psi_\alpha(\sigma)\kappa_\alpha, \quad j = 1, 2, \ldots, p, \end{cases} \\ \begin{cases} |\mathcal{D}^\beta \zeta(\sigma) - \phi_2(\sigma, \mathcal{I}^\alpha \xi(\sigma), \mathcal{I}^\beta \zeta(\sigma))| \le \psi_\beta(\sigma)\kappa_\beta, \quad \sigma \in \omega, \\ |\Delta\zeta(\sigma_k) - \mathcal{E}_k(\zeta(\sigma_k))| \le \psi_\beta(\sigma)\kappa_\beta, \quad k = 1, 2, \ldots, q, \\ |\Delta\zeta'(\sigma_k) - \mathcal{E}_k^*(\zeta(\sigma_k))| \le \psi_\beta(\sigma)\kappa_\beta, \quad k = 1, 2, \ldots, q, \end{cases} \end{cases} \tag{2.2}$$

there exists a solution $(u, v) \in \vartheta$ *with*

$$\|(u, v) - (\xi, \zeta)\|_\vartheta \le \mathcal{N}_{\psi_\alpha,\psi_\beta}\psi_{\alpha,\beta}(\sigma)\kappa, \quad \sigma \in \omega.$$

Definition 2.6. *The given system* (1.1) *is generalized* $\mathcal{HUR}$ *stable with respect to* $\psi_{\alpha,\beta} = \max\{\psi_\alpha, \psi_\beta\}$ *with* $\psi_{\alpha,\beta} \in C(\omega, \mathbb{R})$ *if there exists a constant* $\mathcal{N}_{\psi_\alpha,\psi_\beta} = \max\{\mathcal{N}_{\psi_\alpha}, \mathcal{N}_{\psi_\beta}\} > 0$ *such that, for any approximate solution* $(\xi, \zeta) \in \vartheta$ *of inequality* (2.2)*, there exists a solution* $(u, v) \in \vartheta$ *of* (1.1) *satisfying*

$$\|(u, v) - (\xi, \zeta)\|_\vartheta \le \mathcal{N}_{\psi_\alpha,\psi_\beta}\psi_{\alpha,\beta}(\sigma), \quad \sigma \in \omega.$$

Remark 2.1. *Let* $(\xi, \zeta) \in \vartheta$ *be a solution of inequalities* (2.1) *if there exist functions* $\mathfrak{R}_{\phi_1}, \mathfrak{L}_{\phi_2} \in C(\omega, \mathbb{R})$ *depending on* ξ, ζ *respectively such that*

(1) $|\Re_{\phi_1}(\sigma)| \leq \kappa_\alpha, \ \ |\Omega_{\phi_2}(\sigma)| \leq \kappa_\beta, \ \ \sigma \in \omega;$
(2)

$$\begin{cases} \begin{cases} \mathcal{D}^\alpha \xi(\sigma) = \phi_1(\sigma, \mathcal{I}^\alpha \xi(\sigma), \mathcal{I}^\beta \zeta(\sigma)) + \Re_{\phi_1}(\sigma), \\ \Delta\xi(\sigma_j) = \mathcal{E}_j(\xi(\sigma_j)) + \Re_{\phi_{1j}}, \ \ j = 1, 2, \ldots, p, \\ \Delta\xi'(\sigma_j) = \mathcal{E}_j^*(\xi(\sigma_j)) + \Re_{\phi_{1j}}, \ \ j = 1, 2, \ldots, p, \end{cases} \\ \begin{cases} \mathcal{D}^\beta \zeta(\sigma) = \phi_2(t, \mathcal{I}^\alpha \xi(\sigma), \mathcal{I}^\beta \zeta(\sigma)) + \Omega_{\phi_2}(\sigma), \\ \Delta\zeta(\sigma_k) = \mathcal{E}_k(\zeta(\sigma_k)) + \Omega_{\phi_{2k}}, \ \ k = 1, 2, \ldots, q, \\ \Delta\zeta'(\sigma_k) = \mathcal{E}_k^*(\zeta(\sigma_k)) + \Omega_{\phi_{2k}}, \ \ k = 1, 2, \ldots, q. \end{cases} \end{cases} \tag{2.3}$$

3. Existence and uniqueness

In this section, we discuss the existence and uniqueness of solution of the proposed system (1.1).

Theorem 3.1. *Let* $\alpha, \beta \in (1, 2]$ *and* ϕ_1 *be any linear and continuous function. The fractional impulsive differential equation*

$$\begin{cases} \mathcal{D}^\alpha u(\sigma) = \phi_1(\sigma, \mathcal{I}^\alpha u(\sigma), \mathcal{I}^\beta v(\sigma)), \ \ \sigma \in \omega, \ \ \sigma \neq \sigma_j, \ \ j = 1, 2, \ldots, p, \\ \Delta u(\sigma_j) = \mathcal{E}_j(u(\sigma_j)), \qquad \Delta u'(\sigma_j) = \mathcal{E}_j^*(u(\sigma_j)), \ \ j = 1, 2, \ldots, p, \\ \nu_1 \mathcal{D}^{\alpha-2} u(\sigma)|_{\sigma=0} = u_1, \qquad \mu_1 u(\sigma)|_{\sigma=\mathbb{T}} + \nu_2 \mathcal{I}^{\alpha-1} u(\sigma)|_{\sigma=\mathbb{T}} = u_2, \end{cases} \tag{3.1}$$

has a solution

$$
u(\sigma)=\begin{cases}
\dfrac{\sigma^{\alpha-1}u_2}{\mu_1\mathbb{T}^{\alpha-1}}-\dfrac{\sigma^{\alpha-1}u_1}{\mathbb{T}\nu_1\Gamma(\alpha-1)}+\dfrac{\sigma^{\alpha-2}u_1}{\nu_1\Gamma(\alpha-1)}+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^{\sigma}(\sigma-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
-\dfrac{\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_{\sigma_1}^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi-\dfrac{\nu_2\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\mu_1\Gamma(\alpha-1)}\displaystyle\int_0^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-2}u(\pi)d\pi\\
-\dfrac{\sigma^{\alpha-1}}{\mathbb{T}}\Bigg[\Big((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_1^{-1}\Big)\sigma_1^{2-\alpha}\mathcal{E}_1(u(\sigma_1))\\
+(\mathbb{T}-\sigma_1)\,\sigma_1^{2-\alpha}\mathcal{E}_1^{*}(u(\sigma_1))+\dfrac{(\mathbb{T}-\sigma_1)\,\sigma_1^{2-\alpha}}{\Gamma(\alpha-1)}\displaystyle\int_0^{\sigma_1}(\sigma_1-\pi)^{\alpha-2}\,\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
+\dfrac{\Big((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_1^{-1}\Big)\sigma_1^{2-\alpha}}{\Gamma(\alpha)}\displaystyle\int_0^{\sigma_1}(\sigma_1-\pi)^{\alpha-1}\,\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg],\\
\qquad\sigma\in[0,\sigma_1],\\
\dfrac{\sigma^{\alpha-1}u_2}{\mu_1\mathbb{T}^{\alpha-1}}-\dfrac{\sigma^{\alpha-1}u_1}{\mathbb{T}\nu_1\Gamma(\alpha-1)}+\dfrac{\sigma^{\alpha-2}u_1}{\nu_1\Gamma(\alpha-1)}+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
-\dfrac{\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_{\sigma_z}^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi-\dfrac{\nu_2\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\mu_1\Gamma(\alpha-1)}\displaystyle\int_0^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-2}u(\pi)d\pi\\
-\dfrac{\sigma^{\alpha-1}}{\mathbb{T}}\displaystyle\sum_{j=1}^{z}\Bigg[\Big((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\Big)\sigma_j^{2-\alpha}\mathcal{E}_j(u(\sigma_j))\\
+\Big(\mathbb{T}-\sigma_j\Big)\sigma_j^{2-\alpha}\mathcal{E}_j^{*}(u(\sigma_j))+\dfrac{\Big(\mathbb{T}-\sigma_j\Big)\sigma_j^{2-\alpha}}{\Gamma(\alpha-1)}\displaystyle\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-2}\,\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
+\dfrac{\Big((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\Big)\sigma_j^{2-\alpha}}{\Gamma(\alpha)}\displaystyle\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-1}\,\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg]\\
+\displaystyle\sum_{j=1}^{z}\Bigg[\Big((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}\mathcal{E}_j(u(\sigma_j))\\
+\Big(\sigma-\sigma_j\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}\mathcal{E}_j^{*}(u(\sigma_j))+\dfrac{\Big(\sigma-\sigma_j\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}}{\Gamma(\alpha-1)}\displaystyle\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-2}\,\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
+\dfrac{\Big((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}}{\Gamma(\alpha)}\displaystyle\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-1}\,\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg],\\
\qquad\sigma\in(\sigma_j,\sigma_{j+1}];\quad z=1,2,\ldots,p.
\end{cases}
\tag{3.2}
$$

Proof. Consider

$$
\mathcal{D}^{\alpha}\mathrm{u}(\sigma)=\phi_1(\sigma,\mathcal{I}^{\alpha}\mathrm{u}(\sigma),\mathcal{I}^{\beta}\mathrm{v}(\sigma)),\quad\sigma\in\omega,\quad\alpha\in(1,2].
\tag{3.3}
$$

For $\sigma\in[0,\sigma_1]$, Lemma 2.1 gives

$$
\begin{cases}
\mathrm{u}(\sigma)=\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^{\sigma}(\sigma-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi+a_1\sigma^{\alpha-1}+a_2\sigma^{\alpha-2},\\
\mathrm{u}'(\sigma)=\dfrac{1}{\Gamma(\alpha-1)}\displaystyle\int_0^{\sigma}(\sigma-\pi)^{\alpha-2}\phi_1(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi+a_1(\alpha-1)\sigma^{\alpha-2}+a_2(\alpha-2)\sigma^{\alpha-3}.
\end{cases}
\tag{3.4}
$$

Again, for $\sigma \in (\sigma_1, \sigma_2]$, Lemma 2.1 gives

$$\begin{cases} \mathrm{u}(\sigma) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{\sigma_1}^{\sigma} (\sigma - \pi)^{\alpha-1} \phi_1(\pi, \mathcal{I}^{\alpha}\mathrm{u}(\pi), \mathcal{I}^{\beta}\mathrm{v}(\pi)) d\pi + b_1\sigma^{\alpha-1} + b_2\sigma^{\alpha-2}, \\ \mathrm{u}'(\sigma) = \dfrac{1}{\Gamma(\alpha-1)} \displaystyle\int_{\sigma_1}^{\sigma} (\sigma - \pi)^{\alpha-2} \phi_1(\pi, \mathcal{I}^{\alpha}\mathrm{u}(\pi), \mathcal{I}^{\beta}\mathrm{v}(\pi)) d\pi + b_1(\alpha-1)\sigma^{\alpha-2} + b_2(\alpha-2)\sigma^{\alpha-3}. \end{cases} \tag{3.5}$$

Hence it follows that

$$\begin{cases} \mathrm{u}(\sigma_1^-) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{0}^{\sigma_1} (\sigma_1 - \pi)^{\alpha-1} \phi_1(\pi, \mathcal{I}^{\alpha}\mathrm{u}(\pi), \mathcal{I}^{\beta}\mathrm{v}(\pi)) d\pi + a_1\sigma_1^{\alpha-1} + a_2\sigma_1^{\alpha-2}, \\ \mathrm{u}(\sigma_1^+) = b_1\sigma_1^{\alpha-1} + b_2\sigma_1^{\alpha-2}, \\ \mathrm{u}'(\sigma_1^-) = \dfrac{1}{\Gamma(\alpha-1)} \displaystyle\int_{0}^{\sigma_1} (\sigma_1 - \pi)^{\alpha-2} \phi_1(\pi, \mathcal{I}^{\alpha}\mathrm{u}(\pi), \mathcal{I}^{\beta}\mathrm{v}(\pi)) d\pi + a_1(\alpha-1)\sigma_1^{\alpha-2} + a_2(\alpha-2)\sigma_1^{\alpha-3}, \\ \mathrm{u}'(\sigma_1^+) = b_1(\alpha-1)\sigma_1^{\alpha-2} + b_2(\alpha-2)\sigma_1^{\alpha-3}. \end{cases}$$

Using

$$\begin{cases} \Delta\mathrm{u}(\sigma_1) = \mathrm{u}(\sigma_1^+) - \mathrm{u}(\sigma_1^-) = \mathcal{E}_1(\mathrm{u}(\sigma_1)), \\ \Delta\mathrm{u}'(\sigma_1) = \mathrm{u}'(\sigma_1^+) - \mathrm{u}'(\sigma_1^-) = \mathcal{E}_1^*(\mathrm{u}(\sigma_1)), \end{cases}$$

we obtain

$$\begin{cases} b_1 = a_1 - (\alpha-2)\sigma_1^{1-\alpha}\mathcal{E}_1(\mathrm{u}(\sigma_1)) + \sigma_1^{2-\alpha}\mathcal{E}_1^*(\mathrm{u}(\sigma_1)) + \dfrac{\sigma_1^{2-\alpha}}{\Gamma(\alpha-1)} \displaystyle\int_{0}^{\sigma_1} (\sigma_1 - \pi)^{\alpha-2} \phi_1(\pi, \mathcal{I}^{\alpha}\mathrm{u}(\pi), \mathcal{I}^{\beta}\mathrm{v}(\pi)) d\pi \\ \qquad - \dfrac{(\alpha-2)\sigma_1^{1-\alpha}}{\Gamma(\alpha)} \displaystyle\int_{0}^{\sigma_1} (\sigma_1 - \pi)^{\alpha-1} \phi_1(\pi, \mathcal{I}^{\alpha}\mathrm{u}(\pi), \mathcal{I}^{\beta}\mathrm{v}(\pi)) d\pi, \\ b_2 = a_2 + (\alpha-1)\sigma_1^{2-\alpha}\mathcal{E}_1(\mathrm{u}(\sigma_1)) - \sigma_1^{3-\alpha}\mathcal{E}_1^*(\mathrm{u}(\sigma_1)) - \dfrac{\sigma_1^{3-\alpha}}{\Gamma(\alpha-1)} \displaystyle\int_{0}^{\sigma_1} (\sigma_1 - \pi)^{\alpha-2} \phi_1(\pi, \mathcal{I}^{\alpha}\mathrm{u}(\pi), \mathcal{I}^{\beta}\mathrm{v}(\pi)) d\pi \\ \qquad + \dfrac{(\alpha-1)\sigma_1^{2-\alpha}}{\Gamma(\alpha)} \displaystyle\int_{0}^{\sigma_1} (\sigma_1 - \pi)^{\alpha-1} \phi_1(\pi, \mathcal{I}^{\alpha}\mathrm{u}(\pi), \mathcal{I}^{\beta}\mathrm{v}(\pi)) d\pi. \end{cases}$$

Substituting the values of b_1, b_2 in (3.5), we get

$$
\begin{cases}
\mathrm{u}(\sigma) =a_1\sigma^{\alpha-1} + a_2\sigma^{\alpha-2} + \Big((\alpha-1) - (\alpha-2)\sigma\sigma_1^{-1}\Big)\sigma^{\alpha-2}\sigma_1^{2-\alpha}\mathcal{E}_1(\mathrm{u}(\sigma_1)) \\
\qquad + (\sigma-\sigma_1)\,\sigma^{\alpha-2}\sigma_1^{2-\alpha}\mathcal{E}_1^*(\mathrm{u}(\sigma_1)) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\sigma_1}^{\sigma}(\sigma-\pi)^{\alpha-1}\phi_1(\pi, \mathcal{I}^\alpha\mathrm{u}(\pi), \mathcal{I}^\beta\mathrm{v}(\pi))d\pi \\
\qquad + \dfrac{\Big((\alpha-1) - (\alpha-2)\sigma\sigma_1^{-1}\Big)\sigma^{\alpha-2}\sigma_1^{2-\alpha}}{\Gamma(\alpha)}\displaystyle\int_0^{\sigma_1}(\sigma_1-\pi)^{\alpha-1}\,\phi_1(\pi, \mathcal{I}^\alpha\mathrm{u}(\pi), \mathcal{I}^\beta\mathrm{v}(\pi))d\pi \\
\qquad + \dfrac{(\sigma-\sigma_1)\,\sigma^{\alpha-2}\sigma_1^{2-\alpha}}{\Gamma(\alpha-1)}\displaystyle\int_0^{\sigma_1}(\sigma_1-\pi)^{\alpha-2}\,\phi_1(\pi, \mathcal{I}^\alpha\mathrm{u}(\pi), \mathcal{I}^\beta\mathrm{v}(\pi))d\pi, \\
\mathrm{u}'(\sigma) =a_1(\alpha-1)\sigma^{\alpha-2} + a_2(\alpha-2)\sigma^{\alpha-3} + (\alpha-1)(\alpha-2)\Big(\sigma^{-1} - \sigma_1^{-1}\Big)\sigma^{\alpha-2}\sigma_1^{2-\alpha}\mathcal{E}_1(\mathrm{u}(\sigma_1)) \\
\qquad + \Big((\alpha-1) - (\alpha-2)\sigma^{-1}\sigma_1\Big)\sigma^{\alpha-2}\sigma_1^{2-\alpha}\mathcal{E}_1^*(\mathrm{u}(\sigma_1)) \\
\qquad + \dfrac{1}{\Gamma(\alpha-1)}\displaystyle\int_{\sigma_1}^{\sigma}(\sigma-\pi)^{\alpha-2}\phi_1(\pi, \mathcal{I}^\alpha\mathrm{u}(\pi), \mathcal{I}^\beta\mathrm{v}(\pi))d\pi \\
\qquad + \dfrac{\Big((\alpha-1) - (\alpha-2)\sigma^{-1}\sigma_1\Big)\sigma^{\alpha-2}\sigma_1^{2-\alpha}}{\Gamma(\alpha-1)}\displaystyle\int_0^{\sigma_1}(\sigma_1-\pi)^{\alpha-2}\,\phi_1(\pi, \mathcal{I}^\alpha\mathrm{u}(\pi), \mathcal{I}^\beta\mathrm{v}(\pi))d\pi \\
\qquad + \dfrac{(\alpha-1)(\alpha-2)\Big(\sigma^{-1} - \sigma_1^{-1}\Big)\sigma^{\alpha-2}\sigma_1^{2-\alpha}}{\Gamma(\alpha)}\displaystyle\int_0^{\sigma_1}(\sigma_1-\pi)^{\alpha-1}\,\phi_1(\pi, \mathcal{I}^\alpha\mathrm{u}(\pi), \mathcal{I}^\beta\mathrm{v}(\pi))d\pi.
\end{cases}
$$

