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Research article

Functional inequalities for several classes of q-starlike and q-convex type analytic and multivalent functions using a generalized Bernardi integral operator

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Abstract: In the article we introduce and investigate several new subclasses of q-starlike and q-convex type analytic and multivalent functions involving a generalized Bernardi integral operator, and establish the Fekete-Szegö type functional inequalities for these function classes. Besides, the corresponding bound estimates of the coefficients a_{p+1} and a_{p+2} are obtained.

Keywords: Fekete-Szegö problem; analytic function; multivalent function; q-derivative operator; Bernardi integral operator
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1. Introduction

Let $\mathcal{A}_p$ denote the class of analytic and p-valent functions $f(z)$ with the next form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \quad (p \in \mathbb{N} = \{1, 2, \cdots\}) \tag{1.1}$$

in the open unit disk $\Delta = \{z \in \mathbb{C} :| z |< 1\}$.

For $f \in \mathcal{A}_p$, its q-derivative or the q-difference $\mathcal{D}_q f(z)$ is given by

$$\mathcal{D}_q f(z) = [p]_q z^{p-1} + \sum_{n=1}^{\infty} [n + p]_q a_{n+p} z^{n+p-1}, \quad (0 < q < 1),$$

where the q-derivative operator $\mathcal{D}_q f(z)$ (refer to [13] and [14]) of the function f is defined by

$$\mathcal{D}_q f(z) := \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & (z \neq 0; 0 < q < 1), \\ f'(0), & (z = 0) \end{cases}$$

provided that $f'(0)$ exists, and the q-number $[n]_q$ is just $[\chi]_q$ when $\chi = n \in \mathbb{N}$, here

$$[\chi]_q = \begin{cases} \frac{1-q^\chi}{1-q} & \text{for } \chi \in \mathbb{C}, \\ \sum_{k=0}^{\chi-1} q^k & \text{for } \chi = n \in \mathbb{N}. \end{cases}$$

Note that $\mathcal{D}_q f(z) \longrightarrow f'(z)$ when $q \longrightarrow 1_-$, where f' is the ordinary derivative of the function f.

Consider the generalized Bernardi integral operator $\mathcal{J}^\eta_{p,q} : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ with the next form

$$\mathcal{J}^\eta_{p,q} f(z) = \frac{[p+\eta]_q}{z^\eta} \int_0^z t^{\eta-1} f(t) d_q t, \quad (z \in \Delta, \Re\eta > -1 \text{ and } f \in \mathcal{A}_p). \tag{1.2}$$

Then, for $f \in \mathcal{A}_p$, we obtain that

$$\mathcal{J}^\eta_{p,q} f(z) = z^p + \sum_{n=1}^{\infty} L^\eta_{p,q}(n) a_{n+p} z^{n+p}, \quad (z \in \Delta), \tag{1.3}$$

where

$$L^\eta_{p,q}(n) = \frac{[p+\eta]_q}{[n+\eta]_q} := L_n. \tag{1.4}$$

Here we remark that if $p = 1$, it is exactly q-Bernardi integral operator $\mathcal{J}^\eta_q$ [21]. Further, if $p = 1$ and $q \to 1_-$, obviously it is the classical Bernardi integral operator $\mathcal{J}_\eta$ [5]. In fact, Alexander [1] and Libera [18] integral operators are special versions of $\mathcal{J}_\eta$ for $\eta = 0$ and $\eta = 1$, respectively.

For two analytic functions f and g, if there exists an analytic function h satisfying $h(0) = 0$ and $| h(z) |< 1$ for $z \in \Delta$ so that $f(z) = g(h(z))$, then f is subordinate to g, i.e., $f \prec g$.

Let Λ be the class of all analytic function ϕ via the form

$$\phi(z) = 1 + \sum_{n=1}^{\infty} A_n z^n, \quad (A_1 > 0, z \in \Delta). \tag{1.5}$$

It is well known that the q-calculus [13, 14], even the (p, q)-calculus [6], is a generalization of the ordinary calculus without the limit symbol, and its related theory has been applied into mathematical, physical and engineering fields (see [11,15,25]). Since Ismail et al. [12] firstly utilized the q-derivative operator to investigate the q-calculus of the class of starlike functions in disk, there had a great deal of work in this respect; for example, refer to Rehman et al. [24] for partial sums of generalized q-Mittag-Leffler functions, Srivastava et al. [31] for Fekete-Szegö inequality for classes of (p, q)-starlike and (p, q)-convex functions and [27] for close-to-convexity of a certain family of q-Mittag-Leffler functions, Seoudy and Aouf [26] for the coefficient estimates of q-starlike and q-convex functions and Uçar [33] for the coefficient inequality for q-starlike functions. Besides, by involving some special functions and operators or increasing the complexity of function classes, many new subclasses of analytic functions associated with q-calculus or (p, q)-calculus were considered. Here we may refer to [9, 23], Ahmad et al. [2] for convolution properties for a family of analytic functions involving q-analogue of Ruscheweyh differential operator, Dweby and Darus [8] for subclass of harmonic univalent functions associated with q-analogue of Dziok-Srivastava operator, Mahmmod and Sokól [20] for new subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator, Srivastava et al. [28] for coefficient inequalities for q-starlike functions associated with the Janowski functions, [31] for Fekete-Szegö inequality for classes of (p, q)-starlike and (p, q)-convex functions using the q-Bernardi integral operator and [32] for some results on the q-analogues of the incomplete Fibonacci and Lucas polynomials. For the multivalent functions, Purohit [22] ever studied a new class of multivalently analytic functions associated with fractional q-calculus operators, while Shi et al. [29] investigated the multivalent q-starlike functions connected with circular domain. Moreover, Arif et al. [4] considered a q-analogue of the Ruscheweyh type operator, and Srivastava et al. [30] dealt with basic and fractional q-calculus and associated Fekete-Szegö problems for p-valently q-starlike functions and p-valently q-convex functions of complex order using certain integral operators, and Khan et al. [17] for a new integral operator in q-analog for multivalent functions. Stimulated by the previous results, in the paper we intend to introduce and investigate several new subclasses of q-starlike and q-convex type analytic and multivalent functions involving a generalized Bernardi integral operator, and establish the corresponding Fekete-Szegö type functional inequalities for these function classes. Besides, the corresponding bound estimates of the coefficients a_{p+1} and a_{p+2} are provided.

From now on we introduce some general subclasses of analytic and multivalent functions associated with the q-derivative operator and the generalized Bernardi integral operator.

Definition 1.1. Let $f(z) \in \mathcal{A}_p$ and $\mu, \lambda \geq 0$. If the following subordination

$$(1-\lambda)\left(\frac{\mathcal{J}^{\eta}_{p,q}f(z)}{z^p}\right)^{\mu} + \frac{\lambda \mathcal{D}_q\left(\mathcal{J}^{\eta}_{p,q}f\right)(z)}{[p]_q z^{p-1}}\left(\frac{\mathcal{J}^{\eta}_{p,q}f(z)}{z^p}\right)^{\mu-1} \prec \phi(z) \tag{1.6}$$

is satisfied for $z \in \Delta$, then we call that $f(z)$ belongs to the class $\mathcal{LN}^{\eta}_{p,q}(\mu, \lambda; \phi)$.

