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Research article

Qualar curvatures of pseudo Riemannian manifolds and pseudo Riemannian submanifolds

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Abstract: Some relations involving the qualar and null sectional curvatures for a pseudo Riemannian manifold are obtained. These curvatures are also investigated for pseudo-Riemannian submanifolds. Obtained relations are discussed for some special submanifolds of a Lorentzian manifold.

Keywords: qualar curvature; null sectional curvature; pseudo-Riemannian submanifold; Lorentzian manifold
Mathematics Subject Classification: 53C42, 53C50

1. Introduction

In 1827, C. F. Gauss proved that if two smooth surfaces are isometric, then these surfaces have the same Gaussian curvature at corresponding points. Then the notion of curvature is one of the most attractive and significant topic in differential geometry. For a Riemannian submanifold, the main extrinsic curvature invariant is the squared mean curvature and the main intrinsic curvature invariants include classical curvature invariants namely the sectional curvature which corresponds to the Gaussian curvature in the Euclidean space, the Ricci curvature and the scalar curvature.

In pseudo-Riemannian settings, there exist null plane sections which the sectional curvature map can not be defined as classically on these sections. Therefore, S. Harris [16] introduced the null sectional curvature of a null plane. Later, A. L. Albujer and S. Haesen [1] showed that the null sectional curvature is proportional to the difference in length of the two spacelike closing geodesics which is a generalization of the interpretation of the sectional curvature in the Riemannian case as was observed by T. Levi-Civita [20]. Furthermore, there exist very qualified papers dealing the notion of null sectional curvatures on pseudo-Riemannian manifolds and their submanifolds (cf. [5, 9–11, 13, 17, 18]).

Another curvature invariant for pseudo-Riemannian manifolds and their submanifolds is the qualar curvature. Inspired by the crude mixture of terms quasi and scalar, M. Nardmann [22] introduced the notion of qualar curvature by decomposing scalar curvature. He also give some relations involving the qualar curvature and the scalar curvature for solving the Riemannian prescribed scalar curvature problem in a psedo-Riemannian manifold. Since the qualar curvature is a sum of sectional curvatures of some plane sections, one can consider that this curvature is a intrinsic invariant for a pseudo-Riemannian submanifold as Ricci curvature and scalar curvature.

Motivated by these facts, we investigate to qualar curvatures of pseudo Riemannian manifolds and pseudo-Riemannian submanifolds. Furthermore, we present some relations involving qualar curvatures and null sectional curvatures for these manifolds and their submanifolds. In Section 2, some basic facts related to pseudo-Riemannian manifolds are mentioned. In Section 3, some relations involving the qualar and null curvatures of a pseudo Riemnnian manifolds are obtained. In Section 4, with the help of Gauss formula, some results dealing qualar curvatures of pseudo-Riemannian submanifolds are given.

2. Pseudo Riemannian manifolds

Let $(\widetilde{M},\widetilde{g})$ be an m-dimensional pseudo-Riemannian manifold with the indefinite metric $\widetilde{g}$ of constant index q. The inner product of $\widetilde{g}$ is denoted by $\langle , \rangle$ throughout this paper. If $q = 1$ then $(\widetilde{M},\widetilde{g})$ is called a Lorentzian manifold. A pseudo-Riemannian manifold has constant curvature c is called a pseudo-Riemannian space form and it is usually denoted by $\widetilde{M}(c)$. For any pseudo-Riemannian space form $\widetilde{M}(c)$, there exists the following relation for any X, Y and Z vector fields in the tangent bundle $T\widetilde{M}$:

$$\widetilde{R}(X,Y)Z = c\{\langle Z,X\rangle Y - \langle Z,Y\rangle X\rangle\}. \tag{2.1}$$