Similarly, for $\sigma \in (\sigma_j, \sigma_{j+1}]$,

$$
\begin{aligned}
\mathrm{u}(\sigma) =&a_1\sigma^{\alpha-1} + a_2\sigma^{\alpha-2} + \sum_{j=1}^{z}\Big((\alpha-1) - (\alpha-2)\sigma\sigma_j^{-1}\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}\mathcal{E}_j(\mathrm{u}(\sigma_j)) \\
&+ \sum_{j=1}^{z}\Big(\sigma-\sigma_j\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}\mathcal{E}_j^*(\mathrm{u}(\sigma_j)) + \frac{1}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\alpha-1}\phi_1(\pi, \mathcal{I}^\alpha\mathrm{u}(\pi), \mathcal{I}^\beta\mathrm{v}(\pi))d\pi \\
&+ \sum_{j=1}^{z}\frac{\Big(\sigma-\sigma_j\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-2}\,\phi_1(\pi, \mathcal{I}^\alpha\mathrm{u}(\pi), \mathcal{I}^\beta\mathrm{v}(\pi))d\pi \\
&+ \sum_{j=1}^{z}\frac{\Big((\alpha-1) - (\alpha-2)\sigma\sigma_j^{-1}\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-1}\,\phi_1(\pi, \mathcal{I}^\alpha\mathrm{u}(\pi), \mathcal{I}^\beta\mathrm{v}(\pi))d\pi. \qquad (3.6)
\end{aligned}
$$

Finally, after applying conditions $\nu_1\mathcal{D}^{\alpha-2}\mathrm{u}(\sigma)|_{\sigma=0} = \mathrm{u}_1$, and $\mu_1\mathrm{u}(\sigma)|_{\sigma=\mathbb{T}} + \nu_2\mathcal{I}^{\alpha-1}\mathrm{u}(\sigma)|_{\sigma=\mathbb{T}} = \mathrm{u}_2$ to (3.6) and finding the values of a_1 and a_2, we obtain Eq (2.2). □

Corollary 1. *In view of Theorem 3.1, our coupled system* (1.1) *has the following solution:*

$$
u(\sigma) = \begin{cases}
\dfrac{\sigma^{\alpha-1}u_2}{\mu_1\mathbb{T}^{\alpha-1}} - \dfrac{\sigma^{\alpha-1}u_1}{\mathbb{T}\nu_1\Gamma(\alpha-1)} + \dfrac{\sigma^{\alpha-2}u_1}{\nu_1\Gamma(\alpha-1)} + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^{\sigma}(\sigma-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi \\
-\dfrac{\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_{\sigma_1}^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi - \dfrac{\nu_2\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\mu_1\Gamma(\alpha-1)}\displaystyle\int_0^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-2}u(\pi)d\pi \\
-\dfrac{\sigma^{\alpha-1}}{\mathbb{T}}\Bigg[\Big((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_1^{-1}\Big)\sigma_1^{2-\alpha}\mathcal{E}_1(u(\sigma_1)) + (\mathbb{T}-\sigma_1)\,\sigma_1^{2-\alpha}\mathcal{E}_1^{*}(u(\sigma_1)) \\
+\dfrac{(\mathbb{T}-\sigma_1)\,\sigma_1^{2-\alpha}}{\Gamma(\alpha-1)}\displaystyle\int_0^{\sigma_1}(\sigma_1-\pi)^{\alpha-2}\,\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi \\
+\dfrac{\Big((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_1^{-1}\Big)\sigma_1^{2-\alpha}}{\Gamma(\alpha)}\displaystyle\int_0^{\sigma_1}(\sigma_1-\pi)^{\alpha-1}\,\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg], \\
\qquad \sigma\in[0,\sigma_1], \\
\dfrac{\sigma^{\alpha-1}u_2}{\mu_1\mathbb{T}^{\alpha-1}} - \dfrac{\sigma^{\alpha-1}u_1}{\mathbb{T}\nu_1\Gamma(\alpha-1)} + \dfrac{\sigma^{\alpha-2}u_1}{\nu_1\Gamma(\alpha-1)} + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi \\
-\dfrac{\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_{\sigma_z}^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi - \dfrac{\nu_2\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\mu_1\Gamma(\alpha-1)}\displaystyle\int_0^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-2}u(\pi)d\pi \\
-\dfrac{\sigma^{\alpha-1}}{\mathbb{T}}\displaystyle\sum_{j=1}^{z}\Bigg[\Big((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\Big)\sigma_j^{2-\alpha}\mathcal{E}_j(u(\sigma_j)) + \Big(\mathbb{T}-\sigma_j\Big)\sigma_j^{2-\alpha}\mathcal{E}_j^{*}(u(\sigma_j)) \\
+\dfrac{\Big(\mathbb{T}-\sigma_j\Big)\sigma_j^{2-\alpha}}{\Gamma(\alpha-1)}\displaystyle\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-2}\,\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi \\
+\dfrac{\Big((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\Big)\sigma_j^{2-\alpha}}{\Gamma(\alpha)}\displaystyle\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-1}\,\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg] \\
+\displaystyle\sum_{j=1}^{z}\Bigg[\Big((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}\mathcal{E}_j(u(\sigma_j)) + \Big(\sigma-\sigma_j\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}\mathcal{E}_j^{*}(u(\sigma_j)) \\
+\dfrac{\Big(\sigma-\sigma_j\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}}{\Gamma(\alpha-1)}\displaystyle\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-2}\,\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi \\
+\dfrac{\Big((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}}{\Gamma(\alpha)}\displaystyle\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-1}\,\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg], \\
\qquad \sigma\in(\sigma_j,\sigma_{j+1}];\quad z=1,2,\ldots,p.
\end{cases} \tag{3.7}
$$

and

$$
v(\sigma) = \begin{cases}
\begin{cases}
\dfrac{\sigma^{\beta-1}v_2}{\mu_2\mathbb{T}^{\beta-1}} - \dfrac{\sigma^{\beta-1}v_1}{\mathbb{T}\nu_3\Gamma(\beta-1)} + \dfrac{\sigma^{\beta-2}v_1}{\nu_3\Gamma(\beta-1)} + \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^{\sigma}(\sigma-\pi)^{\beta-1}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi \\
-\dfrac{\sigma^{\beta-1}\mathbb{T}^{1-\beta}}{\Gamma(\beta)}\displaystyle\int_{\sigma_1}^{\mathbb{T}}(\mathbb{T}-\pi)^{\beta-1}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi - \dfrac{\nu_4\sigma^{\beta-1}\mathbb{T}^{1-\beta}}{\mu_2\Gamma(\beta-1)}\displaystyle\int_0^{\mathbb{T}}(\mathbb{T}-\pi)^{\beta-2}v(\pi)d\pi \\
-\dfrac{\sigma^{\beta-1}}{\mathbb{T}}\Bigg[\Big((\beta-1)-(\beta-2)\mathbb{T}\sigma_1^{-1}\Big)\sigma_1^{2-\beta}\mathcal{E}_1(v(\sigma_1)) + (\mathbb{T}-\sigma_1)\,\sigma_1^{2-\beta}\mathcal{E}_1^{*}(v(\sigma_1)) \\
+\dfrac{(\mathbb{T}-\sigma_1)\,\sigma_1^{2-\beta}}{\Gamma(\beta-1)}\displaystyle\int_0^{\sigma_1}(\sigma_1-\pi)^{\beta-2}\,\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi \\
+\dfrac{\Big((\beta-1)-(\beta-2)\mathbb{T}\sigma_1^{-1}\Big)\sigma_1^{2-\beta}}{\Gamma(\beta)}\displaystyle\int_0^{\sigma_1}(\sigma_1-\pi)^{\beta-1}\,\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg], \\
\quad \sigma\in[0,\sigma_1],
\end{cases} \\
\begin{cases}
\dfrac{\sigma^{\beta-1}v_2}{\mu_2\mathbb{T}^{\beta-1}} - \dfrac{\sigma^{\beta-1}v_1}{\mathbb{T}\nu_3\Gamma(\beta-1)} + \dfrac{\sigma^{\beta-2}v_1}{\nu_3\Gamma(\beta-1)} + \dfrac{1}{\Gamma(\beta)}\displaystyle\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\beta-1}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi \\
-\dfrac{\sigma^{\beta-1}\mathbb{T}^{1-\beta}}{\Gamma(\beta)}\displaystyle\int_{\sigma_z}^{\mathbb{T}}(\mathbb{T}-\pi)^{\beta-1}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi - \dfrac{\nu_4\sigma^{\beta-1}\mathbb{T}^{1-\beta}}{\mu_2\Gamma(\beta-1)}\displaystyle\int_0^{\mathbb{T}}(\mathbb{T}-\pi)^{\beta-2}v(\pi)d\pi \\
-\dfrac{\sigma^{\beta-1}}{\mathbb{T}}\displaystyle\sum_{k=1}^{z}\Bigg[\Big((\beta-1)-(\beta-2)\mathbb{T}\sigma_k^{-1}\Big)\sigma_k^{2-\beta}\mathcal{E}_k(v(\sigma_k)) + (\mathbb{T}-\sigma_k)\,\sigma_k^{2-\beta}\mathcal{E}_k^{*}(v(\sigma_k)) \\
+\dfrac{(\mathbb{T}-\sigma_k)\,\sigma_k^{2-\beta}}{\Gamma(\beta-1)}\displaystyle\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-2}\,\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi \\
+\dfrac{\Big((\beta-1)-(\beta-2)\mathbb{T}\sigma_k^{-1}\Big)\sigma_k^{2-\beta}}{\Gamma(\beta)}\displaystyle\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-1}\,\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg] \\
+\displaystyle\sum_{k=1}^{z}\Bigg[\Big((\beta-1)-(\beta-2)\sigma\sigma_k^{-1}\Big)\sigma^{\beta-2}\sigma_k^{2-\beta}\mathcal{E}_k(v(\sigma_k)) + (\sigma-\sigma_k)\,\sigma^{\beta-2}\sigma_k^{2-\beta}\mathcal{E}_k^{*}(v(\sigma_k)) \\
+\dfrac{(\sigma-\sigma_k)\,\sigma^{\beta-2}\sigma_k^{2-\beta}}{\Gamma(\beta-1)}\displaystyle\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-2}\,\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi \\
+\dfrac{\Big((\beta-1)-(\beta-2)\sigma\sigma_k^{-1}\Big)\sigma^{\beta-2}\sigma_k^{2-\beta}}{\Gamma(\beta)}\displaystyle\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-1}\,\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg], \\
\quad \sigma\in(\sigma_k,\sigma_{k+1}]; \quad z=1,2,\ldots,q.
\end{cases}
\end{cases} \tag{3.8}
$$

Now, for transformation of the given system (1.1) into a fixed point problem, let the operators $\mathfrak{I}_1, \mathfrak{I}_2 : \vartheta \to \vartheta$ be define as follows:

$$
\begin{aligned}
\mathfrak{I}_1(u,v)(\sigma) &= (\mathfrak{I}_1^{*}(u(\sigma)), \mathfrak{I}_1^{**}(v(\sigma))), \\
\mathfrak{I}_2(u,v)(\sigma) &= (\mathfrak{I}_2^{*}(u,v)(\sigma), \mathfrak{I}_2^{**}(u,v)(\sigma)),
\end{aligned}
$$

$$\mathfrak{I}_1(\mathrm{u},\mathrm{v})(\sigma)=\begin{cases}\mathfrak{I}_1^{*}(\mathrm{u}(\sigma))=\\ \begin{cases}\dfrac{\sigma^{\alpha-1}\mathrm{u}_2}{\mu_1\mathbb{T}^{\alpha-1}}-\dfrac{\sigma^{\alpha-1}\mathrm{u}_1}{\mathbb{T}\nu_1\Gamma(\alpha-1)}+\dfrac{\sigma^{\alpha-2}\mathrm{u}_1}{\nu_1\Gamma(\alpha-1)}-\dfrac{\nu_2\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\mu_1\Gamma(\alpha-1)}\displaystyle\int_0^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-2}\mathrm{u}(\pi)d\pi\\ -\dfrac{\sigma^{\alpha-1}}{\mathbb{T}}\displaystyle\sum_{j=1}^{z}\Big[\Big((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\Big)\sigma_j^{2-\alpha}\mathcal{E}_j(\mathrm{u}(\sigma_j))+\Big(\mathbb{T}-\sigma_j\Big)\sigma_j^{2-\alpha}\mathcal{E}_j^{*}(\mathrm{u}(\sigma_j))\Big]\\ +\displaystyle\sum_{j=1}^{z}\Big[\Big((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}\mathcal{E}_j(\mathrm{u}(\sigma_j))+\Big(\sigma-\sigma_j\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}\mathcal{E}_j^{*}(\mathrm{u}(\sigma_j))\Big],\\ \quad\sigma\in(\sigma_j,\sigma_{j+1}];\quad z=1,2,\ldots,p,\end{cases}\\ \mathfrak{I}_1^{**}(\mathrm{v}(\sigma))=\\ \begin{cases}\dfrac{\sigma^{\beta-1}\mathrm{v}_2}{\mu_2\mathbb{T}^{\beta-1}}-\dfrac{\sigma^{\beta-1}\mathrm{v}_1}{\mathbb{T}\nu_3\Gamma(\beta-1)}+\dfrac{\sigma^{\beta-2}\mathrm{v}_1}{\nu_3\Gamma(\beta-1)}-\dfrac{\nu_4\sigma^{\beta-1}\mathbb{T}^{1-\beta}}{\mu_2\Gamma(\beta-1)}\displaystyle\int_0^{\mathbb{T}}(\mathbb{T}-\pi)^{\beta-2}\mathrm{v}(\pi)d\pi\\ -\dfrac{\sigma^{\beta-1}}{\mathbb{T}}\displaystyle\sum_{k=1}^{z}\Big[\Big((\beta-1)-(\beta-2)\mathbb{T}\sigma_k^{-1}\Big)\sigma_k^{2-\beta}\mathcal{E}_k(\mathrm{v}(\sigma_k))+(\mathbb{T}-\sigma_k)\,\sigma_k^{2-\beta}\mathcal{E}_k^{*}(\mathrm{v}(\sigma_k))\Big]\\ +\displaystyle\sum_{k=1}^{z}\Big[\Big((\beta-1)-(\beta-2)\sigma\sigma_k^{-1}\Big)\sigma^{\beta-2}\sigma_k^{2-\beta}\mathcal{E}_k(\mathrm{v}(\sigma_k))+(\sigma-\sigma_k)\,\sigma^{\beta-2}\sigma_k^{2-\beta}\mathcal{E}_k^{*}(\mathrm{v}(\sigma_k))\Big],\\ \quad\sigma\in(\sigma_k,\sigma_{k+1}];\quad z=1,2,\ldots,q,\end{cases}\end{cases}\tag{3.9}$$

and

$$\mathfrak{I}_2(\mathrm{u},\mathrm{v})(\sigma)=\begin{cases}\mathfrak{I}_2^{*}(\mathrm{u},\mathrm{v})(\sigma)= \\ \left\{\begin{array}{l}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi \\ -\dfrac{\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_{\sigma_z}^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi \\ -\dfrac{\sigma^{\alpha-1}}{\mathbb{T}}\displaystyle\sum_{j=1}^{z}\left[\dfrac{\left(\mathbb{T}-\sigma_j\right)\sigma_j^{2-\alpha}}{\Gamma(\alpha-1)}\displaystyle\int_{\sigma_{j-1}}^{\sigma_j}\left(\sigma_j-\pi\right)^{\alpha-2}\phi_1(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\right. \\ \left.+\dfrac{\left((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\right)\sigma_j^{2-\alpha}}{\Gamma(\alpha)}\displaystyle\int_{\sigma_{j-1}}^{\sigma_j}\left(\sigma_j-\pi\right)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\right] \\ +\displaystyle\sum_{j=1}^{z}\left[\dfrac{\left(\sigma-\sigma_j\right)\sigma^{\alpha-2}\sigma_j^{2-\alpha}}{\Gamma(\alpha-1)}\displaystyle\int_{\sigma_{j-1}}^{\sigma_j}\left(\sigma_j-\pi\right)^{\alpha-2}\phi_1(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\right. \\ \left.+\dfrac{\left((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\right)\sigma^{\alpha-2}\sigma_j^{2-\alpha}}{\Gamma(\alpha)}\displaystyle\int_{\sigma_{j-1}}^{\sigma_j}\left(\sigma_j-\pi\right)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\right], \\ \quad \sigma\in(\sigma_j,\sigma_{j+1}];\quad z=1,2,\ldots,p,\end{array}\right. \\ \mathfrak{I}_2^{**}(\mathrm{u},\mathrm{v})(\sigma)= \\ \left\{\begin{array}{l}\dfrac{1}{\Gamma(\beta)}\displaystyle\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\beta-1}\phi_2(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi \\ -\dfrac{\sigma^{\beta-1}\mathbb{T}^{1-\beta}}{\Gamma(\beta)}\displaystyle\int_{\sigma_z}^{\mathbb{T}}(\mathbb{T}-\pi)^{\beta-1}\phi_2(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi \\ -\dfrac{\sigma^{\beta-1}}{\mathbb{T}}\displaystyle\sum_{k=1}^{z}\left[\dfrac{(\mathbb{T}-\sigma_k)\,\sigma_k^{2-\beta}}{\Gamma(\beta-1)}\displaystyle\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-2}\,\phi_2(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\right. \\ \left.+\dfrac{\left((\beta-1)-(\beta-2)\mathbb{T}\sigma_k^{-1}\right)\sigma_k^{2-\beta}}{\Gamma(\beta)}\displaystyle\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-1}\,\phi_2(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\right] \\ +\displaystyle\sum_{k=1}^{z}\left[\dfrac{(\sigma-\sigma_k)\,\sigma^{\beta-2}\sigma_k^{2-\beta}}{\Gamma(\beta-1)}\displaystyle\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-2}\,\phi_2(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\right. \\ \left.+\dfrac{\left((\beta-1)-(\beta-2)\sigma\sigma_k^{-1}\right)\sigma^{\beta-2}\sigma_k^{2-\beta}}{\Gamma(\beta)}\displaystyle\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-1}\,\phi_2(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\right], \\ \quad \sigma\in(\sigma_k,\sigma_{k+1}];\quad z=1,2,\ldots,q.\end{array}\right.\end{cases}\tag{3.10}$$

For additional analysis, the following hypothesis needs to hold:

($\mathbf{H}_1$) • For $\sigma\in\omega$ there exist bounded functions $o,\tau,\upsilon\in\vartheta$ such that

$$|\phi_1(\sigma,x_1(\sigma),x_2(\sigma))|\le o(\sigma)+\tau(\sigma)|x_1(\sigma)|+\upsilon(\sigma)|x_2(\sigma)|$$

with $o_1=\sup_{\sigma\in\omega}o(\sigma)$, $\tau_1=\sup_{\sigma\in\omega}\tau(\sigma)$, and $\upsilon_1=\sup_{\sigma\in\omega}\upsilon(\sigma)<1$.

