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Research article

Viscosity-type method for solving pseudomonotone equilibrium problems in a real Hilbert space with applications

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Abstract: The aim of this article is to introduce a new algorithm by integrating a viscosity-type method with the subgradient extragradient algorithm to solve the equilibrium problems involving pseudomonotone and Lipschitz-type continuous bifunction in a real Hilbert space. A strong convergence theorem is proved by the use of certain mild conditions on the bifunction as well as some restrictions on the iterative control parameters. Applications of the main results are also presented to address variational inequalities and fixed-point problems. The computational behaviour of the proposed algorithm on various test problems is described in comparison to other existing algorithms.

Keywords: strong convergence; equilibrium problem; Lipschitz-like conditions; fixed point problems; variational inequality problem **Mathematics Subject Classification:** 47H05, 47H10

1. Introduction

Assume that *C* is a closed and convex subset of a real Hilbert space \mathcal{H} with the inner product and the induced norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Moreover, \mathcal{R} be a set of real numbers

during whole article. Let $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ be a bifunction and satisfy f(v, v) = 0 for all $v \in C$, the *equilibrium problem* (EP) [6,11] for a bifunction f on C is defined in the following way:

Find
$$u^* \in C$$
 such as $f(u^*, v) \ge 0$, $\forall v \in C$. (EP)

Moreover, $S_{EP(f,C)}$ denotes the solution set of an equilibrium problem over the set *C* and u^* is an arbitrary element of $S_{EP(f,C)}$. A *metric projection* $P_C(u)$ of $u \in \mathcal{H}$ onto a closed and convex subset *C* of \mathcal{H} is defined by $P_C(u) = \arg \min ||v - u||$.

In this article, the equilibrium problem is studied based on the following hypothesis:

(a1) A bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ is said to be (see [3,6]) pseudomonotone on C if

$$f(u_1, u_2) \ge 0 \Longrightarrow f(u_2, u_1) \le 0, \quad \forall u_1, u_2 \in C.$$

(a2) A bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ is said to be Lipschitz-type continuous [21] on *C* if there exist two constants $c_1, c_2 > 0$ such that

$$f(u_1, u_3) \le f(u_1, u_2) + f(u_2, u_3) + c_1 ||u_1 - u_2||^2 + c_2 ||u_2 - u_3||^2, \ \forall u_1, u_2, u_3 \in C.$$

- (a3) $\limsup f(u_n, v) \le f(p^*, v)$ for all $v \in C$ and $\{u_n\} \subset C$ satisfies $u_n \rightharpoonup p^*$;
- (a4) $f(u, \cdot)$ is subdifferentiable and convex on \mathcal{H} for every each $u \in \mathcal{H}$.

The above-defined problem (EP) is a general mathematical problem in the sense that it unifies a number of mathematical problems, i.e., the fixed point problems, the vector and scalar minimization problems, the problems of variational inequalities (VIP), the complementarity problems, the saddle point problems, the Nash equilibrium problems in non-cooperative games and the inverse optimization problems [4, 6, 24, 41]. The problem (EP) is also known as the well-known Ky Fan inequality due to his initial contribution [11]. Many authors have developed and generalized many results on the existence and nature of the solution of an equilibrium problem (see for more detail [2, 4, 11]). Due to the significance of the problem (EP) and its applications in both pure and applied sciences, many researchers have studied it extensively in recent years [5, 10, 14].

A proximal point method is used to solve the problem (EP) based on mathematical programming [13]. This method was also known as the two-step extragradient method in [34] due to initial contribution of Korpelevich [18] to solve saddle point problems. Tran et al. in [34] established an iterative sequence $\{u_n\}$ in the following way:

$$\begin{cases} u_0 \in C, \\ v_n = \underset{v \in C}{\arg\min} \{ \rho f(u_n, v) + \frac{1}{2} ||u_n - v||^2 \}, \\ u_{n+1} = \underset{v \in C}{\arg\min} \{ \rho f(v_n, v) + \frac{1}{2} ||u_n - v||^2 \}, \end{cases}$$

where $0 < \rho < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$. Recently, many existing methods have been extended in the case of problem (EP) in finite and infinite-dimensional spaces, such as the proximal point-like methods [13, 22], the extragradient-like methods [17, 19, 26, 27, 32], the subgradient extragradient methods [1, 29, 36, 37], the inertial methods [35, 39] and others in [9, 12, 16, 25, 30, 33, 38].

Inspired by the results in [8, 16, 23], in this paper, we introduce a viscosity-type subgradient extragradient algorithm to solve the equilibrium problems involving pseudomonotone bifunction. A strong convergence theorem for the proposed algorithm is well-established by considering certain mild conditions on bifunction and control parameters. Some applications for our main results are studied to solve two particular classes of an equilibrium problem. In the end, the computational studies show that the new method is more efficient than the existing ones [16, 34].

The remainder of this article has been organized as follows: Section 2 includes some preliminary and basic results. Section 3 contains proposed algorithm and corresponding strong convergence result. Section 4 contains applications of our main results. Section 5 involves the numerical discussion of the proposed method compared to existing ones.

2. Background

A normal cone of C at $u \in C$ is defined by

$$N_C(u) = \{ w \in \mathcal{H} : \langle w, v - u \rangle \le 0, \ \forall v \in C \}.$$

Let $\varphi : C \to \mathcal{R}$ is convex function. The *subdifferential of* φ at $u \in C$ is defined by

$$\partial \varphi(u) = \{ w \in \mathcal{H} : \varphi(v) - \varphi(u) \ge \langle w, v - u \rangle, \ \forall v \in C \}.$$

Lemma 2.1. [15] Assume that $P_C : \mathcal{H} \to C$ is a metric projection such that

(i)

$$||u_1 - P_C(u_2)||^2 + ||P_C(u_2) - u_2||^2 \le ||u_1 - u_2||^2, \ u_1 \in C, u_2 \in \mathcal{H}.$$

(ii) $u_3 = P_C(u_1)$ if and only if

$$\langle u_1 - u_3, u_2 - u_3 \rangle \leq 0, \ \forall u_2 \in C.$$

(iii)

$$|u_1 - P_C(u_1)|| \le ||u_1 - u_2||, \ u_2 \in C, u_1 \in \mathcal{H}.$$

Lemma 2.2. [40] Assume that $\{a_n\} \subset (0, +\infty)$ is a sequence satisfying

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \ \forall n \in \mathbb{N},$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathcal{R}$ satisfies the following conditions:

$$\lim_{n\to\infty}\gamma_n=0, \ \sum_{n=1}^{+\infty}\gamma_n=+\infty \ and \ \limsup_{n\to\infty}\delta_n\leq 0.$$

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. [20] Assume that a sequence $\{a_n\} \subset \mathcal{R}$ and there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_{i+1}}$, for all $i \in \mathbb{N}$. Then, there is a non decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to +\infty$ as $k \to \infty$, and the following conditions are fulfilled by all (sufficiently large) numbers $k \in \mathbb{N}$,

$$a_{m_k} \leq a_{m_{k+1}} \text{ and } a_k \leq a_{m_{k+1}}.$$

In fact m_k is the largest number n in the set $\{1, 2, \dots, k\}$ such that $a_n \leq a_{n+1}$.

