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Research article

Existence of nontrivial solutions for Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian and local nonlinearity

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Abstract: In this paper, we deal with the existence of nontrivial solutions for the following Kirchhoff-type equation

$$M\left(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)_p^s u+V(x)|u|^{p-2}u=\lambda f(x,u),\ \text{ in } \mathbb{R}^N,$$

where $0<s<1<p<\infty$, $sp<N$, $\lambda>0$ is a real parameter, $(-\Delta)_p^s$ is the fractional p-Laplacian operator, $V:\mathbb{R}^N\to\mathbb{R}^+$ is a potential function, M is a Kirchhoff function, the nonlinearity $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a continuous function and just super-linear in a neighborhood of $u=0$. By using an appropriate truncation argument and the mountain pass theorem, we prove the existence of nontrivial solutions for the above equation, provided that λ is sufficiently large. Our results extend and improve the previous ones in the literature.

Keywords: fractional p-Laplacian; Kirchhoff equations; local nonlinearity; Moser iteration method; truncation argument
Mathematics Subject Classification: 35R11, 35J60, 35J20

1. Introduction

In this paper, we investigate the existence of nontrivial solutions for Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian and local nonlinearity. More precisely, we consider the following Kirchhoff-type equation

$$M\left(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)_p^s u+V(x)|u|^{p-2}u=\lambda f(x,u),\ \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $0 < s < 1 < p < \infty$, $sp < N$, $\lambda > 0$ is a real parameter, $(-\Delta)_p^s$ is the fractional p-Laplacian operator which, up to a normalization constant, may be defined as

$$(-\Delta)_p^s \phi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\phi(x) - \phi(y)|^{p-2} (\phi(x) - \phi(y))}{|x - y|^{N+sp}} \mathrm{d}y, \ \forall\, x \in \mathbb{R}^N,$$

along any $\phi \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x - y| < \varepsilon\}$. In fact, many scholars have paid more attention to the fractional and nonlocal operators in recent years. This type of operator occurs naturally in many field of science, such as finance, continuum mechanics, free boundary obstacle problems, population dynamics, plasma physics and anomalous diffusion. For more details on this type of operator, we refer to [2, 5, 6] and the references therein. For the fractional Sobolev spaces and the study of the fractional Laplacian by using variational methods, we refer the readers to [9, 20, 27]. In order to simplify our statements, we first suppose on the potential function V that

(V_1) $V \in C(\mathbb{R}^N)$ and there exists a constant $V_0 > 0$ such that $\inf\limits_{x \in \mathbb{R}^N} V(x) \geq V_0$;

(V_2) there exists $R > 0$ such that

$$\lim_{|y| \to \infty} \mathrm{meas}\left(\{x \in B_R(y) : V(x) \leq d\}\right) = 0 \text{ for any } d \in \mathbb{R}^+.$$

Condition (V_2) was originally from [3], it can be used to solve the problem of the lack of compactness in the whole space $\mathbb{R}^N$. Moreover, the condition (V_2) is weaker than the coercivity condition $\lim\limits_{|x| \to \infty} V(x) = \infty$.

As we all know, Kirchhoff equations was first proposed by Kirchhoff in 1883 (see [15]), which was related to the celebrated D'Alembert wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \mathrm{d}x \right) \frac{\partial^2 u}{\partial x^2} = f(x, u),$$

for free vibrations of elastic strings. Here, ρ is the mass density, P_0 is the initial tension, h is the area of the cross section, E is the Young modulus of the material and L is the length of the string. Bernstein [4] and Pohozaev [24] were the early scholars devoted to study Kirchhoff equations. After Lions [18] proposed an abstract framework for Kirchhoff problems, many scholars have studied Kirchhoff equations by using variational methods. For example, we refer to [12, 19] for Kirchhoff equations involving subcritical nonlinearities; we also collect some articles, see [17, 21] for Kirchhoff equations involving critical and supercritical nonlinearities.

Recently, Fiscella and Valdinoci [11] proposed a stationary Kirchhoff type variational model, which considered the nonlocal aspect of the tension arising from nonlocal measurements of fractional length of the string. More precisely, they studied the following fractional Kirchhoff type problem involving critical growth

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right) (-\Delta)^s u = \lambda f(x, u) + |u|^{2_s^* - 2} u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

By combining a truncated technique with the mountain pass theorem, they obtained the existence of nontrivial solutions for the above equation when λ is large enough. Afterwards, many scholars have studied the existence and multiplicity of nontrivial solutions, ground state solutions, sign-changing solutions for fractional Kirchhoff type equations. For the subcritical case, Pucci et al. [25] obtained the existence of multiple solutions for fractional p-Laplacian Schrödinger-Kirchhoff type equations via the Ekeland variational principle and the mountain pass theorem. For the critical case, Fiscella [10] provided the existence of two solutions for fractional Kirchhoff equation with singular term and critical nonlinearity via variational methods. By using the concentration compactness principle in fractional Sobolev spaces, Xiang et al. [30] established the existence and multiplicity of solutions for a class of fractional p-Laplacian Kirchhoff type problems involving critical exponent. Moreover, there are few results about the existence of solutions for fractional Kirchhoff problems involving the supercritical term. Fortunately, Ambrosio and Servadei [1] first studied the existence of nontrivial solutions for factional Kirchhoff problems with supercritical growth by using a truncation argument, the mountain pass theorem and Moser iterative method. For more related results on fractional Kirchhoff type equations, we refer the interested reader to [23, 26, 29, 31] and the references therein.

However, many scholars usually supposed that the nonlinearity $f(u)$ satisfies the growth condition $|f(u)| \leq C(|u| + |u|^{q-1})$, $q \in (p, p_s^*)$ and the following conditions
$(S1)$ $\frac{f(u)}{|u|^3}$ is an increasing function of $u \in \mathbb{R} \setminus \{0\}$;
$(S2)$ $\lim\limits_{|u| \to +\infty} \frac{F(u)}{u^4} = +\infty$;
or the Ambrosetti-Rabinowitz condition
(AR) there exists $\mu \in (p, p_s^*)$ such that $0 < \mu F(u) \leq f(u)u$ for all $u \in \mathbb{R}$.
Indeed, under the above conditions, it is easy to obtain the existence and multiplicity of solutions for fractional Kirchhoff type equations by using variational methods. For example, Cheng and Gao [7] established the existence of least energy sign-changing solutions when $f(u)$ satisfies $(S1)$ and $(S2)$. Under the nonlinearity $f(u)$ satisfies (AR) condition, Nyamoradi and Zaidan [22], they proved the existence of nontrivial solutions. In addition, some scholars studied the existence of solutions for fractional Kirchhoff type equations when $f(u)$ satisfies the Berestycki-Lions type conditions, see for instance [14, 33].

