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Research article

Definite integral of the logarithm hyperbolic secant function in terms of the Hurwitz zeta function

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Abstract: We evaluate definite integrals of the form given by $\int_0^\infty R(a,x)\log(\cos(\alpha)\text{sech}(x)+1)dx$. The function $R(a,x)$ is a rational function with general complex number parameters. Definite integrals of this form yield closed forms for famous integrals in the books of Bierens de Haan [4] and Gradshteyn and Ryzhik [5].

Keywords: entries of Gradshteyn and Ryzhik; hyperbolic integrals; Bierens de Haan; Hurwitz zeta function
Mathematics Subject Classification: 01A55, 11M06, 11M35, 30-02, 30D10, 30D30, 30E20

1. Introduction

We will derive integrals as indicated in the abstract in terms of special functions. Some special cases of these integrals have been reported in Gradshteyn and Ryzhik [5]. In 1867 David Bierens de Haan derived hyperbolic integrals of the form

$$\int_0^\infty \left((\log(a)-ix)^k+(\log(a)+ix)^k\right)\log(\cos(\alpha)\text{sech}(x)+1)dx \tag{1.1}$$

In our case the constants in the formulas are general complex numbers subject to the restrictions given below. The derivations follow the method used by us in [6]. The generalized Cauchy's integral formula is given by

$$\frac{y^k}{k!}=\frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \tag{1.2}$$

2. Definite integral of the contour integral

We use the method in [6]. Here the contour is in the upper left quadrant with $\Re(w)<0$ and going round the origin with zero radius. Using a generalization of Cauchy's integral formula we first replace

y by $ix + \log(a)$ for the first equation and then y by $-ix + \log(a)$ to get the second equation. Then we add these two equations, followed by multiplying both sides by $\frac{1}{2}\log(\cos(\alpha)\text{sech}(x)+1)$ to get

$$\frac{\left((\log(a)-ix)^k+(\log(a)+ix)^k\right)\log(\cos(\alpha)\text{sech}(x)+1)}{2k!} = \frac{1}{2\pi i}\int_C a^w w^{-k-1}\cos(wx)\log(\cos(\alpha)\text{sech}(x)+1)dw \quad (2.1)$$

where the logarithmic function is defined in Eq (4.1.2) in [2]. We then take the definite integral over $x \in [0, \infty)$ of both sides to get

$$\begin{aligned}
&\int_0^\infty \frac{\left((\log(a)-ix)^k+(\log(a)+ix)^k\right)\log(\cos(\alpha)\text{sech}(x)+1)}{2k!}dx \\
&= \frac{1}{2\pi i}\int_0^\infty\int_C a^w w^{-k-1}\cos(wx)\log(\cos(\alpha)\text{sech}(x)+1)dwdx \\
&= \frac{1}{2\pi i}\int_C\int_0^\infty a^w w^{-k-1}\cos(wx)\log(\cos(\alpha)\text{sech}(x)+1)dxdw \\
&= \frac{1}{2\pi i}\int_C \pi a^w w^{-k-2}\cosh\left(\frac{\pi w}{2}\right)\text{csch}(\pi w)dw - \frac{1}{2\pi i}\int_C \pi a^w w^{-k-2}\text{csch}(\pi w)\cosh(\alpha w)dw
\end{aligned} \quad (2.2)$$

from Eq (1.7.7.120) in [1] and the integral is valid for α, a, and k complex and $|\Re(\alpha)| < \pi$.