Definition 1.2. Let $f(z) \in \mathcal{A}_p$ and $0 \leq \lambda \leq 1$. If the following subordination

$$(1-\lambda)\frac{z\mathcal{D}_q(\mathcal{J}^{\eta}_{p,q}f)(z)}{[p]_q\mathcal{J}^{\eta}_{p,q}f(z)} + \frac{\lambda}{[p]_q}\left(1 + \frac{qz\mathcal{D}_q[\mathcal{D}_q(\mathcal{J}^{\eta}_{p,q}f)](z)}{\mathcal{D}_q(\mathcal{J}^{\eta}_{p,q}f)(z)}\right) \prec \phi(z) \tag{1.7}$$

is satisfied for $z \in \Delta$, then we call that $f(z)$ belongs to the class $\mathcal{LM}^{\eta}_{p,q}(\lambda; \phi)$.

Definition 1.3. Let $f(z) \in \mathcal{A}_p$ and $\mu \geq 0$. If the following subordination

$$\left(\frac{z\mathcal{D}_q\left(\mathcal{J}^{\eta}_{p,q}f\right)(z)}{[p]_q\mathcal{J}^{\eta}_{p,q}f(z)}\right)\left(\frac{\mathcal{J}^{\eta}_{p,q}f(z)}{z^p}\right)^{\mu} \prec \phi(z) \tag{1.8}$$

is satisfied for $z \in \Delta$, then we call that $f(z)$ belongs to the class $\mathcal{NS}^{\eta}_{p,q}(\mu; \phi)$.

Remark 1.4. If we put

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \; for\; 0 < \alpha \leq 1$$

or

$$\phi(z) = \frac{1+(1-2\beta)z}{1-z} \; for\; 0 \leq \beta < 1$$

in Definition (1.1–1.3), then the class $\mathcal{LN}^{\eta}_{p,q}(\mu,\lambda;\phi)$ (res. $\mathcal{LM}^{\eta}_{p,q}(\lambda;\phi)$ and $\mathcal{NS}^{\eta}_{p,q}(\mu;\phi)$) reduces to $\mathcal{LN}^{\eta}_{p,q}(\mu,\lambda;\alpha)$ (res. $\mathcal{LM}^{\eta}_{p,q}(\lambda;\alpha)$ and $\mathcal{NS}^{\eta}_{p,q}(\mu;\alpha)$) or $\mathcal{LN}^{\eta}_{p,q}(\mu,\lambda;\beta)$ (res. $\mathcal{LM}^{\eta}_{p,q}(\lambda;\beta)$ and $\mathcal{NS}^{\eta}_{p,q}(\mu;\beta)$). Without the generalized Bernardi integral operator, the class $\mathcal{LN}^{\eta}_{p,q}(\mu,\lambda;\phi)$ (res. $\mathcal{LM}^{\eta}_{p,q}(\lambda;\phi)$ and $\mathcal{NS}^{\eta}_{p,q}(\mu;\phi)$) is the classical function class $\mathcal{LN}(\mu,\lambda;\phi)$ (res. $\mathcal{LM}(\lambda;\phi)$ and $\mathcal{NS}(\mu;\phi)$) when $p=1$ and $q\to 1_{-}$.

Let Ω be the class of functions $\omega(z)$ denoted by

$$\omega(z)=\sum_{n=1}^{\infty}E_n z^n,\quad (z\in\Delta) \tag{1.9}$$

via the inequality $|\omega(z)|<1 (z\in\Delta)$. Now we recall some necessary Lemmas below.

Lemma 1.5 ([16]). *Let the function $\omega\in\Omega$. Then*

$$|E_2-\tau E_1^2|\le\max\{1,|\tau|\},\quad (\tau\in\mathbb{C}).$$

Specially, the sharp result holds for the next function

$$\omega(z)=z\quad or\quad \omega(z)=z^2,\quad (z\in\Delta).$$

Lemma 1.6 ([7, 10]). *Let $\mathcal{P}$ be the class of all analytic functions $h(z)$ of the following form*

$$h(z)=1+\sum_{n=1}^{\infty}c_n z^n,\quad (z\in\Delta)$$

satisfying $\Re h(z)>0$ and $h(0)=1$. Then there exist the sharp coefficient estimates $|\, c_n \,|\le 2 (n\in\mathbb{N})$. In Particular, the equality holds for all n for the next function

$$h(z)=\frac{1+z}{1-z}=1+\sum_{n=1}^{\infty}2z^n.$$

Lemma 1.7 ([3, 19]). *Let the function $\omega\in\Omega$. Then*

$$|\, E_2-\kappa E_1^2 \,|\le\begin{cases}-\kappa & \text{if } \kappa\le -1,\\ 1 & \text{if } -1\le\kappa\le 1,\\ \kappa & \text{if } \kappa\ge 1.\end{cases}$$

For $\kappa<-1$ or $\kappa>1$, the inequality holds literally if and only if $\omega(z)=z$ or one of its rotations. If $<\kappa<1$, the inequality holds literally if and only if $\omega(z)=z^2$ or one of its rotations. In Particular, if $\kappa=-1$, then the sharp result holds for the next function

$$\omega(z)=\frac{z(z+\xi)}{1+\xi z},\quad (0\le\xi\le 1)$$

or one of its rotations. If $\kappa=1$, then the sharp result holds for the next function

$$\omega(z)=-\frac{z(z+\xi)}{1+\xi z},\quad (0\le\xi\le 1)$$

or one of its rotations. If $-1<\kappa<1$, then the upper bound is sharp as the followings

$$|E_2-\kappa E_1^2|+(\kappa+1)|E_1|^2\le 1,\quad (-1<\kappa\le 0)$$

and

$$|E_2-\kappa E_1^2|+(1-\kappa)|E_1|^2\le 1,\quad (0<\kappa<1).$$

2. Functional estimates for the class $\mathcal{LN}^{\eta}_{p,q}(\mu,\lambda;\phi)$

By (1.9) we give that

$$\begin{aligned} \phi(\omega(z)) &= 1 + A_1E_1z + (A_1E_2 + A_2E_1^2)z^2 \\ &+ (A_1E_3 + 2A_2E_1E_2 + A_3E_1^3)z^3 + \ldots . \end{aligned} \tag{2.1}$$

In the section, with Lemma 1.5 we study Fekete-Szegö functional problem for the class $\mathcal{LN}^{\eta}_{p,q}(\mu,\lambda;\phi)$ and provide the following theorem.