As far as we know, there are no papers dealing with the fractional Kirchhoff type equation with local nonlinearity, which the nonlinearity $f(u)$ is superlinear just in a neighborhood of $u = 0$. There is no doubt that serious difficulties will be encountered, because there is no assumption about the function $f(u)$ at infinity. This difficulty makes the study of fractional Kirchhoff type equations become more interesting and challenging. In order to overcome this difficulty, Li and Su [16] used a truncation argument due to Costa and Wang (see [8]) to get the existence and multiplicity of solutions for Kirchhoff type equations. In [13], Huang and Jia also obtained the existence of positive solutions for quasilinear Schrödinger equations via the truncation argument in [8].

Motivated by the above works, the purpose of this paper is to give existence results for Eq (1.1). To the best of our knowledge, the existence of solutions for fractional Kirchhoff equations with local nonlinearity has not been studied yet. Before stating our result, we consider $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $\mathbb{R}_0^+ = [0, +\infty)$, is supposed to satisfy the following conditions:
(M_1) $M \in C(\mathbb{R}_0^+)$ and there exists $m_0 > 0$ such that $M(t) \geq m_0$ for any $t \in \mathbb{R}_0^+$;
(M_2) the function $t \mapsto M(t)$ is increasing;

(M_3) for each $t_1 \geq t_2 > 0$, it holds

$$\frac{M(t_1)}{t_1} - \frac{M(t_2)}{t_2} \leq m_0 \left(\frac{1}{t_1} - \frac{1}{t_2} \right).$$

Moreover, we assume the following assumptions on f:
(f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $f(x,t) = 0$ for all $t \leq 0$ and there exists $\alpha \in (2p, p_s^*)$ such that

$$\limsup_{t \to 0^+} \frac{f(x,t)}{t^{\alpha-1}} < +\infty;$$

(f_2) there exists $\sigma \in (2p, p_s^*)$ and $\sigma > \alpha$ such that

$$\liminf_{t \to 0^+} \frac{F(x,t)}{t^{\sigma}} > 0;$$

(f_3) there exists $\mu \in (2p, p_s^*)$ such that

$$tf(x,t) \geq \mu F(x,t) > 0 \text{ for } t > 0 \text{ small},$$

where $p_s^* = \frac{Np}{N-sp}$ and $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$.

In the following, let's state our result.

Theorem 1.1. *Suppose that* (V_1)*,* (V_2)*,* (M_1)*–*(M_3) *and* (f_1)*–*(f_3) *are satisfied. Then there exists* $\lambda_0 > 0$ *such that Eq* (1.1) *admits a nontrivial solution for all* $\lambda > \lambda_0$.

The energy functional associated with Eq (1.1) is given by

$$I_\lambda(u) = \frac{1}{p}\left[\mathcal{M}\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \right) + \int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x \right] - \lambda \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x.$$

Due to we assume that the nonlinearity $f(x,u)$ is superlinear only in a neighborhood of $u = 0$, the energy functional may be not well defined. Therefore, we can not directly use the variational method to prove the existence of solutions. In order to prove Theorem 1.1, we will use a truncation argument, which came from [8, 13, 16]. More precisely, we first show that the existence of nontrivial solutions for the revised equation via the mountain pass theorem. By Moser iteration method and L^∞-estimate, we can obtain solution of revised equation, which is the solution of the original Eq (1.1) when λ is sufficiently large. In addition, our assumptions on the nonlinearity f just in a neighborhood of the origin, which are greatly relax. The results of this paper are new and can enrich the previous ones in the literature.

Remark 1.2. *It is worth mentioning that this paper is the first time to assume the nonlinearity* $f(x,u)$ *just in a neighborhood at* $u = 0$ *and discuss existence of solutions for fractional Kirchhoff-type equations. Therefore, our results are new, and enrich the previous ones in the literature.*

This paper is organized as follows. In Section 2, we recall some basic properties of the fractional Sobolev spaces and introduce a truncation argument. In Section 3, we give the proof of Theorem 1.1.

2. Preliminaries

In this section, we first recall some basic results of the fractional Sobolev spaces. Let $0 < s < 1 < p < \infty$ be real numbers and $N > sp$. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is given by

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} < +\infty \right\},$$

where $[u]_{s,p}$ is the Gagliardo semi-norm, namely,

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}$$

and $W^{s,p}(\mathbb{R}^N)$ is equipped with the following norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|_{L^p(\mathbb{R}^N)} + [u]_{s,p}^p \right)^{\frac{1}{p}}.$$

It is well known that $W^{s,p}(\mathbb{R}^N) = \left(W^{s,p}(\mathbb{R}^N), \|\cdot\|_{W^{s,p}(\mathbb{R}^N)} \right)$ is a uniformly convex Banach space.

Moreover, let $p \in [1, \infty)$ be a real number, $L^p(\mathbb{R}^N, V)$ denotes the Lebesgue space of real-valued functions and equipped with the norm

$$\|u\|_{p,V} = \left(\int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x \right)^{\frac{1}{p}} \text{ for all } u \in L^p(\mathbb{R}^N, V).$$

Under the condition (V_1), we can know that $L^p(\mathbb{R}^N, V) = (L^p(\mathbb{R}^N, V), \|\cdot\|_{p,V})$ is a uniformly convex Banach space.

Now, let X denotes the completion of $C_0^\infty(\mathbb{R}^N)$ in $W^{s,p}(\mathbb{R}^N)$ and endowed with the norm

$$\|u\|_X = \left([u]_{s,p}^p + \|u\|_{p,V}^p \right)^{\frac{1}{p}}. \tag{2.1}$$

According to Lemma 10 in the Appendix of [25], we also know that $X = (X, \|\cdot\|_X)$ is a uniformly convex Banach space. Furthermore, X is a reflexive Banach space. The dual space of $(X, \|\cdot\|_X)$ is denoted by $(X^*, \|\cdot\|_{X^*})$.

Denote the best fractional Sobolev constant:

$$S_{s,p}^* = \inf_{u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\displaystyle\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \mathrm{d}x\mathrm{d}y}{\left(\displaystyle\int_{\mathbb{R}^N} |u(x)|^{p_s^*} \mathrm{d}x \right)^{\frac{p}{p_s^*}}}. \tag{2.2}$$

Lemma 2.1. ([25]) *Assume that V satisfies (V_1). If $r \in [p, p_s^*]$, then the embeddings*

$$X \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$$

are continuous. Thus, there exists a constant $C_r > 0$ such that

$$\|u\|_{L^r(\mathbb{R}^N)} \leq C_r \|u\|_X \quad \textit{for all } u \in X. \tag{2.3}$$

Moreover, if $r \in [p, p_s^*)$, then the embedding $X \hookrightarrow\hookrightarrow L^r(B_R)$ is compact for any $R > 0$.