3. Infinite sums of the contour integral

In this section we will again use the generalized Cauchy's integral formula to derive equivalent contour integrals. First we replace y by $y - \pi/2$ for the first equation and y by $y + \pi/2$ for second then add these two equations to get

$$\frac{\left(y-\frac{\pi}{2}\right)^k+\left(y+\frac{\pi}{2}\right)^k}{k!} = \frac{1}{2\pi i}\int_C 2w^{-k-1}e^{wy}\cosh\left(\frac{\pi w}{2}\right)dw \quad (3.1)$$

Next we replace y by $\log(a) + \pi(2p+1)$ then we take the infinite sum over $p \in [0, \infty)$ to get

$$\begin{aligned}
&\sum_{p=0}^\infty \frac{2\pi\left(\left(\log(a)+\pi(2p+1)-\frac{\pi}{2}\right)^k+\left(\log(a)+\pi(2p+1)+\frac{\pi}{2}\right)^k\right)}{k!} \\
&= \frac{1}{2\pi i}\sum_{p=0}^\infty\int_C 4\pi w^{-k-1}\cosh\left(\frac{\pi w}{2}\right)e^{w(\log(a)+\pi(2p+1))}dw \\
&= \frac{1}{2\pi i}\int_C\sum_{p=0}^\infty 4\pi w^{-k-1}\cosh\left(\frac{\pi w}{2}\right)e^{w(\log(a)+\pi(2p+1))}dw
\end{aligned} \quad (3.2)$$

where $\Re(w) < 0$ according to (1.232.3) in [5]. Then we simplify the left-hand side to get the Hurwitz zeta function

$$
\begin{aligned}
&-\frac{2^{k+1}\pi^{k+2}}{(k+1)!}\left(\zeta\left(-k-1,\frac{2\log(a)+\pi}{4\pi}\right)+\zeta\left(-k-1,\frac{2\log(a)+3\pi}{4\pi}\right)\right)\\
&=\frac{1}{2\pi i}\int_C \pi a^w w^{-k-2}\cosh\left(\frac{\pi w}{2}\right)\operatorname{csch}(\pi w)dw
\end{aligned}
\tag{3.3}
$$

Then following the procedure of (3.1) and (3.2) we replace y by $y+\alpha$ and $y-\alpha$ to get the second equation for the contour integral given by

$$
\frac{(y-\alpha)^k+(\alpha+y)^k}{k!}=\frac{1}{2\pi i}\int_C 2w^{-k-1}e^{wy}\cosh(\alpha w)dw \tag{3.4}
$$

next we replace y by $\log(a)+\pi(2p+1)$ and take the infinite sum over $p\in[0,\infty)$ to get

$$
\begin{aligned}
&\sum_{p=0}^{\infty}\frac{(-\alpha+\log(a)+\pi(2p+1))^k+(\alpha+\log(a)+\pi(2p+1))^k}{k!}\\
&=\frac{1}{2\pi i}\sum_{p=0}^{\infty}\int_C 2w^{-k-1}\cosh(\alpha w)e^{w(\log(a)+\pi(2p+1))}dw\\
&=\frac{1}{2\pi i}\int_C\sum_{p=0}^{\infty}2w^{-k-1}\cosh(\alpha w)e^{w(\log(a)+\pi(2p+1))}dw
\end{aligned}
\tag{3.5}
$$

Then we simplify to get

$$
\begin{aligned}
&\frac{2^{k+1}\pi^{k+2}}{(k+1)!}\left(\zeta\left(-k-1,\frac{-\alpha+\log(a)+\pi}{2\pi}\right)+\zeta\left(-k-1,\frac{\alpha+\log(a)+\pi}{2\pi}\right)\right)\\
&=\frac{1}{2\pi i}\int_C \pi\left(-a^w\right)w^{-k-2}\operatorname{csch}(\pi w)\cosh(\alpha w)dw
\end{aligned}
\tag{3.6}
$$

