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Research article

Some integral inequalities for generalized preinvex functions with applications

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Abstract: The main objective of this work is to explore and characterize the idea of s-type preinvex function and related inequalities. Some interesting algebraic properties and logical examples are given to support the newly introduced idea. In addition, we attain the novel version of Hermite-Hadamard type inequality utilizing the introduced preinvexity. Furthermore, we establish two new identities, and employing these, we present some refinements of Hermite-Hadamard-type inequality. Some special cases of the presented results for different preinvex functions are deduced as well. Finally, as applications, some new inequalities for the arithmetic, geometric and harmonic means are established. Results obtained in this paper can be viewed as a significant improvement of previously known results. The awe-inspiring concepts and formidable tools of this paper may invigorate and revitalize for additional research in this worthy and absorbing field.

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1. Introduction

The theory of convex functions has a wide range of potential applications in many fascinating and intriguing fields of research. In addition, this theory also plays a magnificent role in many different areas, such as physics, information theory, coding theory, engineering, optimization, and inequality theory. Currently, this theory has a remarkable contribution to the extensions and improvements of numerous areas of mathematical and applied sciences. Many authors examined, celebrated, and performed their work on the idea of convexity and extended its numerous versions in useful ways using fruitful techniques and innovative ideas. Several new families of classical convex functions have been proposed in the literature. For the attention of the readers, see the references [1–5].

Many authors and scientists have always been trying to do good work and meaningful contributions to the theory of inequalities. The term integral inequalities on convex functions, both derivative and integration, has also been a hot and absorbing topic of discussion for research activities from the current decade. The theory of inequalities has momentous investigations in the variability of applied analysis, for example, geometric function theory, impulsive diffusion equations, coding theory, numerical analysis, and fractional calculus. Recently, Sun [6] and co-researchers [7] generalized the Hermite-Hadamard inequality for harmonically convex function and s-preinvex function using the local fractional integral operator. For the attention of the readers, see the references [8–13].

Recently, several authors have elaborated new variants of inequalities for different kinds of convexities, preinvexities, statistical theory, and so on. Several discussions represent a close relationship between the theory of inequalities and convex functions. In the year 1981, for the very first time invex function concerning bifunction $\eta(.,.)$ was investigated by Hanson (see [14]). After Hanson's work, Ben-Israel and Mond tried to further investigate related invexity and for the first time elaborated the idea of invex sets and preinvex functions (see [15]). Mohan and Neogy [16] commented that the preinvex and invex functions in the form of differentiability are equivalent under suitable conditions. In 2005, Antczak [17] for the first time investigated and explored the properties of the preinvex functions. Many authors have passed the comments, these functions have a wide range of applications in optimization theory and mathematical programming. The concept of preinvexity is more general than convexity. Due to useful and elegant importance, these functions have been generalized and extended in numerous directions. In current time, İşcan et.al explored and investigated s-type convex function in a published article [18]. The magnificent concept of this article may energize and enthusiast for further research activities and be useful to generate the Mandelbrot and Julia sets for a quadratic and cubic polynomial with s-convexity [19, 20].

2. Preliminaries

In this section, we recall some known concepts.

Definition 2.1. [21] Let $\eta : \mathbb{X} \times \mathbb{X} \neq \emptyset \to \mathbb{R}$ be a real valued function, then $\mathbb{X}$ is said to be invex with respect to $\eta(.,.)$ if $\nu_2 + \kappa\eta(\nu_1, \nu_2) \in \mathbb{X}$, for the all mentioned conditions, i.e., for every $\nu_1, \nu_2 \in \mathbb{X}$ and $\kappa \in [0, 1]$.

Note that, in literature the invex set $\mathbb{X}$ is also known as a η-connected set.

We clearly see that, if we choose $\eta(\nu_1, \nu_2) = \nu_1 - \nu_2$, then as a result we attain classical convexity. Therefore, every convex set is an invex with respect to $\eta(\nu_1, \nu_2) = \nu_1 - \nu_2$, but its converse is not true in general, means that $\exists$ invex sets which are not convex (see [17] and [21]).

In the year 1988, mathematicians namely Weir and Mond [22] used the idea of an invex set and to perform and explore the idea of preinvexity.

Definition 2.2. [22] Let $\mathbb{X} \neq \emptyset \in \mathbb{R}$ be an invex set with respect to $\eta : \mathbb{X} \times \mathbb{X} \neq \emptyset \to \mathbb{R}$. Then the function $\psi : \mathbb{X} \to \mathbb{R}$ is said to be preinvex with respect to η if

$$\psi\left(\nu_2 + \kappa\eta\left(\nu_1, \nu_2\right)\right) \leq \kappa\psi\left(\nu_1\right) + (1-\kappa)\psi\left(\nu_2\right), \quad \forall \nu_1, \nu_2 \in \mathbb{X}, \kappa \in [0,1].$$

The function ψ is said to be preincave if and only if ψ is preinvex. We clearly see that, if we choose $\eta(\nu_1, \nu_2) = \nu_1 - \nu_2$, then as a result we attain the classical convex function from the more general form of function namely preinvexity [23].

We also want the following hypothesis regarding the function η which is due to Mohan and Neogy [16].

Condition: Let $\mathbb{X} \subset R^n$ be an open invex subset with respect to $\eta : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$. For any $\nu_1, \nu_2 \in \mathbb{X}$ and $\kappa \in [0,1]$

$$\begin{aligned} \eta\left(\nu_2, \nu_2 + \kappa\, \eta\left(\nu_1, \nu_2\right)\right) &= -\kappa\, \eta\left(\nu_1, \nu_2\right), \\ \eta\left(\nu_1, \nu_2 + \kappa\, \eta\left(\nu_1, \nu_2\right)\right) &= (1-\kappa)\; \eta\left(\nu_1, \nu_2\right). \end{aligned} \tag{2.1}$$

For any $\nu_1, \nu_2 \in \mathbb{X}$ and $\kappa_1, \kappa_2 \in [0,1]$ from condition C, we have

$$\eta\left(\nu_2 + \kappa_2\, \eta\left(\nu_1, \nu_2\right)\,,\ \nu_2 + \kappa_1\, \eta\left(\nu_1, \nu_2\right)\right) = \left(\kappa_2 - \kappa_1\right) \eta\left(\nu_1, \nu_2\right).$$

In the year 2007, Pakistani famous mathematician, Noor [24] established and examined Hermite-Hadamard type inequality for the preinvex functions, which is stated by