Theorem 2.1. *Let $\delta \in \mathbb{C}$. If $f(z) \in \mathcal{A}_p$ belongs to the class $\mathcal{LN}^{\eta}_{p,q}(\mu,\lambda;\phi)$, then*

$$|a_{p+2} - \delta a_{p+1}^2| \le \frac{A_1[p]_q}{|L_2|[\mu[p]_q + \lambda([p+2]_q - [p]_q)]} \max\left\{1; \left|\frac{A_1[p]_q\Theta}{2L_1^2[\mu[p]_q + \lambda([p+1]_q - [p]_q)]^2} - \frac{A_2}{A_1}\right|\right\},$$

where

$$\begin{aligned} \Theta &= 2\delta L_2[\mu[p]_q + \lambda([p+2]_q - [p]_q)] \\ &+ L_1^2[(\mu-1)[\mu - \lambda(\lambda\mu - \mu + 1)][p]_q + 2\lambda(2\mu - \lambda\mu - 1)[p+1]_q]. \end{aligned}$$

Moreover, the sharp result holds for the next function

$$\omega(z) = z \quad or \quad \omega(z) = z^2, \quad (z \in \Delta).$$

Proof. Assume that $f(z) \in \mathcal{LN}^{\eta}_{p,q}(\mu,\lambda;\phi)$. Then, from Definition 1.1, there exists an analytic function $\omega(z) \in \Omega$ such that

$$(1-\lambda)\left(\frac{\mathcal{J}^{\eta}_{p,q}f(z)}{z^p}\right)^{\mu} + \frac{\lambda\mathcal{D}_q\left(\mathcal{J}^{\eta}_{p,q}f\right)(z)}{[p]_q z^{p-1}}\left(\frac{\mathcal{J}^{\eta}_{p,q}f(z)}{z^p}\right)^{\mu-1} = \phi(\omega(z)). \tag{2.2}$$

Part case

$$\begin{aligned} &\left(\frac{\mathcal{J}^{\eta}_{p,q}f(z)}{z^p}\right)^{\mu-1} \\ =\ & 1 + (\mu-1)L_1a_{p+1}z + \left[(\mu-1)L_2a_{p+2} + \frac{(\mu-1)(\mu-2)}{2}L_1^2a_{p+1}^2\right]z^2 \\ +\ & \left[(\mu-1)L_3a_{p+3} + \frac{(\mu-1)(\mu-2)}{2}L_1L_2a_{p+1}a_{p+2} + \frac{(\mu-1)(\mu-2)(\mu-3)}{6}L_1^3a_{p+1}^3\right]z^3 + \ldots, \end{aligned}$$

$$\begin{aligned} &\frac{\lambda\mathcal{D}_q\left(\mathcal{J}^{\eta}_{p,q}f\right)(z)}{[p]_q z^{p-1}} \\ =\ & \lambda + \frac{\lambda L_1a_{p+1}[p+1]_q}{[p]_q}z + \frac{\lambda L_2a_{p+2}[p+2]_q}{[p]_q}z^2 \\ +\ & \frac{\lambda L_3a_{p+3}[p+3]_q}{[p]_q}z^3 + \ldots, \end{aligned}$$

$$
\begin{aligned}
& \frac{\lambda \mathcal{D}_q\left(\mathcal{J}_{p,q}^{\eta} f\right)(z)}{[p]_q z^{p-1}}\left(\frac{\mathcal{J}_{p,q}^{\eta} f(z)}{z^p}\right)^{\mu-1} \\
= \;& \lambda+\lambda\left[\mu+\left(\frac{[p+1]_q}{[p]_q}-1\right)\right] L_1 a_{p+1} z \\
+ \;& \lambda\left\{\left[\mu+\left(\frac{[p+2]_q}{[p]_q}-1\right)\right] L_2 a_{p+2}+(\mu-1)\left[\frac{\mu}{2}+\left(\frac{[p+1]_q}{[p]_q}-1\right)\right] L_1^2 a_{p+1}^2\right\} z^2+\ldots,
\end{aligned}
$$

$$
\begin{aligned}
& (1-\lambda)\left(\frac{\mathcal{J}_{p,q}^{\eta} f(z)}{z^p}\right)^{\mu} \\
= \;& (1-\lambda)+(1-\lambda) \mu L_1 a_{p+1} z+(1-\lambda)\left[\mu L_2 a_{p+2}+\frac{\mu(\mu-1)}{2} L_1^2 a_{p+1}^2\right] z^2 \\
+ \;& (1-\lambda)\left[\mu L_3 a_{p+3}+\frac{\mu(\mu-1)}{2} L_1 L_2 a_{p+1} a_{p+2}+\frac{\mu(\mu-1)(\mu-2)}{6} L_1^3 a_{p+1}^3\right] z^3+\ldots .
\end{aligned}
$$

Since

$$
\begin{aligned}
& (1-\lambda)\left(\frac{\mathcal{J}_{p,q}^{\eta} f(z)}{z^p}\right)^{\mu}+\frac{\lambda \mathcal{D}_q\left(\mathcal{J}_{p,q}^{\eta} f\right)(z)}{[p]_q z^{p-1}}\left(\frac{\mathcal{J}_{p,q}^{\eta} f(z)}{z^p}\right)^{\mu-1} \\
= \;& 1+\left[\mu+\lambda\left(\frac{[p+1]_q}{[p]_q}-1\right)\right] L_1 a_{p+1} z+\left\{\left[\mu+\lambda\left(\frac{[p+2]_q}{[p]_q}-1\right)\right] L_2 a_{p+2}\right. \\
+ \;& \left.\left[\frac{(\mu-1)}{2}[\mu-2\lambda(\lambda\mu-\mu+1)]+\lambda(2\mu-\lambda\mu-1)\frac{[p+1]_q}{[p]_q}\right] L_1^2 a_{p+1}^2\right\} z^2+\ldots,
\end{aligned}
$$

by (2.1) and (2.2) we see that

$$
A_1 E_1=\left[\mu+\lambda\left(\frac{[p+1]_q}{[p]_q}-1\right)\right] L_1 a_{p+1},
$$

$$
\begin{aligned}
A_1 E_2+A_2 E_1^2 \;=\; & \left[\mu+\lambda\left(\frac{[p+2]_q}{[p]_q}-1\right)\right] L_2 a_{p+2} \\
+ \; & \left[\frac{\mu-1}{2}[\mu-2\lambda(\lambda\mu-\mu+1)]+\lambda(2\mu-\lambda\mu-1)\frac{[p+1]_q}{[p]_q}\right] L_1^2 a_{p+1}^2 .
\end{aligned}
$$

Thereby

$$
a_{p+1}=\frac{A_1 E_1[p]_q}{L_1[\mu[p]_q+\lambda([p+1]_q-[p]_q)]} \tag{2.3}
$$

and

$$
\begin{aligned}
a_{p+2} \;=\; & \frac{(A_1 E_2+A_2 E_1^2)[p]_q}{L_2[\mu[p]_q+\lambda([p+2]_q-[p]_q)]} \\
- \; & \frac{A_1^2 E_1^2[p]_q^2\{(\mu-1)[\mu-2\lambda(\lambda\mu-\mu+1)][p]_q+2\lambda(2\mu-\lambda\mu-1)[p+1]_q\}}{2L_2[\mu[p]_q+\lambda([p+1]_q-[p]_q)]^2[\mu[p]_q+\lambda([p+2]_q-[p]_q)]} .
\end{aligned} \tag{2.4}
$$

Further, with (2.3) and (2.4) we obtain that

$$a_{p+2}-\delta a_{p+1}^2=\frac{A_1[p]_q}{L_2[\mu[p]_q+\lambda([p+2]_q-[p]_q)]}[E_2-\hbar E_1^2],$$

where

$$\begin{aligned}\hbar &= \{2\delta L_2[\mu[p]_q+\lambda([p+2]_q-[p]_q)]+L_1^2[(\mu-1)[\mu-\lambda(\lambda\mu-\mu+1)][p]_q\\ &+ 2\lambda(2\mu-\lambda\mu-1)[p+1]_q]\}\frac{A_1[p]_q}{2L_1^2[\mu[p]_q+\lambda([p+1]_q-[p]_q)]^2}-\frac{A_2}{A_1}.\end{aligned}$$