4. Definite integral in terms of the Hurwitz zeta function

Since the right-hand sides of Eqs (2.2), (3.3) and (3.5) are equivalent we can equate the left-hand sides to get

$$
\begin{aligned}
&\int_0^\infty\left((\log(a)-ix)^k+(\log(a)+ix)^k\right)\log(\cos(\alpha)\operatorname{sech}(x)+1)dx\\
&=2\left(\frac{2^{k+1}\pi^{k+2}\left(\zeta\left(-k-1,\frac{-\alpha+\log(a)+\pi}{2\pi}\right)+\zeta\left(-k-1,\frac{\alpha+\log(a)+\pi}{2\pi}\right)\right)}{k+1}\right)\\
&\quad-2\left(\frac{2^{k+1}\pi^{k+2}\left(\zeta\left(-k-1,\frac{2\log(a)+\pi}{4\pi}\right)+\zeta\left(-k-1,\frac{2\log(a)+3\pi}{4\pi}\right)\right)}{k+1}\right)
\end{aligned}
\tag{4.1}
$$

from (9.521) in [5] where $\zeta(z,q)$ is the Hurwitz zeta function. Note the left-hand side of Eq (4.1) converges for all finite k. The integral in Eq (4.1) can be used as an alternative method to evaluating the Hurwitz zeta function. The Hurwitz zeta function has a series representation given by

$$\zeta(z,q)=\sum_{n=0}^{\infty}\frac{1}{(q+n)^z} \tag{4.2}$$

where $\Re(z)>1, q\neq 0,-1,..$ and is continued analytically by (9.541.1) in [5] where $z=1$ is the only singular point.

5. Derivation of special cases

In this section we have evaluated integrals and extended the range of the parameters over which the integrals are valid. The aim of this section is to derive a few integrals in [5] in terms of the Lerch function. We also present errata for one of the integrals and faster converging closed form solutions.

5.1. Derivation of entry 3.531.7 in [5]

Using Eq (4.1) and taking the first partial derivative with respect to α and setting $a=1$ and simplifying the left-hand side we get

$$\int_0^{\infty}\frac{x^k}{\cos(\alpha)+\cosh(x)}dx=2^k\pi^{k+1}\csc(\alpha)\sec\left(\frac{\pi k}{2}\right)\left(\zeta\left(-k,\frac{\pi-\alpha}{2\pi}\right)-\zeta\left(-k,\frac{\alpha+\pi}{2\pi}\right)\right) \tag{5.1}$$

from Eq (7.102) in [3].

5.2. Derivation of entry 4.371.2 in [5]

Using Eq (5.1) and taking the first partial derivative with respect to k and setting $k=0$ and simplifying the left-hand side we get

$$\begin{aligned}\int_0^{\infty}\frac{\log(x)}{\cos(\alpha)+\cosh(x)}dx&=\csc(\alpha)\left(\alpha\log(2\pi)-\pi\log(-\alpha-\pi)+\pi\log(\alpha-\pi)-\pi\log\Gamma\left(-\frac{\alpha+\pi}{2\pi}\right)\right.\\&\qquad\left.+\pi\log\Gamma\left(-\frac{\pi-\alpha}{2\pi}\right)\right)\\&=\csc(\alpha)\left(\alpha\log(2\pi)+\pi\log\left(\frac{\Gamma\left(\frac{\alpha+\pi}{2\pi}\right)}{\Gamma\left(\frac{\pi-\alpha}{2\pi}\right)}\right)\right)\end{aligned} \tag{5.2}$$

from (7.105) in [3].

5.3. Derivation of entry 4.371.1 in [5]

Using Eq (5.2) and setting $\alpha=\pi/2$ and simplifying we get

$$\int_0^{\infty}\log(x)\operatorname{sech}(x)dx=\pi\log\left(\frac{\sqrt{2\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right) \tag{5.3}$$

5.4. Derivation of entry 4.371.3 in [5]

Using Eq (5.2) and taking the first derivative with respect to α and setting $\alpha = \pi/2$ and simplifying we get

$$\int_0^\infty \log(x)\operatorname{sech}^2(x)dx = \log\left(\frac{\pi}{4}\right) - \gamma \tag{5.4}$$

where γ is Euler's constant.