Theorem 2.3. *Let $\psi : \mathbb{X} = [\nu_1, \nu_1 + \eta(\nu_2, \nu_1)] \to (0, \infty)$ be a preinvex functions on the interval of real numbers $\mathbb{X}^\circ$ and $\nu_1, \nu_2 \in \mathbb{X}$ with $\nu_1 < \nu_1 + \eta(\nu_1, \nu_2)$. Then the following Hermite-Hadamard type inequality holds:*

$$\psi\left(\frac{2\nu_1 + \eta(\nu_2, \nu_1)}{2}\right) \leq \frac{1}{\eta(\nu_2, \nu_1)} \int_{\nu_1}^{\nu_1 + \eta(\nu_2, \nu_1)} \psi(x)dx \leq \frac{\psi(\nu_1) + \psi(\nu_2)}{2}.$$

In 2011, another team of mathematicians namely A. Barani, A. G. Gahazanfari and S. S. Dragomir worked on the idea of preinvexity in the published article [25] and first time explored and established Hermite-Hadamard type for the differentiable preinvex function, which is stated by

Theorem 2.4. *Let $\mathbb{X} \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$. Suppose $\psi : \mathbb{X} \to \mathbb{R}$ is a differentiable function. If $|\psi'|$ is preivex on $\mathbb{X}$ then, for every $\nu_1, \nu_2 \in \mathbb{A}$ with $\eta(\nu_2, \nu_1) \neq 0$. Then the following Hermite-Hadamard type inequality holds:*

$$\left|\frac{\psi(\nu_1) + \psi(\nu_1 + \eta(\nu_2, \nu_1)}{2} - \frac{1}{\eta(\nu_2, \nu_1)} \int_{\nu_1}^{\nu_1 + \eta(\nu_2, \nu_1)} \psi(x)dx\right| \leq \frac{|\eta(\nu_2, \nu_1)|}{8} \left[\psi(\nu_1) + \psi(\nu_2)\right].$$

Later on, different authors presented and collaborated there perspectives on the idea of preinvex functions. For author's good work and attractions for the readers, see the published articles [26–29].

Definition 2.5. [30] A nonnegative function $\psi : \mathbb{X} \to \mathbb{R}$ is called s-type convex function if for every $\nu_1, \nu_2 \in \mathbb{X}$, $n \in \mathbb{N}$, $s \in [0,1]$ and $\kappa \in [0,1]$, if

$$\psi\left(\kappa\nu_1 + (1-\kappa)\,\nu_2\right) \leq \left[1 - (s(1-\kappa))\right] \psi\left(\nu_1\right) + \left[1 - s\kappa\right] \psi\left(\nu_2\right). \tag{2.2}$$

Definition 2.6. [31] Two functions ψ and φ are said to be similarly ordered if

$$(\psi(\nu_1) - \psi(\nu_2))(\varphi(\nu_1) - \varphi(\nu_2)) \geq 0, \quad \forall \nu_1, \nu_2 \in \mathbb{R}. \tag{2.3}$$

Owing to the aforementioned trend and inspired by the ongoing activities, the rest of this paper is organized as follows. First of all, in Section 3, we will define and explore the newly introduced idea about preinvex functions and their algebraic properties. In Section 4, we derive the novel version of Hermite-Hadamard type inequality. In Section 5, we will make two new lemmas, and based on these new lemmas and with the help of the newly introduced definition, we will find and attain the refinements of Hermite-Hadamard type inequality. Finally, we will give some applications in support of the newly introduced idea in Section 6 and conclusion.

3. s-type preinvex function and its properties

In this section, we are to add and introduce a new notion for a new family of convex functions namely s-type preinvex function.

Definition 3.1. Let $\mathbb{X} \subset \mathbb{R}$ be a nonempty invex set with respect to $\eta : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$. Then the function $\psi : \mathbb{X} \to \mathbb{R}$ is called s-type preinvex, if

$$\psi(\nu_2 + \kappa\eta(\nu_1, \nu_2)) \leq [1 - (s(1 - \kappa))]\, \psi(\nu_1) + [1 - s\kappa]\, \psi(\nu_2) \tag{3.1}$$

holds for every $\nu_1, \nu_2 \in \mathbb{X}$, $s \in [0, 1]$ and $\kappa \in [0, 1]$.

Remark 1. (i) Taking $s = 1$ in Definition 3.1, then we attain a definition which is called preinvex function which is first time explored by Weir and Mond [22].
(ii) Taking $\eta(\nu_1, \nu_2) = \nu_1 - \nu_2$ in Definition 3.1, then we attain a published definition namely s-type convex function which is first time explored by İşcan et al. [30].
(iii) Taking $s = 1$ and $\eta(\nu_1, \nu_2) = \nu_1 - \nu_2$ in Definition 3.1, then we obtain a definition namely convex function which is investigated by Niculescu et al. [31].

Lemma 3.2. *The following inequalities*

$$\kappa \leq [1 - (s(1 - \kappa))] \quad \textit{and} \quad 1 - \kappa \leq [1 - s\kappa].$$

are holds, if for all $\kappa \in [0, 1]$ *and* $s \in [0, 1]$.

Proof. The rest of the proof is clearly seen. □

Proposition 1. *Every nonnegative preinvex function is s-type preinvex function for* $s \in [0, 1]$.

Proof. By using Lemma 3.2 and definition of preinvexity for $s \in [0, 1]$, we have

$$\psi(\nu_2 + \kappa\eta(\nu_1, \nu_2)) \leq \kappa\psi(\nu_1) + (1 - \kappa)\psi(\nu_2)$$

$$\leq [1 - (s(1 - \kappa))]\, \psi(\nu_1) + [1 - s\kappa]\, \psi(\nu_2).$$

□

Proposition 2. *Every non-negative s-type preinvex function for* $s \in [0,1]$ *is an h-preinvex function with* $h(\kappa) = [1-(s(1-\kappa))]$.

Proof. Using the definition of s-type preinvexity for $s \in [0,1]$ and the mentioned condition $h(\kappa) = [1-(s(1-\kappa))]$, we have

$$\psi(\nu_2 + \kappa\eta(\nu_1,\nu_2)) \leq [1-(s(1-\kappa))]\,\psi(\nu_1) + [1-s\kappa]\,\psi(\nu_2)$$

$$\leq h(\kappa)\psi(\nu_1) + h(1-\kappa)\psi(\nu_2).$$

□

This means that, the new class of s-type preinvexity is very larger with respect to the known class of functions, like preinvex functions and convex functions. This is the beauty of the proposed new Definition 3.1.

Now we make examples in support of the new idea.