- Similarly, for $\sigma \in \omega$ there exist bounded functions $o^*, \tau^*, \upsilon^* \in \vartheta$ such that

$$|\phi_2(\sigma, x_1(\sigma), x_2(\sigma))| \leq o^*(\sigma) + \tau^*(\sigma)|x_1(\sigma)| + \upsilon^*(\sigma)|x_2(\sigma)|$$

with $o_2 = \sup_{\sigma\in\omega} o^*(\sigma)$, $\tau_2 = \sup_{\sigma\in\omega} \tau^*(\sigma)$, and $\upsilon_2 = \sup_{\sigma\in\omega} \upsilon^*(\sigma) < 1$.

($\mathbf{H}_2$) $\mathcal{E}_j, \mathcal{E}_j^* : \mathbb{R} \to \mathbb{R}$ are continuous and there exist constants $\mathcal{G}_\mathcal{E}, \mathcal{G}_{\mathcal{E}^*}, \mathcal{G}'_\mathcal{E}, \mathcal{G}'_{\mathcal{E}^*}, \widehat{\mathcal{G}}_\mathcal{E}, \widehat{\mathcal{G}}_{\mathcal{E}^*}, \widehat{\mathcal{G}}'_\mathcal{E}, \widehat{\mathcal{G}}'_{\mathcal{E}^*} > 0$ such that, for any $(\mathrm{u}, \mathrm{v}) \in \vartheta$,

$$\begin{aligned} |\mathcal{E}_z(\mathrm{u})| &\leq \mathcal{G}_\mathcal{E}|\mathrm{u}| + \mathcal{G}'_\mathcal{E}, & |\mathcal{E}_z(\mathrm{v})| &\leq \widehat{\mathcal{G}}_\mathcal{E}|\mathrm{v}| + \widehat{\mathcal{G}}'_\mathcal{E}, \\ |\mathcal{E}^*_z(\mathrm{u})| &\leq \mathcal{G}_{\mathcal{E}^*}|\mathrm{u}| + \mathcal{G}'_{\mathcal{E}^*}, & |\mathcal{E}^*_z(\mathrm{v})| &\leq \widehat{\mathcal{G}}_{\mathcal{E}^*}|\mathrm{v}| + \widehat{\mathcal{G}}'_{\mathcal{E}^*}, \end{aligned}$$

where $z = 1, 2, \ldots, p$.

($\mathbf{H}_3$)
- For all $x_1, x_2, x_1^*, x_2^* \in \mathbb{R}$ and for each $\sigma \in \omega$, there exist constants $\mathcal{L}_{\phi_1} > 0$, $0 < \mathcal{L}^*_{\phi_1} < 1$ such that

$$|\phi_1(\sigma, x_1, x_2) - \phi_1(\sigma, x_1^*, x_2^*)| \leq \mathcal{L}_{\phi_1}|x_1 - x_1^*| + \mathcal{L}^*_{\phi_1}|x_2 - x_2^*|.$$

- Similarly, for all $x_1, x_2, x_1^*, x_2^* \in \mathbb{R}$ and for each $\sigma \in \omega$, there exist constants $\mathcal{L}_{\phi_2} > 0$, $0 < \mathcal{L}^*_{\phi_2} < 1$ such that

$$|\phi_2(\sigma, x_1, x_2) - \phi_2(\sigma, x_1^*, x_2^*)| \leq \mathcal{L}_{\phi_2}|x_1 - x_1^*| + \mathcal{L}^*_{\phi_2}|x_2 - x_2^*|.$$

($\mathbf{H}_4$) $\mathcal{E}_z, \mathcal{E}_z^* : \mathbb{R} \to \mathbb{R}$ are continuous and there exist constants $\mathcal{L}_\mathcal{E}, \mathcal{L}_{\mathcal{E}^*}, \mathcal{L}^*_\mathcal{E}, \mathcal{L}^*_{\mathcal{E}^*}$ such that, for any $(\mathrm{u}, \mathrm{v}), (\mathrm{u}^*, \mathrm{v}^*) \in \vartheta$,

$$\begin{aligned} |\mathcal{E}_z(\mathrm{u}(\sigma)) - \mathcal{E}_z(\mathrm{u}^*(\sigma))| &\leq \mathcal{L}_\mathcal{E}|\mathrm{u} - \mathrm{u}^*|, & |\mathcal{E}_z(\mathrm{v}(\sigma)) - \mathcal{E}_z(\mathrm{v}^*(\sigma))| &\leq \mathcal{L}^*_\mathcal{E}|\mathrm{v} - \mathrm{v}^*|, \\ |\mathcal{E}^*_z(\mathrm{u}(\sigma)) - \mathcal{E}^*_z(\mathrm{u}^*(\sigma))| &\leq \mathcal{L}_{\mathcal{E}^*}|\mathrm{u} - \mathrm{u}^*|, & |\mathcal{E}^*_z(\mathrm{v}(\sigma)) - \mathcal{E}^*_z(\mathrm{v}^*(\sigma))| &\leq \mathcal{L}^*_{\mathcal{E}^*}|\mathrm{v} - \mathrm{v}^*|. \end{aligned}$$

Here we use Kransnoselskii's fixed point theorem to show that the operator $\mathfrak{I}_1 + \mathfrak{I}_2$ has at least one fixed point. Therefore, we choose a closed ball

$$\vartheta_r = \left\{(\mathrm{u}, \mathrm{v}) \in \vartheta, \|(\mathrm{u}, \mathrm{v})\| \leq r, \|\mathrm{u}\| \leq \frac{r}{2} \text{ and } \|\mathrm{v}\| \leq \frac{r}{2}\right\} \subset \vartheta,$$

where

$$r \geq \frac{\mathcal{G}_1 + \mathcal{G}_1^* + o_1\mathcal{G}_3 + o_2\mathcal{G}_3^*}{1 - (\mathcal{G}_2 + \mathcal{G}_2^* + \mathcal{G}_3\mathcal{G}_4 + \mathcal{G}_3^*\mathcal{G}_4^*)}.$$

Theorem 3.2. *If hypotheses* ($\boldsymbol{H}_1$)–($\boldsymbol{H}_4$) *are hold, then the given system* (1.1) *has at least one solution.*

Proof. 1) For any $(\mathrm{u}, \mathrm{v}) \in \vartheta_r$, we have

$$\|\mathfrak{I}_1(\mathrm{u}, \mathrm{v}) + \mathfrak{I}_2(\mathrm{u}, \mathrm{v})\|_\vartheta \leq \|\mathfrak{I}_1^*(\mathrm{u})\|_{\vartheta_1} + \|\mathfrak{I}_1^{**}(\mathrm{v})\|_{\vartheta_2} + \|\mathfrak{I}_2^*(\mathrm{u}, \mathrm{v})\|_{\vartheta_1} + \|\mathfrak{I}_2^{**}(\mathrm{u}, \mathrm{v})\|_{\vartheta_2}. \tag{3.11}$$

From (3.9), we get

$$|\sigma^{2-\alpha}\mathfrak{I}_1^*(\mathrm{u}(\sigma))| \leq \left|\frac{\sigma\mathrm{u}_2}{\mu_1\mathbb{T}^{\alpha-1}}\right| + \left|\frac{\sigma\mathrm{u}_1}{\mathbb{T}\nu_1\Gamma(\alpha-1)}\right| + \left|\frac{\mathrm{u}_1}{\nu_1\Gamma(\alpha-1)}\right| + \frac{\nu_2\,|\sigma|\left|\mathbb{T}^{1-\alpha}\right|}{\mu_1\Gamma(\alpha-1)}\int_0^\mathbb{T}\left|(\mathbb{T}-\pi)^{\alpha-2}\right||\mathrm{u}(\pi)|\,d\pi$$

$$+\sum_{j=1}^{z}\left|\left((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\right)-\frac{\sigma}{\mathbb{T}}\left((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\right)\right|\left|\sigma_j^{2-\alpha}\right|\left|\mathcal{E}_j(\mathfrak{u}(\sigma_j))\right|$$
$$+\sum_{j=1}^{z}\left|\left(\sigma-\sigma_j\right)-\frac{\sigma}{\mathbb{T}}\left(\mathbb{T}-\sigma_j\right)\right|\left|\sigma_j^{2-\alpha}\right|\left|\mathcal{E}_j^*(\mathfrak{u}(\sigma_j))\right|,$$
$$z=1,2,\ldots,p. \tag{3.12}$$

This implies that

$$\begin{aligned}\|\mathfrak{I}_1^*(\mathfrak{u})\|_{\vartheta_1}\leq&\left|\frac{\sigma\mathfrak{u}_2}{\mu_1\mathbb{T}^{\alpha-1}}\right|+\left|\frac{\sigma\mathfrak{u}_1}{\mathbb{T}\nu_1\Gamma(\alpha-1)}\right|+\left|\frac{\mathfrak{u}_1}{\nu_1\Gamma(\alpha-1)}\right|+\frac{\nu_2|\sigma|}{\mu_1\Gamma(\alpha)}\|\mathfrak{u}\|\\&+z(\alpha-1)\left|\sigma_z^{2-\alpha}\right|\left|1-\frac{\sigma}{\mathbb{T}}\right|(\mathcal{G}_{\mathcal{E}}\|\mathfrak{u}\|+\mathcal{G}'_{\mathcal{E}})+z\left|\sigma_z^{3-\alpha}\right|\left|\frac{\sigma}{\mathbb{T}}-1\right|(\mathcal{G}_{\mathcal{E}^*}\|\mathfrak{u}\|+\mathcal{G}'_{\mathcal{E}^*})\\\leq&\mathcal{G}_1+\mathcal{G}_2\|\mathfrak{u}\|.\end{aligned} \tag{3.13}$$

Similarly, we can obtain

$$\|\mathfrak{I}_1^{**}(\mathfrak{v})\|_{\vartheta_2}\leq\mathcal{G}_1^*+\mathcal{G}_2^*\|\mathfrak{v}\|, \tag{3.14}$$

where

$$\begin{aligned}\mathcal{G}_1&=z\mathcal{G}'_{\mathcal{E}}(\alpha-1)\left|\sigma_z^{2-\alpha}\right|\left|1-\frac{\sigma}{\mathbb{T}}\right|+z\mathcal{G}'_{\mathcal{E}^*}\left|\sigma_z^{3-\alpha}\right|\left|\frac{\sigma}{\mathbb{T}}-1\right|+\left|\frac{\sigma\mathfrak{u}_2}{\mu_1\mathbb{T}^{\alpha-1}}\right|+\left|\frac{\sigma\mathfrak{u}_1}{\mathbb{T}\nu_1\Gamma(\alpha-1)}\right|+\left|\frac{\mathfrak{u}_1}{\nu_1\Gamma(\alpha-1)}\right|,\\\mathcal{G}_2&=z\mathcal{G}_{\mathcal{E}}(\alpha-1)\left|\sigma_z^{2-\alpha}\right|\left|1-\frac{\sigma}{\mathbb{T}}\right|+z\mathcal{G}_{\mathcal{E}^*}\left|\sigma_z^{3-\alpha}\right|\left|\frac{\sigma}{\mathbb{T}}-1\right|+\frac{\nu_2|\sigma|}{\mu_1\Gamma(\alpha)},\ \textit{for}\ z=1,2,\ldots,p,\ \textit{and}\\\mathcal{G}_1^*&=z\widehat{\mathcal{G}}'_{\mathcal{E}}(\beta-1)\left|\sigma_z^{2-\beta}\right|\left|1-\frac{\sigma}{\mathbb{T}}\right|+z\widehat{\mathcal{G}}'_{\mathcal{E}^*}\left|\sigma_z^{3-\beta}\right|\left|\frac{\sigma}{\mathbb{T}}-1\right|+\left|\frac{\sigma\mathfrak{v}_2}{\mu_2\mathbb{T}^{\beta-1}}\right|+\left|\frac{\sigma\mathfrak{v}_1}{\mathbb{T}\nu_3\Gamma(\beta-1)}\right|+\left|\frac{\mathfrak{v}_1}{\nu_3\Gamma(\beta-1)}\right|,\\\mathcal{G}_2^*&=z\widehat{\mathcal{G}}_{\mathcal{E}}(\beta-1)\left|\sigma_z^{2-\beta}\right|\left|1-\frac{\sigma}{\mathbb{T}}\right|+z\widehat{\mathcal{G}}_{\mathcal{E}^*}\left|\sigma_z^{3-\beta}\right|\left|\frac{\sigma}{\mathbb{T}}-1\right|+\frac{\nu_4|\sigma|}{\mu_2\Gamma(\beta)},\ \textit{for}\ z=1,2,\ldots,q.\end{aligned}$$

Also, we have

$$\begin{aligned}\left|\sigma^{2-\alpha}\mathfrak{I}_2^*(\mathfrak{u},\mathfrak{v})\right|\leq&\frac{\left|\sigma^{2-\alpha}\right|}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}\left|(\sigma-\pi)^{\alpha-1}\right||\mathfrak{y}(\pi)|\,d\pi+\frac{|\sigma|\left|\mathbb{T}^{1-\alpha}\right|}{\Gamma(\alpha)}\int_{\sigma_z}^{\mathbb{T}}\left|(\mathbb{T}-\pi)^{\alpha-1}\right||\mathfrak{y}(\pi)|\,d\pi\\&+\frac{\sigma}{\mathbb{T}}\sum_{j=1}^{z}\left[\frac{\left|\left(\mathbb{T}-\sigma_j\right)\right|\left|\sigma_j^{2-\alpha}\right|}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}\left|\left(\sigma_j-\pi\right)^{\alpha-2}\right||\mathfrak{y}(\pi)|\,d\pi\right.\\&\left.+\frac{\left|\left((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\right)\right|\left|\sigma_j^{2-\alpha}\right|}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}\left|\left(\sigma_j-\pi\right)^{\alpha-1}\right||\mathfrak{y}(\pi)|\,d\pi\right]\\&+\sum_{j=1}^{z}\left[\frac{\left|\left(\sigma-\sigma_j\right)\right|\left|\sigma_j^{2-\alpha}\right|}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}\left|\left(\sigma_j-\pi\right)^{\alpha-2}\right||\mathfrak{y}(\pi)|\,d\pi\right.\\&\left.+\frac{\left|\left((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\right)\right|\left|\sigma_j^{2-\alpha}\right|}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}\left|\left(\sigma_j-\pi\right)^{\alpha-1}\right||\mathfrak{y}(\pi)|\,d\pi\right]\end{aligned}$$

$$for\ \ z = 1, 2, \ldots, p. \tag{3.15}$$

Now by $(\mathbf{H}_1)$

$$\begin{aligned}
|\mathfrak{y}(\sigma)| &= |\phi_1(\sigma, \mathcal{I}^{\alpha}\mathfrak{u}(\sigma), \mathcal{I}^{\beta}\mathfrak{v}(\sigma))| \\
&\leq o(\sigma) + \tau(\sigma)|\mathcal{I}^{\alpha}\mathfrak{u}(\sigma)| + \upsilon(\sigma)|\mathcal{I}^{\beta}\mathfrak{v}(\sigma)| \\
&\leq o(\sigma) + \tau(\sigma)\frac{1}{\Gamma(\alpha)}\int_0^{\sigma}\left|(\sigma-\pi)^{\alpha-1}\right||\mathfrak{u}(\pi)|d\pi + \upsilon(\sigma)\frac{1}{\Gamma(\beta)}\int_0^{\sigma}\left|(\sigma-\pi)^{\beta-1}\right||\mathfrak{v}(\pi)|d\pi.
\end{aligned}$$

Now, taking $\sup_{\sigma\in\omega}$ on both sides, we get

$$\|\mathfrak{y}\| \leq o_1 + \tau_1\frac{|\sigma^{\alpha}|\|\mathfrak{u}\|}{\Gamma(\alpha+1)} + \upsilon_1\frac{|\sigma^{\beta}|\|\mathfrak{v}\|}{\Gamma(\beta+1)}. \tag{3.16}$$