Let $\{e_1,\ldots,e_q,e_{q+1},\ldots,e_m\}$ be an orthonormal basis of $T\widetilde{M}$. Suppose that $e_1,\ldots,e_q$ are timelike and $e_{q+1},\ldots,e_m$ are spacelike vectors. Then there exists two orthogonal distribution $\widetilde{V} = \mathrm{Span}\{e_1,\ldots,e_q\}$ and $\widetilde{H} = \mathrm{Span}\{e_{q+1},\ldots,e_m\}$ of $(\widetilde{M},\widetilde{g})$ such that we have the following $\widetilde{g}$-orthogonal decomposition:

$$T\widetilde{M} = \widetilde{V} \oplus \widetilde{H}. \tag{2.2}$$

Here, $\widetilde{V}$ and $\widetilde{H}$ are called as the maximally timelike and maximally spacelike distributions of $(\widetilde{M},\widetilde{g})$ respectively. Note that every maximally timelike and spacelike distributions are isomorphic as smooth vector bundles over on $\widetilde{M}$ [4]. Also, using the decomposition given in (2.2), one can consider this case as a special case of a pseudo-Riemannian almost product structure which is firstly defined by A. Gray [14] and studied by various geometers in [2,3,6,8,12,15,19,24].

Let $\Pi = \mathrm{Span}\{e_i,e_j\}$ be a non-degenerate plane section in $T\widetilde{M}$. The sectional curvature $\widetilde{K}(\Pi)$ of Π is defined by

$$\widetilde{K}(\Pi) \equiv \widetilde{K}(e_i,e_j) = \varepsilon_i\varepsilon_j\langle\widetilde{R}(e_i,e_j)e_j,e_i\rangle, \tag{2.3}$$

where $\varepsilon_i = \langle e_i,e_i\rangle = \mp 1$ for $i \neq j \in \{1,\ldots,m\}$. In the case of Π is a degenerate plane section spanned by a null vector ξ and a unit vector e_i, the sectional curvature so called null sectional curvature is defined by

$$\widetilde{K}(\Pi) \equiv \widetilde{K}(e_i,\xi) = \varepsilon_i\langle\widetilde{R}(e_i,\xi)\xi,e_i\rangle. \tag{2.4}$$

For a fixed $i \in \{1, \dots, m\}$, the Ricci curvature of e_i is defined by

$$\widetilde{\mathrm{Ric}}(e_i) = \sum_{j=1}^{m} \widetilde{K}(e_i, e_j). \tag{2.5}$$

The manifold is called an Einstein manifold if the Ricci curvature (tensor) is constant for all vector fields on $T\widetilde{M}$. The scalar curvature at a point $p \in \widetilde{M}$ is given by

$$\widetilde{\tau}(p) = \sum_{i<j}^{m} \widetilde{K}(e_i, e_j). \tag{2.6}$$

Furthermore, the qualar curvature at a point $p \in \widetilde{M}$, denoted by $\widetilde{\mathrm{qual}}(p)$, is defined by

$$\widetilde{\mathrm{qual}}(p) = 2\sum_{i=1}^{q}\sum_{s=q+1}^{m} \widetilde{K}(e_i, e_s). \tag{2.7}$$

We note that the qualar curvature is equal to the twice mixed scalar curvature of a pseudo Riemannian almost product manifold, for example, see the equation (3.49) in [24].

Let Π_k be a k-dimensional plane section on $T\widetilde{M}$ and $\{e_1, \dots, e_k\}$ be an orthonormal basis of Π_k. The Ricci curvature of a k-dimensional plane section Π_k at a vector field e_i is defined to be

$$\begin{aligned} \widetilde{\mathrm{Ric}}_{\Pi_k}(e_i) &= \sum_{i\neq j=1}^{k} \varepsilon_i\varepsilon_j\langle \widetilde{R}(e_i, e_j)e_j, e_i\rangle \\ &= \sum_{i\neq j=1}^{k} \widetilde{K}(e_i, e_j), \end{aligned} \tag{2.8}$$

where $i \in \{1, \dots, k\}$. The scalar curvature of Π_k at a point $p \in \widetilde{M}$ is defined to be

$$\begin{aligned} \widetilde{\tau}_{\Pi_k}(p) &= \sum_{1\leq i<j\leq k} \varepsilon_i\varepsilon_j\langle \widetilde{R}(e_i, e_j)e_j, e_i\rangle \\ &= \sum_{1\leq i<j\leq k} \widetilde{K}(e_i, e_j). \end{aligned} \tag{2.9}$$