Therefore, according to Lemma 1.5 we finish the proof of Theorem 2.1. □

Corollary 2.2. *If $f(z)\in\mathcal{A}_p$ belongs to the class $\mathcal{LN}^{\eta}_{p,q}(\mu,\lambda;\phi)$, then*

$$\begin{aligned}|a_{p+2}| &\le \frac{A_1[p]_q}{|L_2|[\mu[p]_q+\lambda([p+2]_q-[p]_q)]}\\ &\times \max\left\{1;\left|\frac{(\mu-1)[\mu-\lambda(\lambda\mu-\mu+1)][p]_q+2\lambda(2\mu-\lambda\mu-1)[p+1]_q}{2[\mu[p]_q+\lambda([p+1]_q-[p]_q)]^2}-\frac{A_2}{A_1}\right|\right\}.\end{aligned}$$

Moreover, the sharp result holds for the next function

$$\omega(z)=z \quad or \quad \omega(z)=z^2, \quad (z\in\Delta).$$

When $\phi\in\mathcal{P}$, combining (2.3) and (2.4) with Lemma 1.6 we instantly establish the next corollary for the coefficient bounds of a_{p+1} and a_{p+2}.

Corollary 2.3. *If $f(z)\in\mathcal{A}_p$ belongs to the class $\mathcal{LN}^{\eta}_{p,q}(\mu,\lambda;\phi)$, then*

$$|a_{p+1}|\le\frac{2|E_1|[p]_q}{|L_1|[\mu[p]_q+\lambda([p+1]_q-[p]_q)]}$$

and

$$\begin{aligned}|a_{p+2}| &\le \frac{2(|E_2|+|E_1|^2)[p]_q}{|L_2|[\mu[p]_q+\lambda([p+2]_q-[p]_q)]}\\ &+ \frac{2E_1^2[p]_q^2|(\mu-1)[\mu-\lambda(\lambda\mu-\mu+1)][p]_q+2\lambda(2\mu-\lambda\mu-1)[p+1]_q|}{|L_2|[\mu[p]_q+\lambda([p+1]_q-[p]_q)]^2[\mu[p]_q+\lambda([p+2]_q-[p]_q)]}.\end{aligned}$$

If we choose real δ and η, then by Lemma 1.7 we derive the next result for Fekete-Szegö problem.

Theorem 2.4. *Let $\delta,\eta\in\mathbb{R}$ and $\phi\in\Lambda$ satisfying*

$$\phi(z)=1+\sum_{n=1}^{\infty}A_nz^n, \quad (A_1,A_2>0, z\in\Delta).$$

If $f(z)\in\mathcal{A}_p$ belongs to the class $\mathcal{LN}^{\eta}_{p,q}(\mu,\lambda;\phi)$, then

$$|a_{p+2}-\delta a_{p+1}^2|\le\begin{cases}\frac{[p]_q}{L_2[\mu[p]_q+\lambda([p+2]_q-[p]_q)]}\left\{A_2-\frac{A_1^2[p]_q\Theta}{2L_1^2[\mu[p]_q+\lambda([p+1]_q-[p]_q)]^2}\right\}, & (\delta\le\Upsilon_1);\\ \frac{A_1[p]_q}{L_2[\mu[p]_q+\lambda([p+2]_q-[p]_q)]}, & (\Upsilon_1\le\delta\le\Upsilon_2);\\ \frac{[p]_q}{L_2[\mu[p]_q+\lambda([p+2]_q-[p]_q)]}\left\{-A_2+\frac{A_1^2[p]_q\Theta}{2L_1^2|\mu[p]_q+\lambda([p+1]_q-[p]_q)]^2}\right\}, & (\delta\ge\Upsilon_2),\end{cases}$$

where

$$\begin{aligned}\Upsilon_1 &= \frac{(A_2 - A_1)L_1^2[\mu[p]_q + \lambda([p+1]_q - [p]_q)]^2}{A_1^2 L_2 [p]_q[\mu[p]_q + \lambda([p+2]_q - [p]_q)]} \\ &- \frac{L_1^2[(\mu-1)[\mu - \lambda(\lambda\mu - \mu + 1)][p]_q + 2\lambda(2\mu - \lambda\mu - 1)[p+1]_q]}{2L_2[\mu[p]_q + \lambda([p+2]_q - [p]_q)]}\end{aligned}$$

and

$$\begin{aligned}\Upsilon_2 &= \frac{(A_2 + A_1)L_1^2[\mu[p]_q + \lambda([p+1]_q - [p]_q)]^2}{A_1^2 L_2 [p]_q[\mu[p]_q + \lambda([p+2]_q - [p]_q)]} \\ &- \frac{L_1^2[(\mu-1)[\mu - \lambda(\lambda\mu - \mu + 1)][p]_q + 2\lambda(2\mu - \lambda\mu - 1)[p+1]_q]}{2L_2[\mu[p]_q + \lambda([p+2]_q - [p]_q)]}.\end{aligned}$$

Moreover, we take

$$\begin{aligned}\Upsilon_3 &= \frac{A_2 L_1^2[\mu[p]_q + \lambda([p+1]_q - [p]_q)]^2}{A_1^2 L_2 [p]_q[\mu[p]_q + \lambda([p+2]_q - [p]_q)]} \\ &- \frac{L_1^2[(\mu-1)[\mu - \lambda(\lambda\mu - \mu + 1)][p]_q + 2\lambda(2\mu - \lambda\mu - 1)[p+1]_q]}{2L_2[\mu[p]_q + \lambda([p+2]_q - [p]_q)]}.\end{aligned}$$

Then, each of the following results is true:
(A) For $\delta \in [\Upsilon_1, \Upsilon_3]$,

$$\begin{aligned}|a_{p+2} - \delta a_{p+1}^2| &+ \left\{2(A_1 - A_2)L_1^2[\mu[p]_q + \lambda([p+1]_q - [p]_q)]^2 + A_1^2[p]_q\Theta\right\} \\ &\times \frac{|a_{p+1}|^2}{2A_1^2 L_2[p]_q[\mu[p]_q + \lambda([p+2]_q - [p]_q)]} \leq \frac{A_1[p]_q}{L_2[\mu[p]_q + \lambda([p+2]_q - [p]_q)]};\end{aligned}$$

(B) For $\delta \in [\Upsilon_3, \Upsilon_2]$,

$$\begin{aligned}|a_{p+2} - \delta a_{p+1}^2| &+ \left\{2(A_1 + A_2)L_1^2[\mu[p]_q + \lambda([p+1]_q - [p]_q)]^2 - A_1^2[p]_q\Theta\right\} \\ &\times \frac{|a_{p+1}|^2}{2A_1^2 L_2[p]_q[\mu[p]_q + \lambda([p+2]_q - [p]_q)]} \leq \frac{A_1[p]_q}{L_2[\mu[p]_q + \lambda([p+2]_q - [p]_q)]},\end{aligned}$$

where

$$\begin{aligned}\Theta &= 2\delta L_2[\mu[p]_q + \lambda([p+2]_q - [p]_q)] \\ &+ L_1^2[(\mu-1)[\mu - \lambda(\lambda\mu - \mu + 1)][p]_q + 2\lambda(2\mu - \lambda\mu - 1)[p+1]_q].\end{aligned}$$

Remark 2.5. Fixing the parameter $p = 1$ in Theorems 2.1 and 2.4, we can state the new results for the univalent function classes $\mathcal{LN}_{1,q}^{\eta}(\mu, \lambda; \phi) = \mathcal{LN}_{q}^{\eta}(\mu, \lambda; \phi)$. As Remark 1.4, we may consider $\mathcal{LN}_{p,q}^{\eta}(\mu, \lambda; \alpha)$ or $\mathcal{LN}_{p,q}^{\eta}(\mu, \lambda; \beta)$ to establish latest results. On the other hand, for the different parameters μ and λ, we can deduce new results for $\mathcal{LN}_{p,q}^{\eta}(\mu, \lambda; \phi)$.