Lemma 2.2. ([25]) *Suppose that (V_1) and (V_2) are satisfied. Let $r \in [p, p_s^*)$ be a fixed exponent. If $\{u_n\}$ is a bounded sequence in X, then there exists $u_0 \in X \cap L^r(\mathbb{R}^N)$ such that up to a subsequence,*

$$u_n \to u_0 \textit{ strongly in } L^r(\mathbb{R}^N),$$

as $n \to +\infty$, *for all* $r \in [p, p_s^*)$.

Next, we show that the definitions of $(PS)_c$ condition and the mountain pass theorem.

Definition 2.3. *Let* $\mathcal{E} \in C^1(X, \mathbb{R})$. *We say the* $\mathcal{E}$ *satisfies the* $(PS)_c$ *condition at level* $c \in \mathbb{R}$ *in X, if any* $(PS)_c$ *sequence* $\{u_n\} \subset X$, *that is,* $\mathcal{E}(u_n) \to c$, $\mathcal{E}'(u_n) \to 0$ *in* X^*, *as* $n \to \infty$, *admits a convergent subsequence in X.*

Theorem 2.4. ([32]) *Let X be a real Banach space, suppose* $\mathcal{E} \in C^1(X, \mathbb{R})$ *satisfies the* $(PS)_c$ *condition with* $\mathcal{E}(0) = 0$. *Moreover,*

(i) there exist $\varrho, \eta > 0$ *such that* $\mathcal{E}(u) \geq \eta$ *for all* $u \in X$, *with* $\|u\|_X = \varrho$,

(ii) there exists $e \in X$ *satisfying* $\varrho < \|e\|_X$ *such that* $\mathcal{E}(e) < 0$.

Define

$$\Gamma = \{\gamma \in C^1([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Then

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{E}(\gamma(t))$$

is a critical value of $\mathcal{E}(u)$.

In order to prove Theorem 1.1, we need to modify and extend f to a suitable $\widetilde{f}$. The argument was developed in [8, 13, 16]. Based on this, we make the following truncation argument.

According to (f_1) and (f_2), we know that there exist two positive constants $A, B > 0$ such that

$$F(x, t) \leq At^{\alpha},$$

$$F(x, t) \geq Bt^{\sigma},$$

for $0 < t \leq 2\delta$, where δ is a positive constant that satisfies $0 < \delta < \frac{1}{2}$ and $x \in \mathbb{R}^N$.

For fixed $\delta > 0$, let $d(t) \in C^1(\mathbb{R}, [0, 1])$ be an even cut-off function satisfying $td'(t) \leq 0$,

$$d(t) = \begin{cases} 1, & \text{if } t \leq \delta, \\ 0, & \text{if } t \geq 2\delta, \end{cases} \tag{2.4}$$

and $|td'(t)| \leq \frac{2}{\delta}$ for $t \in \mathbb{R}$. Set

$$\widetilde{F}(x, t) = d(t)F(x, t) + (1 - d(t))F_\infty(x, t), \quad \widetilde{f}(x, t) = \widetilde{F}'(x, t), \tag{2.5}$$

where

$$F_\infty(x, t) = \begin{cases} At^{\alpha}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases} \tag{2.6}$$

It follows from the definition of $d(t)$ and (f_1)–(f_3) that $\widetilde{f}(x, t)$ has the following properties.

Lemma 2.5.([13, 16]) *If* (f_1)–(f_3) *are satisfied, then*

(i) there exists a constant $C > 0$ *such that*

$$\widetilde{f}(x, t) \leq Ct^{\alpha - 1}, \ \forall\ t > 0,$$

(ii) it hold that

$$0 < k\widetilde{F}(x, t) \leq t\widetilde{f}(x, t), \ \forall\ t > 0 \text{ and } k = \min\{\alpha, \mu\}.$$

Finally, we introduced the well-known Simon inequality as follows, which will be used later.

Lemma 2.6.([28]) *There exist constants* $c_p, C_p > 0$ *such that for any* $x, y \in \mathbb{R}^N$*, it holds*

$$|x-y|^p \le \begin{cases} c_p\left(|x|^{p-2}x - |y|^{p-2}y\right)(x-y), & p \ge 2, \\ C_p\left[\left(|x|^{p-2}x - |y|^{p-2}y\right)(x-y)\right]^{\frac{p}{2}}(|x|^p + |y|^p)^{\frac{(2-p)}{2}}, & 1 < p < 2. \end{cases}$$

3. Proof the Theorem 1.1

In this section, we will complete the proof of Theorem 1.1. We first prove that the existence of nontrivial solutions for the modified equation. More precisely, we consider the following fractional Kirchhoff type equation

$$M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y\right)(-\Delta)_p^s u + V(x)|u|^{p-2}u = \lambda \widetilde{f}(x,u), \text{ in } \mathbb{R}^N, \tag{3.1}$$

where $\widetilde{f}$ is given by (2.5). The energy functional $\widetilde{I}_\lambda : X \to \mathbb{R}$ associated with Eq (3.1)

$$\widetilde{I}_\lambda(u) = \frac{1}{p}\left(\mathcal{M}([u]_{s,p}^p) + \|u\|_{p,V}^p\right) - \lambda \int_{\mathbb{R}^N} \widetilde{F}(x,u)\mathrm{d}x, \ u \in X, \tag{3.2}$$

where $\mathcal{M}(t) = \int_0^t M(\tau)\mathrm{d}\tau$. Under the assumptions of Theorem 1.1, by using the similar proof method in [25], we can know that $\widetilde{I}_\lambda \in C^1(X, \mathbb{R})$ and

$$\begin{aligned} \langle \widetilde{I}_\lambda'(u), v\rangle = M([u]_{s,p}^p) &\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\ &+ \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)v(x)\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \widetilde{f}(x,u(x))v(x)\mathrm{d}x, \end{aligned} \tag{3.3}$$

for any $u, v \in X$. Obviously, the critical points of the energy functional $\widetilde{I}_\lambda$ are exactly the weak solutions of Eq (3.1).

In the following, let us first verify that $\widetilde{I}_\lambda$ has mountain pass geometry.

Lemma 3.1. *Suppose that* (V_1), (V_2), (M_1) *and* (f_1)–(f_3) *are satisfied. Then there exist* $\zeta_\lambda, \rho_\lambda > 0$ *such that* $\widetilde{I}_\lambda(u) \ge \zeta_\lambda$ *for any* $u \in X$*, with* $\|u\|_X = \rho_\lambda$.