Now taking $\sup_{\sigma\in\omega}$ of (3.15) and using (3.16) in (3.15), we get

$$\begin{aligned}
\|\mathfrak{I}_2^*(\mathfrak{u},\mathfrak{v})\|_{\vartheta_1} \leq& \left(o_1 + \tau_1\frac{|\sigma^{\alpha}|\|\mathfrak{u}\|}{\Gamma(\alpha+1)} + \upsilon_1\frac{|\sigma^{\beta}|\|\mathfrak{v}\|}{\Gamma(\beta+1)}\right)\left(\frac{\left|\sigma^{2-\alpha}\right||(\sigma-\sigma_z)^{\alpha}|}{\Gamma(\alpha+1)} + \frac{|\sigma|\left|\mathbb{T}^{1-\alpha}\right||(\mathbb{T}-\sigma_z)^{\alpha}|}{\Gamma(\alpha+1)}\right. \\
&+ \frac{z|\sigma|\left|\sigma_z^{2-\alpha}\right|}{\mathbb{T}}\left[\frac{|(\mathbb{T}-\sigma_z)|\left|(\sigma_z-\sigma_{z-1})^{\alpha-1}\right|}{\Gamma(\alpha)} + \frac{\left|\left((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_z^{-1}\right)\right||(\sigma_z-\sigma_{z-1})^{\alpha}|}{\Gamma(\alpha+1)}\right] \\
&\left.+ \frac{z|(\sigma-\sigma_z)|\left|\sigma_z^{2-\alpha}\right|\left|(\sigma_z-\sigma_{z-1})^{\alpha-1}\right|}{\Gamma(\alpha)} + \frac{z\left|\left((\alpha-1)-(\alpha-2)\sigma\sigma_z^{-1}\right)\right|\left|\sigma_z^{2-\alpha}\right||(\sigma_z-\sigma_{z-1})^{\alpha}|}{\Gamma(\alpha+1)}\right) \\
\leq& o_1\mathcal{G}_3 + \tau_1\frac{|\sigma^{\alpha}|\|\mathfrak{u}\|\mathcal{G}_3}{\Gamma(\alpha+1)} + \upsilon_1\frac{|\sigma^{\beta}|\|\mathfrak{v}\|\mathcal{G}_3}{\Gamma(\beta+1)} \\
\leq& o_1\mathcal{G}_3 + \mathcal{G}_3\mathcal{G}_4\|(\mathfrak{u},\mathfrak{v})\|.
\end{aligned} \tag{3.17}$$

Similarly,

$$\|\mathfrak{I}_2^{**}(\mathfrak{u},\mathfrak{v})\|_{\vartheta_2} \leq o_2\mathcal{G}_3^* + \mathcal{G}_3^*\mathcal{G}_4^*\|(\mathfrak{u},\mathfrak{v})\|, \tag{3.18}$$

where

$$\begin{aligned}
\mathcal{G}_3 =& \frac{\left|\sigma^{2-\alpha}\right||(\sigma-\sigma_z)^{\alpha}|}{\Gamma(\alpha+1)} + \frac{|\sigma|\left|\mathbb{T}^{1-\alpha}\right||(\mathbb{T}-\sigma_z)^{\alpha}|}{\Gamma(\alpha+1)} \\
&+ \frac{z|\sigma|\left|\sigma_z^{2-\alpha}\right|}{\mathbb{T}}\left[\frac{|(\mathbb{T}-\sigma_z)|\left|(\sigma_z-\sigma_{z-1})^{\alpha-1}\right|}{\Gamma(\alpha)} + \frac{\left|\left((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_z^{-1}\right)\right||(\sigma_z-\sigma_{z-1})^{\alpha}|}{\Gamma(\alpha+1)}\right] \\
&+ \frac{z|(\sigma-\sigma_z)|\left|\sigma_z^{2-\alpha}\right|\left|(\sigma_z-\sigma_{z-1})^{\alpha-1}\right|}{\Gamma(\alpha)} + \frac{z\left|\left((\alpha-1)-(\alpha-2)\sigma\sigma_z^{-1}\right)\right|\left|\sigma_z^{2-\alpha}\right||(\sigma_z-\sigma_{z-1})^{\alpha}|}{\Gamma(\alpha+1)}, \\
& z = 1, 2, \ldots, p, \\
\mathcal{G}_3^* =& \frac{\left|\sigma^{2-\beta}\right|\left|(\sigma-\sigma_z)^{\beta}\right|}{\Gamma(\beta+1)} + \frac{|\sigma|\left|\mathbb{T}^{1-\beta}\right|\left|(\mathbb{T}-\sigma_z)^{\beta}\right|}{\Gamma(\beta+1)}
\end{aligned}$$

$$+\frac{z|\sigma|\left|\sigma_z^{2-\beta}\right|}{\mathbb{T}}\left[\frac{|(\mathbb{T}-\sigma_z)|\left|(\sigma_z-\sigma_{z-1})^{\beta-1}\right|}{\Gamma(\beta)}+\frac{\left|\left((\beta-1)-(\beta-2)\mathbb{T}\sigma_z^{-1}\right)\right|\left|(\sigma_z-\sigma_{z-1})^{\beta}\right|}{\Gamma(\beta+1)}\right]$$

$$+\frac{z|(\sigma-\sigma_z)|\left|\sigma_z^{2-\beta}\right|\left|(\sigma_z-\sigma_{z-1})^{\beta-1}\right|}{\Gamma(\beta)}+\frac{z\left|\left((\beta-1)-(\beta-2)\sigma\sigma_z^{-1}\right)\right|\left|\sigma_z^{2-\beta}\right|\left|(\sigma_z-\sigma_{z-1})^{\beta}\right|}{\Gamma(\beta+1)},$$

$z=1,2,\ldots,q,$

$$\mathcal{G}_4=\max\left\{\tau_1\frac{|\sigma^{\alpha}|}{\Gamma(\alpha+1)},\upsilon_1\frac{|\sigma^{\beta}|}{\Gamma(\beta+1)}\right\}\ \textit{and}$$

$$\mathcal{G}_4^*=\max\left\{\tau_2\frac{|\sigma^{\alpha}|}{\Gamma(\alpha+1)},\upsilon_2\frac{|\sigma^{\beta}|}{\Gamma(\beta+1)}\right\}.$$

Putting (3.13), (3.14), (3.17) and (3.18) in (3.11), we get

$$\begin{aligned}\|\mathfrak{I}_1(\mathrm{u},\mathrm{v})+\mathfrak{I}_2(\mathrm{u},\mathrm{v})\|_{\vartheta}&\le\mathcal{G}_1+\mathcal{G}_2\|\mathrm{u}\|+\mathcal{G}_1^*+\mathcal{G}_2^*\|\mathrm{v}\|+o_1\mathcal{G}_3+\mathcal{G}_3\mathcal{G}_4\|(\mathrm{u},\mathrm{v})\|+o_2\mathcal{G}_3^*+\mathcal{G}_3^*\mathcal{G}_4^*\|(\mathrm{u},\mathrm{v})\|\\&\le\mathcal{G}_1+\mathcal{G}_1^*+o_1\mathcal{G}_3+o_2\mathcal{G}_3^*+(\mathcal{G}_2+\mathcal{G}_2^*+\mathcal{G}_3\mathcal{G}_4+\mathcal{G}_3^*\mathcal{G}_4^*)\|(\mathrm{u},\mathrm{v})\|\\&\le r.\end{aligned}$$

Hence, $\|\mathfrak{I}_1(\mathrm{u},\mathrm{v})+\mathfrak{I}_2(\mathrm{u},\mathrm{v})\|_{\vartheta}\in\vartheta_r$.

2) Next, for any $\sigma\in\omega$, $(\mathrm{u},\mathrm{v}),(\xi,\zeta)\in\vartheta$

$$\begin{aligned}\|\mathfrak{I}_1(\mathrm{u},\mathrm{v})-\mathfrak{I}_1(\xi,\xi)\|_{\vartheta}\le&\|\mathfrak{I}_1^*(\mathrm{u})-\mathfrak{I}_1^*(\xi)\|_{\vartheta_1}+\|\mathfrak{I}_1^{**}(\mathrm{v})-\mathfrak{I}_1^{**}(\xi)\|_{\vartheta_2}\\\le&\frac{|\nu_2||\sigma|\mathbb{T}^{1-\alpha}|}{|\mu_1|\Gamma(\alpha-1)}\int_0^{\mathbb{T}}\left|(\mathbb{T}-\pi)^{\alpha-2}\right||\mathrm{u}(\pi)-\xi(\pi)|\,d\pi\\&+\sum_{j=1}^{z}\left|\left((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\right)-\frac{\sigma}{\mathbb{T}}\left((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\right)\right|\\&\times|\sigma_j^{2-\alpha}|\left|\mathcal{E}_j(\mathrm{u}(\sigma_j))-\mathcal{E}_j(\xi(\sigma_j))\right|\\&+\sum_{j=1}^{z}\left|\left(\sigma-\sigma_j\right)-\frac{\sigma}{\mathbb{T}}\left(\mathbb{T}-\sigma_j\right)\right||\sigma_j^{2-\alpha}|\left|\mathcal{E}_j^*(\mathrm{u}(\sigma_j))-\mathcal{E}_j^*(\xi(\sigma_j))\right|\\&+\frac{|\nu_4||\mathbb{T}^{1-\beta}|}{|\mu_2|\Gamma(\beta-1)}\int_0^{\mathbb{T}}\left|(\mathbb{T}-\pi)^{\beta-2}\right||\mathrm{v}(\pi)-\zeta(\pi)|\,d\pi\\&+\sum_{k=1}^{z}\left|\left((\beta-1)-(\beta-2)\sigma\sigma_k^{-1}\right)-\frac{\sigma}{\mathbb{T}}\left((\beta-1)-(\beta-2)\mathbb{T}\sigma_k^{-1}\right)\right|\\&\times|\sigma_k^{2-\beta}||\mathcal{E}_k(\mathrm{v}(\sigma_k))-\mathcal{E}_k(\zeta(\sigma_k))|\\&+\sum_{k=1}^{z}\left|(\sigma-\sigma_k)-\frac{\sigma}{\mathbb{T}}(\mathbb{T}-\sigma_k)\right||\sigma_k^{2-\beta}|\left|\mathcal{E}_k^*(\mathrm{v}(\sigma_k))-\mathcal{E}_k^*(\zeta(\sigma_k))\right|\\\le&\left(z(\alpha-1)|\sigma_z^{2-\alpha}|\left|1-\frac{\sigma}{\mathbb{T}}\right|\mathcal{L}_{\mathcal{E}}+z|\sigma_z^{3-\alpha}|\left|\frac{\sigma}{\mathbb{T}}-1\right|\mathcal{L}_{\mathcal{E}^*}+\frac{|\nu_2||\sigma|}{|\mu_1|\Gamma(\alpha)}\right)\|\mathrm{u}-\xi\|\\&+\left(z(\beta-1)|\sigma_z^{2-\beta|}\left|1-\frac{\sigma}{\mathbb{T}}\right|\mathcal{L}_{\mathcal{E}}^*+z|\sigma_z^{3-\beta}|\left|\frac{\sigma}{\mathbb{T}}-1\right|\mathcal{L}_{\mathcal{E}^*}^*+\frac{|\nu_4||\sigma|}{|\mu_2|\Gamma(\beta)}\right)\|\mathrm{v}-\zeta\|\end{aligned}$$

$$\leq \mathcal{L}(\varrho_1 + \varrho_2)\|(\mathrm{u} - \xi, \mathrm{v} - \zeta)\|.$$

Here $\mathcal{L} = \max\{\mathcal{L}_{\mathcal{E}}, \mathcal{L}_{\mathcal{E}^*}, \mathcal{L}^*_{\mathcal{E}}, \mathcal{L}^*_{\mathcal{E}^*}\}$,

$$\varrho_1 = z(\alpha-1)|\sigma_z^{2-\alpha}|\left|1-\frac{\sigma}{\mathbb{T}}\right| + z|\sigma_z^{3-\alpha}|\left|\frac{\sigma}{\mathbb{T}}-1\right| + \frac{|\nu_2||\sigma|}{|\mu_1|\Gamma(\alpha)}, \;\; z = 1, 2, \ldots, p,$$

and

$$\varrho_2 = z(\beta-1)|\sigma_z^{2-\beta|}\left|1-\frac{\sigma}{\mathbb{T}}\right| + z|\sigma_z^{3-\beta}|\left|\frac{\sigma}{\mathbb{T}}-1\right| + \frac{|\nu_4||\sigma|}{|\mu_2|\Gamma(\beta)}, \;\; z = 1, 2, \ldots, q.$$

Therefore, $\mathfrak{I}_1$ is a contractive operator.

3) Now, for the continuity and compactness of $\mathfrak{I}_2$, we make a sequence $T_s = (\mathrm{u}_s, \mathrm{v}_s)$ in ϑ_r such that $(\mathrm{u}_s, \mathrm{v}_s) \to (\mathrm{u}, \mathrm{v})$ for $s \to \infty$ in ϑ_r. Thus, we have

$$\begin{aligned}
&\|\mathfrak{I}_2(\mathrm{u}_s, \mathrm{v}_s) - \mathfrak{I}_2(\mathrm{u}, \mathrm{v})\|_{\vartheta} \\
&\leq \|\mathfrak{I}^*_2(\mathrm{u}_s, \mathrm{v}_s) - \mathfrak{I}^*_2(\mathrm{u}, \mathrm{v})\|_{\vartheta_1} + \|\mathfrak{I}^{**}_2(\mathrm{u}_s, \mathrm{v}_s) - \mathfrak{I}^{**}_2(\mathrm{u}, \mathrm{v})\|_{\vartheta_2} \\
&\leq \left(\frac{\mathcal{L}_{\phi_1}|\sigma^{\alpha}|\|\mathrm{u}_s - \mathrm{u}\|}{\Gamma(\alpha+1)} + \frac{\mathcal{L}^*_{\phi_1}|\sigma^{\beta}|\|\mathrm{v}_s - \mathrm{v}\|}{\Gamma(\beta+1)}\right)\left(\frac{\left|\sigma^{2-\alpha}\right|\left|(\sigma - \sigma_z)^{\alpha}\right|}{\Gamma(\alpha+1)} + \frac{|\sigma|\left|\mathbb{T}^{1-\alpha}\right|\left|(\mathbb{T} - \sigma_z)^{\alpha}\right|}{\Gamma(\alpha+1)}\right. \\
&\quad + \frac{z\,|\sigma|\left|\sigma_z^{2-\alpha}\right|}{\mathbb{T}}\left[\frac{\left|(\mathbb{T} - \sigma_z)\right|\left|(\sigma_z - \sigma_{z-1})^{\alpha-1}\right|}{\Gamma(\alpha)} + \frac{\left|\left((\alpha-1) - (\alpha-2)\mathbb{T}\sigma_z^{-1}\right)\right|\left|(\sigma_z - \sigma_{z-1})^{\alpha}\right|}{\Gamma(\alpha+1)}\right] \\
&\quad \left. + \frac{z\,|(\sigma - \sigma_z)|\left|\sigma_z^{2-\alpha}\right|\left|(\sigma_z - \sigma_{z-1})^{\alpha-1}\right|}{\Gamma(\alpha)} + \frac{z\left|\left((\alpha-1) - (\alpha-2)\sigma\sigma_z^{-1}\right)\right|\left|\sigma_z^{2-\alpha}\right|\left|(\sigma_z - \sigma_{z-1})^{\alpha}\right|}{\Gamma(\alpha+1)}\right) \\
&\quad + \left(\frac{\mathcal{L}_{\phi_2}|\sigma^{\alpha}|\|\mathrm{u}_s - \mathrm{u}\|}{\Gamma(\alpha+1)} + \frac{\mathcal{L}^*_{\phi_2}|\sigma^{\beta}|\|\mathrm{v}_s - \mathrm{v}\|}{\Gamma(\beta+1)}\right)\left(\frac{\left|\sigma^{2-\beta}\right|\left|(\sigma - \sigma_z)^{\beta}\right|}{\Gamma(\beta+1)} + \frac{|\sigma|\left|\mathbb{T}^{1-\beta}\right|\left|(\mathbb{T} - \sigma_z)^{\beta}\right|}{\Gamma(\beta+1)}\right. \\
&\quad + \frac{z\,|\sigma|\left|\sigma_z^{2-\beta}\right|}{\mathbb{T}}\left[\frac{\left|(\mathbb{T} - \sigma_z)\right|\left|(\sigma_z - \sigma_{z-1})^{\beta-1}\right|}{\Gamma(\beta)} + \frac{\left|\left((\beta-1) - (\beta-2)\mathbb{T}\sigma_z^{-1}\right)\right|\left|(\sigma_z - \sigma_{z-1})^{\beta}\right|}{\Gamma(\beta+1)}\right] \\
&\quad \left. + \frac{z\,|(\sigma - \sigma_z)|\left|\sigma_z^{2-\beta}\right|\left|(\sigma_z - \sigma_{z-1})^{\beta-1}\right|}{\Gamma(\beta)} + \frac{z\left|\left((\beta-1) - (\beta-2)\sigma\sigma_z^{-1}\right)\right|\left|\sigma_z^{2-\beta}\right|\left|(\sigma_z - \sigma_{z-1})^{\beta}\right|}{\Gamma(\beta+1)}\right) \\
&\leq \mathcal{G}_3\left(\frac{\mathcal{L}_{\phi_1}|\sigma^{\alpha}|\|\mathrm{u}_s - \mathrm{u}\|}{\Gamma(\alpha+1)} + \frac{\mathcal{L}^*_{\phi_1}|\sigma^{\beta}|\|\mathrm{v}_s - \mathrm{v}\|}{\Gamma(\beta+1)}\right) + \mathcal{G}^*_3\left(\frac{\mathcal{L}_{\phi_2}|\sigma^{\alpha}|\|\mathrm{u}_s - \mathrm{u}\|}{\Gamma(\alpha+1)} + \frac{\mathcal{L}^*_{\phi_2}|\sigma^{\beta}|\|\mathrm{v}_s - \mathrm{v}\|}{\Gamma(\beta+1)}\right).
\end{aligned}$$

This implies $\|\mathfrak{I}_2(\mathrm{u}_s, \mathrm{v}_s) - \mathfrak{I}_2(\mathrm{u}, \mathrm{v})\|_{\vartheta} \to 0$ as $s \to \infty$, therefore $\mathfrak{I}_2$ is continuous.

Next, we show that $\mathfrak{I}_2$ is uniformly bounded on ϑ_r. From (3.17) and (3.18), we have

$$\begin{aligned}
\|\mathfrak{I}_2(\mathrm{u}, \mathrm{v})\|_{\vartheta} &\leq \|\mathfrak{I}^*_2(\mathrm{u}, \mathrm{v})\|_{\vartheta_1} + \|\mathfrak{I}^{**}_2(\mathrm{u}, \mathrm{v})\|_{\vartheta_2} \\
&\leq o_1\mathcal{G}_3 + o_2\mathcal{G}^*_3 + (\mathcal{G}_3\mathcal{G}_4 + \mathcal{G}^*_3\mathcal{G}^*_4)\|(\mathrm{u}, \mathrm{v})\| \\
&\leq r.
\end{aligned}$$

Thus, $\mathfrak{I}_2$ is uniformly bounded on ϑ_r.

