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Research article

On interval-valued $\mathbb{K}$-Riemann integral and Hermite-Hadamard type inequalities

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Abstract: We introduce the concepts of $\mathbb{K}$-Riemann integral and radial $\mathbb{K}$-gH-derivative for interval-valued functions. We also give some important properties of interval-valued $\mathbb{K}$-Riemann integral, and extend interval-valued Hermite-Hadamard type inequalities in the case of $\mathbb{K}$-Riemann integral. Several examples are shown to illustrate the results.

Keywords: interval-valued functions; radial $\mathbb{K}$-gH-derivative; $\mathbb{K}$-Riemann integral; Hermite-Hadamard type inequalities
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1. Introduction

Interval analysis was first proposed in order to reduce errors during mathematical computation. Since the first monograph was written by Moore [2], interval analysis plays an important role in many other fields such as engineering, economics, statistics, and so on. Especially in the engineering field, the dynamics model can solve many dynamic problems, and such systems usually involve multiple uncertain parameters or interval coefficients.

Based on the works of Moore, many researchers hope to establish a complete theoretical system for interval analysis like the real analysis. In 1979, Markov first presented the concept of gH-difference and differentiability for interval-valued functions in [6], which addressed the problems in subtraction between two intervals. In 2009, Stefanini gave some more research on gH-derivative and interval differential equations in [7]. In 2015, Lupulescu studied the fractional calculus for interval-valued functions in [5]. Last year, Chalco-Cano dealt with the algebra of gH-differentiable interval-valued functions in [9]. This year, Ghosh proposed the notions of gH-directional derivative, gH-Gâteaux derivative and gH-Fréchet derivative for interval-valued functions in [18]. The generalization of the theory for interval analysis drew more and more attentions in the past decade.

In 2006, Boros give some notions of a subfield $\mathbb{K}$ of $\mathbb{R}$ and presented the ideas of $\mathbb{Q}$-subdifferential of Jensen-convex functions in [22]. After this, in 2017, Olbryś introduced the concepts of $\mathbb{K}$-Riemann integral and gave a new generalization of classical Hermite-Hadamard inequality in [4]. Motivated by their works, one of the main purposes of this paper is to introduce a new generalization for interval-valued $\mathbb{K}$-Darboux integral and $\mathbb{K}$-Riemann integral.

On the other hand, convex functions take effects in optimal control theory, economic analysis, probability estimate, and many other fields. For the past decades, classical convex has been generalized to many other types. After Costa gave the concepts of interval-valued convex in [1], some classical inequalities for convex functions have been extended to the form of interval-valued especially the Hermite-Hadamard inequality. In fact, if a function is convex then it satisfies the Hermite-Hadamard inequality (More details, see [10–12], [14–17], [19]). Motivated by this, the other purpose of this paper is to generalize some Hermite-Hadamard type inequalities with respect to $\mathbb{K}$-Riemann integral for $\mathbb{K}$-convex and *log*-$\mathbb{K}$-convex interval-valued functions. In addition, the results of this paper may be used as a powerful tool in fuzzy-valued function, interval optimization, and interval-valued differential equations.

In section 2, we provide the basic theory of interval analysis and some properties. The definition of interval-valued $\mathbb{K}$-Darboux, $\mathbb{K}$-Riemann integral and some basic properties are given in section 3. Then we propose the definition of radial $\mathbb{K}$-gH-derivative and the relationship between $\mathbb{K}$-Riemann integral in section 4. In section 5, we give some statements for Hermite-Hadamard type inequalities of $\mathbb{K}$-convex and log-$\mathbb{K}$-convex interval-valued functions. Some examples are also presented.

2. Preliminaries

Let $\mathcal{K}_c = \{I = [a,b] \mid a,b \in \mathbb{R}, a \le b\}$. The length of interval $I = [a,b] \in \mathcal{K}_c$ can be defined by $\ell(I) := b - a$. Moreover, we say I is positive if $a > 0$, and we denote $\mathcal{K}_c^+$ by all positive intervals belong to $\mathcal{K}_c$.

Some basic properties of algebra operations between two intervals can be found in [2]. For two intervals $A = [a^-, a^+]$ and B= $[b^-, b^+]$ belong to $\mathcal{K}_c$, we now give the following important properties.

Definition 2.1. For any $A, B \in \mathcal{K}_c$, the gH-difference between A, B can be defined as

$$A \ominus_g B = \begin{cases} [a^- - b^-, a^+ - b^+], & \text{if } \ell(A) \ge \ell(B), \\ [a^+ - b^+, a^- - b^-], & \text{if } \ell(A) < \ell(B). \end{cases} \tag{2.1}$$

Specially, if $B = b \in \mathbb{R}$ is a constant, then

$$A \ominus_g B = [a^- - b, a^+ - b].$$

More properties of gH-difference can be found in [8].

The partial order relationship "$\le$" between A and B can be defined as

$$A \le B \quad \text{if} \quad a^- \le b^- \quad \text{and} \quad a^+ \le b^+, \tag{2.2}$$

$$A < B \quad \text{if} \quad A \le B \quad \text{and} \quad A \ne B. \tag{2.3}$$

The Hausdorff-Pompeiu distance $\mathcal{H} : \mathcal{K}_c \times \mathcal{K}_c \to [0, \infty)$ between A and B is defined by $\mathcal{H}(A, B) = \max\{|a^- - b^-|, |a^+ - b^+|\}$. Then $(\mathcal{K}_c, \mathcal{H})$ is a complete and separable metric space (see [13]).

Base on this, define a map $\|\cdot\|$: $\mathcal{K}_c \to [0,\infty)$, $\|A\|$:=max$\{|a^-|,|a^+|\} = \mathcal{H}(A,\{0\})$. It's easy to see that $\|\cdot\|$ is a norm on $\mathcal{K}_c$. Hence, $(\mathcal{K}_c,\|\cdot\|)$ is a normed quasi-linear space (see [5]).

In this paper, we use symbols F and G refer to interval-valued functions. For any $F:[a,b]\to\mathcal{K}_c$ where $F=[f^-,f^+]$, we say that F is ℓ-increasing(or ℓ-decreasing) on $[a,b]$ if $\ell(F):[a,b]\to[0,\infty)$ is strictly increasing(or decreasing) on $[a,b]$. If $\ell(F)$ is strictly monotone on $[a,b]$, then we say F is ℓ-monotone on $[a,b]$.

Definition 2.2. $F:[a,b]\to\mathcal{K}_c$ is said to be continuous at $x_0\in[a,b]$, if

$$\lim_{x\to x_0}\|F(x)\ominus_g F(x_0)\| = 0. \tag{2.4}$$

We denote $C([a,b],\mathcal{K}_c)$ by the set of all continuous interval-valued function on $[a,b]$.

Definition 2.3. (Stefanini [7]) Let $F:[a,b]\to\mathcal{K}_c$, we say that $F(x)$ is gH-differential at $x_0\in(a,b)$, if there exist $\mathcal{D}\in\mathcal{K}_c$ such that

$$\lim_{h\to 0}\frac{F(x_0+h)\ominus_g F(x_0)}{h} = \mathcal{D}. \tag{2.5}$$

$\mathcal{D}$ defined in this fashion is denoted the symbol $F'(x_0)$ and $F'(x_0)$ is said to be the gH-derivative of $F(x)$ at x_0.

More properties of gH-derivative can be found in [9].

Now, we introduced the concept of the field $\mathbb{K}$ and number set $[a,b]_{\mathbb{K}}$ given by Olbryś in [4]. Let $\mathbb{Q}\subseteq\mathbb{K}\subseteq\mathbb{R}$, the set $[a,b]_{\mathbb{K}}$ denotes by

$$[a,b]_{\mathbb{K}} = \{\alpha a+(1-\alpha)b : \alpha\in\mathbb{K}\cap[0,1]\}. \tag{2.6}$$

It's obvious that $[a,b]_{\mathbb{K}}\subseteq[a,b]$. Also, we can regard $[a,b]_{\mathbb{K}}$ as a convex combination of $\{a,b\}$ with coefficient from $\mathbb{K}$, i.e. $[a,b]_{\mathbb{K}} = conv_{\mathbb{K}}(\{a,b\})$. For the case $\mathbb{K}=\mathbb{R}$, we use $[a,b]=conv(\{a,b\})$ instead of $[a,b]_{\mathbb{K}} = conv_{\mathbb{K}}(\{a,b\})$. Actually, such subfield $\mathbb{K}$ of the real numbers $\mathbb{R}$ always exists. Since $\mathbb{Q}\subset\mathbb{Q}(\sqrt{2})\subset\ldots\subset\mathbb{Q}(\sqrt{2},\sqrt{3},\ldots,\sqrt{p})\subset\ldots\subset\mathbb{R}$, where p is a prime. There are infinitely many subfields $\mathbb{K}$ satisfies $\mathbb{Q}\subseteq\mathbb{K}\subseteq\mathbb{R}$. Meanwhile, $\mathbb{Q}$ is a subfield of any number field.