Proof. By means of $\alpha > 2p$ and Lemma 2.1, there exists $C_\alpha > 0$ such that $\|u\|_{L^\alpha(\mathbb{R}^N)} \le C_\alpha \|u\|_X$. From (M_1), (2.3), (3.2) and Lemma 2.5 (i), we get

$$\begin{aligned} \widetilde{I}_\lambda(u) &= \frac{1}{p}\left(\mathcal{M}([u]_{s,p}^p) + \|u\|_{p,V}^p\right) - \lambda \int_{\mathbb{R}^N} \widetilde{F}(x,u)\mathrm{d}x \\ &\ge \frac{1}{p}\left(m_0[u]_{s,p}^p + \|u\|_{p,V}^p\right) - \lambda C\|u\|_{L^\alpha(\mathbb{R}^N)}^\alpha \\ &\ge \frac{1}{p}\min\{m_0, 1\}\|u\|_X^p - \lambda C_1\|u\|_X^\alpha. \end{aligned}$$

Since $\alpha > 2p$, we can find $\zeta_\lambda, \rho_\lambda > 0$ such that $\widetilde{I}_\lambda(u) \geq \zeta_\lambda$ for any $u \in X$, with $\|u\|_X = \rho_\lambda$. This completes the proof. □

Lemma 3.2. *Suppose that* (V_1), (V_2), (M_1)–(M_3) *and* (f_1)–(f_3) *are satisfied. Then there exists* $\bar{u} \in X$ *such that* $\widetilde{I}_\lambda(\bar{u}) < 0$.

Proof. It follows from Lemma 2.5 (*ii*) that

$$\widetilde{F}(x, t) \geq Ct^k \text{ for all } t > 0. \tag{3.4}$$

From (M_1)–(M_3), there exists a constant $b > 0$ such that

$$M(t) \leq b(1 + t), \ \forall\, t \geq 0. \tag{3.5}$$

For $u \in C_0^\infty(\mathbb{R}^N)$, with $\|u\|_X = 1$. According to $k > 2p$, (3.4) and (3.5), we get

$$\begin{aligned}
\widetilde{I}_\lambda(tu) &= \frac{1}{p}\left(\mathcal{M}([tu]_{s,p}^p) + \|tu\|_{p,V}^p\right) - \lambda \int_{\mathbb{R}^N} \widetilde{F}(x, tu)\mathrm{d}x \\
&\leq \frac{1}{p}\left(bt^p[u]_{s,p}^p + \frac{b}{2}t^{2p}[u]_{s,p}^{2p} + t^p\|u\|_{p,V}^p\right) - \lambda C_1 t^k \int_{\mathbb{R}^N} |u|^k \mathrm{d}x \\
&\leq C_0 t^{2p} - \lambda C_1 t^k \int_{\mathbb{R}^N} |u|^k \mathrm{d}x.
\end{aligned}$$

Consequently, we can take $\bar{u} = tu \in X$ such that $\widetilde{I}_\lambda(\bar{u}) < 0$ for t sufficiently large. □

Now, similar to the proof of Lemma 6 in [25], we prove that $\widetilde{I}_\lambda$ satisfies $(PS)_c$ condition.

Lemma 3.3. *Suppose that* (V_1), (V_2), (M_1)–(M_3) *and* (f_1)–(f_3) *are satisfied. Then* $\widetilde{I}_\lambda$ *satisfies the* $(PS)_c$ *condition.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset X$ be a $(PS)_c$ sequence for the functional $\widetilde{I}_\lambda$ at level $c \in \mathbb{R}$, that is,

$$\widetilde{I}_\lambda(u_n) \to c, \ \widetilde{I}'_\lambda(u_n) \to 0, \text{ in } X^* \text{ as } n \to \infty. \tag{3.6}$$

Let us first prove that $\{u_n\}$ is bounded in X. For this purpose, we assume that $\|u_n\|_X \to +\infty$, as $n \to \infty$. By (M_1)–(M_3), one has

$$\mathcal{M}(t) \geq \frac{M(t) + m_0}{2} t, \ \forall\, t \geq 0. \tag{3.7}$$

Hence, it follows from (M_1), (3.2), (3.3), (3.7) and Lemma 2.5 (*ii*) that

$$\begin{aligned}
1 + c + \|u_n\|_X &\geq \widetilde{I}_\lambda(u_n) - \frac{1}{k}\langle \widetilde{I}'_\lambda(u_n), u_n\rangle \\
&\geq \frac{1}{p}\left[\frac{\left(M([u_n]_{s,p}^p) + m_0\right)[u_n]_{s,p}^p}{2} + \|u_n\|_{p,V}^p\right] - \frac{1}{k}\left(M([u_n]_{s,p}^p)[u_n]_{s,p}^p + \|u_n\|_{p,V}^p\right) \\
&\geq \frac{1}{2p}\left(M([u_n]_{s,p}^p)[u_n]_{s,p}^p + \|u_n\|_{p,V}^p\right) - \frac{1}{k}\left(M([u_n]_{s,p}^p)[u_n]_{s,p}^p + \|u_n\|_{p,V}^p\right) \\
&\geq \left(\frac{1}{2p} - \frac{1}{k}\right)\left(m_0[u_n]_{s,p}^p + \|u_n\|_{p,V}^p\right) \\
&\geq \left(\frac{1}{2p} - \frac{1}{k}\right)\min\{m_0, 1\}\|u_n\|_X^p,
\end{aligned}$$

as $n \to \infty$. By $k > 2p$, we get a contradiction. Therefore, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X.

Next, we prove that there exists $u \in X$ such that $u_n \to u$ in X, as $n \to \infty$. Combining Lemma 2.1-Lemma 2.2 and Theorem A.1 in [32], there exist a subsequence, still denoted by $\{u_n\}$ and a function u in X such that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } X, \\ u_n \to u, & \text{in } L^r(\mathbb{R}^N),\ r \in [p, p_s^*), \\ u_n \to u, & \text{a.e. in } \mathbb{R}^N, \\ |u_n| \le h_r, & \text{a.e. in } \mathbb{R}^N,\ \text{for some } h_r \in L^r(\mathbb{R}^N), \end{cases} \tag{3.8}$$

as $n \to \infty$. In view of Lemma 2.5 (i) and Hölder's inequality, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \left[\widetilde{f}(x,u_n) - \widetilde{f}(x,u)\right](u_n - u)\mathrm{d}x \right| &\le \int_{\mathbb{R}^N} \left| \left[\widetilde{f}(x,u_n) - \widetilde{f}(x,u)\right](u_n - u) \right| \mathrm{d}x \\ &\le C \int_{\mathbb{R}^N} \left[|u_n|^{\alpha-1} + |u|^{\alpha-1}\right] |u_n - u| \mathrm{d}x \\ &\le C \left(\|u_n\|^{\alpha-1}_{L^\alpha(\mathbb{R}^N)} + \|u\|^{\alpha-1}_{L^\alpha(\mathbb{R}^N)} \right) \|u_n - u\|_{L^\alpha(\mathbb{R}^N)}. \end{aligned}$$