3. Qualar curvatures and null sectional curvatures

Let $(\widetilde{M}, \widetilde{g})$ be an m-dimensional pseudo Riemannian manifold. Then, we can write null vectors using by timelike and spacelike vectors as follows:

$$\xi_i^s = \frac{1}{\sqrt{2}}\{e_i + e_s\},$$

where $i \in \{1, \dots, q\}$ and $s \in \{q+1, \dots, m\}$. In this case, the plane section spanned by ξ_i^s and e_s is a non-degenerate plane section and we have

$$\widetilde{K}(\xi_i^s, e_s) = -\widetilde{K}(e_i, e_s). \tag{3.1}$$

In a similar manner, the plane section spanned by ξ_i^s and e_i is a non-degenerate plane section and we also have

$$\widetilde{K}(\xi_i^s, e_i) = \widetilde{K}(e_i, e_s) \tag{3.2}$$

Let $\ell \in \{1, \ldots, m\}$ and $\ell \neq i$, $\ell \neq s$. The plane section spanned by ξ_i^s and e_ℓ is a degenerate plane section and its null sectional curvature is given by

$$\widetilde{K}(\xi_i^s, e_\ell) = -\frac{1}{2}\widetilde{K}(e_i, e_\ell) + \frac{1}{2}\widetilde{K}(e_s, e_\ell) + \varepsilon_\ell \langle \widetilde{R}(e_i, e_\ell)e_\ell, e_s\rangle. \tag{3.3}$$

Considering these facts, we obtain the following lemma:

Lemma 3.1. *Let $(\widetilde{M}, \widetilde{g})$ be an m-dimensional pseudo Riemannian manifold of index q. Then we have*

$$\widetilde{qual}(p) = \sum_{i=1}^{q} \sum_{s=q+1}^{m} [\widetilde{K}(\xi_i^s, e_i) - \widetilde{K}(\xi_i^s, e_s)]. \tag{3.4}$$

Proof. Putting (3.1) and (3.2) in (2.7), the proof of lemma is straightforward. □

Taking into consideration of (3.1), (3.2) and (3.3), we obtain the followings:

Lemma 3.2. *Let $(\widetilde{M}, \widetilde{g})$ be an m-dimensional pseudo Riemannian manifold of index q. Then the following relations hold:*

i) For $s \in \{q+1, \ldots, m\}$, we have

$$\sum_{\substack{i,\ell=1 \\ i\neq\ell}}^{q} \widetilde{K}(\xi_i^s, e_\ell) = \widetilde{\tau}_{\widetilde{V}}(p) - (q-1)\widetilde{Ric}_{\widetilde{V}}(e_s) + 2\sum_{\substack{i,\ell=1 \\ i\neq\ell}}^{q} \varepsilon_\ell \langle \widetilde{R}(e_i, e_\ell)e_\ell, e_s\rangle. \tag{3.5}$$

ii) For $i \in \{1, \ldots, q\}$, we have

$$\sum_{\substack{s,\ell=q+1 \\ s\neq\ell}}^{m} \widetilde{K}(\xi_i^s, e_\ell) = \widetilde{\tau}_{\widetilde{H}}(p) - (m-q-1)\widetilde{Ric}_{\widetilde{H}}(e_i) + 2\sum_{\substack{s,\ell=q+1 \\ s\neq\ell}}^{m} \varepsilon_\ell \langle \widetilde{R}(e_i, e_\ell)e_\ell, e_s\rangle. \tag{3.6}$$

Corollary 1. *If $(\widetilde{M}, \widetilde{g})$ is an m-dimensional indefinite space form, then we have*

$$\sum_{\substack{i,\ell=1 \\ i\neq\ell}}^{q} \widetilde{K}(\xi_i^t, e_\ell) = 0 \text{ and } \sum_{\substack{s,\ell=q+1 \\ s\neq\ell}}^{m} \widetilde{K}(\xi_j^s, e_\ell) = 0 \tag{3.7}$$

for $t \in \{q+1, \ldots, m\}$ and $j \in \{1, \ldots, q\}$.