Now we can give the following definitions.

Definition 2.4. Let $F:[a,b]\to\mathcal{K}_c$, we say that F is $\mathbb{K}$-linear if F satisfies two properties:
(1) Additive:

$$F(x+y) = F(x)+F(y). \tag{2.7}$$

(2) $\mathbb{K}$-Homogeneity:

$$F(\alpha x) = \alpha F(x), \quad \forall\alpha\in\mathbb{K}. \tag{2.8}$$

Example 2.5. Let $\mathbb{K}=\mathbb{Q}(\sqrt{2})=\{a+\sqrt{2}b \mid a,b\in\mathbb{Q}\}$ and let $F:\mathbb{K}\to\mathcal{K}_c$ be $F(x)=[-\sqrt{2}x,\ \sqrt{2}x]$. We can check that F isn't linear in the usual sense but is $\mathbb{K}$-linear.

Definition 2.6. Let $F:[a,b]\to\mathcal{K}_c$, we say that F is $\mathbb{K}$-convex if for any $\alpha\in\mathbb{K}\cap(0,1)$,

$$\alpha F(x)+(1-\alpha)F(y)\subseteq F(\alpha x+(1-\alpha)y). \tag{2.9}$$

We denote $CX_{\mathbb{K}}([a,b],\mathcal{K}_c)$ by the set of all $\mathbb{K}$-convex interval-valued function on $[a,b]$.

Definition 2.7. Let $F : [a,b] \to \mathcal{K}_c$, we say that F is radial $\mathbb{K}$-continuous at $x_0 \in [a,b]$ if

$$\lim_{\alpha\to 0} \|F((1-\alpha)x_0 + \alpha x) \ominus_g F(x_0)\| = 0, \tag{2.10}$$

where $\alpha \in \mathbb{K}_+$ and $\mathbb{K}_+$ denotes the all positive numbers of $\mathbb{K}$. We denote $C_{\mathbb{K}}([a,b],\mathcal{K}_c)$ by the set of all radial $\mathbb{K}$-continuous interval-valued function on $[a,b]$.

3. Interval-valued $\mathbb{K}$-Riemann integral

We firstly present the concept of bounded, then the definition of supremum and infimum intervals for interval-valued functions can be given.

Definition 3.1. Let $F : [a,b] \to \mathcal{K}_c$, we say that F is bounded on $[a,b]_{\mathbb{K}}$ if there exists $\mathcal{M} > 0$ such that

$$\|F(x)\| \leq \mathcal{M} \quad \text{for any} \quad x \in [a,b]_{\mathbb{K}}.$$

Remark 3.2. For any $F : [a,b] \to \mathcal{K}_c$, F is bounded on $[a,b]_{\mathbb{K}}$ doesn't imply that F is bounded on the whole $[a,b]$. For example, consider $\mathbb{K} = \mathbb{Q}$ and $F : [1,2] \to \mathcal{K}_c$ given by

$$F(x) = \begin{cases} [1,2], & x \in [1,2]_{\mathbb{Q}}, \\ \{0\}, & x = \sqrt{2}, \\ \left[\frac{1}{x-\sqrt{2}} - 1, \frac{1}{x-\sqrt{2}}\right], & \text{others}. \end{cases}$$

One can easily check that F is bounded on $[1,2]_{\mathbb{Q}}$ but not bounded on $[1,2]$.

Definition 3.3. We say that $S \in \mathcal{K}_c$ is the supremum interval of $F : [a,b] \to \mathcal{K}_c$ on $[a,b]_{\mathbb{K}}$ if the following conditions hold.

(1) For any $x \in [a,b]_{\mathbb{K}}$, we have $F(x) \preceq S$.

(2) For any $\varepsilon > 0$, there is $x_0 \in [a,b]_{\mathbb{K}}$ such that $S \ominus_g \varepsilon \prec F(x_0)$.

Moreover,

$$S := \sup_{x\in[a,b]_{\mathbb{K}}} F(x) = [s^-, s^+] = \left[\sup_{x\in[a,b]_{\mathbb{K}}} f^-(x), \sup_{x\in[a,b]_{\mathbb{K}}} f^+(x)\right]. \tag{3.1}$$

Definition 3.4. We say that $T \in \mathcal{K}_c$ is the infimum interval of $F : [a,b] \to \mathcal{K}_c$ on $[a,b]_{\mathbb{K}}$ if the following conditions hold.

(1) For any $x \in [a,b]_{\mathbb{K}}$, we have $T \preceq F(x)$.

(2) For any $\varepsilon > 0$, there is $x_0 \in [a,b]_{\mathbb{K}}$ such that $F(x_0) \prec T + \varepsilon$.

Moreover,

$$T := \inf_{x\in[a,b]_{\mathbb{K}}} F(x) = [t^-, t^+] = \left[\inf_{x\in[a,b]_{\mathbb{K}}} f^-(x), \inf_{x\in[a,b]_{\mathbb{K}}} f^+(x)\right]. \tag{3.2}$$

Example 3.5. Consider $F : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathcal{K}_c$ defined by $F(x) = \left[-\frac{1}{x+2}, \sin x\right]$. Let $\mathbb{K} = \mathbb{Q}$, then we have

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]_{\mathbb{K}} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \cap \mathbb{Q}.$$

Then

$$S = \sup_{x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]_{\mathbb{K}}} F(x) = \left[-\frac{2}{\pi+4}, 1\right],$$

and

$$T = \inf_{x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]_{\mathbb{K}}} F(x) = \left[\frac{2}{\pi-4}, -1\right].$$

For a given $I = [a, b]$, let $I_1 \leq I_2 \leq \ldots \leq I_n$ and $I_i = [a_i, b_i]$, where $a_{i+1} = b_i$, $1 \leq i \leq n, i \in \mathbb{N}$. The finite set of intervals $\tau = \{(I_1, I_2, \ldots, I_n), \bigcup_{i=1}^{n} I_i = I, 1 \leq i \leq n, i \in \mathbb{N}\}$ is called a partition of I. Let $\mathcal{P}_{[a,b]}$ denote by the set of all partitions of I. Through this, we can give the definition of $\mathbb{K}$-partition of I.

$$\mathcal{P}^{\mathbb{K}}_{[a,b]} := \{\tau | \tau = (I_1, I_2, \ldots, I_n) \in \mathcal{P}_{[a,b]}, a_i, b_i \in [a, b]_{\mathbb{K}}, 1 \leq i \leq n, i \in \mathbb{N}\} \tag{3.3}$$

Consider any $\tau = (I_1, I_2, \ldots, I_n) \in \mathcal{P}^{\mathbb{K}}_{[a,b]}$, let

$$S_i = \sup_{x \in [a_i, b_i]_{\mathbb{K}}} F(x), \qquad T_i = \inf_{x \in [a_i, b_i]_{\mathbb{K}}} F(x), \quad 1 \leq i \leq n, i \in \mathbb{N}.$$

Note that

$$T \leq T_i \leq S_i \leq S, \quad 1 \leq i \leq n, i \in \mathbb{N}.$$

If $F : [a, b] \to \mathcal{K}_c$ is bounded on $[a, b]_{\mathbb{K}}$, these supremum and infimum intervals are finite well-defined intervals.

The upper and lower $\mathbb{K}$-Darboux sum associated to F and partition τ can be defined by

$$U_{\mathbb{K}}(F, \tau) := \sum_{i=1}^{n} S_i \ell(I_i),$$

and

$$L_{\mathbb{K}}(F, \tau) := \sum_{i=1}^{n} T_i \ell(I_i).$$

Note that $U_{\mathbb{K}}(F, \tau), L_{\mathbb{K}}(F, \tau) \in \mathcal{K}_c$ and

$$T(b-a) \leq L_{\mathbb{K}}(F, \tau) \leq U_{\mathbb{K}}(F, \tau) \leq S(b-a).$$

Then the upper and lower $\mathbb{K}$-Darboux integral of F on $[a, b]$ can be defined by

$$\overline{\int_a^b} F(x) d_{\mathbb{K}} x := \inf \left\{U_{\mathbb{K}}(F, \tau) : \tau \in \mathcal{P}^{\mathbb{K}}_{[a,b]}\right\},$$

and

$$\underline{\int_a^b} F(x) d_{\mathbb{K}} x := \sup \left\{L_{\mathbb{K}}(F, \tau) : \tau \in \mathcal{P}^{\mathbb{K}}_{[a,b]}\right\}.$$

Since F is bounded on $[a, b]_{\mathbb{K}}$, the above integrals exist.