Then (3.8) implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left[\widetilde{f}(x,u_n) - \widetilde{f}(x,u)\right](u_n - u)\mathrm{d}x = 0. \tag{3.9}$$

Now, let $u \in X$ be fixed and denote by $\mathcal{B}_u$ the linear functional on X defined by

$$\mathcal{B}_u(\varphi) := \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y,\ \forall\, \varphi \in X. \tag{3.10}$$

By Hölder's inequality and the definition of $\mathcal{B}_u(\varphi)$, it is easy to verify that $\mathcal{B}_u$ is continuous and

$$|\mathcal{B}_u(\varphi)| \le \|u\|_X^{p-1} \|\varphi\|_X \text{ for all } \varphi \in X.$$

Obviously, $\mathcal{B}_u$ is bounded. Furthermore, according to $\left\{M([u_n]^p_{s,p}) - M([u]^p_{s,p})\right\}$ is bounded in $\mathbb{R}$ and (3.8), we have

$$\lim_{n\to\infty} \left(\left[M([u_n]^p_{s,p})\right]^{p-1} - \left[M([u]^p_{s,p})\right]^{p-1} \right) \mathcal{B}_u(u_n - u) = 0. \tag{3.11}$$

From $u_n \rightharpoonup u$ in X and $\widetilde{I}'_\lambda(u_n) \to 0$ in X^*, we obtain $\langle \widetilde{I}'_\lambda(u_n) - \widetilde{I}'_\lambda(u), u_n - u \rangle \to 0$ as $n \to \infty$. Therefore, it follows from (3.3) and (3.8)–(3.11) that

$$\begin{aligned} o(1) &= \langle \widetilde{I}'_\lambda(u_n) - \widetilde{I}'_\lambda(u), u_n - u \rangle \\ &= M([u_n]^p_{s,p})\mathcal{B}_{u_n}(u_n - u) - M([u]^p_{s,p})\mathcal{B}_u(u_n - u) \\ &\quad + \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \left[\widetilde{f}(x,u_n) - \widetilde{f}(x,u)\right](u_n - u)\mathrm{d}x \\ &= M([u_n]^p_{s,p})[\mathcal{B}_{u_n}(u_n - u) - \mathcal{B}_u(u_n - u)] + \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\mathrm{d}x + o(1), \end{aligned} \tag{3.12}$$

as $n \to \infty$. By the fact that $M([u_n]^p_{s,p})[\mathcal{B}_{u_n}(u_n - u) - \mathcal{B}_u(u_n - u)] \ge 0$ and $V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) \ge 0$ for all $n \in \mathbb{N}$ by convexity. We get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n - u)\mathrm{d}x = 0, \tag{3.13}$$

$$\lim_{n\to\infty}\left[\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u)\right]=0. \tag{3.14}$$

Finally, we divided into two cases to prove $\|u_n-u\|_X\to 0$, as $n\to\infty$. To this aim, we first assume that $p\geq 2$. In view of Lemma 2.6, (3.10) and (3.14), we have

$$\begin{aligned}
[u_n-u]_{s,p}^p &= \iint_{\mathbb{R}^{2N}}\frac{|(u_n-u)(x)-(u_n-u)(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\\
&\leq c_p\iint_{\mathbb{R}^{2N}}\left[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))-|u(x)-u(y)|^{p-2}(u(x)-u(y))\right]\\
&\quad\cdot(u_n(x)-u(x)-u_n(y)+u(y))|x-y|^{-(N+sp)}\mathrm{d}x\mathrm{d}y\\
&= c_p[\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u)]=o(1),
\end{aligned} \tag{3.15}$$

as $n\to\infty$. By (V_1), (3.13) and Lemma 2.6, we can show that

$$\|u_n-u\|_{p,V}^p\leq c_p\int_{\mathbb{R}^N}V(x)\left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)(u_n-u)\mathrm{d}x=o(1), \tag{3.16}$$

as $n\to\infty$. Hence, combining (3.15) with (3.16), we have $\|u_n-u\|_X\to 0$ as $n\to\infty$.

In the end, we assume that $1<p<2$. Let us introduce the following elementary inequality

$$(a+b)^{\frac{(2-p)}{2}}\leq a^{\frac{(2-p)}{2}}+b^{\frac{(2-p)}{2}},\ \forall\, a,b\geq 0,\ \ 1<p<2. \tag{3.17}$$

From (3.8), there exists a constant $l>0$ such that $[u_n]_{s,p}\leq l$ for all $n\in\mathbb{N}$. According to Hölder's inequality, (3.10), (3.14) and (3.17), we have

$$\begin{aligned}
[u_n-u]_{s,p}^p &\leq C_p\left[\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u)\right]^{\frac{p}{2}}\left([u_n]_{s,p}^p+[u]_{s,p}^p\right)^{\frac{2-p}{2}}\\
&\leq C_p\left[\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u)\right]^{\frac{p}{2}}\left([u_n]_{s,p}^{\frac{p(2-p)}{2}}+[u]_{s,p}^{\frac{p(2-p)}{2}}\right)\\
&\leq \overline{C}\left[\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u)\right]^{\frac{p}{2}}=o(1),
\end{aligned}$$

as $n\to\infty$, where $\overline{C}=2C_pl^{\frac{p(2-p)}{2}}$. Similarly, it follows from (3.8) that there exists a constant $l_*>0$ such that $\|u_n\|_{p,V}\leq l_*$ for all $n\in\mathbb{N}$. Therefore, we conclude from Hölder's inequality and (3.13) that

$$\|u_n-u\|_{p,V}^p\leq\widehat{C}\left[\int_{\mathbb{R}^N}V(x)\left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)(u_n-u)\mathrm{d}x\right]^{\frac{p}{2}}=o(1),$$

as $n\to\infty$, where $\widehat{C}=2C_pl_*^{\frac{p(2-p)}{2}}$. Consequently, we have $\|u_n-u\|_X\to 0$, as $n\to\infty$. This concludes the proof of Lemma 3.3. □

Next, we prove that the existence of nontrivial solutions for Eq (3.1) via the mountain pass theorem.