3. Functional estimates for the class $\mathcal{LM}^{\eta}_{p,q}(\lambda;\phi)$

In the section we mainly consider Fekete-Szegö functional problem for the class $\mathcal{LM}^{\eta}_{p,q}(\lambda;\phi)$ and establish the theorem as follows.

Theorem 3.1. *Let $\delta \in \mathbb{C}$. If $f(z) \in \mathcal{A}_p$ belongs to the class $\mathcal{LM}^{\eta}_{p,q}(\lambda;\phi)$, then*

$$\begin{aligned} |a_{p+2} - \delta a_{p+1}^2| \;\leq\; & \frac{A_1[p]_q^2}{|L_2|([p+2]_q - [p]_q)[\lambda([p+2]_q - [p]_q) + [p]_q]} \\ & \times \;\max\left\{1; \left|\frac{A_1[p]_q\Phi}{L_1^2([p+1]_q - [p]_q)^2[\lambda([p+1]_q - [p]_q) + [p]_q]^2} - \frac{A_2}{A_1}\right|\right\}, \end{aligned}$$

where

$$\Phi = \delta L_2[p]_q([p+2]_q - [p]_q)[\lambda([p+2]_q - [p]_q) + [p]_q] + L_1^2([p+1]_q - [p]_q)[\lambda([p+1]_q^2 - [p]_q^2) + [p]_q^2].$$

Moreover, the sharp result holds for the next function

$$\omega(z) = z \quad or \quad \omega(z) = z^2, \quad (z \in \Delta).$$

Proof. If $f \in \mathcal{LM}^{\eta}_{p,q}(\lambda;\phi)$, from Definition 1.2 there exists an analytic function $\omega(z) \in \Omega$ such that

$$(1-\lambda)\frac{z\mathcal{D}_q(\mathcal{J}^{\eta}_{p,q}f)(z)}{[p]_q\mathcal{J}^{\eta}_{p,q}f(z)} + \frac{\lambda}{[p]_q}\left(1 + \frac{qz\mathcal{D}_q[\mathcal{D}_q(\mathcal{J}^{\eta}_{p,q}f)](z)}{\mathcal{D}_q(\mathcal{J}^{\eta}_{p,q}f)(z)}\right) = \phi(\omega(z)). \tag{3.1}$$

Part case

$$\begin{aligned} & \frac{z\mathcal{D}_q\left(\mathcal{J}^{\eta}_{p,q}f\right)(z)}{[p]_q\mathcal{J}^{\eta}_{p,q}f(z)} \\ = \;& 1 + \left(\frac{[p+1]_q}{[p]_q} - 1\right)L_1 a_{p+1}z + \left[\left(\frac{[p+2]_q}{[p]_q} - 1\right)L_2 a_{p+2} - \left(\frac{[p+1]_q}{[p]_q} - 1\right)L_1^2 a_{p+1}^2\right]z^2 + \ldots, \end{aligned}$$

$$\begin{aligned} & \frac{1}{[p]_q}\left(1 + \frac{qz\mathcal{D}_q[\mathcal{D}_q(\mathcal{J}^{\eta}_{p,q}f)](z)}{\mathcal{D}_q(\mathcal{J}^{\eta}_{p,q}f)(z)}\right) \\ = \;& 1 + \frac{L_1[p+1]_q}{[p]_q}\left(\frac{[p+1]_q}{[p]_q} - 1\right)a_{p+1}z \\ + \;& \left[\frac{L_2[p+2]_q}{[p]_q}\left(\frac{[p+2]_q}{[p]_q} - 1\right)a_{p+2} - \frac{L_1^2[p+1]_q^2}{[p]_q^2}\left(\frac{[p+1]_q}{[p]_q} - 1\right)a_{p+1}^2\right]z^2 + \ldots. \end{aligned}$$

Since

$$\begin{aligned} & (1-\lambda)\frac{z\mathcal{D}_q(\mathcal{J}^{\eta}_{p,q}f)(z)}{[p]_q\mathcal{J}^{\eta}_{p,q}f(z)} + \frac{\lambda}{[p]_q}\left(1 + \frac{qz\mathcal{D}_q[\mathcal{D}_q(\mathcal{J}^{\eta}_{p,q}f)](z)}{\mathcal{D}_q(\mathcal{J}^{\eta}_{p,q}f)(z)}\right) \\ = \;& 1 + \left(\frac{[p+1]_q}{[p]_q} - 1\right)\left[\lambda\left(\frac{[p+1]_q}{[p]_q} - 1\right) + 1\right]L_1 a_{p+1}z \end{aligned}$$

$$+ \quad \left\{\left(\frac{[p+2]_q}{[p]_q}-1\right)\left[\lambda\left(\frac{[p+2]_q}{[p]_q}-1\right)+1\right]L_2a_{p+2}-\left(\frac{[p+1]_q}{[p]_q}-1\right)\left[\lambda\left(\frac{[p+1]_q^2}{[p]_q^2}-1\right)+1\right]L_1^2a_{p+1}^2\right\}z^2+\ldots,$$

by (2.1) and (3.1) we note that

$$A_1E_1=\left(\frac{[p+1]_q}{[p]_q}-1\right)\left[\lambda\left(\frac{[p+1]_q}{[p]_q}-1\right)+1\right]L_1a_{p+1}$$

and

$$\begin{aligned} A_1E_2+A_2E_1^2 &= \left(\frac{[p+2]_q}{[p]_q}-1\right)\left[\lambda\left(\frac{[p+2]_q}{[p]_q}-1\right)+1\right]L_2a_{p+2} \\ &- \left(\frac{[p+1]_q}{[p]_q}-1\right)\left[\lambda\left(\frac{[p+1]_q^2}{[p]_q^2}-1\right)+1\right]L_1^2a_{p+1}^2. \end{aligned}$$

Then, it leads to

$$a_{p+1}=\frac{A_1E_1[p]_q^2}{L_1([p+1]_q-[p]_q)[\lambda([p+1]_q-[p]_q)+[p]_q]} \tag{3.2}$$

and

$$\begin{aligned} a_{p+2} &= \frac{[p]_q^2}{L_2([p+2]_q-[p]_q)[\lambda([p+2]_q-[p]_q)+[p]_q]} \\ &\times \left[(A_1E_2+A_2E_1^2)+\frac{A_1^2E_1^2[p]_q\{\lambda([p+1]_q^2-[p]_q^2)+[p]_q^2\}}{([p+1]_q-[p]_q)[\lambda([p+1]_q-[p]_q)+[p]_q]^2}\right]. \end{aligned} \tag{3.3}$$