Definition 3.6. Let $F : [a,b] \to \mathcal{K}_c$ be bounded on $[a,b]_{\mathbb{K}}$. If we have

$$\overline{\int_a^b} F(x) d_{\mathbb{K}} x = \underline{\int_a^b} F(x) d_{\mathbb{K}} x, \tag{3.4}$$

then F is said to be $\mathbb{K}$-Darboux integrable on $[a,b]$. Let $\int_a^b F(x) d_{\mathbb{K}} x \in \mathcal{K}_c$ refer to the $\mathbb{K}$-Darboux integral of F on $[a,b]$. We denote $D_{\mathbb{K}}([a,b], \mathcal{K}_c)$ by the set of all $\mathbb{K}$-Darboux integrable interval-valued functions on $[a,b]$.

For the special case $\mathbb{K} = \mathbb{R}$, we use $\int_a^b F(x) dx$ instead of $\int_a^b F(x) d_{\mathbb{K}} x$ and $D([a,b], \mathcal{K}_c)$ instead of $D_{\mathbb{K}}([a,b], \mathcal{K}_c)$. The next statement shows that $\mathbb{K}$-Darboux integral is well-defined.

Theorem 3.7. *Let $F : [a,b] \to \mathcal{K}_c$. Then $F \in D_{\mathbb{K}}([a,b], \mathcal{K}_c)$ if and only if for any $\varepsilon > 0$, there exists a partition $\tau \in \mathcal{P}^{\mathbb{K}}_{[a,b]}$ such that*

$$\left\| U_{\mathbb{K}}(F, \tau) \ominus_g L_{\mathbb{K}}(F, \tau) \right\| < \varepsilon. \tag{3.5}$$

Proof. First, consider $F \in D_{\mathbb{K}}([a,b], \mathcal{K}_c)$. For any $\varepsilon > 0$, there exist two partitions $\tau_1, \tau_2 \in \mathcal{P}^{\mathbb{K}}_{[a,b]}$ such that

$$U_{\mathbb{K}}(F, \tau_1) \prec \overline{\int_a^b} F(x) d_{\mathbb{K}} x + \frac{\varepsilon}{2}, \qquad \underline{\int_a^b} F(x) d_{\mathbb{K}} x \ominus_g \frac{\varepsilon}{2} \prec L_{\mathbb{K}}(F, \tau_2).$$

Let $\tau = \tau_1 \cup \tau_2$, then we can get

$$\begin{aligned} &\left\| U_{\mathbb{K}}(F, \tau) \ominus_g L_{\mathbb{K}}(F, \tau) \right\| \le \left\| U_{\mathbb{K}}(F, \tau_1) \ominus_g L_{\mathbb{K}}(F, \tau_2) \right\| \\ =&\left\| \left(U_{\mathbb{K}}(F, \tau_1) \ominus_g \overline{\int_a^b} F(x) d_{\mathbb{K}} x \right) + \left(\underline{\int_a^b} F(x) d_{\mathbb{K}} x \ominus_g L_{\mathbb{K}}(F, \tau_2) \right) \right\| \\ <&\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Conversely, let $\varepsilon > 0$ and choose $\tau \in \mathcal{P}^{\mathbb{K}}_{[a,b]}$ which satisfies (3.5). Since

$$\overline{\int_a^b} F(x) d_{\mathbb{K}} x \preceq U_{\mathbb{K}}(F, \tau) \quad \text{and} \quad L_{\mathbb{K}}(F, \tau) \preceq \underline{\int_a^b} F(x) d_{\mathbb{K}} x,$$

we have

$$0 \le \left\| \overline{\int_a^b} F(x) d_{\mathbb{K}} x \ominus_g \underline{\int_a^b} F(x) d_{\mathbb{K}} x \right\| \le \left\| U_{\mathbb{K}}(F, \tau) \ominus_g L_{\mathbb{K}}(F, \tau) \right\| < \varepsilon.$$

Which finishes the proof. □

By Theorem 3.5 and the properties of classical Darboux integral, we can obtain the following statement.

Corollary 3.8. *Let $F : [a,b] \to \mathcal{K}_c$. Then we say that $F \in D_{\mathbb{K}}([a,b], \mathcal{K}_c)$ if and only if for any sequence $\{\tau_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}^{\mathbb{K}}_{[a,b]}$, where $\tau_n = (I_1^{(n)}, I_2^{(n)}, \ldots, I_{k_n}^{(n)})$ such that*

$$\max_{1\le j\le k_n} \ell(I_j^{(n)}) \to 0 \quad \textit{as} \quad n \to \infty,$$

and for any point $\xi_j^{(n)} \in [a_j^{(n)}, b_j^{(n)}]_{\mathbb{K}}$ *we have*

$$\int_a^b F(x) d_{\mathbb{K}} x = \lim_{n\to\infty} \sum_{j=1}^{k_n} F\left(\xi_j^{(n)}\right) \ell(I_j^{(n)}). \tag{3.6}$$

Now, we can give the definition of interval-valued $\mathbb{K}$-Riemann integral.

Definition 3.9. We say that $F : [a, b] \to \mathcal{K}_c$ is $\mathbb{K}$-Riemann integrable on $[a, b]$, if there exists an $H \in \mathcal{K}_c$ for any $\tau \in \mathcal{P}_{[a,b]}^{\mathbb{K}}$ such that

$$\max_{1\le i\le n} \ell(I_i) \to 0 \quad \text{as} \quad n \to \infty,$$

and $\xi_i \in [a_i, b_i]_{\mathbb{K}}$ for all $1 \le i \le n, i \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{n} F(\xi_i)\ell(I_i) \ominus_g H \right\| = 0. \tag{3.7}$$

We say that interval H is the $\mathbb{K}$-Riemann integral of F and we denote $R_{\mathbb{K}}([a, b], \mathcal{K}_c)$ by the set of all $\mathbb{K}$-Riemann integrable interval-valued functions on $[a, b]$.

Remark 3.10. From Corollary 3.8, it's not hard to see that $\mathbb{K}$-Darboux integral is equivalent to $\mathbb{K}$-Riemann integral of interval-valued functions. We use $\mathbb{K}$-Riemann integral and the symbol $\int_a^b F(x) d_{\mathbb{K}} x$ instead of $\mathbb{K}$-Darboux integral in the rest of this paper.

Theorem 3.11. *Let* $\mathbb{Q} \subseteq \mathbb{K}_1 \subseteq \mathbb{K}_2 \subseteq \mathbb{R}$ *and let* $F : [a, b] \to \mathcal{K}_c$*, if* $F \in R_{\mathbb{K}_2}([a, b], \mathcal{K}_c)$*, then* $F \in R_{\mathbb{K}_1}([a, b], \mathcal{K}_c)$ *and*

$$\int_a^b F(x) d_{\mathbb{K}_1} x = \int_a^b F(x) d_{\mathbb{K}_2} x. \tag{3.8}$$

Proof. Consider $\tau_n = (I_1^{(n)}, I_2^{(n)}, \ldots, I_{k_n}^{(n)}) \in \mathcal{P}_{[a,b]}^{\mathbb{K}_1}$ such that

$$\max_{1\le j\le k_n} \ell(I_j^{(n)}) \to 0 \quad \text{as} \quad n \to \infty,$$

Since $F \in R_{\mathbb{K}_2}([a, b], \mathcal{K}_c)$, for any point $\xi_j^{(n)} \in [a_j^{(n)}, b_j^{(n)}]_{\mathbb{K}_1}$, we have

$$\int_a^b F(x) d_{\mathbb{K}_2} x = \lim_{n\to\infty} \sum_{j=1}^{k_n} F\left(\xi_j^{(n)}\right) \ell(I_j^{(n)}).$$

By the arbitrariness of $\tau_n \in \mathcal{P}_{[a,b]}^{\mathbb{K}_1}$, we have

$$\int_a^b F(x) d_{\mathbb{K}_1} x = \lim_{n\to\infty} \sum_{j=1}^{k_n} F\left(\xi_j^{(n)}\right) \ell(I_j^{(n)}).$$

□

Corollary 3.12. *Let $F : [a, b] \to \mathcal{K}_c$. If $F \in R([a, b], \mathcal{K}_c)$ then for any $\mathbb{K} \subseteq \mathbb{R}$, we have $F \in R_{\mathbb{K}}([a, b], \mathcal{K}_c)$ and*

$$\int_a^b F(x) d_{\mathbb{K}} x = \int_a^b F(x) dx. \tag{3.9}$$

The next example shows that the reverse of the above result is not true.