For equicontinuity, suppose $\eta_1, \eta_2 \in \omega$ with $\eta_1 < \eta_2$, and for any $(\mathfrak{u}, \mathfrak{v}) \in \vartheta_r \subset \vartheta$ where ϑ_r is clearly bounded, we have

$$\begin{aligned}
&\|\mathfrak{I}_2^*(\mathfrak{u},\mathfrak{v})(\eta_1) - \mathfrak{I}_2^*(\mathfrak{u},\mathfrak{v})(\eta_2)\|_{\vartheta_1} \\
&= \max|\sigma^{2-\alpha}(\mathfrak{I}_2^*(\mathfrak{u},\mathfrak{v})(\eta_1) - \mathfrak{I}_2^*(\mathfrak{u},\mathfrak{v})(\eta_2))| \\
&\leq \left(o_1 + \tau_1\frac{|\sigma^\alpha|\|\mathfrak{u}\|}{\Gamma(\alpha+1)} + \upsilon_1\frac{|\sigma^\beta|\|\mathfrak{v}\|}{\Gamma(\beta+1)}\right)\left(\frac{\left|\sigma^{2-\alpha}\right||((\eta_1-\sigma_z)^\alpha - (\eta_2-\sigma_z)^\alpha)|}{\Gamma(\alpha+1)}\right. \\
&\quad + \frac{\left|\sigma^{2-\alpha}\right|\left|\eta_1^{\alpha-1} - \eta_2^{\alpha-1}\right|\left|\mathbb{T}^{1-\alpha}\right||(\mathbb{T}-\sigma_z)^\alpha|}{\Gamma(\alpha+1)} + \left[\left|\left(\eta_1^{\alpha-2} - \eta_2^{\alpha-2}\right)\right| + \frac{\left|\left(\eta_1^{\alpha-1} - \eta_2^{\alpha-1}\right)\right|}{\mathbb{T}}\right] \\
&\quad \left.\times\left[\frac{z\left|\sigma^{2-\alpha}\right|\left|\sigma_z^{3-\alpha}\right|\left|(\sigma_z - \sigma_{z-1})^{\alpha-1}\right|}{\Gamma(\alpha)} + \frac{z(\alpha-1)\left|\sigma^{2-\alpha}\right|\left|\sigma_z^{2-\alpha}\right||(\sigma_z-\sigma_{z-1})^\alpha|}{\Gamma(\alpha+1)}\right]\right).
\end{aligned}$$

This implies that

$$\|\mathfrak{I}_2^*(\mathfrak{u},\mathfrak{v})(\eta_1) - \mathfrak{I}_2^*(\mathfrak{u},\mathfrak{v})(\eta_2)\|_{\vartheta_1} \to 0 \ \ as \ \ \eta_1 \to \eta_2.$$

In the same way, we have

$$\|\mathfrak{I}_2^{**}(\mathfrak{u},\mathfrak{v})(\eta_1) - \mathfrak{I}_2^{**}(\mathfrak{u},\mathfrak{v})(\eta_2)\|_{\vartheta_2} \to 0 \ \ as \ \ \eta_1 \to \eta_2.$$

Hence

$$\|\mathfrak{I}_2(\mathfrak{u},\mathfrak{v})(\eta_1) - \mathfrak{I}_2(\mathfrak{u},\mathfrak{v})(\eta_2)\|_{\vartheta} \to 0 \ \ as \ \ \eta_1 \to \eta_2.$$

Thus, $\mathfrak{I}_2$ is equicontinuous. So $\mathfrak{I}_2$ is relatively compact on ϑ_r. Hence, by the Arzelà–Ascoli Theorem, $\mathfrak{I}_2$ is compact on ϑ_r. Thus all the condition of Theorem 2.1 are satisfied. So the given system (1.1) has at least one solution. □

Theorem 3.3. *Let hypotheses* $(\boldsymbol{H}_3)$, $(\boldsymbol{H}_4)$ *be satisfied with*

$$\Delta_1 + \Delta_3 + \frac{(\Delta_2\mathcal{L}_{\phi_1} + \Delta_4\mathcal{L}_{\phi_2})|\sigma^\alpha|}{\Gamma(\alpha+1)} + \frac{(\Delta_2\mathcal{L}^*_{\phi_1} + \Delta_4\mathcal{L}^*_{\phi_2})|\sigma^\beta|}{\Gamma(\beta+1)} < 1, \tag{3.19}$$

then the given system (1.1) *has unique solution.*

Proof. First we define an operator $\varphi = (\varphi_1, \varphi_2) : \vartheta \to \vartheta$, i.e., $\varphi(\mathfrak{u},\mathfrak{v})(\sigma) = (\varphi_1(\mathfrak{u},\mathfrak{v}), \varphi_2(\mathfrak{u},\mathfrak{v}))(\sigma)$, where

$$\begin{aligned}
\varphi_1(\mathfrak{u},\mathfrak{v})(\sigma) =& \frac{\sigma^{\alpha-1}\mathfrak{u}_2}{\mu_1\mathbb{T}^{\alpha-1}} - \frac{\sigma^{\alpha-1}\mathfrak{u}_1}{\mathbb{T}\nu_1\Gamma(\alpha-1)} + \frac{\sigma^{\alpha-2}\mathfrak{u}_1}{\nu_1\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\alpha-1}\phi_1(\pi, \mathcal{I}^\alpha\mathfrak{u}(\pi), \mathcal{I}^\beta\mathfrak{v}(\pi))d\pi \\
&- \frac{\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\Gamma(\alpha)}\int_{\sigma_z}^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-1}\phi_1(\pi, \mathcal{I}^\alpha\mathfrak{u}(\pi), \mathcal{I}^\beta\mathfrak{v}(\pi))d\pi - \frac{\nu_2\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\mu_1\Gamma(\alpha-1)}\int_0^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-2}\mathfrak{u}(\pi)d\pi \\
&- \frac{\sigma^{\alpha-1}}{\mathbb{T}}\sum_{j=1}^{z}\left[\left((\alpha-1) - (\alpha-2)\mathbb{T}\sigma_j^{-1}\right)\sigma_j^{2-\alpha}\mathcal{E}_j(\mathfrak{u}(\sigma_j)) + \left(\mathbb{T}-\sigma_j\right)\sigma_j^{2-\alpha}\mathcal{E}_j^*(\mathfrak{u}(\sigma_j))\right. \\
&\left. + \frac{\left(\mathbb{T}-\sigma_j\right)\sigma_j^{2-\alpha}}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}\left(\sigma_j-\pi\right)^{\alpha-2}\phi_1(\pi, \mathcal{I}^\alpha\mathfrak{u}(\pi), \mathcal{I}^\beta\mathfrak{v}(\pi))d\pi\right.
\end{aligned}$$

$$+\frac{\left((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\right)\sigma_j^{2-\alpha}}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}\left(\sigma_j-\pi\right)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\Bigg]$$
$$+\sum_{j=1}^{z}\Bigg[\left((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\right)\sigma^{\alpha-2}\sigma_j^{2-\alpha}\mathcal{E}_j(\mathrm{u}(\sigma_j))+\left(\sigma-\sigma_j\right)\sigma^{\alpha-2}\sigma_j^{2-\alpha}\mathcal{E}_j^{*}(\mathrm{u}(\sigma_j))$$
$$+\frac{\left(\sigma-\sigma_j\right)\sigma^{\alpha-2}\sigma_j^{2-\alpha}}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}\left(\sigma_j-\pi\right)^{\alpha-2}\phi_1(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi$$
$$+\frac{\left((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\right)\sigma^{\alpha-2}\sigma_j^{2-\alpha}}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}\left(\sigma_j-\pi\right)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\Bigg],$$
$$\textit{for } z=1,2,\ldots,p,$$

and

$$\begin{aligned}\varphi_2(\mathrm{u},\mathrm{v})(\sigma)=&\frac{\sigma^{\beta-1}\mathrm{v}_2}{\mu_2\mathbb{T}^{\beta-1}}-\frac{\sigma^{\beta-1}\mathrm{v}_1}{\mathbb{T}\nu_3\Gamma(\beta-1)}+\frac{\sigma^{\beta-2}\mathrm{v}_1}{\nu_3\Gamma(\beta-1)}+\frac{1}{\Gamma(\beta)}\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\beta-1}\phi_2(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\\&-\frac{\sigma^{\beta-1}\mathbb{T}^{1-\beta}}{\Gamma(\beta)}\int_{\sigma_z}^{\mathbb{T}}(\mathbb{T}-\pi)^{\beta-1}\phi_2(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi-\frac{\nu_4\sigma^{\beta-1}\mathbb{T}^{1-\beta}}{\mu_2\Gamma(\beta-1)}\int_0^{\mathbb{T}}(\mathbb{T}-\pi)^{\beta-2}\mathrm{v}(\pi)d\pi\\&-\frac{\sigma^{\beta-1}}{\mathbb{T}}\sum_{k=1}^{z}\Bigg[\left((\beta-1)-(\beta-2)\mathbb{T}\sigma_k^{-1}\right)\sigma_k^{2-\beta}\mathcal{E}_k(\mathrm{v}(\sigma_k))+(\mathbb{T}-\sigma_k)\,\sigma_k^{2-\beta}\mathcal{E}_k^{*}(\mathrm{v}(\sigma_k))\\&+\frac{(\mathbb{T}-\sigma_k)\,\sigma_k^{2-\beta}}{\Gamma(\beta-1)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-2}\,\phi_2(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\\&+\frac{\left((\beta-1)-(\beta-2)\mathbb{T}\sigma_k^{-1}\right)\sigma_k^{2-\beta}}{\Gamma(\beta)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-1}\,\phi_2(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\Bigg]\\&+\sum_{k=1}^{z}\Bigg[\left((\beta-1)-(\beta-2)\sigma\sigma_k^{-1}\right)\sigma^{\beta-2}\sigma_k^{2-\beta}\mathcal{E}_k(\mathrm{v}(\sigma_k))+(\sigma-\sigma_k)\,\sigma^{\beta-2}\sigma_k^{2-\beta}\mathcal{E}_k^{*}(\mathrm{v}(\sigma_k))\\&+\frac{(\sigma-\sigma_k)\,\sigma^{\beta-2}\sigma_k^{2-\beta}}{\Gamma(\beta-1)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-2}\,\phi_2(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\\&+\frac{\left((\beta-1)-(\beta-2)\sigma\sigma_k^{-1}\right)\sigma^{\beta-2}\sigma_k^{2-\beta}}{\Gamma(\beta)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-1}\,\phi_2(\pi,\mathcal{I}^{\alpha}\mathrm{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))d\pi\Bigg],\\&\textit{for } z=1,2,\ldots,q.\end{aligned}$$

In view of Theorem 3.2, we have

$$\begin{aligned}&|\sigma^{2-\alpha}(\varphi_1(\mathrm{u},\mathrm{v})-\varphi_1(\xi,\zeta))|\\&\quad\leq\left(\frac{\mathcal{L}_{\phi_1}^{*}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\Bigg(\frac{\left|\sigma^{2-\alpha}\right|\,|(\sigma-\sigma_z)^{\alpha}|}{\Gamma(\alpha+1)}+\frac{|\sigma|\left|\mathbb{T}^{1-\alpha}\right|\,|(\mathbb{T}-\sigma_z)^{\alpha}|}{\Gamma(\alpha+1)}\\&\qquad+\frac{z\,|\sigma|\left|\sigma_z^{2-\alpha}\right|}{\mathbb{T}}\Bigg[\frac{|(\mathbb{T}-\sigma_z)|\left|(\sigma_z-\sigma_{z-1})^{\alpha-1}\right|}{\Gamma(\alpha)}+\frac{\left|\left((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_z^{-1}\right)\right|\,|(\sigma_z-\sigma_{z-1})^{\alpha}|}{\Gamma(\alpha+1)}\Bigg]\end{aligned}$$

$$+\frac{z\,|(\sigma-\sigma_z)|\left|\sigma_z^{2-\alpha}\right|\left|(\sigma_z-\sigma_{z-1})^{\alpha-1}\right|}{\Gamma(\alpha)}+\frac{z\left|\left((\alpha-1)-(\alpha-2)\sigma\sigma_z^{-1}\right)\right|\left|\sigma_z^{2-\alpha}\right|\,|(\sigma_z-\sigma_{z-1})^{\alpha}|}{\Gamma(\alpha+1)}\Bigg)|\mathrm{v}-\zeta|$$
$$+\Bigg[\left(\frac{\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)\Bigg(\frac{\left|\sigma^{2-\alpha}\right|\,|(\sigma-\sigma_z)^{\alpha}|}{\Gamma(\alpha+1)}+\frac{|\sigma|\left|\mathbb{T}^{1-\alpha}\right|\,|(\mathbb{T}-\sigma_z)^{\alpha}|}{\Gamma(\alpha+1)}$$
$$+\frac{z\,|\sigma|\left|\sigma_z^{2-\alpha}\right|}{\mathbb{T}}\left[\frac{|(\mathbb{T}-\sigma_z)|\left|(\sigma_z-\sigma_{z-1})^{\alpha-1}\right|}{\Gamma(\alpha)}+\frac{\left|\left((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_z^{-1}\right)\right|\,|(\sigma_z-\sigma_{z-1})^{\alpha}|}{\Gamma(\alpha+1)}\right]$$
$$+\frac{z\,|(\sigma-\sigma_z)|\left|\sigma_z^{2-\alpha}\right|\left|(\sigma_z-\sigma_{z-1})^{\alpha-1}\right|}{\Gamma(\alpha)}+\frac{z\left|\left((\alpha-1)-(\alpha-2)\sigma\sigma_z^{-1}\right)\right|\left|\sigma_z^{2-\alpha}\right|\,|(\sigma_z-\sigma_{z-1})^{\alpha}|}{\Gamma(\alpha+1)}\Bigg)$$
$$+\left(z(\alpha-1)|\sigma_z^{2-\alpha}|\left|1-\frac{\sigma}{\mathbb{T}}\right|\mathcal{L}_{\mathcal{E}}+z|\sigma_z^{3-\alpha}|\left|\frac{\sigma}{\mathbb{T}}-1\right|\mathcal{L}_{\mathcal{E}^*}+\frac{|\nu_2||\sigma|}{|\mu_1|\Gamma(\alpha)}\right)\Bigg]|\mathrm{u}-\xi|.$$

Taking $\sup_{\sigma\in\omega}$, we get

$$\|\varphi_1(\mathrm{u},\mathrm{v})-\varphi_1(\xi,\zeta)\|_{\vartheta_1}\leq\left(\Delta_1+\frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}+\frac{\Delta_2\mathcal{L}^*_{\phi_1}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\|(\mathrm{u},\mathrm{v})-(\xi,\zeta)\|$$
$$for\ \ z=1,2,\ldots,p,$$

where

$$\Delta_1=z(\alpha-1)|\sigma_z^{2-\alpha}|\left|1-\frac{\sigma}{\mathbb{T}}\right|\mathcal{L}_{\mathcal{E}}+z|\sigma_z^{3-\alpha}|\left|\frac{\sigma}{\mathbb{T}}-1\right|\mathcal{L}_{\mathcal{E}^*}+\frac{|\nu_2||\sigma|}{|\mu_1|\Gamma(\alpha)},$$
$$\Delta_2=\frac{\left|\sigma^{2-\alpha}\right|\,|(\sigma-\sigma_z)^{\alpha}|}{\Gamma(\alpha+1)}+\frac{|\sigma|\left|\mathbb{T}^{1-\alpha}\right|\,|(\mathbb{T}-\sigma_z)^{\alpha}|}{\Gamma(\alpha+1)}$$
$$+\frac{z\,|\sigma|\left|\sigma_z^{2-\alpha}\right|}{\mathbb{T}}\left[\frac{|(\mathbb{T}-\sigma_z)|\left|(\sigma_z-\sigma_{z-1})^{\alpha-1}\right|}{\Gamma(\alpha)}+\frac{\left|\left((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_z^{-1}\right)\right|\,|(\sigma_z-\sigma_{z-1})^{\alpha}|}{\Gamma(\alpha+1)}\right]$$
$$+\frac{z\,|(\sigma-\sigma_z)|\left|\sigma_z^{2-\alpha}\right|\left|(\sigma_z-\sigma_{z-1})^{\alpha-1}\right|}{\Gamma(\alpha)}+\frac{z\left|\left((\alpha-1)-(\alpha-2)\sigma\sigma_z^{-1}\right)\right|\left|\sigma_z^{2-\alpha}\right|\,|(\sigma_z-\sigma_{z-1})^{\alpha}|}{\Gamma(\alpha+1)},$$
$$for\ \ z=1,2,\ldots,p.$$

Similarly,

$$\|\varphi_2(\mathrm{u},\mathrm{v})-\varphi_2(\xi,\zeta)\|_{\vartheta_2}\leq\left(\Delta_3+\frac{\Delta_4\mathcal{L}_{\phi_2}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}+\frac{\Delta_4\mathcal{L}^*_{\phi_2}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\|(\mathrm{u},\mathrm{v})-(\xi,\zeta)\|$$
$$for\ \ z=1,2,\ldots,q,$$

where

$$\Delta_3=z(\beta-1)|\sigma_z^{2-\beta}|\left|1-\frac{\sigma}{\mathbb{T}}\right|\mathcal{L}^*_{\mathcal{E}}+z|\sigma_z^{3-\beta}|\left|\frac{\sigma}{\mathbb{T}}-1\right|\mathcal{L}^*_{\mathcal{E}^*}+\frac{|\nu_4||\sigma|}{|\mu_2|\Gamma(\beta)},$$
$$\Delta_4=\frac{\left|\sigma^{2-\beta}\right|\left|(\sigma-\sigma_z)^{\beta}\right|}{\Gamma(\beta+1)}+\frac{|\sigma|\left|\mathbb{T}^{1-\beta}\right|\left|(\mathbb{T}-\sigma_z)^{\beta}\right|}{\Gamma(\beta+1)}$$

$$+\frac{z\,|\sigma|\left|\sigma_z^{2-\beta}\right|}{\mathbb{T}}\left[\frac{|(\mathbb{T}-\sigma_z)|\left|(\sigma_z-\sigma_{z-1})^{\beta-1}\right|}{\Gamma(\beta)}+\frac{\left|\left((\beta-1)-(\beta-2)\mathbb{T}\sigma_z^{-1}\right)\right|\left|(\sigma_z-\sigma_{z-1})^{\beta}\right|}{\Gamma(\beta+1)}\right]$$

$$+\frac{z\,|(\sigma-\sigma_z)|\left|\sigma_z^{2-\beta}\right|\left|(\sigma_z-\sigma_{z-1})^{\beta-1}\right|}{\Gamma(\beta)}+\frac{z\left|\left((\beta-1)-(\beta-2)\sigma\sigma_z^{-1}\right)\right|\left|\sigma_z^{2-\beta}\right|\left|(\sigma_z-\sigma_{z-1})^{\beta}\right|}{\Gamma(\beta+1)},$$

for $z=1,2,\ldots,q$.