Example 1. $\psi(\nu) = |\nu|, \forall \nu \geq 0$ *is non-negative convex function, so it is non-negative preinvex function because every convex function is preinvex function (see [32]). By using Proposition 1, it is an s-type preinvex function.*

Example 2. $\psi(\nu) = e^{\nu}, \forall \nu \geq 0$ *is non-negative convex function, so it is non-negative preinvex function because every convex function is preinvex function (see [32]). By using Proposition 1, it is an s-type preinvex function.*

Note that, every convex function is a preinvex but the converse is not true. Means that every preinvex function is not a convex function. For example $\psi(\nu) = -|\nu|$.

So next we try to find and make an example of s-type preinvex function with repect to η on $\mathbb{X}$, but it is a non-negative and not convex function.

Example 3. *Define the function* $\psi(\nu) : \mathbb{R}^+ \to \mathbb{R}^+$ *as*

$$\psi(\nu) = \begin{cases} \nu & ; \quad 0 \leq \nu \leq 1 \\ 1 & ; \quad \nu > 1 \end{cases}$$

and

$$\eta(\nu_1,\nu_2) = \begin{cases} \nu_1 - \nu_2 & ; \quad \nu_1 \leq 0, \nu_2 \leq 0 \\ \nu_1 - \nu_2 & ; \quad 0 \leq \nu_1 \leq 1, \nu_2 \leq 1 \\ -2-\nu_2 & ; \quad \nu_1 \leq 0, 0 \leq \nu_2 \leq 1 \\ 2-\nu_2 & ; \quad 0 \leq \nu_1 \leq 1, \nu_2 \leq 0. \end{cases}$$

The above non-negative function $\psi(\nu)$ *is not a convex, but it is a preinvex function with respect to* η. *According to Proposition 1, every non-negative preinvex function is a s-type preinvex functions. Hence the given function* $\psi(\nu)$ *is s-type preinvex function with respect to* η *on X.*

Example 4. *Define the function* $\psi(\nu) : \mathbb{R}^+ \to \mathbb{R}^+$ *as*

$$\psi(\nu) = \begin{cases} \nu + 1 & ; \quad 0 \leq \nu \leq 1 \\ 1 & ; \quad \nu > 1 \end{cases}$$

and

$$\eta(\nu_1,\nu_2)=\begin{cases}\nu_1+\nu_2 & ; \quad \nu_1\le\nu_2\\ 2(\nu_1+\nu_2); & \nu_1>\nu_2\end{cases}$$

$\forall \nu_1,\nu_2\in\mathbb{R}^+=[0,+\infty)$. *The above non-negative function $\psi(\nu)$ is not a convex, but it is a preinvex function with respect to η. According to Proposition 1, every non-negative preinvex function is a s-type preinvex functions. Hence the given function $\psi(\nu)$ is s-type preinvex function with respect to η on $\mathbb{X}$.*

Now, we will discuss and explore the some properties in the support of the newly introduced idea.

Theorem 3.3. *Let $\psi,\varphi:\mathbb{X}=[\nu_1,\nu_2]\to\mathbb{R}$. If $\psi,\ \varphi$ be two s-type preinvex functions with respect to same η, then*
(i) $\psi+\varphi$ is an s-type preinvex function with respect to η.
(ii) For $c\in\mathbb{R}(c\ge 0)$, then $c\psi$ is an s-type preinvex function with respect to η.

Proof. (i) Let $\psi,\ \varphi$ be s-type preinvex function with respect to same η, then for all $\nu_1,\nu_2\in\mathbb{X}$, $s\in[0,1]$ and $\kappa\in[0,1]$, we have

$$\begin{aligned}(\psi+\varphi)(\nu_2+\kappa\eta(\nu_1,\nu_2))&=\psi(\nu_2+\kappa\eta(\nu_1,\nu_2))+\varphi(\nu_2+\kappa\eta(\nu_1,\nu_2))\\&\le[1-(s(1-\kappa))]\,\psi(\nu_1)+[1-s\kappa]\,\psi(\nu_2)\\&+[1-(s(1-\kappa))]\,\varphi(\nu_1)+[1-s\kappa]\,\varphi(\nu_2)\\&=[1-(s(1-\kappa))]\,[\psi(\nu_1)+\varphi(\nu_1)]+[1-s\kappa]\,[\psi(\nu_2)+\varphi(\nu_2)]\\&=[1-(s(1-\kappa))]\,(\psi+\varphi)(\nu_1)+[1-s\kappa]\,(\psi+\varphi)(\nu_2).\end{aligned}$$

(ii) Let ψ be an s-type preinvex function with respect to η, then for all $\nu_1,\nu_2\in\mathbb{X}$, $s\in[0,1]$, $c\in\mathbb{R}(c\ge 0)$ and $\kappa\in[0,1]$, we have

$$\begin{aligned}(c\psi)(\nu_2+\kappa\eta(\nu_1,\nu_2))&\le c\Big[[1-(s(1-\kappa))]\,\psi(\nu_1)+[1-s\kappa]\,\psi(\nu_2)\Big]\\&=[1-(s(1-\kappa))]\,c\psi(\nu_1)+[1-s\kappa]\,c\psi(\nu_2)\\&=[1-(s(1-\kappa))]\,(c\psi)(\nu_1)+[1-s\kappa]\,(c\psi)(\nu_2).\end{aligned}$$

This is the required proof. □

Remark 2. (i) Choosing $s=1$ in Theorem 3.3, then we get $\psi+\varphi$ and $c\psi$ as preinvex functions.
(ii) Choosing $\eta(\nu_1,\nu_2)=\nu_1-\nu_2$ in Theorem 3.3, then we get $\psi+\varphi$ and $c\psi$ as s-type convex functions.
(iii) Choosing $\eta(\nu_1,\nu_2)=\nu_1-\nu_2$ and $s=1$ in Theorem 3.3, then we get $\psi+\varphi$ and $c\psi$ as convex functions.