Theorem 3.4. *Suppose that* (V_1), (V_2), (M_1)–(M_3) *and* (f_1)–(f_3) *are satisfied. Then Eq* (3.1) *has a nontrivial solution.*

Proof. By Lemma 3.1 and Lemma 3.2, we know that the functional $\widetilde{I_\lambda}$ satisfies the geometry of the mountain pass theorem. Moreover, in view of Lemma 3.3 and $\widetilde{I_\lambda}(0) = 0$, we know that $\widetilde{I_\lambda}$ satisfies all the condition of Theorem 2.4. Thus, we can see that $\widetilde{I_\lambda}$ has a critical value c_λ and

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} \widetilde{I_\lambda}(\gamma(t))$$

where

$$\Gamma_\lambda = \{\gamma \in C^1([0, 1], X) : \gamma(0) = 0, \gamma(1) = \bar{u}\},$$

and $\bar{u}$ is given by Lemma 3.2. Therefore, the Eq (3.1) has a nontrivial solution $u_\lambda \in X$ with $\widetilde{I_\lambda}(u_\lambda) = c_\lambda$. □

From the truncation argument in Section 2, we know that if the solution u_λ of Eq (3.1) satisfies $||u_\lambda||_{L^\infty(\mathbb{R}^N)} \leq \delta$, then u_λ is a nontrivial solution of the original Eq (1.1). Before we do that, we will use the following two lemmas to find an uniform boundedness of $||u_\lambda||_X^p$ and the critical level c_λ.

Lemma 3.5. *Let u_λ be a nontrivial solution of Eq* (3.1)*. Then there exists a constant* $\Pi > 0$ *such that*

$$||u_\lambda||_X^p \leq \Pi c_\lambda.$$

Proof. By u_λ is a critical point of $\widetilde{I_\lambda}$, (M_1), $k > 2p$, Lemma 2.5 (*ii*) and (3.7), we get

$$\begin{aligned}
c_\lambda &= \widetilde{I_\lambda}(u_\lambda) - \frac{1}{k}\langle \widetilde{I'_\lambda}(u_\lambda), u_\lambda\rangle \\
&\geq \frac{1}{p}\left(\frac{M([u_n]_{s,p}^p)[u_n]_{s,p}^p}{2} + \frac{m_0[u_n]_{s,p}^p}{2} + ||u_n||_{p,V}^p\right) - \frac{1}{k}\left(M([u_n]_{s,p}^p)[u_n]_{s,p}^p + ||u_n||_{p,V}^p\right) \\
&\geq \frac{1}{2p}\left(M([u_n]_{s,p}^p)[u_n]_{s,p}^p + ||u_n||_{p,V}^p\right) - \frac{1}{k}\left(M([u_n]_{s,p}^p)[u_n]_{s,p}^p + ||u_n||_{p,V}^p\right) \\
&\geq \left(\frac{1}{2p} - \frac{1}{k}\right)\left(m_0[u_n]_{s,p}^p + ||u_n||_{p,V}^p\right) \\
&\geq \left(\frac{1}{2p} - \frac{1}{k}\right)\min\{m_0, 1\}||u_n||_X^p,
\end{aligned}$$

Thus, there exists $\Pi > 0$ is a constant such

$$||u_\lambda||_X^p \leq \Pi c_\lambda.$$

The proof is completed. □

Lemma 3.6. *There exists a constant* $\Theta > 0$ *independent of λ such that*

$$c_\lambda \leq \Theta\lambda^{-\frac{p}{k-p}},$$

for all sufficiently large λ.

Proof. In view of (3.4), (3.5) and Theorem 3.4, we have

$$\begin{aligned}
c_\lambda &\leq \max_{t\in[0,1]} \widetilde{I}_\lambda(t\bar{u}) \\
&= \max_{t\in[0,1]} \left[\frac{1}{p}\left(\mathcal{M}([t\bar{u}]_{s,p}^p) + \|t\bar{u}\|_{p,V}^p\right) - \lambda \int_{\mathbb{R}^N} \widetilde{F}(x, t\bar{u})\mathrm{d}x\right] \\
&\leq \max_{t\in[0,1]} \left[\frac{1}{p}\left(bt^p[\bar{u}]_{s,p}^p + \frac{b}{2}t^{2p}[\bar{u}]_{s,p}^{2p} + t^p\|\bar{u}\|_{p,V}^p\right) - \lambda C t^k \int_{\mathbb{R}^N} \bar{u}^k \mathrm{d}x\right] \\
&\leq \max_{t\in[0,1]} \left[\frac{t^p}{p}\left(\max\{b,1\}\|\bar{u}\|_X^p + \frac{b}{2}\|\bar{u}\|_X^{2p}\right) - \lambda C t^k \int_{\mathbb{R}^N} \bar{u}^k \mathrm{d}x\right] \\
&\leq \Theta \lambda^{-\frac{p}{k-p}},
\end{aligned}$$

for all sufficiently large λ, where $\bar{u}$ is given by Lemma 3.2 and $\Theta > 0$ is a constant. □

Now, similar to the proof of Theorem 1.1 in [1], we use the Moser iteration technique to estimate $\|u_\lambda\|_{L^\infty(\mathbb{R}^N)}$.

Lemma 3.7. *If $u \in X$ is a nontrivial solution of Eq (3.1), then $u \in L^\infty(\mathbb{R}^N)$. In addition, there exists a constant $\Xi > 0$ such that*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq \Xi \lambda^{\frac{k-p_s^*}{(k-p)(p_s^*-\alpha)}},$$

for all λ sufficiently large.

Proof. Let $L > 0$, we define

$$\xi(u) = u u_L^{p(\beta-1)},$$

$$\Phi(u) = \int_0^u (\xi'(t))^{\frac{1}{p}} \mathrm{d}t,$$

and

$$\psi(u) = \frac{u^p}{p},$$

where $\beta > 1$ and $u_L = \min\{u, L\}$. Hence, we conclude from the functions introduced above that

$$\psi'(x-y)\,(\xi(x) - \xi(y)) \geq |\Phi(x) - \Phi(y)|^p, \ \ \forall\ x, y \in [0, +\infty), \tag{3.18}$$

$$\Phi(u) \geq \frac{1}{\beta} u u_L^{\beta-1}, \ \ \forall\, t \geq 0. \tag{3.19}$$

By the definition of $\xi(u)$, we have $\left|u u_L^{p(\beta-1)}\right| \leq L^{p(\beta-1)} u$ in $\mathbb{R}^N$. Thus, $\xi(u) \in X$. Considering $\xi(u)$ as a test function in (3.1), we get

$$\begin{aligned}
&M([u]_{s,p}^p) \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u u_L^{p(\beta-1)}(x) - u u_L^{p(\beta-1)}(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x) u u_L^{p(\beta-1)}(x)\mathrm{d}x = \lambda \int_{\mathbb{R}^N} \widetilde{f}(x, u(x)) u u_L^{p(\beta-1)}(x)\mathrm{d}x.
\end{aligned} \tag{3.20}$$

Now, we define

$$w_L := u u_L^{\beta-1}. \tag{3.21}$$

From (2.2) that

$$S^*_{s,p}\|w_L\|^p_{L^{p^*_s}(\mathbb{R}^N)} \le [w_L]^p_{s,p} \le \|w_L\|^p_X.$$