Example 3.13. Let $\mathbb{K}_1 \subseteq \mathbb{K}_2 \subseteq \mathbb{R}$, and $\mathbb{K}_1 \neq \mathbb{K}_2$. Consider $F : [a, b] \to \mathcal{K}_c$ given by

$$F(x) = \begin{cases} [0, 1], & x \in [a, b]_{\mathbb{K}_1}, \\ [-1, 0], & x \in [a, b] \setminus [a, b]_{\mathbb{K}_1}. \end{cases}$$

It's obviously that $F \in R_{\mathbb{K}_1}([a, b], \mathcal{K}_c)$ and

$$\int_a^b F(x) d_{\mathbb{K}_1} = [0, b - a].$$

Now, we consider any partition $\tau \in \mathcal{P}^{\mathbb{K}_2}_{[a,b]} \setminus \mathcal{P}^{\mathbb{K}_1}_{[a,b]}$. Since $[a_i, b_i]_{\mathbb{K}_1} \subset [a_i, b_i]_{\mathbb{K}_2} \subset [a_i, b_i]$, we obtain

$$S_i = \sup_{x \in [a_i, b_i]_{\mathbb{K}_2}} F(x) = [0, 1], \qquad T_i = \inf_{x \in [a_i, b_i]_{\mathbb{K}_2}} F(x) = [-1, 0].$$

It implies

$$U_{\mathbb{K}_2}(F, \tau) = [0, b - a], \quad L_{\mathbb{K}_2}(F, \tau) = [a - b, 0].$$

Thus

$$[0, b - a] = \overline{\int_a^b} F(x) d_{\mathbb{K}_2} x \neq \underline{\int_a^b} F(x) d_{\mathbb{K}_2} x = [a - b, 0],$$

which illustrates that $F \notin R_{\mathbb{K}_2}([a, b], \mathcal{K}_c)$. Moreover, if we pick another subfield $\mathbb{K}_3$ with $\mathbb{Q} \subseteq \mathbb{K}_3 \subset \mathbb{K}_2$, one can shows that $F \in R_{\mathbb{K}_3}([a, b], \mathcal{K}_c)$.

The following results can be obtained easily by the case of real-valued functions(see [3]).

Theorem 3.14. *Let $F : [a, b] \to \mathcal{K}_c$. If $F \in CX_{\mathbb{K}}([a, b], \mathcal{K}_c)$, then there always have a unique $\Psi : [a, b] \to \mathcal{K}_c$ and*

$$\Psi(x) = F(x), \quad \textit{for any} \quad x \in [a, b]_{\mathbb{K}}. \tag{3.10}$$

Moreover, Ψ is convex and $\Psi \in C([a, b], \mathcal{K}_c)$.

Proposition 3.15. *Let $F, G : [a, b] \to \mathcal{K}_c$ and $F, G \in R_{\mathbb{K}}([a, b], \mathcal{K}_c)$.*

(i) For any $\alpha, \beta \in \mathbb{R}$, if $\alpha F + \beta G \in IR^{\mathbb{K}}_{[a,b]}$, then

$$\int_a^b [\alpha F(x) + \beta G(x)]\, d_{\mathbb{K}} x = \alpha \int_a^b F(x) d_{\mathbb{K}} x + \beta \int_a^b G(x) d_{\mathbb{K}} x. \tag{3.11}$$

(ii) If $F \subseteq G$, then

$$\int_a^b F(x) d_{\mathbb{K}} x \subseteq \int_a^b G(x) d_{\mathbb{K}} x. \tag{3.12}$$

(iii) If $\|F(x)\| : [a,b] \to [0,\infty)$ *is* $\mathbb{K}$*-Riemann integrable and*

$$\left\| \int_a^b F(x) d_{\mathbb{K}} x \right\| \leq \int_a^b \|F(x)\| \, d_{\mathbb{K}} x. \tag{3.13}$$

Theorem 3.16. *Let* $F, G : [a,b] \to \mathcal{K}_c$*. If* $F, G \in R_{\mathbb{K}}([a,b], \mathcal{K}_c)$*, then*

$$\int_a^b F(x) d_{\mathbb{K}} x \ominus_g \int_a^b G(x) d_{\mathbb{K}} x \subseteq \int_a^b (F \ominus_g G)(x) d_{\mathbb{K}} x. \tag{3.14}$$

Moreover, if $\ell(F) - \ell(G)$ *has a constant sign on* $[a,b]$*, then*

$$\int_a^b F(x) d_{\mathbb{K}} x \ominus_g \int_a^b G(x) d_{\mathbb{K}} x = \int_a^b (F \ominus_g G)(x) d_{\mathbb{K}} x. \tag{3.15}$$

Proof. First, let $\Phi^- := f^- - g^-$, $\Phi^+ := f^+ - g^+$, then we have

$$\begin{aligned} \int_a^b \min\{\phi^-, \phi^+\} \, d_{\mathbb{K}} x &\leq \min\left\{ \int_a^b \phi^- d_{\mathbb{K}} x, \int_a^b \phi^+ d_{\mathbb{K}} x \right\} \\ &\leq \max\left\{ \int_a^b \phi^- d_{\mathbb{K}} x, \int_a^b \phi^+ d_{\mathbb{K}} x \right\} \leq \int_a^b \max\{\phi^-, \phi^+\} \, d_{\mathbb{K}} x. \end{aligned}$$

It implies that

$$\begin{aligned} \int_a^b F(x) d_{\mathbb{K}} x \ominus_g \int_a^b G(x) d_{\mathbb{K}} x &= \left[\min\left\{ \int_a^b \phi^- d_{\mathbb{K}} x, \int_a^b \phi^+ d_{\mathbb{K}} x \right\}, \max\left\{ \int_a^b \phi^- d_{\mathbb{K}} x, \int_a^b \phi^+ d_{\mathbb{K}} x \right\} \right] \\ &\subseteq \left[\int_a^b \min\{\phi^-, \phi^+\} d_{\mathbb{K}} x, \int_a^b \max\{\phi^-, \phi^+\} d_{\mathbb{K}} x \right] \\ &= \int_a^b (F \ominus_g G)(x) d_{\mathbb{K}} x. \end{aligned}$$

Moreover, if $\ell(F) - \ell(G)$ has constant sign on $[a,b]$, then $F \ominus_g G = [\phi^-, \phi^+]$, if $\ell(F) \geq \ell(G)$, or $F \ominus_g G = [\phi^+, \phi^-]$, if $\ell(F) < \ell(G)$. We now assume that $\ell(F) \geq \ell(G)$ on $[a,b]$, and $F \ominus_g G = [\phi^-, \phi^+]$. Thus, we have $\int_a^b \phi^- d_{\mathbb{K}} x \leq \int_a^b \phi^+ d_{\mathbb{K}} x$, which shows that

$$\int_a^b (F \ominus_g G)(x) d_{\mathbb{K}} x = \int_a^b F(x) d_{\mathbb{K}} x \ominus_g \int_a^b G(x) d_{\mathbb{K}} x.$$

The other case $\ell(F) < \ell(G)$ can be proved in the same way. □

Theorem 3.17. *Let* $c \in [a,b]_{\mathbb{K}}$*. For any* $F : [a,b] \to \mathcal{K}_c$*, if* $F \in R_{\mathbb{K}}([a,b], \mathcal{K}_c)$*, then* $F \in R_{\mathbb{K}}([a,c], \mathcal{K}_c)$ *and* $F \in R_{\mathbb{K}}([c,b], \mathcal{K}_c)$*. Moreover*

$$\int_a^b F(x) d_{\mathbb{K}} x = \int_a^c F(x) d_{\mathbb{K}} x + \int_c^b F(x) d_{\mathbb{K}} x. \tag{3.16}$$

Proof. Suppose $F \in R_{\mathbb{K}}([a,b],\mathcal{K}_c)$. For any $\varepsilon > 0$ there exists $\tau \in \mathcal{P}^{\mathbb{K}}_{[a,b]}$ such that

$$\left\| U_{\mathbb{K}}(F,\tau) \ominus_g L_{\mathbb{K}}(F,\tau) \right\| < \varepsilon.$$