Theorem 3.4. *Let $\mathbb{J}$ be a non-negative and non empty set, $\psi:\mathbb{X}\to\mathbb{J}$ be an s-type preinvex function with respect to η and $\varphi:\mathbb{J}\to\mathbb{R}$ is non-decreasing function. Then the function $\varphi\circ\psi$ is an s-type preinvex function with respect to same η.*

Proof. For all $\nu_1,\nu_2\in\mathbb{X}$, $s\in[0,1]$ and $\kappa\in[0,1]$, we have

$$\begin{aligned}(\varphi\circ\psi)(\nu_2+\kappa\eta(\nu_1,\nu_2))&=\varphi(\psi(\nu_2+\kappa\eta(\nu_1,\nu_2)))\\&\le\varphi\Big[[1-(s(1-\kappa))]\,\psi(\nu_1)+[1-s\kappa]\,\psi(\nu_2)\Big]\end{aligned}$$

$$\leq [1-(s(1-\kappa))]\,\varphi(\psi(\nu_1)) + [1-s\kappa]\,\varphi(\psi(\nu_2))$$
$$= [1-(s(1-\kappa))]\,(\varphi\circ\psi)(\nu_1) + [1-s\kappa]\,(\varphi\circ\psi)(\nu_2)\,.$$

This is the required proof. □

Remark 3. (i) If $s=1$ in Theorem 3.4, then

$$(\varphi\circ\psi)(\nu_2+\kappa\eta(\nu_1,\nu_2)) \leq \kappa(\varphi\circ\psi)(\nu_1) + (1-\kappa)(\varphi\circ\psi)(\nu_2)\,.$$

(ii) If we put $\eta(\nu_1,\nu_2)=\nu_1-\nu_2$ in Theorem 3.4, then

$$(\varphi\circ\psi)(\kappa\nu_1+(1-\kappa)\nu_2) \leq [1-(s(1-\kappa))](\varphi\circ\psi)(\nu_1) + [1-s\kappa](\varphi\circ\psi)(\nu_2)\,.$$

Theorem 3.5. *Let* $0<\nu_1<\nu_2$, $\psi_j : \mathbb{X}=[\nu_1,\nu_2]\to[0,+\infty)$ *be a class of s-type preinvex function with respect to same* η *and* $\psi(u)=\sup_j \psi_j(u)$*. Then* ψ *is an s-type preinvex function with respect to* η *and* $U=\{\nu\in[\nu_1,\nu_2] : \psi(\nu_i)<\infty\}$ *is an interval.*

Proof. Let $\nu_1,\nu_2\in U$, $s\in[0,1]$ and $\kappa\in[0,1]$, then

$$\begin{aligned}\psi(\nu_2+\kappa\eta(\nu_1,\nu_2)) &= \sup_j \psi_j(\nu_2+\kappa\eta(\nu_1,\nu_2))\\ &\leq [1-(s(1-\kappa))]\sup_j \psi_j(\nu_1) + [1-s\kappa]\sup_j \psi_j(\nu_2)\\ &= [1-(s(1-\kappa))]\,\psi(\nu_1) + [1-s\kappa]\,\psi(\nu_2) < \infty.\end{aligned}$$

This is the required proof. □

Theorem 3.6. *Let* $\psi,\varphi : \mathbb{X}=[\nu_1,\nu_2]\to\mathbb{R}$*. If* ψ, φ *be two s-type preinvex function with respect to same* η *and* ψ, φ *are similarly ordered functions and* $[1-(s(1-\kappa))]+[1-s\kappa]\leq 1$, *then the product* $\psi\varphi$ *is an s-type preinvex function with respect to* η.

Proof. Let ψ, φ be an s-type preinvex function with respect to same η, $s\in[0,1]$ and $\kappa\in[0,1]$, then

$$\begin{aligned}\psi(\nu_2+\kappa\eta(\nu_1,\nu_2))\varphi(\nu_2+\kappa\eta(\nu_1,\nu_2)) &\leq \Big[[1-(s(1-\kappa))]\,\psi(\nu_1) + [1-s\kappa]\,\psi(\nu_2)\Big]\\ &\times\Big[[1-(s(1-\kappa))]\,\varphi(\nu_1) + [1-s\kappa]\,\varphi(\nu_2)\,.\Big]\\ &\leq [1-(s(1-\kappa))]^2\,\psi(\nu_1)\varphi(\nu_1) + [1-s\kappa]^2\,\psi(\nu_2)\varphi(\nu_2)\\ &+ [1-(s(1-\kappa))]\,[1-s\kappa]\,[\psi(\nu_1)\varphi(\nu_2)+\psi(\nu_2)\varphi(\nu_1)]\\ &\leq [1-(s(1-\kappa))]^2\psi(\nu_1)\varphi(\nu_1) + [1-s\kappa]^2\,\psi(\nu_2)\varphi(\nu_2)\\ &+ [1-(s(1-\kappa))]\,[1-(s\kappa)]\,[\psi(\nu_1)\varphi(\nu_1)+\psi(\nu_2)\varphi(\nu_2)]\\ &= \Big[[1-(s(1-\kappa))]\psi(\nu_1)\varphi(\nu_1) + [1-s\kappa]\,\psi(\nu_2)\varphi(\nu_2)\Big]\\ &\times\Big([1-(s(1-\kappa))]+[1-s\kappa]\Big)\\ &\leq [1-(s(1-\kappa))]\,\psi(\nu_1)\varphi(\nu_1) + [1-s\kappa]\,\psi(\nu_2)\varphi(\nu_2).\end{aligned}$$

This shows that the product of two s-type preinvex functions with respect to same η is again an s-type preinvex function with respect to η. □

Remark 4. Taking $\eta(\nu_1, \nu_2) = \nu_1 - \nu_2$ in Theorem 3.6, then we attain the new inequality namely the product of s-type convex functions

$$\psi(\kappa\nu_1 + (1-\kappa)\nu_2)\varphi(\kappa\nu_1 + (1-\kappa)\nu_2) \leq [1 - (s(1-\kappa))]\psi(\nu_1)\varphi(\nu_1) + [1 - s\kappa]\psi(\nu_2)\varphi(\nu_2).$$

4. Hermite-Hadamard type inequality via s-type preinvex function

The principal intention and main aim of this section is to establish novel version of Hermite-Hadamard type inequality in the mode of newly discussed concept namely s-type preinvex functions.

Theorem 4.1. *Let $\mathbb{X}^\circ \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \mathbb{X}^\circ \times \mathbb{X}^\circ \to \mathbb{R}$ and $\nu_1, \nu_2 \in \mathbb{X}^\circ$ with $\nu_2 + \eta(\nu_1, \nu_2) \leq \nu_2$. Suppose $\psi : [\nu_2 + \eta(\nu_1, \nu_2), \nu_2]$ and satisfies Condition-C then the following Hermite-Hadamard type inequalities hold*

$$\frac{1}{2-s}\psi\left(\nu_2 + \frac{1}{2}\eta(\nu_1, \nu_2)\right) \leq \frac{1}{\eta(\nu_1, \nu_2)}\int_{\nu_2}^{\nu_2+\eta(\nu_1,\nu_2)} \psi(x)dx \leq \frac{\psi(\nu_1) + \psi(\nu_2)}{2}(2-s).$$