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Lemma 2.4. [15] For each $u_1, u_2 \in \mathcal{H}$ and $\delta \in \mathcal{R}$, then the following relationships are true.

(i)

$$\|\delta u_1 + (1-\delta)u_2\|^2 = \delta \|u_1\|^2 + (1-\delta)\|u_2\|^2 - \delta(1-\delta)\|u_1 - u_2\|^2$$

(ii)

$$||u_1 + u_2||^2 \le ||u_1||^2 + 2\langle u_2, u_1 + u_2 \rangle$$

Lemma 2.5. (Theorem 27.4 [28]) Let $\varphi : C \to \mathcal{R}$ be a proper convex, subdifferentiable and lower semi-continuous function on *C*. An element $u \in C$ is a minimizer of a function φ iff

$$0 \in \partial \varphi(u) + N_C(u),$$

where $\partial \varphi(u)$ stands for the sub-differential of φ at $u \in C$ and $N_C(u)$ the normal cone of C at u.

3. Main results

In this section, we present an iterative scheme for solving pseudomonotone equilibrium problems that is based on Tran et al. in [34] and viscosity scheme [23]. It is important to note that the proposed method has a straightforward structure for achieving strong convergence. Suppose that $g : \mathcal{H} \to \mathcal{H}$ be a strict contraction function with constant $\xi \in [0, 1)$. The main algorithm has been presented as follows:

Algorithm 1 (A Viscosity Method for Pseudomonotone Equilibrium Problems)

Step 0: Let $u_0 \in C$, $0 < \rho < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$ and a sequence $\chi_n \subset (0, 1)$ satisfies the conditions, i.e.,

$$\lim_{n\to\infty}\chi_n=0 \text{ and } \sum_n^{+\infty}\chi_n=+\infty.$$

Step 1: Compute

$$v_n = \arg\min_{v \in C} \{\rho f(u_n, v) + \frac{1}{2} ||u_n - v||^2 \}.$$

If $u_n = v_n$, then stop the sequence. Otherwise, go to **Step 2**. **Step 2**: Compute

$$\mathcal{H}_n = \{ w \in \mathcal{H} : \langle u_n - \rho \omega_n - v_n, w - v_n \rangle \le 0 \},\$$

where $\omega_n \in \partial_2 f(u_n, v_n)$ and evaluate

$$t_n = \arg\min_{v \in \mathcal{H}_n} \{ \rho f(v_n, v) + \frac{1}{2} ||u_n - v||^2 \}.$$

Step 3: Compute

$$u_{n+1} = \chi_n g(u_n) + (1 - \chi_n) t_n,$$

where g is a contraction. Set n := n + 1 and go back to Step 1.

Remark 3.1. It can be easily prove that $C \subset \mathcal{H}_n$. By v_n and Lemma 2.5, we have

$$0 \in \partial_2 \Big\{ \rho f(u_n, v) + \frac{1}{2} ||u_n - v||^2 \Big\} (v_n) + N_C(v_n).$$

Indeed, for $\omega_n \in \partial f(u_n, v_n)$ there exists $\overline{\omega}_n \in N_C(v_n)$ such that

$$\rho\omega_n + v_n - u_n + \overline{\omega}_n = 0.$$

Thus, we have

$$\langle u_n - v_n, v - v_n \rangle = \rho \langle \omega_n, v - v_n \rangle + \langle \overline{\omega}_n, v - v_n \rangle, \ \forall \ v \in C.$$

Due to $\overline{\omega}_n \in N_C(v_n)$ means that $\langle \overline{\omega}_n, v - v_n \rangle \leq 0$, for all $v \in C$. It implies that

$$\langle u_n - v_n, v - v_n \rangle \leq \rho \langle \omega_n, v - v_n \rangle, \ \forall \ v \in C,$$

which imply that $\langle u_n - \rho \omega_n - v_n, v - v_n \rangle \le 0, \forall v \in C$. It proves that $C \subset \mathcal{H}_n$ for each $n \in \mathbb{N}$.

Theorem 3.1. Assume that $\{u_n\}$ is a sequence generated by Algorithm 1 and for some $u^* \in S_{EP(f,C)} \neq \emptyset$. Then, $\{u_n\}$ converges strongly to $u^* = P_{S_{EP(f,C)}} \circ g(u^*)$.

Proof. Claim 1: The $\{u_n\}$ sequence is bounded. By Lemma 2.5, we have

$$0 \in \partial_2 \Big(\rho f(v_n, v) + \frac{1}{2} ||u_n - v||^2 \Big)(t_n) + N_{\mathcal{H}_n}(t_n).$$

Thus, there exists $\omega_n \in \partial f(v_n, t_n)$ and $\overline{\omega}_n \in N_{\mathcal{H}_n}(t_n)$ such that $\rho \omega_n + t_n - u_n + \overline{\omega}_n = 0$. Thus,

$$\langle u_n - t_n, v - t_n \rangle = \rho \langle \omega_n, v - t_n \rangle + \langle \overline{\omega}_n, v - t_n \rangle, \ \forall v \in \mathcal{H}_n.$$

Since $\overline{\omega}_n \in N_{\mathcal{H}_n}(t_n)$ follows that $\langle \overline{\omega}_n, v - t_n \rangle \leq 0$, for all $v \in \mathcal{H}_n$. Thus, we have

$$\rho\langle\omega_n, v - t_n\rangle \ge \langle u_n - t_n, v - t_n\rangle, \quad \forall v \in \mathcal{H}_n.$$
(3.1)

Since $\omega_n \in \partial f(v_n, t_n)$ and using the subdifferential definition, we get

$$f(v_n, v) - f(v_n, t_n) \ge \langle \omega_n, v - t_n \rangle, \quad \forall v \in \mathcal{H}.$$
(3.2)

Combining expressions (3.1) and (3.2), we obtain

$$\rho f(v_n, v) - \rho f(v_n, t_n) \ge \langle u_n - t_n, v - t_n \rangle, \quad \forall v \in \mathcal{H}_n.$$
(3.3)

Substituting $v = u^*$ in expression (3.3), we obtain

$$\rho f(v_n, u^*) - \rho f(v_n, t_n) \ge \langle u_n - t_n, u^* - t_n \rangle.$$
(3.4)