5.5. Derivation of entry 4.376.1 in [5]

Using Eq (5.1) and taking the first partial derivative with respect to k then setting $k = -1/2$ and $\alpha = \pi/2$ and simplifying we get

$$\int_0^\infty \frac{\log(x)\operatorname{sech}(x)}{\sqrt{x}}dx = \frac{1}{2}\sqrt{\pi}\left(-2\zeta'\left(\frac{1}{2},\frac{1}{4}\right) + 2\zeta'\left(\frac{1}{2},\frac{3}{4}\right) + \left(\zeta\left(\frac{1}{2},\frac{3}{4}\right) - \zeta\left(\frac{1}{2},\frac{1}{4}\right)\right)\left(\pi + \log\left(\frac{1}{4\pi^2}\right)\right)\right) \tag{5.5}$$

The expression in [4] is correct but converges much slower than Eq (5.5).

5.6. Derivation of Eq (8) Table 257 in [4]

Using Eq (5.1) and taking the first partial derivative with respect to α we get

$$\begin{aligned}\int_0^\infty \frac{x^k}{(\cos(\alpha)+\cosh(x))^2}dx = &-2^{k-1}\pi^k \csc^2(\alpha)\sec\left(\frac{\pi k}{2}\right)\left(k\left(\zeta\left(1-k,\frac{\pi-\alpha}{2\pi}\right) + \zeta\left(1-k,\frac{\alpha+\pi}{2\pi}\right)\right)\right) \\ &- 2^{k-1}\pi^k \csc^2(\alpha)\sec\left(\frac{\pi k}{2}\right)\left(2\pi\cot(\alpha)\left(\zeta\left(-k,\frac{\pi-\alpha}{2\pi}\right) - \zeta\left(-k,\frac{\alpha+\pi}{2\pi}\right)\right)\right)\end{aligned} \tag{5.6}$$

from (7.102) in [3]. Next we use L'Hopital's rule and take the limit as $\alpha \to 0$ to get

$$\int_0^\infty \frac{x^k}{(\cosh(x)+1)^2}dx = -\frac{1}{3}2^{1-k}\left(\left(2^k-8\right)\zeta(k-2) - \left(2^k-2\right)\zeta(k)\right)\Gamma(k+1) \tag{5.7}$$

Then we take the first partial derivative with respect to k to get

$$\begin{aligned}\int_0^\infty \frac{x^k\log(x)}{(\cosh(x)+1)^2}dx = &\frac{1}{3}2^{4-k}k\Gamma(k)\zeta'(k-2) - \frac{2}{3}k\Gamma(k)\zeta'(k-2) - \frac{1}{3}2^{2-k}k\Gamma(k)\zeta'(k) \\ &+ \frac{2}{3}k\Gamma(k)\zeta'(k) - \frac{1}{3}2^{1-k}k\log(256)\zeta(k-2)\Gamma(k) + \frac{1}{3}2^{1-k}k\log(4)\zeta(k)\Gamma(k) \\ &+ \frac{1}{3}2^{4-k}k\zeta(k-2)\Gamma(k)\psi^{(0)}(k+1) - \frac{2}{3}k\zeta(k-2)\Gamma(k)\psi^{(0)}(k+1) \\ &- \frac{1}{3}2^{2-k}k\zeta(k)\Gamma(k)\psi^{(0)}(k+1) + \frac{2}{3}k\zeta(k)\Gamma(k)\psi^{(0)}(k+1)\end{aligned} \tag{5.8}$$

Finally we set $k = 0$ to get

$$\int_0^\infty \frac{\log(x)}{(\cosh(x)+1)^2}dx = \frac{1}{3}\left(14\zeta'(-2) - \gamma + \log\left(\frac{\pi}{2}\right)\right) \tag{5.9}$$

The integral listed in [4] appears with an error in the integrand.