Proof. Since $\nu_1, \nu_2 \in \mathbb{X}^\circ$ and I° is an invex set with respect to η, for every $\kappa \in [0, 1]$, we have $\nu_2 + \kappa\eta(\nu_1, \nu_2) \in \mathbb{X}^\circ$. From the definition of s-type preinvex function of ψ, we have that

$$\psi(\nu_2 + \kappa\eta(\nu_1, \nu_2)) \leq [1 - (s(1-\kappa))]\psi(\nu_1) + [1 - s\kappa]\psi(\nu_2).$$

Integrating both the sides with respect to κ over [0,1], we have

$$\int_0^1 \psi(\nu_2 + \kappa\eta(\nu_1, \nu_2))d\kappa \leq \psi(\nu_1)\int_0^1 [1 - (s(1-\kappa))]d\kappa + \psi(\nu_2)\int_0^1 [1 - s\kappa]d\kappa.$$

We also have

$$\int_0^1 \psi(\nu_2 + \kappa\eta(\nu_1, \nu_2))d\kappa = \frac{1}{\eta(\nu_1, \nu_2)}\int_{\nu_2}^{\nu_2+\eta(\nu_1,\nu_2)} \psi(x)dx.$$

Hence,

$$\frac{1}{\eta(\nu_1, \nu_2)}\int_{\nu_2}^{\nu_2+\eta(\nu_1,\nu_2)} \psi(x)dx \leq \frac{\psi(\nu_1) + \psi(\nu_2)}{2}(2-s).$$

This completes the proof of the right hand side of above desired inequality. For the left hand side, we use the Definition 3.1, putting $\kappa = \frac{1}{2}$, one has

$$\psi(y + \kappa\eta(x, y)) \leq [1 - (s(1-\kappa))]\psi(x) + [1 - (s\kappa)]\psi(y),$$
$$\psi\left(y + \frac{1}{2}\eta(x, y)\right) \leq \left[1 - \left(\frac{s}{2}\right)\right]\left[\psi(x) + \psi(y)\right],$$

putting $y = \nu_2 + \kappa\eta(\nu_1, \nu_2)$ and $x = \nu_2 + (1-\kappa)\eta(\nu_1, \nu_2)$ in above inequality, so first we prove the L. H. S of above inequality

$$\psi\left(y + \frac{1}{2}\eta(x, y)\right) = \psi\left(\nu_2 + \frac{1}{2}\eta(\nu_1, \nu_2)\right).$$

So after putting the value of x and y, we get

$$\psi\left(y+\frac{1}{2}\eta(x,y)\right)=\psi(\nu_2+\kappa\eta(\nu_1,\nu_2)+\frac{1}{2}\eta(\nu_2+(1-\kappa)\eta(\nu_1,\nu_2),\nu_2+\kappa\eta(\nu_1,\nu_2))). \tag{4.1}$$

Now by using Condition C, we have

$$\eta(\nu_2+(1-\kappa)\eta(\nu_1,\nu_2),\nu_2+\kappa\eta(\nu_1,\nu_2))=(1-\kappa-\kappa)\eta(\nu_1,\nu_2),$$
$$\eta(\nu_2+(1-\kappa)\eta(\nu_1,\nu_2),\nu_2+\kappa\eta(\nu_1,\nu_2))=(1-2\kappa)\eta(\nu_1,\nu_2).$$

Now we put the value of η in 4.1, then as a result, we get

$$\psi\left(y+\frac{1}{2}\eta(x,y)\right)=\psi(\nu_2+\kappa\eta(\nu_1,\nu_2)+\frac{1}{2}(1-2\kappa)\eta(\nu_1,\nu_2)),$$
$$\psi\left(y+\frac{1}{2}\eta(x,y)\right)=\psi\left(\nu_2+\left(\kappa+\frac{1}{2}-\kappa\right)\eta(\nu_1,\nu_2)\right),$$
$$\psi\left(y+\frac{1}{2}\eta(x,y)\right)=\psi\left(\nu_2+\frac{1}{2}\eta(\nu_1,\nu_2)\right).$$

Now,

$$\begin{aligned}&\psi\left(\nu_2+\frac{1}{2}\eta(\nu_1,\nu_2)\right)\\ &\leq\left[1-\left(\frac{s}{2}\right)\right]\left[\int_0^1\psi(\nu_2+(1-\kappa)\eta(\nu_1,\nu_2))d\kappa+\int_0^1\psi(\nu_2+\kappa\eta(\nu_1,\nu_2))d\kappa\right]\\ &\leq\left[1-\left(\frac{s}{2}\right)\right]\frac{2}{\eta(\nu_1,\nu_2)}\int_{\nu_2}^{\nu_2+\eta(\nu_1,\nu_2)}\psi(x)dx\\ &\leq 2\left[1-\left(\frac{s}{2}\right)\right]\frac{1}{\eta(\nu_1,\nu_2)}\int_{\nu_2}^{\nu_2+\eta(\nu_1,\nu_2)}\psi(x)dx.\end{aligned}$$

This is the required proof. □

Corollary 1. *If we put $s=1$ and $\eta(\nu_1,\nu_2)=\nu_1-\nu_2$ in Theorem 4.1, then we get Hermite-Hadamard inequality in [33].*

Note: Moreover we assumed the condition C in the 2^{nd} part of proof of Theorem 4.1, so it is necessary to give an example as above, where η fulfils the condition C.

Example 5. *Remembering example 2.2 of published article (see [34]) as follows*
Let

$$\psi(\nu)=-|\nu|,\quad \forall\ \nu\in\mathbb{X} \tag{4.2}$$

$$\eta(\nu_1,\nu_2)=\begin{cases}\nu_1-\nu_2, & if\ \nu_1\geq 0,\ \nu_2\geq 0;\\ \nu_1-\nu_2, & if\ \nu_1\leq 0,\ \nu_2\leq 0;\\ -2-\nu_2, & if\ \nu_1>0,\ \nu_2\leq 0;\\ 2-\nu_2, & if\ \ \nu_1\leq 0,\ \nu_2>0.\end{cases}$$

From the above example we claim that or easily investigate that ψ is an s-type preinvex function with respect to η if $s=1$ and that ψ and η satisfies the condition C. However ψ is not a convex function.

5. Refinements of Hermite-Hadamard type inequality

The aim of this section is to investigate the refinements of Hermite-Hadamard inequality by using the newly introduced definition. In order to attain the refinements of Hermite-Hadamard inequality, we need the following lemmas.