Hence, it follows from (M_1) and (3.18)–(3.21) that

$$\begin{aligned}
&M([u]^p_{s,p})\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(uu_L^{p(\beta-1)}(x)-uu_L^{p(\beta-1)}(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\\
&\quad+\int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)uu_L^{p(\beta-1)}(x)\mathrm{d}x\\
&\ge m_0[\Phi(u)]^p_{s,p}+\int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)uu_L^{p(\beta-1)}(x)\mathrm{d}x\\
&\ge \frac{m_0}{\beta^p}\,[w_L]^p_{s,p}+\int_{\mathbb{R}^N} V(x)\,|w_L|^p\,\mathrm{d}x\\
&\ge \min\{\frac{m_0}{\beta^p},1\}\|w_L\|^p_X\\
&\ge \min\{\frac{m_0}{\beta^p},1\}S^*_{s,p}\|w_L\|^p_{L^{p^*_s}(\mathbb{R}^N)}.
\end{aligned} \tag{3.22}$$

Assume first that $\frac{m_0}{\beta^p}<1$. According to Lemma 2.5 (i), (3.20)–(3.22) and Hölder's inequality, we have

$$\begin{aligned}
\|w_L\|^p_{L^{p^*_s}(\mathbb{R}^N)} &\le \frac{\lambda\beta^p}{m_0S^*_{s,p}}\int_{\mathbb{R}^N}\widetilde{f}(x,u(x))uu_L^{p(\beta-1)}(x)\mathrm{d}x\\
&\le \frac{\lambda\beta^p}{m_0S^*_{s,p}}\int_{\mathbb{R}^N} u^{\alpha-p}\,(w_L)^p\,\mathrm{d}x\\
&\le \frac{\lambda\beta^p}{m_0S^*_{s,p}}\left(\int_{\mathbb{R}^N}u^{p^*_s}\mathrm{d}x\right)^{\frac{\alpha-p}{p^*_s}}\left(\int_{\mathbb{R}^N}w_L^{\frac{pp^*_s}{p^*_s-\alpha+p}}\mathrm{d}x\right)^{\frac{p^*_s-\alpha+p}{p^*_s}}.
\end{aligned} \tag{3.23}$$

Since $\alpha\in(2p,p^*_s)$ and $p>1$, by simple calculation, we get

$$p<\nu^*_s:=\frac{pp^*_s}{p^*_s-\alpha+p}<p^*_s. \tag{3.24}$$

From Lemma 3.5 and Sobolev inequality, we obtain

$$\|u\|^p_{L^{p^*_s}(\mathbb{R}^N)} \le \left(S^*_{s,p}\right)^{-1}\|u\|^p_X \le \left(S^*_{s,p}\right)^{-1}\Pi c_\lambda. \tag{3.25}$$

It follows from the definition of w_L, we can show that $u_L\le u$ in $\mathbb{R}^N$. By (3.23)–(3.25), we have

$$\|w_L\|^p_{L^{p^*_s}(\mathbb{R}^N)} \le \frac{\lambda\beta^p}{m_0}\left(S^*_{s,p}\right)^{-\frac{\alpha}{p}}(\Pi c_\lambda)^{\frac{\alpha-p}{p}}\|w_L\|^p_{L^{\nu^*_s}(\mathbb{R}^N)} \le \beta^pC_\lambda\left(\int_{\mathbb{R}^N}|u^{\beta\nu^*_s}|\mathrm{d}x\right)^{\frac{p}{\nu^*_s}}, \tag{3.26}$$

where $C_\lambda=\frac{\lambda}{m_0}\left(S^*_{s,p}\right)^{-\frac{\alpha}{p}}(\Pi c_\lambda)^{\frac{\alpha-p}{p}}$. Letting $L\to+\infty$ in (3.26) and using the Fatou's Lemma, we have

$$\|u\|_{L^{\beta p^*_s}(\mathbb{R}^N)} \le \beta^{\frac{1}{\beta}}C_\lambda^{\frac{1}{\beta p}}\|u\|_{L^{\beta\nu^*_s}(\mathbb{R}^N)}. \tag{3.27}$$

Thus, we get

$$u \in L^{\beta \nu_s^*}(\mathbb{R}^N) \ \Rightarrow \ u \in L^{\beta p_s^*}(\mathbb{R}^N). \tag{3.28}$$

Taking $\vartheta = \frac{p_s^*}{\nu_s^*} > 1$ and $\beta = \vartheta$ in (3.27), we obtain

$$\|u\|_{L^{\vartheta p_s^*}(\mathbb{R}^N)} \leq \vartheta^{\frac{1}{\vartheta}} C_\lambda^{\frac{1}{\vartheta p}} \|u\|_{L^{p_s^*}(\mathbb{R}^N)}. \tag{3.29}$$

Taking $\beta = \vartheta^2$ in (3.27), we get

$$\|u\|_{L^{\vartheta^2 p_s^*}(\mathbb{R}^N)} \leq \left(\vartheta^2\right)^{\frac{1}{\vartheta^2}} C_\lambda^{\frac{1}{\vartheta^2 p}} \|u\|_{L^{\vartheta p_s^*}(\mathbb{R}^N)}. \tag{3.30}$$

Combining (3.29) with (3.30), one has

$$\|u\|_{L^{\vartheta^2 p_s^*}(\mathbb{R}^N)} \leq \vartheta^{\left(\frac{1}{\vartheta}+\frac{2}{\vartheta^2}\right)} C_\lambda^{\frac{1}{p}\left(\frac{1}{\vartheta}+\frac{1}{\vartheta^2}\right)} \|u\|_{L^{p_s^*}(\mathbb{R}^N)}. \tag{3.31}$$

We proceed the i times iterations, by taking $\beta = \vartheta^i$, $i = 1, 2, \ldots,$. Then, by letting $i \to \infty$, we have

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq \beta^{\frac{\beta}{(\beta-1)^2}} C_\lambda^{\frac{1}{p(\beta-1)}} \|u\|_{L^{p_s^*}(\mathbb{R}^N)}. \tag{3.32}$$

Next, we assume that $\frac{m_0}{\beta^p} \geq 1$. Similarly, we get

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq C_\lambda^{\frac{1}{p(\beta-1)}} \|u\|_{L^{p_s^*}(\mathbb{R}^N)}. \tag{3.33}$$

Therefore, by $u \in L^{p_s^*}(\mathbb{R}^N)$, we have $u \in L^\infty(\mathbb{R}^N)$.