Proof. Under the assumption, we have from the Eq (2.1) that

$$\langle \widetilde{R}(e_i, e_\ell)e_\ell, e_s\rangle = 0. \tag{3.8}$$

Taking into account of (2.8), (2.9), (3.8) and Lemma 3.2, the proof is straightforward. □

Corollary 2. *If $(\widetilde{M}, \widetilde{g})$ is an m-dimensional Lorentzian manifold, then the following relation holds:*

$$\sum_{1\neq\ell=2}^{m} \widetilde{K}(\xi_1^s, e_\ell) = \tau_{\widetilde{H}}(p) - (m-2)\widetilde{Ric}(e_1) \tag{3.9}$$

for $s \in \{q+1, \dots, m\}$.

Now we recall the following proposition of S. G. Harris (cf. Proposition 2.3 in [16]):

Proposition 3.3. *[16] A Lorentzian manifold of dimension at least three has constant curvature if and only if it has null sectional curvature everywhere zero.*

As a generalization of the result of Proposition 3.3 of S. G. Harris, we obtain the following:

Proposition 3.4. *Let $(\widetilde{M}, \widetilde{g})$ is an m-dimensional $(m > 3)$ Lorentzian manifold. Then we have the following situations:*

i. If $(\widetilde{M}, \widetilde{g})$ is an Einstein manifold, then there exists an orthonormal basis $\{e_1, \dots, e_q, e_{q+1}, \dots, e_m\}$ on $T\widetilde{M}$ satisfying

$$\sum_{1\neq\ell=2}^{m} \widetilde{K}(\xi_1^s, e_\ell) = 0. \tag{3.10}$$

ii. If $(\widetilde{M}, \widetilde{g})$ has null sectional curvature everywhere zero then it is an Einstein manifold.

Proof. Suppose that $(\widetilde{M}, \widetilde{g})$ is an Einstein manifold. In this case, we have

$$\widetilde{\tau}_{\widetilde{H}}(p) = (m-2)\,\widetilde{\mathrm{Ric}}(X) \tag{3.11}$$

for any unit timelike vector field $X \in T\widetilde{M}$. Putting (3.11) in (3.9), we obtain (3.10) which implies the statement (i).

The proof of statement (ii) is straightforward from Proposition 3.3.

□

4. Qualar curvatures of pseudo Riemannian submanifolds

Let (M, g) be n-dimensional pseudo Riemannian submanifold of $(\widetilde{M}, \widetilde{g})$ with the induced metric g of constant index q. The submanifold (M, g) is called as

i. a spacelike submanifold if $q = 0$,
ii. a timelike submanifold if $n = q$,
iii. a pseudo-Riemannian submanifold if it is neither spacelike nor timelike.

Let us denote the Riemannian connections with respect to the pseudo Riemannian metric $\widetilde{g}$ by $\widetilde{\nabla}$ and the induced pseudo Riemannian connection on M by ∇ respectively. The Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{4.1}$$
$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{4.2}$$

for any tangent vector fields X and Y and N normal to M, where σ denotes the second fundamental form and $\nabla^{\perp}$ the normal connection and A the shape operator of M. Also, it is known that the tensors A and σ are related by the following equation:

$$\langle A_N X, Y\rangle = \langle \sigma(X, Y), N\rangle. \tag{4.3}$$

Let R denotes the Riemannian curvature tensor of the submanifold. The equation of Gauss is given by

$$\langle R(X, Y)Z, W\rangle = \langle \widetilde{R}(X, Y)Z, W\rangle + \langle \sigma(X, W), \sigma(Y, Z)\rangle - \langle \sigma(X, Z), \sigma(Y, W)\rangle \tag{4.4}$$

for any $X, Y, Z, W \in TM$. Using the Gauss equation, one has

$$\begin{aligned} \varepsilon_i\varepsilon_j\langle R(e_j, e_i)e_i, e_j\rangle &= \varepsilon_i\varepsilon_j\langle \widetilde{R}(e_j, e_i)e_i, e_j\rangle + \varepsilon_i\varepsilon_j \left\langle \sigma(e_i, e_i), \sigma(e_j, e_j)\right\rangle \\ &\quad -\varepsilon_i\varepsilon_j \left\langle \sigma(e_i, e_j), \sigma(e_j, e_i)\right\rangle. \end{aligned} \tag{4.5}$$