Lemma 5.1. *Let* $\psi : \left[\nu_1, \nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right] \to \mathbb{R}$ *be a differentiable mapping on* $\left(\nu_1, \nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)$ *with* $0 < c \leq 1$ *and* $\nu_1 + \eta(\nu_2, \nu_1) > \nu_1 > 0$. *If* $\psi' \in \mathcal{L}\left(\nu_1, \nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)$, *then the following equality holds in the preinvex setting:*

$$\begin{aligned}
&\frac{\psi(\nu_1) + \psi(\nu_1 + \eta(\frac{\nu_2}{c}, \nu_1))}{2} - \frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta\left(\frac{\nu_2}{c},\nu_1\right)} \psi(x)dx \\
&= \frac{\eta(\nu_2, c\nu_1)}{2c} \int_0^1 (1-2\kappa)\psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)d\kappa. \qquad (5.1)
\end{aligned}$$

Proof. Let $\nu_1, \nu_2 \in \mathbb{X}$. Since $\mathbb{X}$ is an invex set w.r.t η , $\forall\ \kappa \in [0, 1]$, $\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c}) \in \mathbb{X}$ Integrating by parts

$$\begin{aligned}
&\frac{\eta(\nu_2, c\nu_1)}{2c} \int_0^1 (1-2\kappa)\psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)d\kappa \\
&= \frac{\eta(\nu_2, c\nu_1)}{2c}\left[\frac{(1-2\kappa)\psi\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)}{\eta(\nu_1, \frac{\nu_2}{c})}\Bigg|_0^1 + 2\int_0^1 \frac{\psi\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)}{\eta(\nu_1, \frac{\nu_2}{c})}d\kappa\right] \\
&= \frac{\eta(\nu_2, c\nu_1)}{2c}\left[\frac{c(\psi(\nu_1) + \psi(\frac{\nu_2}{c}))}{\eta(\nu_2, c\nu_1)} - \frac{2c}{\eta(\nu_2, c\nu_1)}\int_0^1 \psi\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)d\kappa\right] \\
&= \frac{\psi(\nu_1) + \psi(\nu_1 + \eta(\frac{\nu_2}{c}, \nu_1))}{2} - \frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta\left(\frac{\nu_2}{c},\nu_1\right)} \psi(x)dx.
\end{aligned}$$

Which completes the proof. □

Lemma 5.2. *Let* $\psi : \left[\nu_1, \nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right] \to \mathbb{R}$ *be a differentiable mapping on* $\left(\nu_1, \nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)$ *with* $0 < c \leq 1$ *and* $\nu_1 + \eta(\nu_2, \nu_1) > \nu_1 > 0$. *If* $\psi' \in \mathcal{L}\left(\nu_1, \nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)$, *then the following equality holds in the preinvex setting:*

$$\begin{aligned}
&\frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta\left(\frac{\nu_2}{c},\nu_1\right)} \psi(x)dx - \psi\left(\frac{2\nu_1 + \eta(\nu_2, \nu_1)}{2c}\right) \\
&= \frac{\eta(\nu_2, c\nu_1)}{c}\left\{\int_0^1 \kappa\psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)d\kappa - \int_{1/2}^1 \psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)d\kappa\right\}. \qquad (5.2)
\end{aligned}$$

Proof. Let $\nu_1, \nu_2 \in \mathbb{X}$. Since $\mathbb{X}$ is an invex set w.r.t η , $\forall \kappa \in [0, 1]$, $\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c}) \in \mathbb{X}$. Integrating by parts,

$$\frac{\eta(\nu_2, c\nu_1)}{c}\left\{\int_0^1 \kappa\psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)d\kappa - \int_{1/2}^1 \psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)d\kappa\right\}$$

$$
\begin{aligned}
=&\frac{\eta(\nu_2, c\nu_1)}{c} \times \left[\frac{\kappa\psi\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)}{\eta(\nu_1, \frac{\nu_2}{c})}\Bigg|_0^1 - \int_0^1 \frac{\psi\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)}{\eta(\nu_1, \frac{\nu_2}{c})} d\kappa - \frac{\psi\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)}{\eta(\nu_1, \frac{\nu_2}{c})} d\kappa\Bigg|_{\frac{1}{2}}^1 \right]\\
=&\frac{\eta(\nu_2, c\nu_1)}{c} \left[\frac{c\psi(\nu_1)}{\eta(c\nu_1, \nu_2)} - \frac{c}{\eta(c\nu_1, \nu_2)} \int_0^1 \psi\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right) d\kappa \right.\\
&\left. - \frac{c}{\eta(c\nu_1, \nu_2)} \left(\psi(\nu_1) - \psi\left(\frac{(2\nu_1 + \eta(\nu_2, \nu_1))}{2c}\right)\right)\right]\\
=&\frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c}, \nu_1)} \psi(x)dx - \psi\left(\frac{2\nu_1 + \eta(\nu_2, \nu_1)}{2c}\right).
\end{aligned}
$$

In this way the proof is completed. □

Theorem 5.3. *Let $\mathbb{X}^\circ \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \mathbb{X}^\circ \times \mathbb{X}^\circ \to \mathbb{R}$ and $\nu_1, \nu_2 \in \mathbb{X}^\circ$ with $\nu_2 + \eta(\nu_1, \nu_2) \leq \nu_2$. Suppose $\psi : [\nu_2 + \eta(\nu_1, \nu_2), \nu_2]$ be a differentiable mapping on $\mathbb{X}^\circ$. If $|\psi'|$ is s-type preinvex function on $(\nu_1, \nu_1 + \eta(\nu_2, \nu_1))$ for $s \in [0, 1]$, then the following inequality holds:*

$$
\begin{aligned}
&\left| \frac{\psi(\nu_1) + \psi\left(\nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)}{2} - \frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c}, \nu_1)} \psi(x)dx \right|\\
&\leq \frac{\eta(\nu_2, c\nu_1)}{2c} \left\{ \frac{2-s}{4} \left[|\psi'(\nu_1)| + |\psi'\left(\frac{\nu_2}{c}\right)| \right] \right\}.
\end{aligned} \tag{5.3}
$$

Proof. Using Lemma 5.1, one has

$$
\begin{aligned}
&\left| \frac{\psi(\nu_1) + \psi\left(\nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)}{2} - \frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c}, \nu_1)} \psi(x)dx \right|\\
&= \frac{\eta(\nu_2, c\nu_1)}{2c} \int_0^1 |1 - 2\kappa||\psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)|d\kappa.
\end{aligned}
$$