Since $u^* \in S_{EP(f,C)}$, it follows that $f(u^*, v_n) \ge 0$, implies that $f(v_n, u^*) \le 0$ due to the pseudomonotonicity of the bifunction f. From expression (3.4), we have

$$\langle u_n - t_n, t_n - u^* \rangle \ge \rho f(v_n, t_n). \tag{3.5}$$

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Due to the Lipschitz-type condition on a bifunction f, we obtain

$$f(u_n, t_n) \le f(u_n, v_n) + f(v_n, t_n) + c_1 ||u_n - v_n||^2 + c_2 ||v_n - t_n||^2.$$
(3.6)

Combining expressions (3.5) and (3.6), we have

$$\langle u_n - t_n, t_n - u^* \rangle \ge \rho \{ f(u_n, t_n) - f(u_n, v_n) \} - c_1 \rho ||u_n - v_n||^2 - c_2 \rho ||v_n - t_n||^2.$$
(3.7)

Since $t_n \in \mathcal{H}_n$ and it gives that $\langle u_n - \rho \omega_n - v_n, t_n - v_n \rangle \leq 0$, which implies that

$$\langle u_n - v_n, t_n - v_n \rangle \le \rho \langle \omega_n, t_n - v_n \rangle.$$
(3.8)

Since $\omega_n \in \partial_2 f(u_n, v_n)$ and using the subdifferential definition, we obtain

$$f(u_n, v) - f(u_n, v_n) \ge \langle \omega_n, v - v_n \rangle, \ \forall v \in \mathcal{H}.$$

Substituting $v = t_n$ in the above expression, we get

$$f(u_n, t_n) - f(u_n, v_n) \ge \langle \omega_n, t_n - v_n \rangle.$$
(3.9)

It follows from inequalities (3.8) and (3.9) that

$$\rho\{f(u_n, t_n) - f(u_n, v_n)\} \ge \langle u_n - v_n, t_n - v_n \rangle.$$
(3.10)

From (3.7) and (3.10), we have

$$\langle u_n - t_n, t_n - u^* \rangle \ge \langle u_n - v_n, t_n - v_n \rangle - c_1 \rho ||u_n - v_n||^2 - c_2 \rho ||v_n - t_n||^2.$$
(3.11)

We have the following equalities:

$$2\langle u_n - t_n, t_n - u^* \rangle = ||u_n - u^*||^2 - ||t_n - u_n||^2 - ||t_n - u^*||^2$$

and

$$2\langle v_n - u_n, v_n - t_n \rangle = ||u_n - v_n||^2 + ||t_n - v_n||^2 - ||u_n - t_n||^2.$$

The above facts and (3.11) implies that

$$||t_n - u^*||^2 \le ||u_n - u^*||^2 - (1 - 2c_1\rho)||u_n - v_n||^2 - (1 - 2c_2\rho)||t_n - v_n||^2.$$
(3.12)

It is given that $u^* \in S_{EP(f,C)}$. From the definition of sequence $\{u_{n+1}\}$ and due to the fact that g is a contraction with $\xi \in [0, 1)$, we have

$$\begin{aligned} \left\| u_{n+1} - u^* \right\| &= \left\| \chi_n g(u_n) + (1 - \chi_n) t_n - u^* \right\| \\ &= \left\| \chi_n [g(u_n) - u^*] + (1 - \chi_n) [t_n - u^*] \right\| \\ &= \left\| \chi_n [g(u_n) + g(u^*) - g(u^*) - u^*] + (1 - \chi_n) [t_n - u^*] \right\| \\ &\leq \chi_n \left\| g(u_n) - g(u^*) \right\| + \chi_n \left\| g(u^*) - u^* \right\| + (1 - \chi_n) \left\| t_n - u^* \right\| \\ &\leq \chi_n \xi \left\| u_n - u^* \right\| + \chi_n \left\| g(u^*) - u^* \right\| + (1 - \chi_n) \left\| t_n - u^* \right\|. \end{aligned}$$
(3.13)

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Combining expressions (3.12) and (3.13) and $\chi_n \subset (0, 1)$, we deduce that

$$\begin{aligned} \|u_{n+1} - u^*\| &\leq \chi_n \xi \|u_n - u^*\| + \chi_n \|g(u^*) - u^*\| + (1 - \chi_n) \|u_n - u^*\| \\ &= [1 - \chi_n + \xi \chi_n] \|u_n - u^*\| + \chi_n (1 - \xi) \frac{\|g(u^*) - u^*\|}{(1 - \xi)} \\ &\leq \max \left\{ \|u_n - u^*\|, \frac{\|g(u^*) - u^*\|}{(1 - \xi)} \right\} \\ &\leq \max \left\{ \|u_0 - u^*\|, \frac{\|g(u^*) - u^*\|}{(1 - \xi)} \right\}. \end{aligned}$$
(3.14)

Thus, we conclude that the $\{u_n\}$ is bounded sequence. Due to the reflexivity of \mathcal{H} and the boundedness of $\{u_n\}$ guarantees that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\} \rightarrow u^* \in \mathcal{H}$ as $k \rightarrow \infty$. **Claim 2:** The sequence $\{u_n\}$ is strongly convergent.

Next, we prove the strong convergence of the iterative sequence $\{u_n\}$ generated by Algorithm 1. The Lipschitz-continuity and pseudomonotone property of the bifunction f implies that the solution set $S_{EP(f,C)}$ is a closed and convex set (for more details see [34]). Since the mapping is a contraction and so does $P_{S_{EP(f,C)}} \circ g$. Now, we are in position to use the Banach contraction theorem for the existence of a unique fixed point $u^* \in S_{EP(f,C)}$ such that

$$u^* = P_{S_{EP(f,C)}}(g(u^*))$$

By using Lemma 2.1 (ii), we have

$$\langle g(u^*) - u^*, v - u^* \rangle \le 0, \ \forall v \in S_{EP(f,C)}.$$
 (3.15)

By Lemma 2.4 (i) and (3.12), we have

$$\begin{aligned} \left\| u_{n+1} - u^* \right\|^2 &= \left\| \chi_n g(u_n) + (1 - \chi_n) t_n - u^* \right\|^2 \\ &= \left\| \chi_n [g(u_n) - u^*] + (1 - \chi_n) [t_n - u^*] \right\|^2 \\ &= \chi_n \|g(u_n) - u^*\|^2 + (1 - \chi_n) \|t_n - u^*\|^2 - \chi_n (1 - \chi_n) \|g(u_n) - t_n\|^2 \\ &\leq \chi_n \|g(u_n) - u^*\|^2 + (1 - \chi_n) \left[\|u_n - u^*\|^2 - (1 - 2c_1\rho) \|u_n - v_n\|^2 \\ &- (1 - 2c_2\rho) \|t_n - v_n\|^2 \right] - \chi_n (1 - \chi_n) \|g(u_n) - t_n\|^2 \\ &\leq \chi_n \|g(u_n) - u^*\|^2 + \|u_n - u^*\|^2 \\ &- (1 - \chi_n) (1 - 2c_1\rho) \|u_n - v_n\|^2 - (1 - \chi_n) (1 - 2c_2\rho) \|t_n - v_n\|^2. \end{aligned}$$
(3.16)