Hence

$$\|\varphi(\mathrm{u},\mathrm{v})-\varphi(\xi,\zeta)\|_{\vartheta}\leq\left(\Delta_1+\Delta_3+\frac{(\Delta_2\mathcal{L}_{\phi_1}+\Delta_4\mathcal{L}_{\phi_2})|\sigma^{\alpha}|}{\Gamma(\alpha+1)}+\frac{(\Delta_2\mathcal{L}^*_{\phi_1}+\Delta_4\mathcal{L}^*_{\phi_2})|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\|(\mathrm{u},\mathrm{v})-(\xi,\zeta)\|.$$

This implies that the operator φ is a contraction. Therefore, (1.1) has a unique solution. □

4. Ulam's stability analysis

In this section, we study different kinds of stabilities, like $\mathcal{HU}$, generalized $\mathcal{HU}$, $\mathcal{HUR}$, and generalized $\mathcal{HUR}$ stability of the proposed system.

Theorem 4.1. *If assumptions* $(\boldsymbol{H}_3)$, $(\boldsymbol{H}_4)$ *and inequality* (3.19) *are satisfied and*

$$\mathcal{F}=1-\frac{\left(\frac{\Delta_2\mathcal{L}^*_{\phi_1}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\left(\frac{\Delta_4\mathcal{L}_{\phi_2}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)}{\left[1-\left(\Delta_1+\frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)\right]\left[1-\left(\Delta_3+\frac{\Delta_4\mathcal{L}^*_{\phi_2}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\right]}>0,$$

then the unique solution of the coupled system (1.1) *is* $\mathcal{HU}$ *stable and consequently generalized* $\mathcal{HU}$ *stable.*

Proof. Let $(\xi,\zeta)\in\vartheta$ is a solution of inequality (2.1), and let $(\mathrm{u},\mathrm{v})\in\vartheta$ be the unique solution of the coupled system given by

$$\begin{cases}\begin{cases}\mathcal{D}^{\alpha}\mathrm{u}(\sigma)-\phi_1(\sigma,\mathcal{I}^{\alpha}\mathrm{u}(\sigma),\mathcal{I}^{\beta}\mathrm{v}(\sigma))=0,\ \ \sigma\in\omega,\ \ \sigma\neq\sigma_j,\ \ j=1,2,\ldots,p,\\ \Delta\mathrm{u}(\sigma_j)-\mathcal{E}_j(\mathrm{u}(\sigma_j))=0,\qquad \Delta\mathrm{u}'(\sigma_j)-\mathcal{E}^*_j(\mathrm{u}(\sigma_j))=0,\ \ j=1,2,\ldots,p,\\ \nu_1\mathcal{D}^{\alpha-2}\mathrm{u}(\sigma)|_{\sigma=0}=\mathrm{u}_1,\qquad \mu_1\mathrm{u}(\sigma)|_{\sigma=\mathbb{T}}+\nu_2\mathcal{I}^{\alpha-1}\mathrm{u}(\sigma)|_{\sigma=\mathbb{T}}=\mathrm{u}_2,\end{cases}\\ \begin{cases}\mathcal{D}^{\beta}\mathrm{v}(\sigma)-\phi_2(\sigma,\mathcal{I}^{\alpha}\mathrm{u}(\sigma),\mathcal{I}^{\beta}\mathrm{v}(\sigma))=0,\ \ \sigma\in\omega,\ \ \sigma\neq\sigma_k,\ \ k=1,2,\ldots,q,\\ \Delta\mathrm{v}(\sigma_k)-\mathcal{E}_k(\mathrm{v}(\sigma_k))=0,\qquad \Delta\mathrm{v}'(\sigma_k)-\mathcal{E}^*_k(\mathrm{v}(\sigma_k))=0,\ \ k=1,2,\ldots,q,\\ \nu_3\mathcal{D}^{\beta-2}\mathrm{v}(\sigma)|_{\sigma=0}=\mathrm{v}_1,\qquad \mu_2\mathrm{v}(\sigma)|_{\sigma=\mathbb{T}}+\nu_4\mathcal{I}^{\beta-1}\mathrm{v}(\sigma)|_{\sigma=\mathbb{T}}=\mathrm{v}_2.\end{cases}\end{cases}\tag{4.1}$$

By Remark 2.1 we have

$$\begin{cases}\begin{cases}\mathcal{D}^{\alpha}\xi(\sigma)=\phi_1(\sigma,\mathcal{I}^{\alpha}\xi(\sigma),\mathcal{I}^{\beta}\zeta(\sigma))+\Re_{\phi_1}(\sigma),\\ \Delta\xi(\sigma_j)=\mathcal{E}_j(\xi(\sigma_j))+\Re_{\phi_{1j}},\ \ j=1,2,\ldots,p,\\ \Delta\xi'(\sigma_j)=\mathcal{E}^*_j(\xi(\sigma_j))+\Re_{\phi_{1j}},\ \ j=1,2,\ldots,p,\end{cases}\\ \begin{cases}\mathcal{D}^{\beta}\zeta(\sigma)=\phi_2(\sigma,\mathcal{I}^{\alpha}\xi(\sigma),\mathcal{I}^{\beta}\zeta(\sigma))+\mathfrak{L}_{\phi_2}(\sigma),\\ \Delta\zeta(\sigma_k)=\mathcal{E}_k(\zeta(\sigma_k))+\mathfrak{L}_{\phi_{2k}},\ \ k=1,2,\ldots,q,\\ \Delta\zeta'(\sigma_k)=\mathcal{E}^*_k(\zeta(\sigma_k))+\mathfrak{L}_{\phi_{2k}},\ \ k=1,2,\ldots,q.\end{cases}\end{cases}\tag{4.2}$$

By Corollary 1, the solution of problem (4.2) is

$$\begin{aligned}\xi(\sigma) =&\frac{\sigma^{\alpha-1}\mathfrak{u}_2}{\mu_1\mathbb{T}^{\alpha-1}} - \frac{\sigma^{\alpha-1}\mathfrak{u}_1}{\mathbb{T}\nu_1\Gamma(\alpha-1)} + \frac{\sigma^{\alpha-2}\mathfrak{u}_1}{\nu_1\Gamma(\alpha-1)} - \frac{\nu_2\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\mu_1\Gamma(\alpha-1)}\int_0^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-2}\xi(\pi)d\pi\\
&+\frac{1}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\alpha-1}(\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\Re_{\phi_1}(\pi))d\pi\\
&-\frac{\sigma^{\alpha-1}\mathbb{T}^{1-\alpha}}{\Gamma(\alpha)}\int_{\sigma_z}^{\mathbb{T}}(\mathbb{T}-\pi)^{\alpha-1}(\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\Re_{\phi_1}(\pi))d\pi\\
&-\frac{\sigma^{\alpha-1}}{\mathbb{T}}\sum_{j=1}^{z}\Bigg[\Big((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\Big)\sigma_j^{2-\alpha}(\mathcal{E}_j(\xi(\sigma_j))+\Re_{\phi_{1j}})+\Big(\mathbb{T}-\sigma_j\Big)\sigma_j^{2-\alpha}(\mathcal{E}_j^{*}(\xi(\sigma_j))+\Re_{\phi_{1j}})\\
&+\frac{\Big(\mathbb{T}-\sigma_j\Big)\sigma_j^{2-\alpha}}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-2}(\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\Re_{\phi_1}(\pi))d\pi\\
&+\frac{\Big((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\Big)\sigma_j^{2-\alpha}}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-1}(\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\Re_{\phi_1}(\pi))d\pi\Bigg]\\
&+\sum_{j=1}^{z}\Bigg[\Big((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}(\mathcal{E}_j(\xi(\sigma_j))+\Re_{\phi_{1j}})+\Big(\sigma-\sigma_j\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}(\mathcal{E}_j^{*}(\xi(\sigma_j))+\Re_{\phi_{1j}})\\
&+\frac{\Big(\sigma-\sigma_j\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-2}(\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\Re_{\phi_1}(\pi))d\pi\\
&+\frac{\Big((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\Big)\sigma^{\alpha-2}\sigma_j^{2-\alpha}}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}\Big(\sigma_j-\pi\Big)^{\alpha-1}(\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\Re_{\phi_1}(\pi))d\pi\Bigg],\\
&z=1,2,\ldots,p,\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}\zeta(\sigma) =&\frac{\sigma^{\beta-1}\mathfrak{v}_2}{\mu_2\mathbb{T}^{\beta-1}} - \frac{\sigma^{\beta-1}\mathfrak{v}_1}{\mathbb{T}\nu_3\Gamma(\beta-1)} + \frac{\sigma^{\beta-2}\mathfrak{v}_1}{\nu_3\Gamma(\beta-1)} - \frac{\nu_4\sigma^{\beta-1}\mathbb{T}^{1-\beta}}{\mu_2\Gamma(\beta-1)}\int_0^{\mathbb{T}}(\mathbb{T}-\pi)^{\beta-2}\zeta(\pi)d\pi\\
&+\frac{1}{\Gamma(\beta)}\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\beta-1}(\phi_2(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\mathfrak{Q}_{\phi_2}(\pi))d\pi\\
&-\frac{\sigma^{\beta-1}\mathbb{T}^{1-\beta}}{\Gamma(\beta)}\int_{\sigma_z}^{\mathbb{T}}(\mathbb{T}-\pi)^{\beta-1}(\phi_2(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\mathfrak{Q}_{\phi_2}(\pi))d\pi\\
&-\frac{\sigma^{\beta-1}}{\mathbb{T}}\sum_{k=1}^{z}\Bigg[\Big((\beta-1)-(\beta-2)\mathbb{T}\sigma_k^{-1}\Big)\sigma_k^{2-\beta}(\mathcal{E}_k(\zeta(\sigma_k))+\mathfrak{Q}_{\phi_{2k}})+(\mathbb{T}-\sigma_k)\,\sigma_k^{2-\beta}(\mathcal{E}_k^{*}(\zeta(\sigma_k))+\mathfrak{Q}_{\phi_{2k}})\\
&+\frac{(\mathbb{T}-\sigma_k)\,\sigma_k^{2-\beta}}{\Gamma(\beta-1)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-2}(\phi_2(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\mathfrak{Q}_{\phi_2}(\pi))d\pi\\
&+\frac{\Big((\beta-1)-(\beta-2)\mathbb{T}\sigma_k^{-1}\Big)\sigma_k^{2-\beta}}{\Gamma(\beta)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-1}(\phi_2(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\mathfrak{Q}_{\phi_2}(\pi))d\pi\Bigg]\\
&+\sum_{k=1}^{z}\Bigg[\Big((\beta-1)-(\beta-2)\sigma\sigma_k^{-1}\Big)\sigma_k^{2-\beta}(\mathcal{E}_k(\zeta(\sigma_k))+\mathfrak{Q}_{\phi_{2k}})+(\sigma-\sigma_k)\,\sigma^{\beta-2}\sigma_k^{2-\beta}(\mathcal{E}_k^{*}(\zeta(\sigma_k))+\mathfrak{Q}_{\phi_{2k}})\end{aligned}$$

$$+\frac{(\sigma-\sigma_k)\,\sigma^{\beta-2}\sigma_k^{2-\beta}}{\Gamma(\beta-1)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-2}\left(\phi_2(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\mathfrak{Q}_{\phi_2}(\pi)\right)d\pi$$
$$+\frac{\left((\beta-1)-(\beta-2)\sigma\sigma_k^{-1}\right)\sigma^{\beta-2}\sigma_k^{2-\beta}}{\Gamma(\beta)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-1}\left(\phi_2(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\mathfrak{Q}_{\phi_2}(\pi)\right)d\pi\Bigg],$$
$$z=1,2,\ldots,q. \tag{4.4}$$

We consider

$$\begin{aligned}
|\sigma^{2-\alpha}(\mathfrak{u}(\sigma)-\xi(\sigma))| \leq & \frac{\left|\sigma^{2-\alpha}\right|}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}\left|(\sigma-\pi)^{\alpha-1}\right||\phi_1(\pi,\mathcal{I}^{\alpha}\mathfrak{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))-\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))|d\pi\\
&+\frac{|\sigma||\mathbb{T}^{1-\alpha}|}{\Gamma(\alpha)}\int_{\sigma_z}^{\mathbb{T}}\left|(\mathbb{T}-\pi)^{\alpha-1}\right||\phi_1(\pi,\mathcal{I}^{\alpha}\mathfrak{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))-\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))|d\pi\\
&+\frac{|\nu_2||\sigma||\mathbb{T}^{1-\alpha}|}{|\mu_1|\Gamma(\alpha-1)}\int_0^{\mathbb{T}}\left|(\mathbb{T}-\pi)^{\alpha-2}\right||\mathfrak{u}(\pi)-\xi(\pi)|d\pi\\
&+\sum_{j=1}^{z}\left|\left((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\right)-\frac{\sigma}{\mathbb{T}}\left((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\right)\right|\\
&\times|\sigma_j^{2-\alpha}|\left|\mathcal{E}_j(\mathfrak{u}(\sigma_j))-\mathcal{E}_j(\xi(\sigma_j))\right|\\
&+\sum_{j=1}^{z}\left|\left(\sigma-\sigma_j\right)-\frac{\sigma}{\mathbb{T}}\left(\mathbb{T}-\sigma_j\right)\right||\sigma_j^{2-\alpha}|\left|\mathcal{E}_j^{*}(\mathfrak{u}(\sigma_j))-\mathcal{E}_j^{*}(\xi(\sigma_j))\right|\\
&+\frac{|\sigma|}{\mathbb{T}}\sum_{j=1}^{z}\Bigg[\frac{|\left(\mathbb{T}-\sigma_j\right)||\sigma_j^{2-\alpha}|}{\Gamma(\alpha-1)}\\
&\times\int_{\sigma_{j-1}}^{\sigma_j}\left|\left(\sigma_j-\pi\right)^{\alpha-2}\right||\phi_1(\pi,\mathcal{I}^{\alpha}\mathfrak{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))-\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))|d\pi\\
&+\frac{\left|\left((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\right)\right||\sigma_j^{2-\alpha}|}{\Gamma(\alpha)}\\
&\times\int_{\sigma_{j-1}}^{\sigma_j}\left|\left(\sigma_j-\pi\right)^{\alpha-1}\right||\phi_1(\pi,\mathcal{I}^{\alpha}\mathfrak{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))-\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))|d\pi\Bigg]\\
&+\sum_{j=1}^{z}\Bigg[\frac{\left|\left(\sigma-\sigma_j\right)\right||\sigma_j^{2-\alpha}|}{\Gamma(\alpha-1)}\\
&\times\int_{\sigma_{j-1}}^{\sigma_j}\left|\left(\sigma_j-\pi\right)^{\alpha-2}\right||\phi_1(\pi,\mathcal{I}^{\alpha}\mathfrak{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))-\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))|d\pi\\
&+\frac{\left|\left((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\right)\right||\sigma_j^{2-\alpha}|}{\Gamma(\alpha)}\\
&\times\int_{\sigma_{j-1}}^{\sigma_j}\left|\left(\sigma_j-\pi\right)^{\alpha-1}\right||\phi_1(\pi,\mathcal{I}^{\alpha}\mathfrak{u}(\pi),\mathcal{I}^{\beta}\mathrm{v}(\pi))-\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))|d\pi\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\left|\sigma^{2-\alpha}\right|}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}\left|(\sigma-\pi)^{\alpha-1}\right|\left|\Re_{\phi_1}(\pi)\right|d\pi+\frac{|\sigma|\left|\mathbb{T}^{1-\alpha}\right|}{\Gamma(\alpha)}\int_{\sigma_z}^{\mathbb{T}}\left|(\mathbb{T}-\pi)^{\alpha-1}\right|\left|\Re_{\phi_1}(\pi)\right|d\pi\\
&+\sum_{j=1}^{z}\left|\left((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\right)-\frac{\sigma}{\mathbb{T}}\left((\alpha-1)-(\alpha-2)\mathbb{T}\sigma_j^{-1}\right)\right||\sigma_j^{2-\alpha}|\left|\Re_{\phi_{1j}}\right|\\
&+\sum_{j=1}^{z}\left|\left(\sigma-\sigma_j\right)-\frac{\sigma}{\mathbb{T}}\left(\mathbb{T}-\sigma_j\right)\right||\sigma_j^{2-\alpha}|\left|\Re_{\phi_{1j}}\right|\\
&+\frac{\sigma}{\mathbb{T}}\sum_{j=1}^{z}\left[\frac{\left|\left(\mathbb{T}-\sigma_j\right)\right|\left|\sigma_j^{2-\alpha}\right|}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}\left|\left(\sigma_j-\pi\right)^{\alpha-2}\right|\left|\Re_{\phi_1}(\pi)\right|d\pi\right.\\
&\left.+\frac{\left|(\alpha-1)-(\alpha-2)\mathbb{T}\left|\sigma_j^{-1}\right|\right|\left|\sigma_j^{2-\alpha}\right|}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}\left|\left(\sigma_j-\pi\right)^{\alpha-1}\right|\left|\Re_{\phi_1}(\pi)\right|d\pi\right]\\
&+\sum_{j=1}^{z}\left[\frac{\left|\left(\sigma-\sigma_j\right)\right|\left|\sigma_j^{2-\alpha}\right|}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}\left|\left(\sigma_j-\pi\right)^{\alpha-2}\right|\left|\Re_{\phi_1}(\pi)\right|d\pi\right.\\
&\left.+\frac{\left|\left((\alpha-1)-(\alpha-2)\sigma\sigma_j^{-1}\right)\right|\left|\sigma_j^{2-\alpha}\right|}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}\left|\left(\sigma_j-\pi\right)^{\alpha-1}\right|\left|\Re_{\phi_1}(\pi)\right|d\pi\right].
\end{aligned}
$$