Finally, by employing (3.32), (3.33), Lemma 3.6 and the Sobolev inequality, there exists a constant $\Xi > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq \Xi \lambda^{\frac{k-p_s^*}{(k-p)(p_s^*-\alpha)}},$$

for all sufficiently large λ. The lemma is now proved. □

Proof of Theorem 1.1. Since $k, \alpha \in (2p, p_s^*)$, we obtain

$$\frac{k - p_s^*}{(k-p)(p_s^* - \alpha)} < 0. \tag{3.34}$$

Hence, in view of (3.34) and Lemma 3.7, there exists $\lambda_0 > 0$ such that

$$\|u_\lambda\|_{L^\infty(\mathbb{R}^N)} \leq \delta,$$

for all $\lambda > \lambda_0$, where δ is fixed in (2.4). Therefore, u_λ is a nontrivial solution of Eq (1.1) for $\lambda > \lambda_0$. □

4. Conclusions

In this paper, we have considered a class of fractional Kirchhoff type equation. Under suitable assumptions on V and M, using a truncation argument and the mountain pass theorem, we have established the existence of nontrivial solutions. It is expected that the results proved in this paper may be starting point further research in this field.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. V. Ambrosio, R. Servadei, Supercritical fractional Kirchhoff type problems, *Fract. Calc. Appl. Anal.*, **22** (2019), 1351–1377.
2. G. Autuori, P. Pucci, Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$, *J. Differ. Equ.*, **255** (2013), 2340–2362.
3. T. Bartsch, Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problem on $\mathbb{R}^N$, *Commun. Partial Differ. Equ.*, **20** (1995), 1725–1741.
4. S. Bernstein, Sur une classe d'équations fonctionnelles aux dérivées partielles, *Bull. Acad. Sci. URSS. Sér. Math.*, **4** (1983), 17–26.
5. L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, *Commun. Partial Differ. Equ.*, **32** (2007), 1245–1260.
6. X. J. Chang, Z. Q. Wang, Nodal and multiple solutions of nonlinear problems involoving the fractional Laplacian, *J. Differ. Equ.*, **256** (2014), 479–494.
7. K. Cheng, Q. Gao, Sign-changing solutions for the stationary Kirchhoff problems involving the fractional laplacian in $\mathbb{R}^N$, *Acta Math. Sci.*, **38** (2018), 1712–1730.
8. D. G. Costa, Z. Q. Wang, Multiplicity results for a class of superlinear elliptic problems, *Proc. Amer. Math. Soc.*, **133** (2005), 787–794.
9. E. D. Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev space, *Bull. Sci. Math.*, **136** (2012), 521–573.
10. A. Fiscella, A fractional Kirchhoff problem involving a singular term and a critical nonlinearity, *Adv. Nonlinear Anal.*, **8** (2017), 645–660.
11. A. Fiscella, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, *Adv. Nonlinear Anal.*, **94** (2014), 156–170.
12. G. Z. Gu, X. H. Tang, The concentration behavior of ground states for a class of Kirchhoff-type problems with Hartree-type nonlinearity, *Adv. Nonlinear Stud.*, **19** (2019), 779–795.
13. C. Huang, G. Jia, Existence of positive solutions for supercritical quasilinear Schrödinger elliptic equations, *J. Math. Anal. Appl.*, **472** (2019), 705–727.
14. H. Jin, W. B. Liu, Fractional Kirchhoff equation with a general critical nonlinearity, *Appl. Math. Lett.*, **74** (2017), 140–146.
15. G. Kirchhoff, *Vorlesungen Uber Mechanik*, Teubner, Leipzig, 1883.

16. A. R. Li, J. B. Su, Existence and multiplicity of solutions for Kirchhoff-type equation with radial potentials in $\mathbb{R}^3$, *Z. Angew. Math. Phys.*, **66** (2015), 3147–3158.
17. Q. Q. Li, K. M. Teng, X. Wu, Existence of nontrivial solutions for Schrödinger-Kirchhoff type equations with critical or supercritical growth, *Math. Methods Appl. Sci.*, **41** (2017), 1136–1144.
18. J. L. Lions, *On some questions in boundary value problems of mathematical physics*, In: Contemporary Developments in Continuum Mechanics and Partial Differential Equations, North-Holland, Amsterdam, 1978.
19. G. B. Li, H. Y. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, *J. Differ. Equ.*, **257** (2014), 566–600.
20. G. Molica Bisci, V. D. Rădulescu, R. Servadei, *Variational methods for nonlocal fractional problems*, Cambridge Univ. Press., Cambridge, 2016.
21. D. Naimen, M. Shibata, Two positive solutions for the Kirchhoff type elliptic problem with critical nonlinearity in high dimension, *Nonlinear Anal.*, **186** (2019), 187–208.
22. N. Nyamoradi, L. I. Zaidan, Existence and multiplicity of solutions for fractional p-Laplacian Schrödinger-Kirchhoff type equations, *Complex Var. Elliptic Equ.*, **63** (2017), 346–359.
23. R. C. Pei, Y. Zhang, J. H. Zhang, Ground state solutions for fractional p-Kirchhoff equation with subcritical and critical exponential growth, *Bull. Malays. Math. Sci. Soc.*, **43** (2020), 355–377.
24. S. I. Pohozǎev, A certain class of quasilinear hyperbolic equations, *Mat. Sb.*, **96** (1975), 152–166.
25. P. Pucci, M. Q. Xiang, B. L. Zhang, Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$, *Calc. Var. Partial Differ. Equ.*, **54** (2015), 2785–2806.
26. P. Pucci, M. Q. Xiang, B. L. Zhang, Existence results for Schrödinger-Choquard-Kirchhoff equations involving the fractional p-Laplacian, *Adv. Calc. Var.*, **12** (2019), 253–275.
27. R. Servadei, E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacia, *Trans. Amer. Math. Soc.*, **367** (2015), 67–102.
28. J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$, *Lecture Notes in Math.*, **665** (1978), 205–227.
29. M. Q. Xiang, B. L. Zhang, X. Y. Guo, Infinitely many solutions for a fractional Kirchhoff type problem via fountain theorem, *Nonlinear Anal.*, **120** (2015), 299–313.
30. M. Q. Xiang, B. L. Zhang, V. D. Rădulescu, Superlinear Schrödinger-Kirchhoff type problems involving the fractional p-Laplacian and critical exponent, *Adv. Nonlinear Anal.*, **9** (2020), 690–709.
31. M. Q. Xiang, B. L. Zhang, D. Repovš, Existence and multiplicity of solutions for fractional Schrödinger-Kirchhoff equations with Truding-Moser nonlinearity, *Nonlinear Anal.*, **186** (2019), 74–98.
32. M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.
33. J. Zhang, Z. L. Lou, Y. J. Ji, W. Shao, Ground state of Kirchhoff type fractional Schrödinger equations with critical growth, *J. Math. Anal. Appl.*, **462** (2018), 57–83.