The remainder of the proof shall be split into the following two parts: **Case 1:** Assume that there is a fixed number $N_1 \in \mathbb{N}$ such that

$$||u_{n+1} - u^*|| \le ||u_n - u^*||, \ \forall n \ge N_1.$$
(3.17)

Thus, above implies that $\lim_{n\to\infty} ||u_n - u^*||$ exists and let $\lim_{n\to\infty} ||u_n - u^*|| = l$. From (3.16), we get

$$(1-\chi_n)(1-2c_1\rho)\|u_n-v_n\|^2+(1-\chi_n)(1-2c_2\rho)\|t_n-v_n\|^2$$

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$$\leq \chi_n \|g(u_n) - u^*\|^2 + \|u_n - u^*\|^2 - \|u_{n+1} - u^*\|^2.$$
(3.18)

Due to the existence of $\lim_{n\to\infty} ||u_n - u^*|| = l$ and $\chi_n \to 0$, we infer that

$$\lim_{n \to \infty} ||u_n - v_n|| = \lim_{n \to \infty} ||t_n - v_n|| = 0.$$
(3.19)

It follows that

$$\lim_{n \to \infty} ||u_n - t_n|| \le \lim_{n \to \infty} ||u_n - v_n|| + \lim_{n \to \infty} ||v_n - t_n|| = 0.$$
(3.20)

We can also obtain

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|\chi_n g(u_n) + (1 - \chi_n) t_n - u_n\| \\ &= \|\chi_n [g(u_n) - u_n] + (1 - \chi_n) [t_n - u_n]\| \\ &\leq \chi_n \|g(u_n) - u_n\| + (1 - \chi_n) \|t_n - u_n\| \longrightarrow 0. \end{aligned}$$
(3.21)

The above expression implies that

$$\lim_{n \to \infty} \|u_{n+1} - u_n\| = 0.$$
(3.22)

Thus, implies that the sequences $\{v_n\}$ and $\{t_n\}$ are bounded. Let $\{u_{n_k}\}$ be subsequence of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $\hat{u} \in \mathcal{H}$. Next, we need to prove that $\hat{u} \in S_{EP(f,C)}$. Due to the inequality (3.3), the Lipschitz-like condition of f and (3.10), we obtain

$$\rho f(v_{n_k}, v) \ge \rho f(v_{n_k}, t_{n_k}) + \langle u_{n_k} - t_{n_k}, v - t_{n_k} \rangle
\ge \rho f(u_{n_k}, t_{n_k}) - \rho f(u_{n_k}, v_{n_k}) - c_1 \rho ||u_{n_k} - v_{n_k}||^2
- c_2 \rho ||v_{n_k} - t_{n_k}||^2 + \langle u_{n_k} - t_{n_k}, v - t_{n_k} \rangle
\ge \langle u_{n_k} - v_{n_k}, t_{n_k} - v_{n_k} \rangle - c_1 \rho ||u_{n_k} - v_{n_k}||^2
- c_2 \rho ||v_{n_k} - t_{n_k}||^2 + \langle u_{n_k} - t_{n_k}, v - t_{n_k} \rangle,$$
(3.23)

where *v* is an arbitrary member in \mathcal{H}_n . From (3.19), (3.20) and the boundedness of $\{u_n\}$ imply that right-hand side goes to zero. From $\rho > 0$, the condition (a3) and $v_{n_k} \rightarrow \hat{u}$, we obtain

$$0 \le \limsup_{k \to \infty} f(v_{n_k}, v) \le f(\hat{u}, v), \ \forall v \in \mathcal{H}_n.$$
(3.24)

It follows that $f(\hat{u}, v) \ge 0$, for all $v \in C$, and hence $\hat{u} \in S_{EP(f,C)}$. Next, we consider

$$\lim_{n \to \infty} \sup_{u \to \infty} \langle g(u^*) - u^*, u_n - u^* \rangle$$

=
$$\lim_{k \to \infty} \sup_{u \to \infty} \langle g(u^*) - u^*, u_{n_k} - u^* \rangle = \langle g(u^*) - u^*, \hat{u} - u^* \rangle \le 0.$$
(3.25)

We have $\lim_{n\to\infty} ||u_{n+1} - u_n|| = 0$. We can deduce that

$$\lim_{n \to \infty} \sup_{u \to \infty} \langle g(u^*) - u^*, u_{n+1} - u^* \rangle$$

$$\leq \limsup_{n \to \infty} \langle g(u^*) - u^*, u_{n+1} - u_n \rangle + \limsup_{n \to \infty} \langle g(u^*) - u^*, u_n - u^* \rangle \leq 0.$$
(3.26)

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From Lemma 2.4(ii) and (3.12), we have

$$\begin{aligned} \left\| u_{n+1} - u^* \right\|^2 \\ &= \left\| \chi_n g(u_n) + (1 - \chi_n) t_n - u^* \right\|^2 \\ &= \left\| \chi_n [g(u_n) - u^*] + (1 - \chi_n) [t_n - u^*] \right\|^2 \\ &\leq (1 - \chi_n)^2 \left\| t_n - u^* \right\|^2 + 2\chi_n \langle g(u_n) - u^*, (1 - \chi_n) [t_n - u^*] + \chi_n [g(u_n) - u^*] \rangle \\ &= (1 - \chi_n)^2 \left\| t_n - u^* \right\|^2 + 2\chi_n \langle g(u_n) - g(u^*) + g(u^*) - u^*, u_{n+1} - u^* \rangle \\ &= (1 - \chi_n)^2 \left\| t_n - u^* \right\|^2 + 2\chi_n \langle g(u_n) - g(u^*), u_{n+1} - u^* \rangle + 2\chi_n \langle g(u^*) - u^*, u_{n+1} - u^* \rangle \\ &\leq (1 - \chi_n)^2 \left\| t_n - u^* \right\|^2 + 2\chi_n \xi \left\| u_n - u^* \right\| \left\| u_{n+1} - u^* \right\| + 2\chi_n \langle g(u^*) - u^*, u_{n+1} - u^* \rangle \\ &\leq (1 + \chi_n^2 - 2\chi_n) \left\| u_n - u^* \right\|^2 + 2\chi_n \xi \left\| u_n - u^* \right\|^2 + 2\chi_n \langle g(u^*) - u^*, u_{n+1} - u^* \rangle \\ &= (1 - 2\chi_n) \left\| u_n - u^* \right\|^2 + \chi_n^2 \left\| u_n - u^* \right\|^2 + 2\chi_n \xi \left\| u_n - u^* \right\|^2 + 2\chi_n \langle g(u^*) - u^*, u_{n+1} - u^* \rangle \\ &= [1 - 2\chi_n(1 - \xi)] \left\| u_n - u^* \right\|^2 + 2\chi_n(1 - \xi) \left[\frac{\chi_n \left\| u_n - u^* \right\|^2}{2(1 - \xi)} + \frac{\langle g(u^*) - u^*, u_{n+1} - u^* \rangle}{1 - \xi} \right]. \end{aligned}$$
(3.27)