By adding point c to τ write as τ_c such that $\tau_c = (I_1,\ldots,I_k,I_c,I_{k+1},\ldots,I_n)$, where $I_k = [a_k,c]$ and $I_c = [c,b_k]$. Consider $\tau_c = \tau_1 \cup \tau_2$ and

$$\tau_1 := \tau_c \cap [a,c]_{\mathbb{K}} \in \mathcal{P}^{\mathbb{K}}_{[a,c]}, \quad \tau_2 := \tau_c \cap [c,b]_{\mathbb{K}} \in \mathcal{P}^{\mathbb{K}}_{[c,b]}.$$

Since

$$U_{\mathbb{K}}(F,\tau_c) = U_{\mathbb{K}}(F,\tau_1) + U_{\mathbb{K}}(F,\tau_2), \quad L_{\mathbb{K}}(F,\tau_c) = L_{\mathbb{K}}(F,\tau_1) + L_{\mathbb{K}}(F,\tau_2).$$

It seems that

$$\begin{aligned} &\left\| U_{\mathbb{K}}(F,\tau_1) \ominus_g L_{\mathbb{K}}(F,\tau_1) \right\| \\ =&\left\| \left(U_{\mathbb{K}}(F,\tau_c) \ominus_g U_{\mathbb{K}}(F,\tau_2)\right) \ominus_g \left(L_{\mathbb{K}}(F,\tau_c) \ominus_g L_{\mathbb{K}}(F,\tau_2)\right) \right\| \\ \leq&\left\| U_{\mathbb{K}}(F,\tau) \ominus_g L_{\mathbb{K}}(F,\tau) \right\| < \varepsilon. \end{aligned}$$

Similarly, we obtain

$$\left\| U_{\mathbb{K}}(F,\tau_2) \ominus_g L_{\mathbb{K}}(F,\tau_2) \right\| < \varepsilon.$$

This illustrates $F \in R_{\mathbb{K}}([a,c],\mathcal{K}_c)$ and $F \in R_{\mathbb{K}}([c,b],\mathcal{K}_c)$.

Finally, throughout above result, we obtain

$$\begin{aligned} \int_a^b F(x)d_{\mathbb{K}}x \preceq U_{\mathbb{K}}(F,\tau_c) &= U_{\mathbb{K}}(F,\tau_1) + U_{\mathbb{K}}(F,\tau_2) \\ &\prec L_{\mathbb{K}}(F,\tau_1) + L_{\mathbb{K}}(F,\tau_2) + \varepsilon \\ &\prec \int_a^c F(x)d_{\mathbb{K}}x + \int_c^b F(x)d_{\mathbb{K}}x + \varepsilon. \end{aligned}$$

Similarly, we have

$$\int_a^c F(x)d_{\mathbb{K}}x + \int_c^b F(x)d_{\mathbb{K}}x \ominus_g \varepsilon \prec \int_a^b F(x)d_{\mathbb{K}}x.$$

It seems (3.16) holds. □

Next, we give a calculation of the integral for $\mathbb{K}$-linear interval-valued functions. Note that the $\mathbb{K}$-linear interval-valued functions may not continuous at all point and the set of discontinuous points may not countable, so the classical Riemann integrability has failed here.

Proposition 3.18. *Let $F : [a,b] \to \mathcal{K}_c$. If F is $\mathbb{K}$-linear, then $F \in R_{\mathbb{K}}([a,b],\mathcal{K}_c)$. Moreover,*

$$\int_a^b F(x)d_{\mathbb{K}}x = F\left(\frac{a+b}{2}\right)(b-a). \tag{3.17}$$

Proof. It's obviously that if F is $\mathbb{K}$-linear then $F \in R_{\mathbb{K}}([a,b],\mathcal{K}_c)$. Consider partitions $\tau_n = \left(I_1^{(n)}, I_2^{(n)}, \ldots, I_{k_n}^{(n)}\right) \in \mathcal{P}^{\mathbb{K}_1}_{[a,b]}$ and $\xi_j^{(n)} = a + \frac{j}{n}(b-a)$. By Corollary 3.8,

$$\begin{aligned}
\int_a^b F(x)d_{\mathbb{K}}x &= \lim_{n\to\infty}\sum_{j=1}^{n} F\left(\xi_j^{(n)}\right)\frac{1}{n}(b-a)\\
&= \lim_{n\to\infty} F\left(na + \frac{n(n+1)}{2n}(b-a)\right)\frac{1}{n}(b-a)\\
&= \lim_{n\to\infty} F\left(a + \frac{n+1}{2n}(b-a)\right)(b-a)\\
&= \lim_{n\to\infty}\left[F(a) + \frac{n+1}{2n}F(b-a)\right](b-a)\\
&= \left[F(a) + F\left(\frac{b-a}{2}\right)\right](b-a)\\
&= F\left(\frac{a+b}{2}\right)(b-a).
\end{aligned}$$

□

Example 3.19. Let $\mathbb{K} = \mathbb{Q}$. Consider $F : [0,1] \to \mathcal{K}_c$ such that $F(x) = [-x, x]$. Note that F is $\mathbb{K}$-linear. By Proposition 3.18, we have

$$\int_0^1 F(x)d_{\mathbb{K}}x = F\left(\frac{1+0}{2}\right)(1-0) = \left[-\frac{1}{2}, \frac{1}{2}\right].$$

On the other hand, for a given partition $\tau = (I_1, I_2, \ldots, I_n) \in \mathcal{P}^{\mathbb{K}}_{[0,1]}$ such that $\ell(I_i) = \frac{1}{n}$, and $a_i, b_i \in [0,1]_{\mathbb{K}}$, $1 \le i \le n, i \in \mathbb{N}$. Then

$$U_{\mathbb{K}}(F,\tau) = \sum_{i=1}^{n} S_i\ell(I_i) = \sum_{i=1}^{n}\frac{[-a_i, b_i]}{n} = \left[\frac{1-n}{2n}, \frac{1+n}{2n}\right],$$

and

$$L_{\mathbb{K}}(F,\tau) = \sum_{i=1}^{n} T_i\ell(I_i) = \sum_{i=1}^{n}\frac{[-b_i, a_i]}{n} = \left[-\frac{1+n}{2n}, \frac{n-1}{2n}\right].$$

Thus

$$\overline{\int_0^1} F(x)d_{\mathbb{K}}x = \inf\{U_{\mathbb{K}}(F,\tau)\} = \left[-\frac{1}{2}, \frac{1}{2}\right] = \sup\{L_{\mathbb{K}}(F,\tau)\} = \underline{\int_0^1} F(x)d_{\mathbb{K}}x.$$

Consequently, Proposition 3.18 is verified.

The next example shows that if $c \in (a,b)$, Theorem 3.17 may not hold.

Example 3.20. Consider $F : [a,b] \to \mathcal{K}_c$ is $\mathbb{K}$-linear, let $c = \alpha a + (1-\alpha)b$, where $\alpha \in (0,1)$. If $\mathbb{K} \neq \mathbb{R}$, then

$$\begin{aligned}
&\int_a^b F(x)d_{\mathbb{K}}x \ominus_g \left(\int_a^c F(x)d_{\mathbb{K}}x + \int_c^b F(x)d_{\mathbb{K}}x\right)\\
=&F\left(\frac{a+b}{2}\right)(b-a) \ominus_g \left(F\left(\frac{a+c}{2}\right)(c-a) + F\left(\frac{c+b}{2}\right)(b-c)\right)\\
=&\frac{1}{2}\Big(F(\alpha(a-b)) \ominus_g \alpha F(a-b)\Big)(a-b).
\end{aligned} \tag{3.18}$$

Obviously, if we have $\alpha \in [0,1]_{\mathbb{K}}$, (3.18) is equal to zero. But for some $\alpha \in (0,1) \setminus \mathbb{K}$, i.e. $c \in [a,b] \setminus [a,b]_{\mathbb{K}}$, (3.18) is different from zero.

4. Radial gH-$\mathbb{K}$-derivative for interval-valued functions

In this section, we introduce the definition of interval-valued radial gH-$\mathbb{K}$-derivative.

Definition 4.1. Let $F : [a,b] \to \mathcal{K}_c$, we say that $F(x)$ is radial gH-$\mathbb{K}$ differential at x_0 in the direction λ, if there exists $\mathcal{I} \in \mathcal{K}_c$ for any $h \in \mathbb{K}_+$ such that

$$\lim_{h\to 0} \frac{F(x_0 + h\lambda) \ominus_g F(x_0)}{h} = \mathcal{I}. \tag{4.1}$$

$\mathcal{I}$ defined in this fashion is denoted by the symbol $D^{\lambda}_{\mathbb{K}}F(x_0)$ and $D^{\lambda}_{\mathbb{K}}F(x_0)$ is said to be the radial gH-$\mathbb{K}$-derivative of $F(x)$ at x_0 in the direction λ.