Furthermore, in accordance with (3.2) and (3.3) we gain that

$$a_{p+2}-\delta a_{p+1}^2=\frac{A_1[p]_q^2}{L_2([p+2]_q-[p]_q)[\lambda([p+2]_q-[p]_q)+[p]_q]}[E_2-\varrho E_1^2],$$

where

$$\varrho=\frac{A_1[p]_q\Phi}{L_1^2([p+1]_q-[p]_q)^2[\lambda([p+1]_q-[p]_q)+[p]_q]^2}-\frac{A_2}{A_1}.$$

Thus, from Lemma 1.5 we give the Fekete-Szegö functional inequality in Theorem 3.1. □

Corollary 3.2. *If $f(z)\in\mathcal{A}_p$ belongs to the class $\mathcal{LM}_{p,q}^{\eta}(\lambda;\phi)$, then*

$$\begin{aligned} |a_{p+2}| &\le \frac{A_1[p]_q^2}{|L_2|([p+2]_q-[p]_q)[\lambda([p+2]_q-[p]_q)+[p]_q]} \\ &\times \max\left\{1;\left|\frac{A_1[p]_q[\lambda([p+1]_q^2-[p]_q^2)+[p]_q^2]}{([p+1]_q-[p]_q)[\lambda([p+1]_q-[p]_q)+[p]_q]^2}-\frac{A_2}{A_1}\right|\right\}. \end{aligned}$$

Moreover, the sharp result holds for the next function

$$\omega(z)=z \quad or \quad \omega(z)=z^2, \quad (z\in\Delta).$$

If $\phi \in \mathcal{P}$, by (3.2) and (3.3) we take Lemma 1.6 to prove the next corollary for the coefficient bounds of a_{p+1} and a_{p+2}.

Corollary 3.3. *If $f(z) \in \mathcal{A}_p$ belongs to the class $\mathcal{LM}^{\eta}_{p,q}(\lambda;\phi)$, then*

$$|a_{p+1}| \leq \frac{2|E_1|[p]_q^2}{|L_1|([p+1]_q - [p]_q)[\lambda([p+1]_q - [p]_q) + [p]_q]}$$

and

$$\begin{aligned} |a_{p+2}| \;\; &\leq \;\; \frac{2[p]_q^2}{|L_2|([p+2]_q - [p]_q)[\lambda([p+2]_q - [p]_q) + [p]_q]} \\ &\times \;\; \left[(|E_2| + |E_1|^2) + \frac{2|E_1|^2[p]_q\{\lambda([p+1]_q^2 - [p]_q^2) + [p]_q^2\}}{([p+1]_q - [p]_q)[\lambda([p+1]_q - [p]_q) + [p]_q]^2}\right]. \end{aligned}$$

On the other hand, if we take real δ and η, then by Lemma 1.7 we give the next result for Fekete-Szegö problem.

Theorem 3.4. *Let $\delta, \eta \in \mathbb{R}$ and $\phi \in \Lambda$ satisfying*

$$\phi(z) = 1 + \sum_{n=1}^{\infty} A_n z^n, \;\; (A_1, A_2 > 0, z \in \Delta).$$

If $f(z) \in \mathcal{A}_p$ belongs to the class $\mathcal{LM}^{\eta}_{p,q}(\lambda;\phi)$, then

$$|a_{p+2} - \delta a_{p+1}^2| \leq$$

$$\begin{cases} \frac{[p]_q^2}{|L_2|([p+2]_q-[p]_q)[\lambda([p+2]_q-[p]_q)+[p]_q]}\left\{A_2 - \frac{A_1^2[p]_q\Phi}{L_1^2([p+1]_q-[p]_q)^2[\lambda([p+1]_q-[p]_q)+[p]_q]^2}\right\}, \; (\delta \leq \Gamma_1); \\ \frac{A_1[p]_q^2}{|L_2|([p+2]_q-[p]_q)[\lambda([p+2]_q-[p]_q)+[p]_q]}, \; (\Gamma_1 \leq \delta \leq \Gamma_2); \\ \frac{[p]_q^2}{|L_2|([p+2]_q-[p]_q)[\lambda([p+2]_q-[p]_q)+[p]_q]}\left\{-A_2 + \frac{A_1^2[p]_q\Phi}{L_1^2([p+1]_q-[p]_q)^2[\lambda([p+1]_q-[p]_q)+[p]_q]^2}\right\}, \; (\delta \geq \Gamma_2), \end{cases}$$

where

$$\begin{aligned} \Gamma_1 \;\; &= \;\; \{(A_2 - A_1)([p+1]_q - [p]_q)\{\lambda([p+1]_q - [p]_q) + [p]_q\}^2 - A_1^2[p]_q[\lambda([p+1]_q^2 - [p]_q^2) + [p]_q^2)]\} \\ &\times \;\; \frac{L_1^2([p+1]_q - [p]_q)}{L_2 A_1^2[p]_q^2([p+2]_q - [p]_q)[\lambda([p+2]_q - [p]_q) + [p]_q]} \end{aligned}$$

and

$$\begin{aligned} \Gamma_2 \;\; &= \;\; \{(A_2 + A_1)([p+1]_q - [p]_q)\{\lambda([p+1]_q - [p]_q) + [p]_q\}^2 - A_1^2[p]_q[\lambda([p+1]_q^2 - [p]_q^2) + [p]_q^2)]\} \\ &\times \;\; \frac{L_1^2([p+1]_q - [p]_q)}{L_2 A_1^2[p]_q^2([p+2]_q - [p]_q)[\lambda([p+2]_q - [p]_q) + [p]_q]}. \end{aligned}$$

Moreover, we choose

$$\Gamma_3 \;\; = \;\; \{A_2([p+1]_q - [p]_q)\{\lambda([p+1]_q - [p]_q) + [p]_q\}^2 - A_1^2[p]_q[\lambda([p+1]_q^2 - [p]_q^2) + [p]_q^2)]\}$$

$$\times \quad \frac{L_1^2([p+1]_q - [p]_q)}{L_2 A_1^2 [p]_q^2 ([p+2]_q - [p]_q)[\lambda([p+2]_q - [p]_q) + [p]_q]}.$$

Then, each of the following results is true:
(A) For $\delta \in [\Gamma_1, \Gamma_3]$,

$$\begin{aligned} |a_{p+2} - \delta a_{p+1}^2| &+ \frac{(A_1 - A_2) L_1^2 ([p+1]_q - [p]_q)^2 [\lambda([p+1]_q - [p]_q) + [p]_q]^2 + A_1^2 [p]_q \Phi}{A_1^2 L_2 [p]_q^2 ([p+2]_q - [p]_q)[\lambda([p+2]_q - [p]_q) + [p]_q]} |a_{p+1}|^2 \\ &\leq \frac{A_1 [p]_q^2}{L_2([p+2]_q - [p]_q)[\lambda([p+2]_q - [p]_q) + [p]_q]}; \end{aligned}$$