It follows from expressions (3.26) and (3.27), we obtain

$$\limsup_{n \to \infty} \left[\frac{\chi_n \left\| u_n - u^* \right\|^2}{2(1 - \xi)} + \frac{\langle g(u^*) - u^*, u_{n+1} - u^* \rangle}{1 - \xi} \right] \le 0.$$
(3.28)

Choose $n \ge N_2 \in \mathbb{N}$ $(N_2 \ge N_1)$ large enough such that $2\chi_n(1 - \xi) < 1$. By using (3.27), (3.28) and applying Lemma 2.2, we conclude that $||u_n - u^*|| \to 0$ as $n \to \infty$. **Case 2:** Assume there is a subsequence $\{n_i\}$ of $\{n\}$ such that

$$||u_{n_i} - u^*|| \le ||u_{n_{i+1}} - u^*||, \ \forall i \in \mathbb{N}.$$

Thus, by Lemma 2.3 there is a sequence $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \to \infty$, such that

$$||u_{m_k} - u^*|| \le ||u_{m_{k+1}} - u^*||$$
 and $||u_k - u^*|| \le ||u_{m_{k+1}} - u^*||$, for all $k \in \mathbb{N}$. (3.29)

Similar to Case 1, the expression (3.16) provides that

$$(1 - \chi_{m_k})(1 - 2c_1\rho) \|u_{m_k} - v_{m_k}\|^2 + (1 - \chi_{m_k})(1 - 2c_2\rho) \|t_{m_k} - v_{m_k}\|^2$$

$$\leq \chi_{m_k} \|g(u_{m_k}) - u^*\|^2 + \|u_{m_k} - u^*\|^2 - \|u_{m_{k+1}} - u^*\|^2.$$
(3.30)

Due to $\chi_{m_k} \rightarrow 0$, we deduce the following:

$$\lim_{k \to \infty} \|u_{m_k} - v_{m_k}\| = \lim_{k \to \infty} \|t_{m_k} - v_{m_k}\| = 0.$$
(3.31)

Also, we can obtain

$$\begin{aligned} \left\| u_{m_k+1} - u_{m_k} \right\| &= \left\| \chi_{m_k} g(u_{m_k}) + (1 - \chi_{m_k}) t_{m_k} - u_{m_k} \right\| \\ &= \left\| \chi_{m_k} [g(u_{m_k}) - u_{m_k}] + (1 - \chi_{m_k}) [t_{m_k} - u_{m_k}] \right\| \end{aligned}$$

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$$\leq \chi_{m_k} \| g(u_{m_k}) - u_{m_k} \| + (1 - \chi_{m_k}) \| t_{m_k} - u_{m_k} \| \longrightarrow 0.$$
 (3.32)

We have to use the same justification as in the Case 1, such that

$$\limsup_{k \to \infty} \langle g(u^*) - u^*, u_{m_k+1} - u^* \rangle \le 0.$$
(3.33)

By using expressions (3.27) and (3.29), we have

$$\begin{aligned} \left\| u_{m_{k}+1} - u^{*} \right\|^{2} &\leq \left[1 - 2\chi_{m_{k}}(1-\xi) \right] \left\| u_{m_{k}} - u^{*} \right\|^{2} \\ &+ 2\chi_{m_{k}}(1-\xi) \left[\frac{\chi_{m_{k}} \left\| u_{m_{k}} - u^{*} \right\|^{2}}{2(1-\xi)} + \frac{\langle g(u^{*}) - u^{*}, u_{m_{k}+1} - u^{*} \rangle}{1-\xi} \right] \\ &\leq \left[1 - 2\chi_{m_{k}}(1-\xi) \right] \left\| u_{m_{k}+1} - u^{*} \right\|^{2} \\ &+ 2\chi_{m_{k}}(1-\xi) \left[\frac{\chi_{m_{k}} \left\| u_{m_{k}} - u^{*} \right\|^{2}}{2(1-\xi)} + \frac{\langle g(u^{*}) - u^{*}, u_{m_{k}+1} - u^{*} \rangle}{1-\xi} \right]. \end{aligned}$$
(3.34)

It follows that

$$\left\|u_{m_{k+1}} - u^*\right\|^2 \le \frac{\chi_{m_k} \left\|u_{m_k} - u^*\right\|^2}{2(1-\xi)} + \frac{\langle g(u^*) - u^*, u_{m_{k+1}} - u^* \rangle}{1-\xi}.$$
(3.35)

Since $\chi_{m_k} \to 0$, and $||u_{m_k} - u^*||$ is a bounded. Thus, expressions (3.33) and (3.35) implies that

$$||u_{m_k+1} - u^*||^2 \to 0$$
, as $k \to \infty$. (3.36)

The above implies that

$$\lim_{k \to \infty} \|u_k - u^*\|^2 \le \lim_{k \to \infty} \|u_{m_k + 1} - u^*\|^2 \le 0.$$
(3.37)

Consequently, $u_n \rightarrow u^*$. This completes the proof of the theorem.

Remark 3.2. If we define $g(u) = u_0$ in Algorithm 1, we obtain the Halpern subgradient extragradient method in [16].

4. Applications

Now, we consider the application of our main results to solve the problem of classic variational inequalities [31] for an operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is defined in the following way:

Fins
$$u^* \in C$$
 such that $\langle \mathcal{G}(u^*), v - u^* \rangle \ge 0, \ \forall v \in C.$ (VIP)

We consider the following conditions to study variational inequalities.

(b1) The solution set of the problem (VIP) denoted by $VI(\mathcal{G}, C)$ is nonempty. (b2) $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is said to be a pseudomonotone, i.e.,

$$\langle \mathcal{G}(u), v - u \rangle \ge 0 \Longrightarrow \langle \mathcal{G}(v), u - v \rangle \le 0, \ \forall u, v \in C.$$

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(b3) $\mathcal{G}: \mathcal{H} \to \mathcal{H}$ is said to be a *Lipschitz continuous* if there exits a constants L > 0 such that

$$||\mathcal{G}(u) - \mathcal{G}(v)|| \le L||u - v||, \ \forall u, v \in C;$$

(b4) $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is called to be *sequentially weakly continuous*, i.e., $\{\mathcal{G}(u_n)\}$ weakly converges to $\mathcal{G}(u)$ for every sequence $\{u_n\}$ converges weakly to u.