Since $|\psi'|$ is s-type preinvex on $(\nu_1, \nu_1 + \eta(\nu_2, \nu_1))$, we have

$$
\begin{aligned}
&\left| \frac{\psi(\nu_1) + \psi\left(\nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)}{2} - \frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c}, \nu_1)} \psi(x)dx \right|\\
\leq&\frac{\eta(\nu_2, c\nu_1)}{2c} \int_0^1 |1 - 2\kappa| \left[(1 - s(1-\kappa))\psi'(\nu_1)| + (1 - s\kappa)|\psi'\left(\frac{\nu_2}{c}\right)| \right] d\kappa\\
\leq&\frac{\eta(\nu_2, c\nu_1)}{2c} \left\{ |\psi'(\nu_1)| \int_0^1 |1 - 2\kappa|1 - s(1-\kappa)d\kappa + |\psi'\left(\frac{\nu_2}{c}\right)| \int_0^1 |1 - 2\kappa|(1 - s\kappa)d\kappa \right\}.
\end{aligned} \tag{5.4}
$$

Since,

$$
\begin{aligned}
&\int_0^1 (1 - s(1-\kappa))|1 - 2\kappa|d\kappa\\
=&\int_0^1 (1 - s\kappa)|1 - 2\kappa|d\kappa = -\frac{s-2}{4}.
\end{aligned}
$$

The proof of the Theorem is completed by using these computations in (5.4). □

Corollary 2. *If we choose $s = 1$ in (5.3), then we attain the following inequality*

$$\left|\frac{\psi(\nu_1)+\psi\left(\nu_1+\eta(\frac{\nu_2}{c},\nu_1)\right)}{2}-\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx\right|$$
$$\leq\frac{\eta(\nu_2,c\nu_1)}{8c}\left\{\left[|\psi'(\nu_1)|+4|\psi'\left(\frac{\nu_2}{c}\right)|\right]\right\}.$$

Corollary 3. *If we choose $\eta(\nu_1,\nu_2)=\nu_1-\nu_2$ (5.3), then we attain the following inequality*

$$\left|\frac{\psi(\nu_1)+\psi(\frac{\nu_2}{c})}{2}-\frac{c}{(\nu_2-c\nu_1)}\int_{\nu_1}^{\frac{\nu_2}{c}}\psi(x)dx\right|\leq\frac{(\nu_2-c\nu_1)}{2c}\left\{\frac{2-s}{4}\left[|\psi'(\nu_1)|+|\psi'\left(\frac{\nu_2}{c}\right)|\right]\right\}.$$

Corollary 4. *If we choose $s = 1$ and $\eta(\nu_1,\nu_2)=\nu_1-\nu_2$ (5.3), then we attain the following inequality*

$$\left|\frac{\psi(\nu_1)+\psi(\frac{\nu_2}{c})}{2}-\frac{c}{(\nu_2-c\nu_1)}\int_{\nu_1}^{\frac{\nu_2}{c}}\psi(x)dx\right|\leq\frac{(\nu_2-c\nu_1)}{8c}\left\{\left[|\psi'(\nu_1)|+4|\psi'\left(\frac{\nu_2}{c}\right)|\right]\right\}.$$

Theorem 5.4. *Let $\mathbb{X}^\circ\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:\mathbb{X}^\circ\times\mathbb{X}^\circ\to\mathbb{R}$ and $\nu_1,\nu_2\in\mathbb{X}^\circ$ with $\nu_2+\eta(\nu_1,\nu_2)\leq\nu_2$. Suppose $\psi:[\nu_2+\eta(\nu_1,\nu_2),\nu_2]$ be a differentiable mapping on $\mathbb{X}^\circ$. If $|\psi'|^q$ is s-type preinvex on $(\nu_1,\nu_1+\eta(\nu_2,\nu_1))$ for $p,q>1$, $\frac{1}{q}+\frac{1}{p}=1$ and $s\in[0,1]$, then the following inequality holds:*

$$\left|\frac{\psi(\nu_1)+\psi\left(\nu_1+\eta(\frac{\nu_2}{c},\nu_1)\right)}{2}-\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx\right|$$
$$\leq\frac{\eta(\nu_2,c\nu_1)}{2c}\left(\frac{1}{p+1}\right)^{1/p}\left\{\frac{2-s}{2}\left[|\psi'(\nu_1)|^q+|\psi'\left(\frac{\nu_2}{c}\right)|^q\right]\right\}^{1/q}. \tag{5.5}$$

Proof. Using Lemma 5.1 and Hölder's inequality, one has

$$\left|\frac{\psi(\nu_1)+\psi\left(\nu_1+\eta(\frac{\nu_2}{c},\nu_1)\right)}{2}-\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx\right|$$
$$=\frac{\eta(\nu_2,c\nu_1)}{2c}\int_0^1|1-2\kappa||\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)|d\kappa$$
$$\leq\frac{\eta(\nu_2,c\nu_1)}{2c}\left(\int_0^1|1-2\kappa|^pd\kappa\right)^{1/p}\left(\int_0^1|\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)|^qd\kappa\right)^{1/q}. \tag{5.6}$$

Since $|\psi'|^q$ is s-type preinvex on $(\nu_1,\nu_1+\eta(\nu_2,\nu_1))$, we have

$$\int_0^1|\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)|^qd\kappa=|\psi'(\nu_1)|^q\int_0^1(1-s(1-\kappa))d\kappa+|\psi'\left(\frac{\nu_2}{c}\right)|^q\int_0^1(1-s\kappa)d\kappa.$$

Now, Eq (5.6) becomes

$$\left|\frac{\psi(\nu_1)+\psi\left(\nu_1+\eta(\frac{\nu_2}{c},\nu_1)\right)}{2}-\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx\right|$$

$$\leq \frac{\eta(\nu_2, c\nu_1)}{2c}\left(\frac{1}{p+1}\right)^{1/p}\left(|\psi'(\nu_1)|^q \int_0^1 (1-s(1-\kappa))d\kappa + |\psi'\left(\frac{\nu_2}{c}\right)|^q \int_0^1 (1-s\kappa)d\kappa\right)^{1/q}. \quad (5.7)$$

Since,

$$\int_0^1 (1-s(1-\kappa))\,d\kappa = \int_0^1 (1-s\kappa)\,d\kappa = -\frac{s-2}{2},$$
$$\int_0^1 |1-2\kappa|^p d\kappa = \frac{1}{p+1}.$$

The proof of the Theorem is completed by using the above computations in (5.7). □

Corollary 5. *If we choose $s = 1$ in* (5.5)*, then we attain the following inequality*

$$\left|\frac{\psi(\nu_1) + \psi\left(\nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)}{2} - \frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)} \psi(x)dx\right|$$
$$\leq \frac{\eta(\nu_2, c\nu_1)}{2^{\frac{1+q}{q}} c}\left(\frac{1}{p+1}\right)^{1/p}\left(|\psi'(\nu_1)|^q + |\psi'\left(\frac{\nu_2}{c}\right)|^q\right)^{1/q}.$$