(B) For $\delta \in [\Gamma_3, \Gamma_2]$,

$$\begin{aligned} |a_{p+2} - \delta a_{p+1}^2| &+ \frac{(A_1 + A_2) L_1^2 ([p+1]_q - [p]_q)^2 [\lambda([p+1]_q - [p]_q) + [p]_q]^2 - A_1^2 [p]_q \Phi}{A_1^2 L_2 [p]_q^2 ([p+2]_q - [p]_q)[\lambda([p+2]_q - [p]_q) + [p]_q]} |a_{p+1}|^2 \\ &\leq \frac{A_1 [p]_q^2}{L_2([p+2]_q - [p]_q)[\lambda([p+2]_q - [p]_q) + [p]_q]}, \end{aligned}$$

where

$$\Phi = \delta L_2 [p]_q ([p+2]_q - [p]_q)[\lambda([p+2]_q - [p]_q) + [p]_q] + L_1^2 [p+1]_q ([p+1]_q - [p]_q)[\lambda([p+1]_q - [p]_q) + [p]_q].$$

Remark 3.5. Similarly, by taking the parameter $p = 1$ in Theorems 3.1 and 3.4, we can obtain the new results for the univalent function classes $\mathcal{LM}_{1,q}^{\eta}(\lambda;\phi) = \mathcal{LM}_{q}^{\eta}(\lambda;\phi)$. As Remark 1.4, we may consider $\mathcal{LM}_{p,q}^{\eta}(\lambda;\alpha)$ or $\mathcal{LM}_{p,q}^{\eta}(\lambda;\beta)$ to establish latest results. Clearly, for special parameter λ, we can still imply new results for $\mathcal{LM}_{p,q}^{\eta}(\lambda;\phi)$.

4. Functional estimates for the class $\mathcal{NS}_{p,q}^{\eta}(\mu;\phi)$

In the section we investigate Fekete-Szegö functional problem for the class $\mathcal{NS}_{p,q}^{\eta}(\mu;\phi)$ and obtain the corresponding theorem below.

Theorem 4.1. *Let $\delta \in \mathbb{C}$. If $f(z) \in \mathcal{A}_p$ belongs to the class $\mathcal{NS}_{p,q}^{\eta}(\mu;\phi)$, then*

$$\begin{aligned} |a_{p+2} - \delta a_{p+1}^2| &\leq \frac{A_1 [p]_q}{|L_2|(\mu[p]_q + [p+2]_q - [p]_q)} \\ &\times \max\left\{1; \left| \frac{A_1 [p]_q \Psi}{2L_1^2(\mu[p]_q + [p+1]_q - [p]_q)^2} - \frac{A_2}{A_1} \right| \right\}, \end{aligned}$$

where

$$\Psi = 2\delta L_2 [p]_q (\mu[p]_q + [p+2]_q - [p]_q) + (\mu - 1)[\mu[p]_q + 2([p+1]_q - [p]_q)] L_1^2.$$

Moreover, the sharp result holds for the next function

$$\omega(z) = z \quad or \quad \omega(z) = z^2, \quad (z \in \Delta).$$

Proof. Since $f \in \mathcal{NS}^{\eta}_{p,q}(\mu;\phi)$, from Definition 1.3 there exists an analytic function $\omega(z) \in \Omega$ such that

$$\left(\frac{z\mathcal{D}_q\left(\mathcal{J}^{\eta}_{p,q}f\right)(z)}{[p]_q\mathcal{J}^{\eta}_{p,q}f(z)}\right)\left(\frac{\mathcal{J}^{\eta}_{p,q}f(z)}{z^p}\right)^{\mu} = \phi(\omega(z)). \tag{4.1}$$

Part case

$$\begin{aligned}&\frac{z\mathcal{D}_q\left(\mathcal{J}^{\eta}_{p,q}f\right)(z)}{[p]_q\mathcal{J}^{\eta}_{p,q}f(z)}\\ =\ & 1+\left(\frac{[p+1]_q}{[p]_q}-1\right)L_1a_{p+1}z+\left[\left(\frac{[p+2]_q}{[p]_q}-1\right)L_2a_{p+2}-\left(\frac{[p+1]_q}{[p]_q}-1\right)L_1^2a_{p+1}^2\right]z^2+\ldots,\end{aligned}$$

$$\begin{aligned}&\left(\frac{\mathcal{J}^{\eta}_{p,q}f(z)}{z^p}\right)^{\mu}\\ =\ & 1+\mu L_1a_{p+1}z+\left[\mu L_2a_{p+2}+\frac{\mu(\mu-1)}{2}L_1^2a_{p+1}^2\right]z^2\\ +\ & \left[\mu L_3a_{p+3}+\frac{\mu(\mu-1)}{2}L_1L_2a_{p+1}a_{p+2}+\frac{\mu(\mu-1)(\mu-2)}{6}L_1^3a_{p+1}^3\right]z^3+\ldots.\end{aligned}$$

Since

$$\left(\frac{z\mathcal{D}_q\left(\mathcal{J}^{\eta}_{p,q}f\right)(z)}{[p]_q\mathcal{J}^{\eta}_{p,q}f(z)}\right)\left(\frac{\mathcal{J}^{\eta}_{p,q}f(z)}{z^p}\right)^{\mu} = 1+\left(\frac{[p+1]_q}{[p]_q}-1+\mu\right)L_1a_{p+1}z$$
$$+\left\{\left(\frac{[p+2]_q}{[p]_q}-1+\mu\right)L_2a_{p+2}+(\mu-1)\left[\left(\frac{[p+1]_q}{[p]_q}-1\right)+\frac{\mu}{2}\right]L_1^2a_{p+1}^2\right\}z^2+\ldots,$$

from (2.1) and (4.1) we know that

$$A_1E_1 = \left(\frac{[p+1]_q}{[p]_q}-1+\mu\right)L_1a_{p+1}$$

and

$$\begin{aligned}A_1E_2+A_2E_1^2 &= \left(\frac{[p+2]_q}{[p]_q}-1+\mu\right)L_2a_{p+2}\\ &+ (\mu-1)\left[\left(\frac{[p+1]_q}{[p]_q}-1\right)+\frac{\mu}{2}\right]L_1^2a_{p+1}^2.\end{aligned}$$

Thus it deduces that

$$a_{p+1} = \frac{A_1E_1[p]_q}{L_1(\mu[p]_q+[p+1]_q-[p]_q)} \tag{4.2}$$

and

$$a_{p+2} = \frac{(A_1E_2+A_2E_1^2)[p]_q}{L_2(\mu[p]_q+[p+2]_q-[p]_q)}$$

$$- \frac{(\mu-1)A_1^2E_1^2[p]_q^2[\mu[p]_q+2([p+1]_q-[p]_q)]}{2L_2(\mu[p]_q+[p+2]_q-[p]_q)(\mu[p]_q+[p+1]_q-[p]_q)^2}. \tag{4.3}$$

Moreover, in the light of (4.2) and (4.3) we know that

$$a_{p+2}-\delta a_{p+1}^2 = \frac{A_1[p]_q}{L_2(\mu[p]_q+[p+2]_q-[p]_q)}[E_2-\aleph E_1^2],$$

where

$$\aleph = \frac{2\delta L_2[p]_q(\mu[p]_q+[p+2]_q-[p]_q)+(\mu-1)[\mu[p]_q+2([p+1]_q-[p]_q)]L_1^2}{2L_1^2(\mu[p]_q+[p+1]_q-[p]_q)^2}A_1[p]_q-\frac{A_2}{A_1}.$$