Thus, we have

$$\tau(p) = \widetilde{\tau}_{T_pM}(p) + \sum_{r,s=n+1}^{m} \widetilde{\varepsilon}_r\widetilde{\varepsilon}_s \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j\sigma^r_{ii}\sigma^s_{jj} - \sum_{r=n+1}^{m} \widetilde{\varepsilon}_r \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j(\sigma^r_{ij})^2, \tag{4.6}$$

where

$$\sigma(e_i, e_j) = \sum_{r=n+1}^{m} \varepsilon_r\sigma^r_{ij}. \tag{4.7}$$

Here, $\varepsilon_r = \langle e_r, e_e\rangle$ for any $r \in \{n+1, \ldots, m\}$. The mean curvature vector $\hbar(p)$ at $p \in M$ is defined by

$$\hbar(p) = \frac{1}{n}\text{trace}(A_N) = \frac{1}{n}\sum_{j=1}^{n} \varepsilon_j\sigma(e_j, e_j). \tag{4.8}$$

Now we can write

$$\begin{aligned} \sigma(X, Y) &= \sigma^{\widetilde{V}}(X, Y) + \sigma^{\widetilde{H}}(X, Y), \quad X, Y \in TM, \\ \hbar(p) &= \hbar|_{\widetilde{V}}(p) + \hbar|_{\widetilde{H}}(p), \end{aligned} \tag{4.9}$$

where $\sigma^{\widetilde{V}}(X, Y)$, $\hbar|_{\widetilde{V}}(p) \in \widetilde{V}$ belong to the vertical distribution $\widetilde{V}$ and $\sigma^{\widetilde{H}}(X, Y)$, $\hbar|_{\widetilde{H}}(p)$ belong to the horizontal distribution $\widetilde{H}$. Note that these decompositions are unique.

The submanifold M is called totally geodesic if $\sigma = 0$, and it is called minimal if $\hbar = 0$. If $\sigma(X, Y) = \langle X, Y\rangle \hbar$ for all $X, Y \in TM$, then M is called totally umbilical. Also, M is called quasi-minimal if $\hbar \neq 0$ and $\langle \hbar(p), \hbar(p)\rangle = 0$ at each point $p \in M$. For more details, we refer to [7, 8, 23].

Let $\{e_1, \ldots, e_q, e_{q+1}, \ldots, e_n\}$ be an orthonormal basis on TM, where $e_1, \ldots, e_q$ are timelike and $e_{q+1}, \ldots, e_n$ are spacelike vectors. Since (M, g) is a pseudo Riemannian submanifold, we can write TM as orhogonal direct sum of its maximally spacelike distribution H and maximally timelike distribution V as follows:

$$TM = V \oplus H, \tag{4.10}$$

where $V = \mathrm{Span}\{e_1, \ldots, e_q\}$ and $H = \mathrm{Span}\{e_{q+1}, \ldots, e_n\}$.

More specifically, M is called timelike $\mathtt{V}$-geodesic if $\sigma^{\widetilde{V}}\Big|_V = 0$, timelike H-geodesic if $\sigma^{\widetilde{V}}\Big|_H = 0$, timelike mixed geodesic if $\sigma^{\widetilde{V}}\Big|_{V\times H} = 0$, timelike geodesic if $\sigma^{\widetilde{V}} = 0$, spacelike V-geodesic if $\sigma^{\widetilde{H}}\Big|_V = 0$, spacelike H-geodesic if $\sigma^{\widetilde{H}}\Big|_H = 0$, spacelike mixed geodesic if $\sigma^{\widetilde{H}}\Big|_{V\times H} = 0$, spacelike geodesic if $\sigma^{\widetilde{H}} = 0$, mixed geodesic if $\sigma|_{V\times H} = 0$ [25].