Check that if $F : [a,b] \to \mathcal{K}_c$ is $\mathbb{K}$-linear, then we have

$$D^{\lambda}_{\mathbb{K}}F(x) = F(\lambda). \tag{4.2}$$

If $F : [a,b] \to \mathcal{K}_c$ is gH-differential at $x_0 \in [a,b]$, then F is radial gH-$\mathbb{K}$-differential at x_0 in the direction λ and

$$D^{\lambda}_{\mathbb{K}}F(x) = \lambda F'(x). \tag{4.3}$$

The next statement shows the relationship between interval-valued $\mathbb{K}$-Riemann integral and radial gH-$\mathbb{K}$-derivative.

Theorem 4.2. *Let $F : [a,b] \to \mathcal{K}_c$, and $F \in R_{\mathbb{K}}([a,x], \mathcal{K}_c)$, for any $x \in (a,b]$. We define $\Phi : [a,b] \to \mathcal{K}_c$ by*

$$\Phi(x) := \int_a^x F(t)d_{\mathbb{K}}t. \tag{4.4}$$

If $F \in C_{\mathbb{K}}([a,b], \mathcal{K}_c)$, then Φ is radial gH-$\mathbb{K}$-differential at x in the direction $x-a$ and

$$D^{x-a}_{\mathbb{K}}\Phi(x) = F(x)(x-a). \tag{4.5}$$

Proof. Fix $x \in (a,b]$, for any $\varepsilon > 0$ and $h \in \mathbb{K}_+$, we have

$$\begin{aligned}
&\left\| \frac{\Phi(x + h(x-a)) \ominus_g \Phi(x)}{h} \ominus_g F(x)(x-a) \right\| \\
=&\left\| \frac{1}{h} \left(\int_a^{x+h(x-a)} F(t)d_{\mathbb{K}}t \ominus_g \int_a^x F(t)d_{\mathbb{K}}t \right) \ominus_g \frac{1}{h} \int_x^{x+h(x-a)} F(x)d_{\mathbb{K}}t \right\| \\
=&\left\| \frac{1}{h} \int_x^{x+h(x-a)} F(t)d_{\mathbb{K}}t \ominus_g \frac{1}{h} \int_x^{x+h(x-a)} F(x)d_{\mathbb{K}}t \right\| \\
=&\frac{1}{h} \left\| \int_x^{x+h(x-a)} (F(t) \ominus_g F(x))d_{\mathbb{K}} \right\| \leq \frac{1}{h} \int_x^{x+h(x-a)} \left\| F(t) \ominus_g F(x) \right\| d_{\mathbb{K}}t.
\end{aligned}$$

Let $h \in \mathbb{K}_+$ be small enough, then

$$\left\| F(t) \ominus_g F(x) \right\| < \frac{\varepsilon}{x-a}, \quad \text{where} \quad t \in [x, x + h(x-a)]_{\mathbb{K}}.$$

Thus

$$\begin{aligned}
&\frac{1}{h}\int_x^{x+h(x-a)} \left\|F(t)\ominus_g F(x)\right\| d_{\mathbb{K}}t \\
\le&\frac{1}{h}\int_x^{x+h(x-a)} \frac{\varepsilon}{x-a} d_{\mathbb{K}}t = \frac{1}{h}\cdot h(x-a)\cdot\frac{\varepsilon}{x-a} = \varepsilon.
\end{aligned}$$

□

Now, we can give the next characterization of $\mathbb{K}$-convex interval-valued functions.

Theorem 4.3. *Let $F : [a,b] \to \mathcal{K}_c$. If $F \in CX_{\mathbb{K}}([a,b],\mathcal{K}_c)$ and F is ℓ-monotone on $[a,b]$, then for any $x \in [a,b]$, we have*

$$F(x)\ominus_g F(a) = \frac{1}{x-a}\int_a^x D_{\mathbb{K}}^{x-a}F(t)d_{\mathbb{K}}t. \tag{4.6}$$

Proof. Since $F \in CX_{\mathbb{K}}([a,b],\mathcal{K}_c)$, by Theorem 3.14, there exist a unique continuous $\Psi : [a,b] \to \mathcal{K}_c$ and

$$F(t) = \Psi(t), \qquad \text{for any} \quad t \in [a,x]_{\mathbb{K}}.$$

Then we obtain

$$\begin{aligned}
D_{\mathbb{K}}^{x-a}F(t) &= \lim_{h\to 0}\frac{F(t+h(x-a))\ominus_g F(t)}{h} \\
&= (x-a)\cdot\lim_{h\to 0}\frac{\Psi(t+h(x-a))\ominus_g \Psi(t)}{h(x-a)} = (x-a)\Psi'_+(t),
\end{aligned}$$

where $h \in \mathbb{K}_+$. Thus,

$$\frac{1}{x-a}\int_a^x D_{\mathbb{K}}^{x-a}F(t)d_{\mathbb{K}}t = \int_a^x \Psi'_+(t)dt = \Psi(x)\ominus_g\Psi(a) = F(x)\ominus_g F(a).$$

□

The following example illustrates that if F is not ℓ-monotone, then (4.6) is false.

Example 4.4. Consider $F : [0,2] \to \mathcal{K}_c$ such that

$$F(x) = \left[\frac{1}{2}x^2, 2\ln(x+1)\right] \quad \text{for any} \quad x \in [0,2].$$

We see that $F \in CX_{\mathbb{K}}([0,2],\mathcal{K}_c)$ for any subfield $\mathbb{K} \subseteq \mathbb{R}$. It's obviously that $F(x)$ is radial gH-$\mathbb{K}$-differential on $[0,2]$, and

$$D_{\mathbb{K}}^2F(x) = 2F'(x) = \begin{cases} \left[2x, \frac{4}{x+1}\right] & \text{if } x \in [0,1), \\ \{2\} & \text{if } x = 1, \\ \left[\frac{4}{x+1}, 2x\right] & \text{if } x \in (1,2]. \end{cases}$$

Since $D_{\mathbb{K}}^2F(x) \in R([0,2],\mathcal{K}_c)$, then we have $D_{\mathbb{K}}^2F(x) \in R_{\mathbb{K}}([0,2],\mathcal{K}_c)$. Check that $F(x)$ is ℓ-increasing on $[0,1]$ and ℓ-decreasing on $[1,2]$, then we obtain

$$\begin{aligned}
\int_0^2 D_{\mathbb{K}}^2F(x)d_{\mathbb{K}}x &= \int_0^1\left[2x, \frac{4}{x+1}\right]d_{\mathbb{K}}x + \int_1^2\left[\frac{4}{x+1}, 2x\right]d_{\mathbb{K}}x \\
&= \left[1+4\ln\frac{3}{2}, 3+4\ln 2\right],
\end{aligned}$$

and

$$F(2) \ominus_g F(0) = [2, 2\ln 3] \ominus_g \{0\} = [2, 2\ln 3].$$

Consequently,

$$\frac{1}{2-0}\int_0^2 D^2_{\mathbb{K}} F(x) d_{\mathbb{K}} x = \left[\frac{1}{2} + 2\ln\frac{3}{2}, \frac{3}{2} + 2\ln 2\right] \neq [2, 2\ln 3] = F(2) \ominus_g F(0).$$

But if we only consider $F(x)$ on $[0, 1]$, then

$$\frac{1}{1-0}\int_0^2 D^1_{\mathbb{K}} F(x) d_{\mathbb{K}} x = \int_0^1 \left[x, \frac{2}{x+1}\right] d_{\mathbb{K}} x = \left[\frac{1}{2}, 2\ln 2\right],$$

and

$$F(1) \ominus_g F(0) = \left[\frac{1}{2}, 2\ln 2\right] \ominus_g \{0\} = \left[\frac{1}{2}, 2\ln 2\right].$$

Thus, Theorem 4.3 is verified.

5. Hermite-Hadamard type inequalities

Above all, let's study the standard Hermite-Hadamard type inequality in the case of $\mathbb{K}$-Riemann integral.