As in Theorem 3.3, we get

$$
\begin{aligned}
\|\mathfrak{u}-\xi\|_{\vartheta_1}\leq&\left(\Delta_1+\frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)\|\mathfrak{u}-\xi\|_{\vartheta_1}+\left(\frac{\Delta_2\mathcal{L}^{*}_{\phi_1}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\|\mathfrak{v}-\zeta\|_{\vartheta_1}\\
&+\left(\Delta_2+z(\alpha-1)|\sigma_z^{2-\alpha}|\left|1-\frac{\sigma}{\mathbb{T}}\right|+z|\sigma_z^{3-\alpha}|\left|\frac{\sigma}{\mathbb{T}}-1\right|+\frac{|\nu_2||\sigma|}{|\mu_1|\Gamma(\alpha)}\right)\kappa_{\alpha},\\
&z=1,2,\ldots,p,
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
\|\mathfrak{v}-\zeta\|_{\vartheta_2}\leq&\left(\frac{\Delta_4\mathcal{L}_{\phi_2}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)\|\mathfrak{u}-\xi\|_{\vartheta_2}+\left(\Delta_3+\frac{\Delta_4\mathcal{L}^{*}_{\phi_2}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\|\mathfrak{v}-\zeta\|_{\vartheta_2}\\
&+\left(\Delta_4+z(\beta-1)|\sigma_z^{2-\beta}|\left|1-\frac{\sigma}{\mathbb{T}}\right|+z|\sigma_z^{3-\beta}|\left|\frac{\sigma}{\mathbb{T}}-1\right|+\frac{|\nu_4||\sigma|}{|\mu_2|\Gamma(\beta)}\right)\kappa_{\beta},\\
&z=1,2,\ldots,q.
\end{aligned}
\tag{4.6}
$$

From (4.5) and (4.6), we have

$$
\begin{aligned}
&\|\mathfrak{u}-\xi\|_{\vartheta_1}-\frac{\left(\frac{\Delta_2\mathcal{L}^{*}_{\phi_1}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)}{1-\left(\Delta_1+\frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)}\|\mathfrak{v}-\zeta\|_{\vartheta_1}\\
&\qquad\leq\frac{\left(\Delta_2+z(\alpha-1)|\sigma_z^{2-\alpha}|\left|1-\frac{\sigma}{\mathbb{T}}\right|+z|\sigma_z^{3-\alpha}|\left|\frac{\sigma}{\mathbb{T}}-1\right|+\frac{|\nu_2||\sigma|}{|\mu_1|\Gamma(\alpha)}\right)}{1-\left(\Delta_1+\frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)}\kappa_{\alpha}
\end{aligned}
$$

and

$$\|\mathrm{v}-\zeta\|_{\vartheta_2}-\frac{\left(\frac{\Delta_4\mathcal{L}_{\phi_2}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)}{1-\left(\Delta_3+\frac{\Delta_4\mathcal{L}^{*}_{\phi_2}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)}\|\mathrm{u}-\xi\|_{\vartheta_2}$$
$$\leq\frac{\left(\Delta_4+z(\beta-1)|\sigma_z^{2-\beta}|\left|1-\frac{\sigma}{\mathbb{T}}\right|+z|\sigma_z^{3-\beta}|\left|\frac{\sigma}{\mathbb{T}}-1\right|+\frac{|\nu_4||\sigma|}{|\mu_2|\Gamma(\beta)}\right)}{1-\left(\Delta_3+\frac{\Delta_4\mathcal{L}^{*}_{\phi_2}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)}\kappa_\beta$$

respectively. Let

$$\mathcal{P}_1=\frac{\left(\frac{\Delta_2\mathcal{L}^{*}_{\phi_1}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)}{1-\left(\Delta_1+\frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)},\qquad \mathcal{P}_2=\frac{\left(\Delta_2+z(\alpha-1)|\sigma_z^{2-\alpha}|\left|1-\frac{\sigma}{\mathbb{T}}\right|+z|\sigma_z^{3-\alpha}|\left|\frac{\sigma}{\mathbb{T}}-1\right|+\frac{|\nu_2||\sigma|}{|\mu_1|\Gamma(\alpha)}\right)}{1-\left(\Delta_1+\frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)},$$
$$\mathcal{P}_3=\frac{\left(\frac{\Delta_4\mathcal{L}_{\phi_2}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)}{1-\left(\Delta_3+\frac{\Delta_4\mathcal{L}^{*}_{\phi_2}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)},\ \textit{and}\ \mathcal{P}_4=\frac{\left(\Delta_4+z(\beta-1)|\sigma_z^{2-\beta}|\left|1-\frac{\sigma}{\mathbb{T}}\right|+z|\sigma_z^{3-\beta}|\left|\frac{\sigma}{\mathbb{T}}-1\right|+\frac{|\nu_4||\sigma|}{|\mu_2|\Gamma(\beta)}\right)}{1-\left(\Delta_3+\frac{\Delta_4\mathcal{L}^{*}_{\phi_2}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)}.$$

Then the last two inequalities can be written in a matrix form as follows:

$$\begin{bmatrix}1 & -\mathcal{P}_1\\ -\mathcal{P}_3 & 1\end{bmatrix}\begin{bmatrix}\|\mathrm{u}-\xi\|_{\vartheta_1}\\ \|\mathrm{v}-\zeta\|_{\vartheta_2}\end{bmatrix}\leq\begin{bmatrix}\mathcal{P}_2\kappa_\alpha\\ \mathcal{P}_4\kappa_\beta\end{bmatrix}$$

$$\begin{bmatrix}\|\mathrm{u}-\xi\|_{\vartheta_1}\\ \|\mathrm{v}-\zeta\|_{\vartheta_2}\end{bmatrix}\leq\begin{bmatrix}\frac{1}{\mathcal{F}} & \frac{\mathcal{P}_1}{\mathcal{F}}\\ \frac{\mathcal{P}_3}{\mathcal{F}} & \frac{1}{\mathcal{F}}\end{bmatrix}\begin{bmatrix}\mathcal{P}_2\kappa_\alpha\\ \mathcal{P}_4\kappa_\beta\end{bmatrix},\tag{4.7}$$

where

$$\mathcal{F}=1-\frac{\left(\frac{\Delta_2\mathcal{L}^{*}_{\phi_1}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\left(\frac{\Delta_4\mathcal{L}_{\phi_2}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)}{\left[1-\left(\Delta_1+\frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)\right]\left[1-\left(\Delta_3+\frac{\Delta_4\mathcal{L}^{*}_{\phi_2}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\right]}>0.$$

From system (4.7) we have

$$\|\mathrm{u}-\xi\|_{\vartheta_1}\leq\frac{\mathcal{P}_2\kappa_\alpha}{\mathcal{F}}+\frac{\mathcal{P}_1\mathcal{P}_4\kappa_\beta}{\mathcal{F}},$$
$$\|\mathrm{v}-\zeta\|_{\vartheta_2}\leq\frac{\mathcal{P}_2\mathcal{P}_3\kappa_\alpha}{\mathcal{F}}+\frac{\mathcal{P}_4\kappa_\beta}{\mathcal{F}},$$

which implies that

$$\|\mathrm{u}-\xi\|_{\vartheta_1}+\|\mathrm{v}-\zeta\|_{\vartheta_2}\leq\frac{\mathcal{P}_2\kappa_\alpha}{\mathcal{F}}+\frac{\mathcal{P}_1\mathcal{P}_4\kappa_\beta}{\mathcal{F}}+\frac{\mathcal{P}_2\mathcal{P}_3\kappa_\alpha}{\mathcal{F}}+\frac{\mathcal{P}_4\kappa_\beta}{\mathcal{F}}.$$

If $\kappa = \max\{\kappa_\alpha, \kappa_\beta\}$ and $\mathcal{N}_{\alpha,\beta} = \frac{\mathcal{P}_2}{\mathcal{F}} + \frac{\mathcal{P}_1\mathcal{P}_4}{\mathcal{F}} + \frac{\mathcal{P}_2\mathcal{P}_3}{\mathcal{F}} + \frac{\mathcal{P}_4}{\mathcal{F}}$, then

$$\|(\mathrm{u}, \mathrm{v}) - (\xi, \zeta)\|_\vartheta \leq \mathcal{N}_{\alpha,\beta}\kappa.$$

Thus system (1.1) is $\mathcal{HU}$ stable. Also, if

$$\|(\mathrm{u}, \mathrm{v}) - (\xi, \zeta)\|_\vartheta \leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'(\kappa),$$

with $\mathcal{N}'(0) = 0$, then the given system (1.1) is generalized $\mathcal{HU}$ stable. □

For the next result, we assume the following:

($\mathbf{H}_5$) Let there exists two nondecreasing functions $w_\alpha, w_\beta \in C(\omega, \mathbb{R}^+)$ such that

$$\mathcal{I}^\alpha w_\alpha(\sigma) \leq \mathcal{L}_\alpha w_\alpha(\sigma) \quad \textit{and} \quad \mathcal{I}^\beta w_\beta(\sigma) \leq \mathcal{L}_\beta w_\beta(\sigma), \quad \textit{where} \quad \mathcal{L}_\alpha, \mathcal{L}_\beta > 0. \tag{4.8}$$

Theorem 4.2. *If assumptions* ($\boldsymbol{H}_3$)–($\boldsymbol{H}_5$) *and inequality* (3.19) *are satisfied and*

$$\mathcal{F} = 1 - \frac{\left(\frac{\Delta_2\mathcal{L}^*_{\phi_1}|\sigma^\beta|}{\Gamma(\beta+1)}\right)\left(\frac{\Delta_4\mathcal{L}_{\phi_2}|\sigma^\alpha|}{\Gamma(\alpha+1)}\right)}{\left[1 - \left(\Delta_1 + \frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^\alpha|}{\Gamma(\alpha+1)}\right)\right]\left[1 - \left(\Delta_3 + \frac{\Delta_4\mathcal{L}^*_{\phi_2}|\sigma^\beta|}{\Gamma(\beta+1)}\right)\right]} > 0,$$

then the unique solution of the given system (1.1) *is* $\mathcal{HUR}$ *stable and accordingly generalized* $\mathcal{HUR}$ *stable.*

Proof. With the help of Definitions 2.5 and 2.6, we can achieve our result doing the same steps as in Theorem 4.1. □

5. Example

Here we present a specific example, as follows.

Example 5.1. *Let*

$$\begin{cases} \begin{cases} \mathcal{D}^{\frac{6}{5}}u(\sigma) - \dfrac{2 + \mathcal{I}^{\frac{6}{5}}u(\sigma) + \mathcal{I}^{\frac{5}{4}}v(\sigma)}{80e^{\sigma+90}(1 + \mathcal{I}^{\frac{6}{5}}u(\sigma) + \mathcal{I}^{\frac{5}{4}}v(\sigma))} = 0, \quad \sigma \neq \dfrac{3}{2}, \\ \Delta u\left(\dfrac{3}{2}\right) = \mathcal{E}_1\left(u\left(\dfrac{3}{2}\right)\right) = \dfrac{|u(\frac{3}{2})|}{70 + |u(\frac{3}{2})|}, \\ \Delta u'\left(\dfrac{3}{2}\right) = \mathcal{E}_1^*\left(u\left(\dfrac{3}{2}\right)\right) = \dfrac{|u(\frac{3}{2})|}{70 + |u(\frac{3}{2})|}, \\ \mathcal{D}^{-\frac{4}{5}}u(\sigma)|_{\sigma=0} = u_1, \qquad -50u(\sigma)|_{\sigma=e} + \dfrac{1}{85}\mathcal{I}^{\frac{1}{5}}u(\sigma)|_{\sigma=e} = u_2, \end{cases} \\ \begin{cases} \mathcal{D}^{\frac{5}{4}}v(\sigma) - \dfrac{\sigma\cos(u(\sigma)) - v(\sigma)\sin(\sigma)}{95} - \dfrac{u(\sigma)}{95 + u(\sigma)} = 0, \quad \sigma \neq \dfrac{3}{2}, \\ \Delta v\left(\dfrac{3}{2}\right) = \mathcal{E}_1\left(v\left(\dfrac{3}{2}\right)\right) = \dfrac{|v(\frac{3}{2})|}{70 + |v(\frac{3}{2})|}, \\ \Delta v'\left(\dfrac{3}{2}\right) = \mathcal{E}_1^*\left(v\left(\dfrac{3}{2}\right)\right) = \dfrac{|v(\frac{3}{2})|}{70 + |v(\frac{3}{2})|}, \\ \mathcal{D}^{-\frac{3}{4}}v(\sigma)|_{\sigma=0} = v_1, \qquad -50v(\sigma)|_{\sigma=e} + \dfrac{1}{85}\mathcal{I}^{\frac{1}{4}}v(\sigma)|_{\sigma=e} = v_2. \end{cases} \end{cases} \tag{5.1}$$

From system (5.1)*, we see that* $\alpha = \frac{6}{5}$, $\beta = \frac{5}{4}$, $\mu_1 = \mu_2 = -50$, $\nu_1 = \nu_3 = 1$, $\nu_2 = \nu_4 = \frac{1}{85}$, $\mathbb{T} = e$, $\sigma_1 = \frac{3}{2}$, *and* $u_1, u_2, v_1, v_2 \in \mathbb{R}$.

Set

$$\phi_1(\sigma, u, v) = \frac{2 + \mathcal{I}^{\frac{6}{5}}u(\sigma) + \mathcal{I}^{\frac{5}{4}}v(\sigma)}{80e^{\sigma+90}(1 + \mathcal{I}^{\frac{6}{5}}u(\sigma) + \mathcal{I}^{\frac{5}{4}}v(\sigma))},$$
$$\phi_2(\sigma, u, v) = \frac{\sigma\cos(u(\sigma)) - v(\sigma)\sin(\sigma)}{95} - \frac{u(\sigma)}{95 + u(\sigma)}.$$

Now, for all $u, u^*, v, v^* \in \mathbb{R}$*, and* $\sigma \in [0, e]$*, we obtain*

$$|\phi_1(\sigma, u, v) - \phi_1(\sigma, u^*, v^*)| = \frac{1}{80e^{90}}|u - u^*| + \frac{1}{80e^{90}}|v - v^*|$$

and

$$|\phi_2(\sigma, u, v) - \phi_1(\sigma, u^*, v^*)| = \frac{1}{95}|u - u^*| + \frac{1}{95}|v - v^*|.$$

These satisfy condition $(\boldsymbol{H}_3)$ *with* $\mathcal{L}_{\phi_1} = \mathcal{L}^*_{\phi_1} = \frac{1}{80e^{90}}$, $\mathcal{L}_{\phi_2} = \mathcal{L}^*_{\phi_2} = \frac{1}{95}$.

Set

$$\mathcal{E}_1\left(u\left(\frac{3}{2}\right)\right) = \frac{|u(\frac{3}{2})|}{70 + |u(\frac{3}{2})|}, \qquad \mathcal{E}_1^*\left(u\left(\frac{3}{2}\right)\right) = \frac{|u(\frac{3}{2})|}{70 + |u(\frac{3}{2})|},$$
$$\mathcal{E}_1\left(v\left(\frac{3}{2}\right)\right) = \frac{|v(\frac{3}{2})|}{70 + |v(\frac{3}{2})|} \quad \text{and} \quad \mathcal{E}_1^*\left(v\left(\frac{3}{2}\right)\right) = \frac{|v(\frac{3}{2})|}{70 + |v(\frac{3}{2})|}.$$

Then we have

$$\left|\mathcal{E}_1\left(u\left(\frac{3}{2}\right)\right)-\mathcal{E}_1\left(u^*\left(\frac{3}{2}\right)\right)\right|=\frac{1}{70}|u-u^*|, \qquad \left|\mathcal{E}_1^*\left(u\left(\frac{3}{2}\right)\right)-\mathcal{E}_1^*\left(u^*\left(\frac{3}{2}\right)\right)\right|=\frac{1}{70}|u-u^*|,$$
$$\left|\mathcal{E}_1\left(v\left(\frac{3}{2}\right)\right)-\mathcal{E}_1\left(v^*\left(\frac{3}{2}\right)\right)\right|=\frac{1}{70}|v-v^*| \quad \text{and} \quad \left|\mathcal{E}_1^*\left(v\left(\frac{3}{2}\right)\right)-\mathcal{E}_1^*\left(v^*\left(\frac{3}{2}\right)\right)\right|=\frac{1}{70}|v-v^*|.$$