Proposition 4.1. *Let M be an n-dimensional pseudo Riemannian submanifold of $(\widetilde{M}, \widetilde{g})$ and $p \in M$. Then we have*

$$qual(p) \;=\; \widetilde{qual}_{T_pM}(p) + 2n\langle trace_V\sigma, \hbar\rangle - 2|trace_V\sigma|^2 - 2|\sigma_{V\times H}|^2, \tag{4.11}$$

where $trace_V$ denotes the trace with respect to the maximally timelike distribution V of M.

Proof. From (4.5), we get

$$\langle R(e_i, e_s)e_s, e_i\rangle \;=\; \langle \widetilde{R}(e_i, e_s)e_s, e_i\rangle - \langle \sigma(e_s, e_s), \sigma(e_i, e_i)\rangle + \langle \sigma(e_i, e_s), \sigma(e_i, e_s)\rangle$$

for $i \in \{1, \ldots, q\}$ and $s \in \{q + 1, \ldots, n\}$. Therefore, we can write

$$\begin{aligned}\sum_{i=1}^{q}\sum_{s=q+1}^{n}\langle R(e_i, e_s)e_s, e_i\rangle \;=\; & \sum_{i=1}^{q}\sum_{s=q+1}^{n}\langle \widetilde{R}(e_i, e_s)e_s, e_i\rangle + \sum_{i=1}^{q}\sum_{s=1}^{n}\langle \sigma(e_s, e_s), \sigma(e_i, e_i)\rangle \\ & - \sum_{i,s=1}^{q}\langle \sigma(e_s, e_s), \sigma(e_i, e_i)\rangle - \sum_{i=1}^{q}\sum_{s=q+1}^{n}\langle \sigma(e_i, e_s), \sigma(e_i, e_s)\rangle .\end{aligned} \tag{4.12}$$

Using (2.7) in (4.12), we obtain

$$\begin{aligned}\mathrm{qual}(p) \;=\; & \widetilde{\mathrm{qual}}_{T_pM}(p) + 2\sum_{i=1}^{q}\sum_{s=1}^{n}\langle \sigma(e_s, e_s), \sigma(e_i, e_i)\rangle - 2\sum_{i,s=1}^{q}\langle \sigma(e_s, e_s), \sigma(e_i, e_i)\rangle \\ & -2\sum_{i=1}^{q}\sum_{s=q+1}^{n}\langle \sigma(e_i, e_s), \sigma(e_i, e_s)\rangle ,\end{aligned}$$

which implies (4.11). □

For the special case $q = 1$ of Lemma 4.1, we get the following corollary:

Corollary 3. *Let M be an n-dimensional pseudo Riemannian submanifold with index 1 of $(\widetilde{M}, \widetilde{g})$. For any unit timelike vector X in T_pM, the following relation satisfies:*

$$qual(p) \;=\; \widetilde{qual}_{T_pM}(p) + 2n\langle\sigma(X,X),\hbar\rangle - 2|\sigma(X,X)|^2 - 2|\sigma_{\{X\}\times H}|^2. \tag{4.13}$$

For the special case $q = \widetilde{q} = 1$ of Lemma 4.1, we get the following corollary:

Corollary 4. *Let M be an n-dimensional timelike submanifold of a Lorentzian manifold $\widetilde{M}$. For any unit timelike vector X in T_pM, the following relation satisfies:*

$$qual(p) = \widetilde{qual}_{T_pM}(p) + 2n\langle\sigma(X,X),\hbar\rangle - 2\|\sigma(X,X)\|^2 - 2\|\sigma_{\{X\}\times H}\|^2. \tag{4.14}$$

Proof. Since the indexes of both submanifold and ambient manifold are equal to each other, we see that the second fundamental form which lies in the normal space becomes a spacelike vector. Using this fact, the proof of corollary is straightforward from Corollary 3. □