If we define $f(u, v) := \langle \mathcal{G}(u), v - u \rangle$, for all $u, v \in C$. Then, the equilibrium problem becomes the problem of variational inequalities described above where $L = 2c_1 = 2c_2$. From the above value of the bifunction f, we have

$$\begin{cases} v_n = \underset{v \in C}{\arg\min} \{ \rho f(u_n, v) + \frac{1}{2} ||u_n - v||^2 \} = P_C(u_n - \rho \mathcal{G}(u_n)), \\ t_n = \underset{v \in \mathcal{H}_n}{\arg\min} \{ \rho f(v_n, v) + \frac{1}{2} ||u_n - v||^2 \} = P_{\mathcal{H}_n}(u_n - \rho \mathcal{G}(v_n)). \end{cases}$$
(4.1)

As a consequence of the results in Section 3, we have the following results:

Corollary 4.1. Let $\mathcal{G} : C \to \mathcal{H}$ be a mapping satisfying the conditions (b1)–(b4). Choose $u_0 \in C$, $0 < \rho < \frac{1}{I}$ and a sequence $\chi_n \subset (0, 1)$ satisfying the conditions, i.e.,

$$\lim_{n\to\infty}\chi_n=0 \ and \ \sum_n^{+\infty}\chi_n=+\infty$$

Assume that $\{u_n\}$ is the sequence generated in the following way:

$$\begin{cases} v_n = P_C(u_n - \rho \mathcal{G}(u_n)), \\ t_n = P_{\mathcal{H}_n}(u_n - \rho \mathcal{G}(v_n)), \\ u_{n+1} = \chi_n g(u_n) + (1 - \chi_n) t_n, \end{cases}$$

where

$$\mathcal{H}_n = \{ z \in \mathcal{H} : \langle u_n - \rho \mathcal{G}(u_n) - v_n, z - v_n \rangle \le 0 \}$$

Then, the sequence $\{u_n\}$ converges strongly to $u^* \in VI(\mathcal{G}, \mathcal{C})$.

Note that condition (b4) can be deleted in the case when G is monotone. Indeed, this condition is a particular case of condition (a3) is used to prove (3.24). Without condition (b4), the inequality (3.23) is obtained by imposing the monotonocity on G. In that case, we obtain

$$\langle \mathcal{G}(v), v - v_n \rangle \ge \langle \mathcal{G}(v_n), v - v_n \rangle, \quad \forall v \in C.$$
 (4.2)

By the use $f(u, v) = \langle \mathcal{G}(u), v - u \rangle$ in (3.23), we get

$$\limsup_{k \to \infty} \langle \mathcal{G}(v_{n_k}), v - v_{n_k} \rangle \ge 0, \ \forall v \in \mathcal{H}_n.$$
(4.3)

Combining (4.2) with (4.3), we obtain

$$\limsup_{k \to \infty} \langle \mathcal{G}(v), v - v_{n_k} \rangle \ge 0, \quad \forall v \in C.$$
(4.4)

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Let $v_s = (1 - s)\hat{u} + sy$, for all $s \in [0, 1]$. Due to convexity of *C* implies that $v_s \in C$ for any $s \in (0, 1)$. Since $v_{n_k} \rightharpoonup \hat{u} \in C$ and $\langle \mathcal{G}(v), v - \hat{u} \rangle \ge 0$ for every $v \in C$. Thus, we have

$$0 \le \langle \mathcal{G}(v_s), v_s - \hat{u} \rangle = s \langle \mathcal{G}(v_s), v - \hat{u} \rangle.$$
(4.5)

Therefore, $\langle \mathcal{G}(v_s), v - \hat{u} \rangle \ge 0$, for all $s \in (0, 1)$. Since $v_s \to \hat{u}$ as $s \to 0$ and due to continuity of \mathcal{G} , we have $\langle \mathcal{G}(\hat{u}), v - \hat{u} \rangle \ge 0$, for all $v \in C$, which implies that $\hat{u} \in VI(\mathcal{G}, C)$.

Remark 4.1. From the above discussion, it can be concluded that the Corollary 4.1 still holds, even if we remove the condition (b4) in case of monotone variational inequality.

Next, we consider the application of our results to solve fixed-point problems involving κ -strict pseudocontraction mapping and the *fixed point problem* for an operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is defined in the following way:

Find
$$u^* \in C$$
 such that $\mathcal{T}(u^*) = u^*$. (FPP)

We assume that the following requirements have been met to study fixed point problems.

(c1) $\mathcal{T}: C \to C$ is said to be a κ -strict pseudo-contraction [7] on C if

$$||Tu - Tv||^{2} \le ||u - v||^{2} + \kappa ||(u - Tu) - (v - Tv)||^{2}, \ \forall u, v \in C;$$

(c2) weakly sequentially continuous on C if

$$\mathcal{T}(u_n) \rightarrow \mathcal{T}(u^*)$$
 for any sequence in *C* satisfy $u_n \rightarrow u^*$.

If we consider that the mapping \mathcal{T} is a κ -strict pseudocontraction and weakly continuous then $f(u, v) = \langle u - \mathcal{T}u, v - u \rangle$ satisfies the conditions (a1)–(a4) and $2c_1 = 2c_2 = \frac{3-2\kappa}{1-\kappa}$. The values of v_n and t_n in Algorithm 1, we have

$$\begin{cases} v_n = \underset{v \in C}{\arg\min} \{ \rho f(u_n, v) + \frac{1}{2} ||u_n - v||^2 \} = P_C[u_n - \rho(u_n - \mathcal{T}(u_n))], \\ t_n = \underset{v \in \mathcal{H}_n}{\arg\min} \{ \rho f(v_n, v) + \frac{1}{2} ||u_n - v||^2 \} = P_{\mathcal{H}_n}[u_n - \rho(v_n - \mathcal{T}(v_n))]. \end{cases}$$
(4.6)

Corollary 4.2. Let C be a nonempty, convex and closed subset of a Hilbert space \mathcal{H} and $\mathcal{T} : C \to C$ is a κ -strict pseudocontraction and weakly continuous with solution set $Fix(\mathcal{T}) \neq \emptyset$. Choose $u_0 \in C$, $0 < \rho < \frac{1-\kappa}{3-2\kappa}$ and sequence $\chi_n \subset (0,1)$ satisfies the conditions, i.e.,

$$\lim_{n\to\infty}\chi_n=0 \ and \ \sum_n^{+\infty}\chi_n=+\infty.$$

Let $\{u_n\}$ be the sequence generated in the following way:

$$\begin{cases} v_n = P_C[u_n - \rho(u_n - \mathcal{T}(u_n))], \\ t_n = P_{\mathcal{H}_n}[u_n - \rho(v_n - \mathcal{T}(v_n))], \\ u_{n+1} = \chi_n g(u_n) + (1 - \chi_n)t_n. \end{cases}$$

where

$$\mathcal{H}_n = \{ w \in \mathcal{H} : \langle (1 - \rho)u_n + \rho \mathcal{T}(u_n) - v_n, w - v_n \rangle \le 0 \}.$$

Then, sequence $\{u_n\}$ strongly converges to $u^* \in Fix(\mathcal{T})$.