These satisfy condition $(\boldsymbol{H}_4)$ *with* $\mathcal{L}_{\mathcal{E}}=\mathcal{L}_{\mathcal{E}}^*=\mathcal{L}_{\mathcal{E}^*}=\mathcal{L}_{\mathcal{E}^*}^*=\frac{1}{70}$.
From Theorem 3.3, we use the inequality and get

$$\Delta_1+\Delta_3+\frac{(\Delta_2\mathcal{L}_{\phi_1}+\Delta_4\mathcal{L}_{\phi_2})|\sigma^{\alpha}|}{\Gamma(\alpha+1)}+\frac{(\Delta_2\mathcal{L}_{\phi_1}^*+\Delta_4\mathcal{L}_{\phi_2}^*)|\sigma^{\beta}|}{\Gamma(\beta+1)}\approx 0.976847<1,$$

hence (5.1) *has a unique solution, so* (5.1) *has a solution* $(\mathfrak{u},\mathfrak{v})\in\vartheta$. *The solution of* (5.1) *is given by*

$$
u(\sigma)=\begin{cases}
\dfrac{\sigma^{\frac{1}{5}}u_2}{-50e^{\frac{1}{5}}}-\dfrac{\sigma^{\frac{1}{5}}u_1}{e\Gamma(\frac{1}{5})}+\dfrac{\sigma^{-\frac{4}{5}}u_1}{\Gamma(\frac{1}{5})}+\dfrac{1}{\Gamma(\frac{6}{5})}\displaystyle\int_0^{\sigma}(\sigma-\pi)^{\frac{1}{5}}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
-\dfrac{\sigma^{\frac{1}{5}}e^{-\frac{1}{5}}}{\Gamma(\frac{6}{5})}\displaystyle\int_{\frac{3}{2}}^{e}(e-\pi)^{\frac{1}{5}}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi+\dfrac{\frac{1}{85}\sigma^{\frac{1}{5}}e^{-\frac{1}{5}}}{50\Gamma(\frac{1}{5})}\displaystyle\int_0^{e}(e-\pi)^{-\frac{4}{5}}u(\pi)d\pi\\
-\dfrac{\sigma^{\frac{1}{5}}}{e}\Bigg[\left(\left(\dfrac{1}{5}\right)+e\left(\dfrac{4}{5}\right)\left(\dfrac{3}{2}\right)^{-1}\right)\left(\dfrac{3}{2}\right)^{\frac{4}{5}}\mathcal{E}_1\left(u\left(\dfrac{3}{2}\right)\right)+\left(e-\dfrac{3}{2}\right)\left(\dfrac{3}{2}\right)^{\frac{4}{5}}\mathcal{E}_1^{*}\left(u\left(\dfrac{3}{2}\right)\right)\\
+\dfrac{\left(e-\frac{3}{2}\right)\left(\frac{3}{2}\right)^{\frac{4}{5}}}{\Gamma(\frac{1}{5})}\displaystyle\int_0^{\frac{3}{2}}\left(\dfrac{3}{2}-\pi\right)^{-\frac{4}{5}}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
+\dfrac{\left((\frac{1}{5})+e(\frac{4}{5})\left(\frac{3}{2}\right)^{-1}\right)\left(\frac{3}{2}\right)^{\frac{4}{5}}}{\Gamma(\frac{6}{5})}\displaystyle\int_0^{\frac{3}{2}}\left(\dfrac{3}{2}-\pi\right)^{\frac{1}{5}}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg],\\
\qquad\sigma\in\left[0,\dfrac{3}{2}\right],\\
\dfrac{\sigma^{\frac{1}{5}}u_2}{-50e^{\frac{1}{5}}}-\dfrac{\sigma^{\frac{1}{5}}u_1}{e\Gamma(\frac{1}{5})}+\dfrac{\sigma^{-\frac{4}{5}}u_1}{\Gamma(\frac{1}{5})}+\dfrac{1}{\Gamma(\frac{6}{5})}\displaystyle\int_{\frac{3}{2}}^{\sigma}(\sigma-\pi)^{\frac{1}{5}}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
-\dfrac{\sigma^{\frac{1}{5}}e^{-\frac{1}{5}}}{\Gamma(\frac{6}{5})}\displaystyle\int_{\frac{3}{2}}^{e}(e-\pi)^{\frac{1}{5}}\dfrac{2+\mathcal{I}^{\frac{6}{5}}u(\pi)+\mathcal{I}^{\frac{5}{4}}v(\pi)}{80e^{\pi+90}(1+\mathcal{I}^{\frac{6}{5}}u(\pi)+\mathcal{I}^{\frac{5}{4}}v(\pi))}d\pi+\dfrac{\frac{1}{85}\sigma^{\frac{1}{5}}e^{-\frac{1}{5}}}{50\Gamma(\frac{1}{5})}\displaystyle\int_0^{e}(e-\pi)^{-\frac{4}{5}}u(\pi)d\pi\\
-\dfrac{\sigma^{\frac{1}{5}}}{e}\Bigg[\left(\left(\dfrac{1}{5}\right)+e\left(\dfrac{4}{5}\right)\left(\dfrac{3}{2}\right)^{-1}\right)\left(\dfrac{3}{2}\right)^{\frac{4}{5}}\mathcal{E}_1\left(u\left(\dfrac{3}{2}\right)\right)+\left(e-\dfrac{3}{2}\right)\left(\dfrac{3}{2}\right)^{\frac{4}{5}}\mathcal{E}_1^{*}\left(u\left(\dfrac{3}{2}\right)\right)\\
+\dfrac{\left(e-\frac{3}{2}\right)\left(\frac{3}{2}\right)^{\frac{4}{5}}}{\Gamma(\frac{1}{5})}\displaystyle\int_0^{\frac{3}{2}}\left(\dfrac{3}{2}-\pi\right)^{-\frac{4}{5}}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
+\dfrac{\left((\frac{1}{5})+e(\frac{4}{5})\left(\frac{3}{2}\right)^{-1}\right)\left(\frac{3}{2}\right)^{\frac{4}{5}}}{\Gamma(\frac{6}{5})}\displaystyle\int_0^{\frac{3}{2}}\left(\dfrac{3}{2}-\pi\right)^{\frac{1}{5}}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg]\\
+\Bigg[\left(\left(\dfrac{1}{5}\right)+\sigma\left(\dfrac{4}{5}\right)\left(\dfrac{3}{2}\right)^{-1}\right)\left(\dfrac{3}{2}\right)^{\frac{4}{5}}\sigma^{-\frac{4}{5}}\mathcal{E}_1\left(u\left(\dfrac{3}{2}\right)\right)+\left(\sigma-\dfrac{3}{2}\right)\left(\dfrac{3}{2}\right)^{\frac{4}{5}}\sigma^{-\frac{4}{5}}\mathcal{E}_1^{*}\left(u\left(\dfrac{3}{2}\right)\right)\\
+\dfrac{\left(\sigma-\frac{3}{2}\right)\left(\frac{3}{2}\right)^{\frac{4}{5}}\sigma^{-\frac{4}{5}}}{\Gamma(\frac{1}{5})}\displaystyle\int_0^{\frac{3}{2}}\left(\dfrac{3}{2}-\pi\right)^{-\frac{4}{5}}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
+\dfrac{\left((\frac{1}{5})+\sigma(\frac{4}{5})\left(\frac{3}{2}\right)^{-1}\right)\left(\frac{3}{2}\right)^{\frac{4}{5}}\sigma^{-\frac{4}{5}}}{\Gamma(\frac{6}{5})}\displaystyle\int_0^{\frac{3}{2}}\left(\dfrac{3}{2}-\pi\right)^{\frac{1}{5}}\phi_1(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg],\\
\qquad\sigma\in\left(\dfrac{3}{2},e\right]
\end{cases}
$$

and

$$
v(\sigma)=\begin{cases}
\begin{cases}
\frac{\sigma^{\frac{1}{4}}v_2}{-50e^{\frac{1}{4}}}-\frac{\sigma^{\frac{1}{4}}v_1}{e\Gamma(\frac{1}{4})}+\frac{\sigma^{-\frac{3}{4}}v_1}{\Gamma(\frac{1}{4})}+\frac{1}{\Gamma(\frac{5}{4})}\int_0^{\sigma}(\sigma-\pi)^{\frac{1}{4}}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
-\frac{\sigma^{\frac{1}{4}}e^{-\frac{1}{4}}}{\Gamma(\frac{5}{4})}\int_{\frac{3}{2}}^{e}(e-\pi)^{\frac{1}{4}}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi+\frac{\frac{1}{85}\sigma^{\frac{1}{4}}e^{-\frac{1}{4}}}{50\Gamma(\frac{1}{4})}\int_0^{e}(e-\pi)^{-\frac{3}{4}}v(\pi)d\pi\\
-\frac{\sigma^{\frac{1}{4}}}{e}\Bigg[\left(\left(\frac{1}{4}\right)+e\left(\frac{3}{4}\right)\left(\frac{3}{2}\right)^{-1}\right)\left(\frac{3}{2}\right)^{\frac{3}{4}}\mathcal{E}_1\left(v\left(\frac{3}{2}\right)\right)+\left(e-\frac{3}{2}\right)\left(\frac{3}{2}\right)^{\frac{3}{4}}\mathcal{E}_1^{*}\left(v\left(\frac{3}{2}\right)\right)\\
+\frac{\left(e-\frac{3}{2}\right)\left(\frac{3}{2}\right)^{\frac{3}{4}}}{\Gamma(\frac{1}{4})}\int_0^{\frac{3}{2}}\left(\frac{3}{2}-\pi\right)^{-\frac{3}{4}}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
+\frac{\left(\left(\frac{1}{4}\right)+e\left(\frac{3}{4}\right)\left(\frac{3}{2}\right)^{-1}\right)\left(\frac{3}{2}\right)^{\frac{3}{4}}}{\Gamma(\frac{5}{4})}\int_0^{\frac{3}{2}}\left(\frac{3}{2}-\pi\right)^{\frac{1}{4}}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg],\\
\sigma\in\left[0,\frac{3}{2}\right],
\end{cases}\\
\begin{cases}
\frac{\sigma^{\frac{1}{4}}v_2}{-50e^{\frac{1}{4}}}-\frac{\sigma^{\frac{1}{4}}v_1}{e\Gamma(\frac{1}{4})}+\frac{\sigma^{-\frac{3}{4}}v_1}{\Gamma(\frac{1}{4})}+\frac{1}{\Gamma(\frac{5}{4})}\int_{\frac{3}{2}}^{\sigma}(\sigma-\pi)^{\frac{1}{4}}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
-\frac{\sigma^{\frac{1}{4}}e^{-\frac{1}{4}}}{\Gamma(\frac{5}{4})}\int_{\frac{3}{2}}^{e}(e-\pi)^{\frac{1}{4}}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi-\frac{\frac{1}{85}\sigma^{\frac{1}{4}}e^{-\frac{1}{4}}}{-50\Gamma(\frac{1}{4})}\int_0^{e}(e-\pi)^{-\frac{3}{4}}v(\pi)d\pi\\
-\frac{\sigma^{\frac{1}{4}}}{e}\Bigg[\left(\left(\frac{1}{4}\right)+e\left(\frac{3}{4}\right)\left(\frac{3}{2}\right)^{-1}\right)\left(\frac{3}{2}\right)^{\frac{3}{4}}\mathcal{E}_1\left(v\left(\frac{3}{2}\right)\right)+\left(e-\frac{3}{2}\right)\left(\frac{3}{2}\right)^{\frac{3}{4}}\mathcal{E}_1^{*}\left(v\left(\frac{3}{2}\right)\right)\\
+\frac{\left(e-\frac{3}{2}\right)\left(\frac{3}{2}\right)^{\frac{3}{4}}}{\Gamma(\frac{1}{4})}\int_0^{\frac{3}{2}}\left(\frac{3}{2}-\pi\right)^{-\frac{3}{4}}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
+\frac{\left(\left(\frac{1}{4}\right)+e\left(\frac{3}{4}\right)\left(\frac{3}{2}\right)^{-1}\right)\left(\frac{3}{2}\right)^{\frac{3}{4}}}{\Gamma(\frac{5}{4})}\int_0^{\frac{3}{2}}\left(\frac{3}{2}-\pi\right)^{\frac{1}{4}}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg]\\
+\Bigg[\left(\left(\frac{1}{4}\right)+\sigma\left(\frac{3}{4}\right)\left(\frac{3}{2}\right)^{-1}\right)\left(\frac{3}{2}\right)^{\frac{3}{4}}\sigma^{-\frac{3}{4}}\mathcal{E}_1\left(v\left(\frac{3}{2}\right)\right)+\left(\sigma-\frac{3}{2}\right)\sigma^{-\frac{3}{4}}\left(\frac{3}{2}\right)^{\frac{3}{4}}\mathcal{E}_1^{*}\left(v\left(\frac{3}{2}\right)\right)\\
+\frac{\left(\sigma-\frac{3}{2}\right)\left(\frac{3}{2}\right)^{\frac{3}{4}}\sigma^{-\frac{3}{4}}}{\Gamma(\frac{1}{4})}\int_0^{\frac{3}{2}}\left(\frac{3}{2}-\pi\right)^{-\frac{3}{4}}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\\
+\frac{\left(\left(\frac{1}{4}\right)+\sigma\left(\frac{3}{4}\right)\left(\frac{3}{2}\right)^{-1}\right)\left(\frac{3}{2}\right)^{\frac{3}{4}}\sigma^{-\frac{3}{4}}}{\Gamma(\frac{5}{4})}\int_0^{\frac{3}{2}}\left(\frac{3}{2}-\pi\right)^{\frac{1}{4}}\phi_2(\pi,\mathcal{I}^{\alpha}u(\pi),\mathcal{I}^{\beta}v(\pi))d\pi\Bigg],\\
\sigma\in\left(\frac{3}{2},e\right].
\end{cases}
\end{cases}
$$

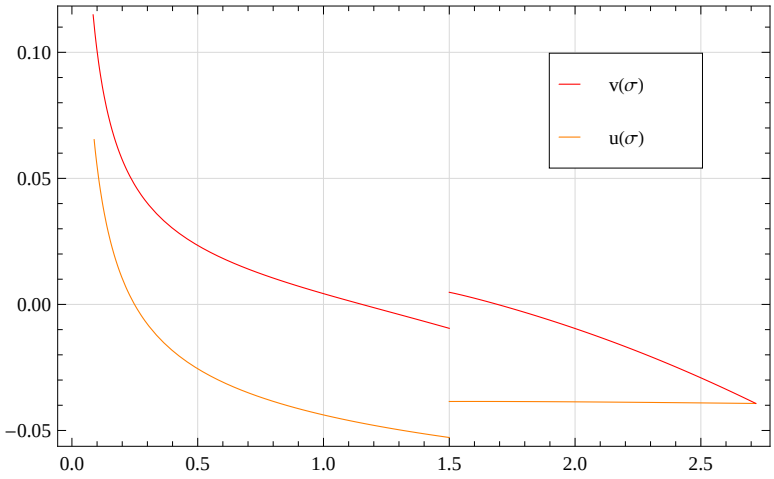

Figure 1. The graph of the solution in case (i).

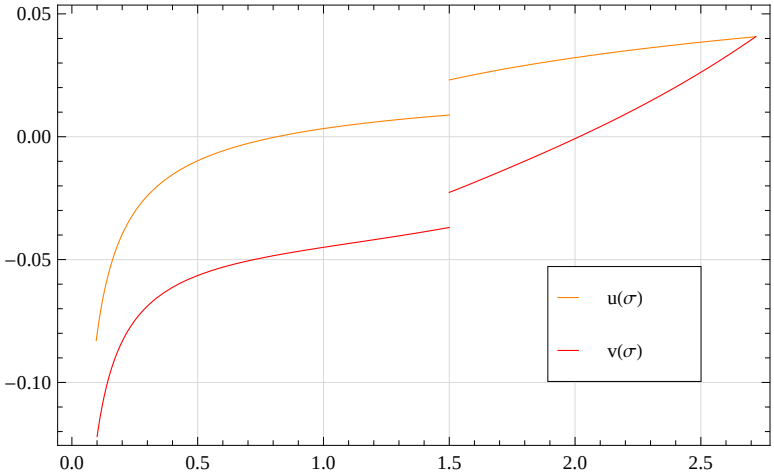

Figure 2. The graph of the solution in case (ii).

(i) If we take $\phi_1(\sigma, \mathcal{I}^\alpha u(\sigma), \mathcal{I}^\beta v(\sigma)) = \dfrac{1}{80e^{\sigma+90}}$, $\phi_2(\sigma, \mathcal{I}^\alpha u(\sigma), \mathcal{I}^\beta v(\sigma)) = \dfrac{\sigma\cos(\sigma) - \sin(\sigma)}{95} - \dfrac{1}{95}$, $\mathcal{E}_1\left(u\left(\frac{3}{2}\right)\right) = \mathcal{E}_1^*\left(u\left(\frac{3}{2}\right)\right) = \mathcal{E}_1\left(v\left(\frac{3}{2}\right)\right) = \mathcal{E}_1^*\left(v\left(\frac{3}{2}\right)\right) = \frac{1}{70}$, *and* $u(\sigma) = v(\sigma) = \sigma$ *then with the constant values* $u_1 = v_1 = \frac{1}{15}$, $u_2 = v_2 = 2$, *the graph of the solution is shown in Figure 1.*

(ii) If we take $\phi_1(\sigma, \mathcal{I}^\alpha u(\sigma), \mathcal{I}^\beta v(\sigma)) = \dfrac{\sigma+1}{80e^{\sigma+90}}$, $\phi_2(\sigma, \mathcal{I}^\alpha u(\sigma), \mathcal{I}^\beta v(\sigma)) = \dfrac{\sigma^2+1}{95} - \dfrac{\sigma}{95}$, $\mathcal{E}_1\left(u\left(\frac{3}{2}\right)\right) = \mathcal{E}_1^*\left(u\left(\frac{3}{2}\right)\right) = \mathcal{E}_1\left(v\left(\frac{3}{2}\right)\right) = \mathcal{E}_1^*\left(v\left(\frac{3}{2}\right)\right) = \frac{1}{70}$, *and* $u(\sigma) = v(\sigma) = \sigma$ *then with the constant values* $u_1 = v_1 = -\frac{1}{15}$, $u_2 = v_2 = -2$, *the graph of the solution is shown in Figure 2.*

From Theorem 4.1, we use the inequality and get

$$\mathcal{F} = 1 - \frac{\left(\frac{\Delta_2\mathcal{L}_{\phi_1}^*|\sigma^\beta|}{\Gamma(\beta+1)}\right)\left(\frac{\Delta_4\mathcal{L}_{\phi_2}|\sigma^\alpha|}{\Gamma(\alpha+1)}\right)}{\left[1 - \left(\Delta_1 + \frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^\alpha|}{\Gamma(\alpha+1)}\right)\right]\left[1 - \left(\Delta_3 + \frac{\Delta_4\mathcal{L}_{\phi_2}^*|\sigma^\beta|}{\Gamma(\beta+1)}\right)\right]} \approx 1 > 0,$$

thus, the given system (5.1) *is* $\mathcal{HU}$ *stable and also generalized* $\mathcal{HU}$ *stable. Likewise, we can justify the condition of Theorems 3.2 and 4.2.*

6. Conclusion

In this article, we used the Kransnoselskii's fixed point theorem and acquired the necessary cases for the existence and uniqueness of solution for the given fractional integro-differential Eqs (1.1). Furthermore, under specific assumptions and conditions, we proved different kinds of Ulam's stability of system (1.1). The concept of Ulam's stability is very important because it gives a relationship between approximate and exact solutions, so our results may be very helpful in approximation theory and numerical analysis. The mentioned stability is rarely investigated for impulsive fractional integro-differential equations. Finally, we illustrated the main results by giving a suitable example.

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Conflict of interest

The authors declare that they have no conflict of interest.

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