Theorem 4.2. *Let M be a timelike submanifold of a Lorentzian manifold. For any unit timelike vector X in T_pM, the following relation satisfies:*

$$qual(p) \leq \widetilde{qual}_{T_pM}(p) + 2n\langle\sigma(X,X),\hbar\rangle. \tag{4.15}$$

The equality case of (4.15) *holds for all unit vectors X in T_pM if and only if M is timelike V-geodesic and mixed geodesic.*

Proof. Using (4.14), we obtain the Eq (4.15). The equality case of (4.15) holds for all unit vectors X in T_pM if and only if

$$\sigma(X,X) = 0, \quad \text{and} \quad \sigma_{\{X\}\times H} = 0 \tag{4.16}$$

which imply that M is timelike V-geodesic and mixed geodesic. □

Corollary 5. *Let M be a timelike submanifold of an n-dimensional Minkowski space. For any unit timelike vector X in TM, the following inequality satisfies:*

$$qual(p) \leq 2n\langle\sigma(X,X),\hbar\rangle. \tag{4.17}$$

The equality case of (4.17) *holds for all unit vectors X in TM if and only if M is timelike V-geodesic and mixed geodesic.*

Corollary 6. *Let M be a minimal timelike submanifold of a Minkowski space. Then we have*

$$qual(p) \leq 0. \tag{4.18}$$

The equality case of (4.18) *holds for all $p \in M$ if and only if M is timelike V-geodesic and mixed geodesic.*

Corollary 7. *Every minimal timelike submanifold of a Minkowski space is of constant curvature if and only if it is timelike V-geodesic and mixed geodesic.*

Theorem 4.3. *Let M be a totally umbilical timelike submanifold of a Lorentzian manifold. Then we have*

$$2n\|\hbar(p)\|^2 < qual(p) - \widetilde{qual}_{T_pM}(p). \tag{4.19}$$

Proof. Since M is totally umbilical, we have from (4.14) that

$$2n\|\hbar(p)\|^2 \leq \mathrm{qual}(p) - \widetilde{\mathrm{qual}}_{T_pM}(p) \tag{4.20}$$

with the equality if and only if M is timelike V-geodesic and mixed geodesic. Therefore, M becomes totally geodesic which contradicts that the submanifold is totally umbilical. □

Corollary 8. *Let M be a totally umbilical timelike submanifold of n-dimensional Minkowski space. Then we have the following inequality for all $p \in M$:*

$$2n\|\hbar(p)\|^2 < qual(p). \tag{4.21}$$

Now, we shall recall the following Lemma of B. Y. Chen (cf. Lemma 3.2 in [8]).

Lemma 4.4. *Let $\varphi : M \to R^m_q(c)$ be an isometric immersion of a pseudo Riemannian manifold M into an indefinite real space form $R^m_q(c)$. If M is totally umbilical, then*

i) *$\hbar$ is a parallel normal vector field, i.e., $\widetilde{\nabla}\hbar = 0$;*
ii) *$\langle\hbar, \hbar\rangle$ is constant;*
iii) *ϕ is a parallel immersion, i.e., $\widetilde{\nabla}\sigma = 0$ identically on M;*
iv) *M is of constant curvature $c + \langle\hbar, \hbar\rangle$;*
v) *$A_H = \langle\hbar, \hbar\rangle I$, where I denotes the identity transformation;*
vi) *M is a parallel submanifold.*

From the *iv*) statement of Lemma 4.4, Corollary 8, we see a contradiction for totally umbilical timelike submanifold in a Minkowski space. Thus, we get the following result:

Corollary 9. *There exist no totally umbilical timelike submanifold in a Minkowski space.*

Now, we shall recall the following theorem of J. Li [21]

Theorem 4.5. *If M is a totally umbilical hypersurface of a Minkowski space, then either M is a Riemannian space form or a locally Minkowski space.*

Remark 1. We note that Corollary 9 is an another proof way of Theorem 4.5 of J. Li [21].

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Conflict of interest

The author declares that there is no competing interest.

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