Theorem 5.1. *Let $F : [a, b] \to \mathcal{K}_c$. If $F \in CX_{\mathbb{K}}([a, b], \mathcal{K}_c)$ then*

$$\frac{F(a) + F(b)}{2} \subseteq \frac{1}{b-a}\int_a^b F(x) d_{\mathbb{K}} x \subseteq F\left(\frac{a+b}{2}\right). \tag{5.1}$$

Proof. Since $F \in CX_{\mathbb{K}}([a, b], \mathcal{K}_c)$, by Theorem 3.14, there exist a unique continuous $\Psi : [a, b] \to \mathcal{K}_c$ and

$$F(t) = \Psi(t), \qquad \text{for any} \quad t \in [a, x]_{\mathbb{K}}.$$

Note that $\Psi(x)$ is convex and satisfied the classical Hermite-Hadamard inequalities. Thus

$$\frac{F(a) + F(b)}{2} = \frac{\Psi(a) + \Psi(b)}{2} \subseteq \frac{1}{b-a}\int_a^b \Psi(x) dx \subseteq \Psi\left(\frac{a+b}{2}\right) = F\left(\frac{a+b}{2}\right).$$

By Corollary 3.12, we obtain

$$\int_a^b F(x) d_{\mathbb{K}} x = \int_a^b \Psi(x) dx,$$

which finishes the proof. □

Example 5.2. Consider $F : [0, 1] \to \mathcal{K}_c$ defined by $F(x) = [x^2, \sqrt{x} + 1]$. Let $\mathbb{K} = \mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$. Then we have

$$\frac{F(0) + F(1)}{2} = \frac{[0, 1] + [1, 2]}{2} = \left[\frac{1}{2}, \frac{3}{2}\right], \quad \text{and} \quad F\left(\frac{0+1}{2}\right) = \left[\frac{1}{4}, \frac{2 + \sqrt{2}}{2}\right].$$

Since $F \in R([0,1], \mathcal{K}_c)$, by Corollary 3.12, we have $F \in R_{\mathbb{K}}([0,1], \mathcal{K}_c)$, and

$$\int_0^1 F(x) d_{\mathbb{K}} x = \int_0^1 F(x) dx = \int_0^1 [x^2, \sqrt{x}+1] dx = \left[\frac{1}{3}, \frac{5}{3}\right].$$

It seems that

$$\left[\frac{1}{2}, \frac{3}{2}\right] \subseteq \left[\frac{1}{3}, \frac{5}{3}\right] \subseteq \left[\frac{1}{4}, \frac{2+\sqrt{2}}{2}\right],$$

which illustrates

$$\frac{F(0)+F(1)}{2} \subseteq \frac{1}{1-0} \int_0^1 F(x) d_{\mathbb{K}} x \subseteq F\left(\frac{0+1}{2}\right).$$

Consequently, Theorem 5.1 is verified.

Example 5.3. Let $\mathbb{K} \subset \mathbb{R}$, and $\mathbb{K} \neq \mathbb{Q}$. Consider $F : [0,1] \to \mathcal{K}_c$ given by

$$F(x) = \begin{cases} [e^x - 1, \sqrt{x}+2], & x \in [0,1]_{\mathbb{K}}, \\ [x^2 - 4, 1 + \ln(1+x)], & x \in [0,1] \setminus [0,1]_{\mathbb{K}}. \end{cases}$$

Then

$$\frac{F(0)+F(1)}{2} = \frac{[0,2]+[e-1,3]}{2} = \left[\frac{e-1}{2}, \frac{5}{2}\right],$$

and

$$F\left(\frac{0+1}{2}\right) = \left[\sqrt{e}-1, \frac{4+\sqrt{2}}{2}\right].$$

Apparently, $F \notin R([0,1], \mathcal{K}_c)$ but $F \in R_{\mathbb{K}}([0,1], \mathcal{K}_c)$, by Theorem 4.3 we have

$$\int_0^1 F(x) d_{\mathbb{K}} x = \left[e-1, \frac{8}{3}\right] \ominus_g [1,0] = \left[e-2, \frac{8}{3}\right].$$

Finally, we have

$$\left[\frac{e-1}{2}, \frac{5}{2}\right] \subseteq \left[e-2, \frac{8}{3}\right] \subseteq \left[\sqrt{e}-1, \frac{4+\sqrt{2}}{2}\right].$$

Consequently, Theorem 5.1 is verified.

Definition 5.4. Let $F : [a,b] \to \mathcal{K}_c^+$. We say that F is *log*-$\mathbb{K}$-convex if for any $\alpha \in \mathbb{K} \cap (0,1)$,

$$[F(x)]^{\alpha} \cdot [F(y)]^{1-\alpha} \subseteq F(\alpha x + (1-\alpha) y). \tag{5.2}$$

We denote $CX_{\mathbb{K}}^{log}([a,b], \mathcal{K}_c)$ by the set of all *log*-$\mathbb{K}$-convex interval-valued function on $[a,b]$.

Remark 5.5. For any interval $A = [a^-, a^+] \in \mathcal{K}_c^+$, and $\alpha > 0$, we have

$$A^{\alpha} = [a^-, a^+]^{\alpha} = \left[(a^-)^{\alpha}, (a^+)^{\alpha}\right].$$

Proposition 5.6. *Let $F : [a,b] \to \mathcal{K}_c^+$. If $F \in CX_{\mathbb{K}}^{log}([a,b], \mathcal{K}_c)$, then we have*

$$\sqrt{F(a)F(b)} \subseteq \sqrt{F(x)F(a+b-x)} \subseteq F\left(\frac{a+b}{2}\right). \tag{5.3}$$

After that, we can give the following results about log-$\mathbb{K}$-convex interval-valued functions.

Theorem 5.7. *Let* $F : [a, b] \to \mathcal{K}_c^+$. *If* $F \in CX_{\mathbb{K}}^{log}([a, b], \mathcal{K}_c)$, *then for any* $\alpha \in \mathbb{K} \cap (0, 1) \setminus \{\frac{1}{2}\}$,

$$\int_a^b [F(x)]^{1-\alpha}[F(a + b - x)]^{\alpha} d_{\mathbb{K}}x \subseteq \frac{1}{1 - 2\alpha} \int_{(1-\alpha)a+\alpha b}^{\alpha a+(1-\alpha)b} F(t)d_{\mathbb{K}}t. \tag{5.4}$$

Proof. The cases $\alpha = 0, 1$ are trivial. Since $F \in CX_{\mathbb{K}}^{log}([a, b], \mathcal{K}_c)$, then

$$[F(a + b - x)]^{\alpha} \cdot [F(x)]^{1-\alpha} \subseteq F(\alpha(a + b - x) + (1 - \alpha)x) = F((1 - 2\alpha)x + \alpha(a + b)).$$

By integrating the above inequality, we obtain

$$\int_a^b [F(a + b - x)]^{\alpha} \cdot [F(x)]^{1-\alpha} d_{\mathbb{K}}x \subseteq \int_a^b F((1 - 2\alpha)x + \alpha(a + b))d_{\mathbb{K}}x.$$

We define $t = (1 - 2\alpha)x + \alpha(a + b)$, and $d_{\mathbb{K}}t = (1 - 2\alpha)d_{\mathbb{K}}x$.

If $x = a$, we have $t = (1 - \alpha)a + \alpha b$ and if $x = b$, we have $t = \alpha a + (1 - \alpha)b$. Thus,

$$\int_a^b F((1 - 2\alpha)x + \alpha(a + b))d_{\mathbb{K}}x = \frac{1}{1 - 2\alpha} \int_{(1-\alpha)a+\alpha b}^{\alpha a+(1-\alpha)b} F(t)d_{\mathbb{K}}t.$$

□

Remark 5.8. If $\alpha = \frac{1}{2}$, it's obvious that

$$\int_a^b [F(x)]^{1-\alpha}[F(a + b - x)]^{\alpha} d_{\mathbb{K}}x \subseteq \frac{1}{b - a} F\left(\frac{a + b}{2}\right). \tag{5.5}$$

Theorem 5.9. *Let* $F : [a, b] \to \mathcal{K}_c^+$, *and* $F \in CX_{\mathbb{K}}^{log}([a, b], \mathcal{K}_c)$. *If* $\varphi : [a, b] \to [0, +\infty)$ *is* $\mathbb{K}$*-Riemann integrable on* $[a, b]$ *with* $\int_a^b \varphi(x)d_{\mathbb{K}}x > 0$, *then for any* $p > 0$ *we have*

$$\sqrt{F(a)F(b)} \subseteq \left(\frac{\int_a^b \varphi(x)F^p(x)F^p(a + b - x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}\right)^{\frac{1}{2p}} \subseteq F\left(\frac{a + b}{2}\right). \tag{5.6}$$