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5. Numerical illustrations

Numerical results are presented in this section to show the efficiency of the proposed method. The MATLAB codes was run in MATLAB version 9.5 (R2018b) on the Intel(R) Core(TM)i5-6200 CPU PC @ 2.30GHz 2.40GHz, RAM 8.00 GB.

Example 5.1. Assume that $f : C \times C \rightarrow \mathcal{R}$ is defined by

$$f(u, v) = \langle Pu + Qv + c, v - u \rangle, \quad \forall u, v \in C,$$

where $c \in \mathbb{R}^n$ and P, Q are matrices of order n. The matrix P is symmetric positive semi-definite and the matrix Q - P is symmetric negative semi-definite with Lipschitz-type constants $c_1 = c_2 = \frac{1}{2}||P - Q||$ (see [34] for details). The matrices P, Q and vector c are defined by

$$P = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \qquad Q = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}.$$

The constraint set $C \subset \mathbb{R}^n$ is considered as $C := \{u \in \mathbb{R}^n : -5 \le u_i \le 5\}$. Furthermore, control parameters conditions are taken as follows: $\rho = \frac{1}{4c_1}$ and $D_n = ||u_n - v_n||$ for Algorithm 1 (EgM) in [34]; $\rho = \frac{1}{4c_1}, \chi_n = \frac{1}{100(n+2)}$ and $D_n = ||u_n - v_n||$ for Algorithm 2 (H-EgM) in [16]; $\rho = \frac{1}{4c_1}, \chi_n = \frac{1}{100(n+2)}, g(u) = \frac{u}{2}$ and $D_n = ||u_n - v_n||$ for Algorithm 1 (V-EgM). The numerical and graphical results of three methods are shown in Figures 1–4 and Table 1.



Figure 1. Numerical behaviour of Algorithm 1 with Algorithm 1 in [34] and Algorithm 3.2 in [16] while $u_0 = (0, 0, 0, 0, 0)^T$.

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Figure 2. Numerical behaviour of Algorithm 1 with Algorithm 1 in [34] and Algorithm 3.2 in [16] while $u_0 = (2, 2, 2, 2, 2)^T$.



Figure 3. Numerical behaviour of Algorithm 1 with Algorithm 1 in [34] and Algorithm 3.2 in [16] while $u_0 = (1, 0, -1, 2, 1)^T$.

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Figure 4. Numerical behaviour of Algorithm 1 with Algorithm 1 in [34] and Algorithm 3.2 in [16] while $u_0 = (2, -1, 3, -4, 5)^T$.

	Number of Iterations			Execution Time in Seconds		
<i>u</i> ₀	EgM	H-EgM	V-EgM	EgM	H-EgM	V-EgM
$(0, 0, 0, 0, 0)^T$	27	32	21	0.217359	0.242564	0.226170
$(2, 2, 2, 2, 2)^T$	27	60	21	0.264663	0.553107	0.175542
$(1, 0, -1, 2, 1)^T$	31	49	24	0.292574	0.576171	0.219645
$(2, -1, 3, -4, 5)^T$	32	80	25	0.303238	0.635728	0.204971

Table 1. Numerical results values for Figures 1–4.

Example 5.2. Let $f : C \times C \rightarrow \mathcal{R}$ defined in the following way:

$$f(u, v) = \sum_{i=2}^{5} (v_i - u_i) ||u||, \ \forall u, v \in \mathcal{R}^5,$$

where $C = \{(u_1, \dots, u_5) : u_1 \ge -1, u_i \ge 1, i = 2, \dots, 5\}$. Thus, the bifunction f is Lipschitz-type continuous with $c_1 = c_2 = 2$, and satisfies the conditions (a1)–(a4). The solution set of an equilibrium problem is $EP(f, C) = \{(u_1, 1, 1, 1, 1) : u_1 > 1\}$. Furthermore, the control conditions $\rho = \frac{1}{6c_1}$ for Algorithm 1 (EgM) in [34]; $\rho = \frac{1}{6c_1}$ and $\chi_n = \frac{1}{200(n+2)}$ for Algorithm 2 (H-EgM) in [16]; $\rho = \frac{1}{6c_1}$, $\chi_n = \frac{1}{100(n+2)}$ and $g(u) = \frac{u}{3}$ for Algorithm 1 (V-EgM). The numerical results of three methods are shown in Tables 2–4.

Iter (n)	<i>u</i> ₁	<i>u</i> ₂	<i>u</i> ₃	<i>u</i> ₄	<i>u</i> ₅
1	1.9999999902	8.9034597770	11.903459749	3.9034598398	1.9034598209
2	2.0000001112	7.9193730581	10.919372983	2.9193733082	1.0000084500
3	2.0000001050	7.0300765337	10.030076426	2.0300768272	1.000000224
4	2.0000002139	6.2245554051	9.2245552403	1.2245575794	1.0000004995
5	2.000002018	5.4892537351	8.4892535376	1.000000391	1.000000272
6	2.0000001984	4.8171736152	7.8171733928	1.000000297	1.000000297
÷	:	:	:	:	÷
26	2.0000015507	1.0000019717	1.0384612829	1.0000019717	1.0000019717
27	2.0000016692	1.0000021759	1.0000028248	1.0000021759	1.0000021759
28	2.0000017877	1.0000021759	1.0000021759	1.0000021759	1.0000021759
CPU time is seconds	0.850380				

Table 2. Example 5.2: Numerical study of Algorithm 1 in [34] with TOL= 10^{-3} and $u_0 = (2, 10, 13, 5, 3)^T$.

Table 3. Example 5.2: Numerical study of Algorithm 3.2 in [16] with TOL= 10^{-3} and $u_0 = (2, 10, 13, 5, 3)^T$.