Hence, in view of Lemma 1.5 we get the Fekete-Szegö functional inequality in Theorem 4.1. □

Corollary 4.2. *If $f(z)\in\mathcal{A}_p$ belongs to the class $\mathcal{NS}_{p,q}^{\eta}(\mu;\phi)$, then*

$$\begin{aligned}|a_{p+2}| &\leq \frac{A_1[p]_q}{|L_2|(\mu[p]_q+[p+2]_q-[p]_q)} \\ &\times \max\left\{1;\left|\frac{A_1(\mu-1)[p]_q\{\mu[p]_q+2([p+1]_q-[p]_q)\}}{2(\mu[p]_q+[p+1]_q-[p]_q)^2}-\frac{A_2}{A_1}\right|\right\}.\end{aligned}$$

Moreover, the sharp result holds for the next function

$$\omega(z)=z \quad or \quad \omega(z)=z^2, \quad (z\in\Delta).$$

Once $\phi\in\mathcal{P}$, together with (4.2) and (4.3) we apply Lemma 1.6 to prove the next corollary for the coefficient bounds of a_{p+1} and a_{p+2}.

Corollary 4.3. *If $f(z)\in\mathcal{A}_p$ belongs to the class $\mathcal{NS}_{p,q}^{\eta}(\mu;\phi)$, then*

$$|a_{p+1}|\leq\frac{2E_1[p]_q}{|L_1|(\mu[p]_q+[p+1]_q-[p]_q)}$$

and

$$\begin{aligned}|a_{p+2}| &\leq \frac{2(|E_2|+|E_1|^2)[p]_q}{|L_2|(\mu[p]_q+[p+2]_q-[p]_q)} \\ &+ \frac{2|\mu-1||E_1|^2[p]_q^2[\mu[p]_q+2([p+1]_q-[p]_q)]}{|L_2|(\mu[p]_q+[p+2]_q-[p]_q)(\mu[p]_q+[p+1]_q-[p]_q)^2}.\end{aligned}$$

Clearly, if we let δ and η be real, then from Lemma 1.7 we also show the following result for Fekete-Szegö problem.

Theorem 4.4. *Let $\delta,\eta\in\mathbb{R}$ and $\phi\in\Lambda$ satisfying*

$$\phi(z)=1+\sum_{n=1}^{\infty}A_nz^n, \quad (A_1,A_2>0, z\in\Delta).$$

If $f(z) \in \mathcal{A}_p$ belongs to the class $\mathcal{NS}^{\eta}_{p,q}(\mu;\phi)$, then

$$|a_{p+2}-\delta a_{p+1}^2| \leq \begin{cases} \frac{[p]_q}{L_2(\mu[p]_q+[p+2]_q-[p]_q)}\left\{A_2 - \frac{A_1^2[p]_q\Psi}{2L_1^2(\mu[p]_q+[p+1]_q-[p]_q)^2}\right\}, & (\delta \leq \Pi_1); \\ \frac{A_1[p]_q}{L_2(\mu[p]_q+[p+2]_q-[p]_q)}; \\ \frac{[p]_q}{L_2(\mu[p]_q+[p+2]_q-[p]_q)}\left\{-A_2 + \frac{A_1^2[p]_q\Psi}{2L_1^2(\mu[p]_q+[p+1]_q-[p]_q)^2}\right\}, & (\delta \geq \Pi_2), \end{cases}$$

where

$$\Pi_1 = \frac{2(A_2-A_1)L_1^2(\mu[p]_q+[p+1]_q-[p]_q)^2 - A_1^2(\mu-1)L_1^2[p]_q[\mu[p]_q+2([p+1]_q-[p]_q)]}{2L_2A_1^2[p]_q^2(\mu[p]_q+[p+2]_q-[p]_q)}$$

and

$$\Pi_2 = \frac{2(A_2+A_1)L_1^2(\mu[p]_q+[p+1]_q-[p]_q)^2 - A_1^2(\mu-1)L_1^2[p]_q[\mu[p]_q+2([p+1]_q-[p]_q)]}{2L_2A_1^2[p]_q^2(\mu[p]_q+[p+2]_q-[p]_q)}.$$

Moreover, we put

$$\Pi_1 = \frac{2A_2L_1^2(\mu[p]_q+[p+1]_q-[p]_q)^2 - A_1^2(\mu-1)L_1^2[p]_q[\mu[p]_q+2([p+1]_q-[p]_q)]}{2L_2A_1^2[p]_q^2(\mu[p]_q+[p+2]_q-[p]_q)}.$$

Then, each of the following results is true:
(A) For $\delta \in [\Pi_1, \Pi_3]$,

$$\begin{aligned} |a_{p+2}-\delta a_{p+1}^2| &+ \frac{2(A_1-A_2)L_1^2(\mu[p]_q+[p+1]_q-[p]_q)^2 + A_1^2[p]_q\Psi}{2L_2A_1^2[p]_q[\mu[p]_q+([p+2]_q-[p]_q)]}|a_{p+1}|^2 \\ &\leq \frac{A_1[p]_q}{L_2(\mu[p]_q+[p+2]_q-[p]_q)}; \end{aligned}$$

(B) For $\delta \in [\Pi_3, \Pi_2]$,

$$\begin{aligned} |a_{p+2}-\delta a_{p+1}^2| &+ \frac{2(A_1+A_2)L_1^2(\mu[p]_q+[p+1]_q-[p]_q)^2 - A_1^2[p]_q\Psi}{2L_2A_1^2[p]_q[\mu[p]_q+([p+2]_q-[p]_q)]}|a_{p+1}|^2 \\ &\leq \frac{A_1[p]_q}{L_2(\mu[p]_q+[p+2]_q-[p]_q)}, \end{aligned}$$

where

$$\Psi = 2\delta L_2[p]_q(\mu[p]_q+[p+2]_q-[p]_q) + (\mu-1)[\mu[p]_q+2([p+1]_q-[p]_q)]L_1^2.$$

Remark 4.5. Similarly, by choose the parameter $p = 1$ in Theorems 4.1 and 4.4, we can provide the new results for the univalent function classes $\mathcal{NS}^{\eta}_{1,q}(\mu;\phi) = \mathcal{NS}^{\eta}_{q}(\mu;\phi)$. As Remark 1.4, we may consider $\mathcal{NS}^{\eta}_{p,q}(\mu;\alpha)$ or $\mathcal{NS}^{\eta}_{p,q}(\mu;\beta)$ to establish latest results. Besides, for the fixed parameter μ, we can still infer new results for $\mathcal{NS}^{\eta}_{p,q}(\mu;\phi)$.

5. Conclusion

By involving a generalized Bernardi integral operator, several new subclasses of q-starlike and q-convex type analytic and multivalent functions are introduced to generalize the classical starlike and convex functions. Meanwhile, for these classes we may know integral operator and q-derivative as well as multivalency how to change the coefficients of functions. In our main results, we establish the Fekete-Szegö type functional inequalities for these function classes. Further, the corresponding bound estimates of the coefficients a_{p+1} and a_{p+2} are interpreted. In fact, if we use the other integral operators or take (p, q)-operator when certain function is univalent but not multivalent, we may get many similar results as in this article.

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Conflict of interest

The authors declare no conflict of interest.

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