Proof. For any $p > 0$, if $F \in CX_{\mathbb{K}}^{log}([a, b], \mathcal{K}_c)$, then we have $F^{2p} \in CX_{\mathbb{K}}^{log}([a, b], \mathcal{K}_c)$. By Proposition 5.6,

$$F^p(a)F^p(b) \subseteq F^p(x)F^p(a + b - x) \subseteq F^{2p}\left(\frac{a + b}{2}\right).$$

Multiply the above inequality with $\varphi(x)$ and then integrate, we have

$$\begin{aligned} &F^p(a)F^p(b) \int_a^b \varphi(x)d_{\mathbb{K}}x \\ &\subseteq \int_a^b \varphi(x)F^p(x)F^p(a + b - x)d_{\mathbb{K}}x \subseteq F^{2p}\left(\frac{a + b}{2}\right) \int_a^b \varphi(x)d_{\mathbb{K}}x. \end{aligned}$$

Thus,

$$\sqrt{F(a)F(b)} \subseteq \left(\frac{\int_a^b \varphi(x)F^p(x)F^p(a + b - x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}\right)^{\frac{1}{2p}} \subseteq F\left(\frac{a + b}{2}\right).$$

□

Theorem 5.10. *Let $F : [a,b] \to \mathcal{K}_c^+$, and $F \in CX_{\mathbb{K}}^{log}([a,b],\mathcal{K}_c)$. If $\varphi : [a,b] \to [0,+\infty)$ is $\mathbb{K}$-Riemann integrable on $[a,b]$ with $\int_a^b \varphi(x)d_{\mathbb{K}}x > 0$, then we have*

$$\begin{aligned}
{}^{\frac{1}{b-a}\left(\frac{\int_a^b x\varphi(x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}-a\right)}[F(a)]^{\frac{1}{b-a}\left(b-\frac{\int_a^b x\varphi(x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}\right)} &\subseteq \exp\left(\frac{\int_a^b \varphi(x)\ln F(x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}\right) \\
&\subseteq F\left(\frac{\int_a^b \varphi(x)xd_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}\right).
\end{aligned} \tag{5.7}$$

Proof. Firstly, since $F \in CX_{\mathbb{K}}^{log}([a,b],\mathcal{K}_c)$, consider Jensen's inequality for interval-valued functions(see [20, 21]), we obtain

$$\frac{\int_a^b \varphi(x)\ln F(x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x} \subseteq \ln F\left(\frac{\int_a^b \varphi(x)xd_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}\right).$$

By taking the exponential in the above inequality, we get the first inequality.

Eventually, we have

$$\frac{x-a}{b-a}\ln F(b) + \frac{b-x}{b-a}\ln F(a) \subseteq \ln F\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a\right) = \ln F(x).$$

It seems that

$$\begin{aligned}
&\ln\left([F(b)]^{\frac{1}{b-a}\left(\frac{\int_a^b x\varphi(x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}-a\right)}[F(a)]^{\frac{1}{b-a}\left(b-\frac{\int_a^b x\varphi(x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}\right)}\right) \\
=&\frac{1}{b-a}\left[\left(\frac{\int_a^b x\varphi(x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x} - a\right)\ln F(b) + \left(b - \frac{\int_a^b x\varphi(x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}\right)\ln F(a)\right] \\
\subseteq&\frac{\int_a^b \varphi(x)\ln F(x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}.
\end{aligned}$$

By taking the exponential in the above inequality, we get the second inequality. □

Example 5.11. Let $\mathbb{K} = \mathbb{Q}$. Consider $F : [1,2] \to \mathcal{K}_c^+$ given by $F(x) = [\frac{1}{x}, x]$. Check that $F \in CX_{\mathbb{K}}^{log}([1,2],\mathcal{K}_c)$. Since $F \in R_{\mathbb{K}}([1,2],\mathcal{K}_c)$, if $\alpha = \frac{1}{3}$, we have

$$\int_a^b [F(x)]^{1-\alpha}[F(a+b-x)]^{\alpha}d_{\mathbb{K}}x = \int_1^2 \left[\frac{1}{x}, x\right]^{\frac{2}{3}} \cdot \left[\frac{1}{3-x}, 3-x\right]^{\frac{1}{3}} d_{\mathbb{K}}x = [u,v],$$

where

$$u = \sqrt{3}\left(\arctan\frac{\sqrt[3]{16}-1}{\sqrt{3}} - \arctan\frac{2-\sqrt[3]{2}}{\sqrt{3}\cdot\sqrt[3]{2}}\right),$$

and

$$v = \frac{1}{\sqrt[3]{2}} + \sqrt{3}\left(\arctan\frac{\sqrt[3]{16}-1}{\sqrt{3}} - \arctan\frac{2-\sqrt[3]{2}}{\sqrt{3}\cdot\sqrt[3]{2}}\right).$$

Also,

$$\frac{1}{1-2\alpha}\int_{(1-\alpha)a+\alpha b}^{\alpha a+(1-\alpha)b} F(t)d_{\mathbb{K}}t = 3\int_{\frac{4}{3}}^{\frac{5}{3}}\left[\frac{1}{t}, t\right]d_{\mathbb{K}}t = \left[3\ln\frac{5}{4}, \frac{3}{2}\right].$$

Thus, we obtain

$$[u, v] \subseteq \left[3\ln\frac{5}{4}, \frac{3}{2}\right].$$

Consequently, Theorem 5.7 is verified.

Now, consider $p = 1$, given $\varphi(x) = x$ with $\int_1^2 \varphi(x)d_{\mathbb{K}}x = \frac{3}{2}$, then

$$\begin{aligned}&\left(\frac{\int_a^b \varphi(x)F^p(x)F^p(a+b-x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}\right)^{\frac{1}{2p}}\\ =&\left(\frac{\int_1^2 x\left[\frac{1}{x}, x\right]\left[\frac{1}{3-x}, 3-x\right]d_{\mathbb{K}}x}{\int_1^2 xd_{\mathbb{K}}x}\right)^{\frac{1}{2}} = \left[\frac{2}{3}\ln 2, \frac{13}{6}\right]^{\frac{1}{2}}.\end{aligned}$$

Also we have

$$F\left(\frac{a+b}{2}\right) = \left[\frac{2}{3}, \frac{3}{2}\right], \quad \text{and} \quad \sqrt{F(a)F(b)} = \left[\frac{\sqrt{2}}{2}, \sqrt{2}\right].$$

Thus, we obtain

$$\left[\frac{\sqrt{2}}{2}, \sqrt{2}\right] \subseteq \left[\frac{2}{3}\ln 2, \frac{13}{6}\right]^{\frac{1}{2}} \subseteq \left[\frac{2}{3}, \frac{3}{2}\right].$$

Consequently, Theorem 5.9 is verified.

Next, compute that

$$F\left(\frac{\int_a^b \varphi(x)xd_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}\right) = F\left(\frac{14}{9}\right) = \left[\frac{9}{14}, \frac{14}{9}\right],$$

$$\exp\left(\frac{\int_a^b \varphi(x)\ln F(x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}\right) = \left[e^{\frac{6-16\ln 2}{12}}, e^{\frac{16\ln 2-6}{12}}\right],$$

and

$$[F(b)]^{\frac{1}{b-a}\left(\frac{\int_a^b x\varphi(x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}-a\right)}[F(a)]^{\frac{1}{b-a}\left(b-\frac{\int_a^b x\varphi(x)d_{\mathbb{K}}x}{\int_a^b \varphi(x)d_{\mathbb{K}}x}\right)} = [F(2)]^{\frac{5}{9}} = \left[\frac{1}{2}, 2\right]^{\frac{5}{9}}.$$

Thus, we obtain

$$\left[\frac{1}{2}, 2\right]^{\frac{5}{9}} \subseteq \left[e^{\frac{6-16\ln 2}{12}}, e^{\frac{16\ln 2-6}{12}}\right] \subseteq \left[\frac{9}{14}, \frac{14}{9}\right].$$

Consequently, Theorem 5.10 is verified.

6. Conclusion

In this work, the concepts of interval-valued $\mathbb{K}$-Riemann integral and radial $\mathbb{K}$-gH-derivative are introduced and some basic properties are discussed. Furthermore, some new Hermite-Hadamard type inequalities for $\mathbb{K}$-convex and *log*-$\mathbb{K}$-convex interval-valued functions are established. The main result of this paper give a new type of integral for interval-valued functions, which may be used in the further study of fuzzy-valued functions, interval optimization and interval-valued differential equations. In the future, we intend to study some applications in interval optimizations and interval-valued differential equations by using $\mathbb{K}$-Riemann integral and $\mathbb{K}$-gH-derivative.

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Conflict of interest

The authors declare that they have no competing interests.

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