Iter (n)	u_1	<i>u</i> ₂	<i>u</i> ₃	u_4	<i>u</i> ₅
1	1.999999942	8.7515611443	11.751561124	3.7515611399	1.7515612532
2	1.999999938	7.6461082189	10.646108180	2.6461082675	1.0016667229
3	1.999999939	6.6610509620	9.6610509019	1.6610510862	1.0012500202
4	2.00000061	5.7776478420	8.7776477159	1.0020019229	1.0010004543
5	2.00000055	4.9810038039	7.9810036568	1.0016666917	1.0008333584
6	2.00000048	4.2622453450	7.2622451730	1.0014285992	1.0007143134
÷	:	:	:	:	÷
38	2.0000028210	1.0005787475	1.0007710569	1.0002582317	1.0001300255
39	2.0000029393	1.0005643243	1.0007518260	1.0002518214	1.0001268203
40	2.0000030576	1.0005506046	1.0007335331	1.0002457238	1.0001237715
CPU time is seconds	1.191645				

Example 5.3. Suppose that $\mathcal{H} = L^2([0, 1])$ is a Hilbert space with the inner product $\langle u, v \rangle = \int_0^1 u(t)v(t)dt$, for all $u, v \in \mathcal{H}$ and the induced norm is

$$||u|| = \sqrt{\int_0^1 |u(t)|^2 dt}.$$

Moreover, assume that $C := \{u \in L^2([0, 1]) : ||u|| \le 1\}$. Assume that $\mathcal{G} : C \to \mathcal{H}$ is defined by

$$\mathcal{G}(u)(t) = \int_0^1 \left[u(t) - H(t,s)f(u(s)) \right] ds + g(t)$$

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where $H(t, s) = \frac{2tse^{(t+s)}}{e\sqrt{e^2-1}}$, $f(u) = \cos(u)$ and $g(t) = \frac{2te^t}{e\sqrt{e^2-1}}$. We consider the bifunction $f(u, v) = \langle \mathcal{G}(u), v - u \rangle$ with the Lipschitz-type continuous with $c_1 = c_2 = 1$. Furthermore, control conditions $\rho = \frac{1}{5c_1}$ for Algorithm 1 (EgM) in [34]; $\rho = \frac{1}{5c_1}$ and $\chi_n = \frac{1}{300(n+2)}$ for Algorithm 2 (H-EgM) in [16]; $\rho = \frac{1}{5c_1}$, $\chi_n = \frac{1}{100(n+2)}$ and $g(u) = \frac{u}{2}$ for Algorithm 1 (V-EgM). The numerical results of three methods are shown in Figures 5 and 6 and Table 5.

Table 4. Example 5.2: Numerical study of Algorithm 1 with TOL= 10^{-3} and $u_0 = (2, 10, 13, 5, 3)^T$.

Iter (n)	u_1	<i>u</i> ₂	<i>u</i> ₃	u_4	<i>u</i> ₅
1	1.9950000119	8.5308876762	11.523387660	3.5433877312	1.54838779261
2	1.9916750267	7.2594419489	10.246954421	2.2802545702	0.99924734095
3	1.9891855417	6.1501278861	9.1339059323	1.1771670518	0.99874942183
4	1.9871963569	5.1700590512	8.1508532923	0.9991771920	0.99899876985
5	1.9855403611	4.3036279625	7.2819381754	0.9991660040	0.99916585540
6	1.9841221147	3.5358595487	6.5120423683	0.9992851446	0.99928514452
:	:	:	:	:	:
19	1.9741765217	0.99975135569	1.0030932482	0.99975135569	0.99975135569
20	1.9737065994	0.99976333843	0.9997641538	0.99976333843	0.99976333843
21	1.9732581496	0.99977416665	0.99977416684	0.99977416665	0.99977416665
CPU time is seconds	0.711785				



Figure 5. Numerical study of Algorithm 1 with Algorithm 1 in [34] and Algorithm 3.2 in [16].

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Figure 6. Numerical study of Algorithm 1 with Algorithm 1 in [34] and Algorithm 3.2 in [16].

	Number of Iterations			Execution Time in Seconds			
u_0	EgM H-EgM V-EgM		EgM	H-EgM	V-EgM		
t	49	106	41	0.058515	0.107587	0.052661	
$3t^{2}$	52	54	40	0.057655	0.058550	0.042137	
sin(t)	52	76	41	0.056489	0.083268	0.057777	
$\exp(t)$	55	97	52	0.130014	0.106885	0.061795	

Table 5. Numerical results values for Figures 5 and 6.

Example 5.4. Assume that $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ is defined by

$$f(u, v) = (5 - ||u||)\langle u, v - u \rangle, \ \forall u, v \in \mathcal{H},$$

where $\mathcal{H} = l_2$ is a real Hilbert space having the elements are square-summable sequences and $C = \{u \in \mathcal{H} : ||u|| \le 3\}$. The bifunction f is Lipschitz-type continuous and value of Lipschitz-constants are $c_1 = c_2 = \frac{11}{2}$. Furthermore, control conditions $\rho = \frac{1}{4c_1}$ for Algorithm 1 (EgM) in [34]; $\rho = \frac{1}{4c_1}$ and $\chi_n = \frac{1}{200(n+2)}$ for Algorithm 2 (H-EgM) in [16]; $\rho = \frac{1}{4c_1}$, $\chi_n = \frac{1}{100(n+2)}$ and $g(u) = \frac{u}{2}$ for Algorithm 1 (V-EgM). The numerical results of three methods are shown in Figures 7 and 8 and Table 6. The projection onto *C* is evaluated in the following way:

$$P_C(u) = \begin{cases} u & \text{if } ||u|| \le 3, \\ \frac{3u}{||u||}, & \text{else.} \end{cases}$$



Figure 7. Numerical study of Algorithm 1 with Algorithm 1 in [34] and Algorithm 3.2 in [16] when $u_0 = (1, 1, \dots, 1_{500}, 0, 0, \dots)$.



Figure 8. Numerical study of Algorithm 1 with Algorithm 1 in [34] and Algorithm 3.2 in [16] when $u_0 = (1, 2, \dots, 1000, 0, 0, \dots)$.

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	Number of Iterations			Execution Time in Seconds		
u_0	EgM	H-EgM	V-EgM	EgM	H-EgM	V-EgM
$(1, 1, \cdots, 1_{500}, 0, 0, \cdots)$	39	59	33	1.233697	1.849991	1.8499918
$(1, 2, \cdots, 1000, 0, 0, \cdots)$	55	68	49	1.773837	2.065840	1.5172537

Table 6. Numerical results values for Figures 7 and 8.

6. Conclusion

We have studied a viscosity type extragradient-like method for determining the solution of pseudomonotone equilibrium problem in real Hilbert spaces and also prove that the generated sequence converges strongly to the solution. Numerical conclusions were drawn to explain the numerical efficiency of our algorithms in comparison to other methods. These numerical studies showed that viscosity influences improve the efficiency of the iterative sequence in this context.

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Conflict of interest

No potential conflict of interest was reported by the author(s).

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