5.7. Derivation of entry 3.522.6 in [5]

Using Eq (4.1) we first take the limit as $k \to -1$ by applying L'Hopital's rule and using (7.105) in [3] and simplifying the right-hand side we get

$$\int_0^\infty \frac{\log(\cos(\alpha)\operatorname{sech}(x)+1)}{a^2+x^2}dx = \frac{\pi}{a}\log\left(\frac{\sqrt{\pi}2^{\frac{1}{2}-\frac{a}{\pi}}\Gamma\left(\frac{a}{\pi}+\frac{1}{2}\right)}{\Gamma\left(\frac{a-\alpha+\pi}{2\pi}\right)\Gamma\left(\frac{a+\alpha+\pi}{2\pi}\right)}\right) \tag{5.10}$$

Next we take the first partial derivative with respect to α and set $\alpha = 0$ to get

$$\int_0^\infty \frac{\operatorname{sech}(x)}{a^2+x^2}dx = -\frac{\psi^{(0)}\left(\frac{2a+\pi}{4\pi}\right) - \psi^{(0)}\left(\frac{2a+3\pi}{4\pi}\right)}{2a} \tag{5.11}$$

from Eq (8.360.1) in [5] where $\Re(a) > 0$, next we replace x with bx to get

$$\int_0^\infty \frac{\operatorname{sech}(bx)}{a^2+b^2x^2}dx = -\frac{\psi^{(0)}\left(\frac{2a+\pi}{4\pi}\right) - \psi^{(0)}\left(\frac{2a+3\pi}{4\pi}\right)}{2ab} \tag{5.12}$$

Next we set $a = b = \pi$ to get

$$\int_0^\infty \frac{\operatorname{sech}(\pi x)}{x^2+1}dx = \frac{1}{2}\left(\psi^{(0)}\left(\frac{5}{4}\right) - \psi^{(0)}\left(\frac{3}{4}\right)\right) = 2 - \frac{\pi}{2} \tag{5.13}$$

from Eq (8.363.8) in [5].

5.8. Derivation of entry 3.522.8 in [5]

Using Eq (5.12) and setting $a = b = \pi/2$ we get

$$\int_0^\infty \frac{\operatorname{sech}\left(\frac{\pi x}{2}\right)}{x^2+1}dx = \frac{1}{2}\left(-\gamma - \psi^{(0)}\left(\frac{1}{2}\right)\right) = \log(2) \tag{5.14}$$

from Eq (8.363.8) in [5].

5.9. Derivation of entry 3.522.10 in [5]

Using Eq (5.12) and setting $a = b = \pi/4$ we get

$$\int_0^\infty \frac{\operatorname{sech}\left(\frac{\pi x}{4}\right)}{x^2+1}dx = \frac{1}{2}\left(\psi^{(0)}\left(\frac{7}{8}\right) - \psi^{(0)}\left(\frac{3}{8}\right)\right) = \frac{\pi - 2\coth^{-1}\left(\sqrt{2}\right)}{\sqrt{2}} \tag{5.15}$$

from Eq (8.363.8) in [5].

5.10. Derivation of a definite integral in terms of the trigamma function

Using Eq (5.10) and taking the second partial derivative with respect to α we get

$$\int_0^\infty \frac{\operatorname{sech}^2(x)}{a^2+x^2}dx = \frac{\psi^{(1)}\left(\frac{a}{\pi}+\frac{1}{2}\right)}{\pi a} \tag{5.16}$$

from Eq (8.363.8) in [5] where $\Re(a) > 0$.

6. Discussion

In this paper we were able to present errata and express our closed form solutions in terms of special functions and fundamental constants such π, Euler's constant and log(2). The use of the trigamma function is quite often necessary in statistical problems involving beta or gamma distributions. This work provides both an accurate and extended range for the solutions of the integrals derived.

7. Conclusion

We have presented a novel method for deriving some interesting definite integrals by Bierens de Haan using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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Conflict of interest

The authors declare there are no conflicts of interest.

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