Corollary 6. *If we choose $\eta(\nu_1, \nu_2) = \nu_1 - \nu_2$ in* (5.5)*, then we attain the following inequality*

$$\left|\frac{\psi(\nu_1) + \psi(\frac{\nu_2}{c})}{2} - \frac{c}{(\nu_2 - c\nu_1)} \int_{\nu_1}^{\frac{\nu_2}{c}} \psi(x)dx\right| \leq \frac{(\nu_2 - c\nu_1)}{2c}\left(\frac{1}{p+1}\right)^{1/p}\left\{\frac{2-s}{2}\left[|\psi'(\nu_1)|^q + |\psi'\left(\frac{\nu_2}{c}\right)|^q\right]\right\}^{1/q}.$$

Corollary 7. *If we choose $s = 1$ and $\eta(\nu_1, \nu_2) = \nu_1 - \nu_2$ in* (5.5)*, then we attain the following inequality*

$$\left|\frac{\psi(\nu_1) + \psi(\frac{\nu_2}{c})}{2} - \frac{c}{(\nu_2 - c\nu_1)} \int_{\nu_1}^{\frac{\nu_2}{c}} \psi(x)dx\right| \leq \frac{(\nu_2 - c\nu_1)}{2^{\frac{1+q}{q}} c}\left(\frac{1}{p+1}\right)^{1/p}\left(|\psi'(\nu_1)|^q + |\psi'\left(\frac{\nu_2}{c}\right)|^q\right)^{1/q}.$$

Theorem 5.5. *Let $\mathbb{X}^\circ \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \mathbb{X}^\circ \times \mathbb{X}^\circ \to \mathbb{R}$ and $\nu_1, \nu_2 \in \mathbb{X}^\circ$ with $\nu_2 + \eta(\nu_1, \nu_2) \leq \nu_2$. Suppose $\psi : [\nu_2 + \eta(\nu_1, \nu_2), \nu_2]$ be a differentiable mapping on $\mathbb{X}^\circ$. If $|\psi'|^q$ is s-type preinvex on $(\nu_1, \nu_1 + \eta(\nu_2, \nu_1))$ for $p, q > 1$, $\frac{1}{q} + \frac{1}{p} = 1$ and $s \in [0, 1]$, then the following inequality holds:*

$$\left|\frac{\psi(\nu_1) + \psi\left(\nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)}{2} - \frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)} \psi(x)dx\right|$$
$$\leq \frac{\eta(\nu_2, c\nu_1)}{c}\left[\frac{1}{2^{(\frac{p+1}{p})}}\right]\left\{\frac{2-s}{4}\left[|\psi'(\nu_1)|^q + |\psi'\left(\frac{\nu_2}{c}\right)|^q\right]\right\}^{1/q}. \quad (5.8)$$

Proof. Using Lemma 5.1 and Hölder's inequality, one has

$$\left|\frac{\psi(\nu_1) + \psi\left(\nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)}{2} - \frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)} \psi(x)dx\right|$$

$$
\begin{aligned}
&\leq \frac{\eta(\nu_2, c\nu_1)}{2c} \int_0^1 |1-2\kappa||\psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)|d\kappa \\
&= \frac{\eta(\nu_2, c\nu_1)}{2c} \int_0^1 |1-2\kappa|^{1/p}|1-2\kappa|^{1/q}|\psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)|d\kappa \\
&\leq \frac{\eta(\nu_2, c\nu_1)}{2c} \left(\int_0^1 |1-2\kappa|d\kappa\right)^{1/p} \left(\int_0^1 |1-2\kappa||\psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)|^q d\kappa\right)^{1/q}. \qquad (5.9)
\end{aligned}
$$

Since $|\psi'|^q$ is s-type preinvex on $(\nu_1, \nu_1 + \eta(\nu_2, \nu_1))$, we have

$$
\int_0^1 |\psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)|^q d\kappa = |\psi'(\nu_1)|^q \int_0^1 (1 - s(1-\kappa))d\kappa + |\psi'\left(\frac{\nu_2}{c}\right)|^q \int_0^1 (1 - s\kappa)d\kappa.
$$

Now, Eq (5.9) becomes

$$
\begin{aligned}
&\left|\frac{\psi(\nu_1) + \psi\left(\nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)}{2} - \frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)} \psi(x)dx\right| \\
&\leq \frac{\eta(\nu_2, c\nu_1)}{2c} \left(\int_0^1 |1-2\kappa|d\kappa\right)^{1/p} \left(|\psi'(\nu_1)|^q \int_0^1 |1-2\kappa|(1-s(1-\kappa))d\kappa + \left|\psi'\left(\frac{\nu_2}{c}\right)\right|^q \int_0^1 |1-2\kappa|(1-sk)d\kappa\right)^{1/q}. \qquad (5.10)
\end{aligned}
$$

Since,

$$
\int_0^1 |1-2\kappa|\,(1 - s\,(1-\kappa))\,d\kappa = \int_0^1 |1-2\kappa|\,(1-s\kappa)\,d\kappa = -\frac{s-2}{4},
$$
$$
\int_0^1 |1-2\kappa|d\kappa = \frac{1}{2}.
$$

The proof of the Theorem gets completed by using these computations in (5.10). □

Corollary 8. *If we choose $s = 1$ in* (5.8)*, then we attain the following inequality*

$$
\left|\frac{\psi(\nu_1) + \psi\left(\nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)}{2} - \frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)} \psi(x)dx\right| \leq \frac{\eta(\nu_2, c\nu_1)}{c} \left[\frac{1}{2^{\left(\frac{2q+1}{q}\right)}}\right] \left(|\psi'(\nu_1)|^q + |\psi'\left(\frac{\nu_2}{c}\right)|^q\right)^{1/q}.
$$

Corollary 9. *If we choose $\eta(\nu_1, \nu_2) = \nu_1 - \nu_2$ in* (5.8)*, then we attain the following inequality*

$$
\left|\frac{\psi(\nu_1) + \psi(\frac{\nu_2}{c})}{2} - \frac{c}{(\nu_2 - c\nu_1)} \int_{\nu_1}^{\frac{\nu_2}{c}} \psi(x)dx\right| \leq \frac{(\nu_2 - c\nu_1)}{c} \left[\frac{1}{2^{\left(\frac{p+1}{p}\right)}}\right] \left\{\frac{2-s}{4}\left[|\psi'(\nu_1)|^q + |\psi'\left(\frac{\nu_2}{c}\right)|^q\right]\right\}^{1/q}.
$$

Corollary 10. *If we choose $s = 1$ and $\eta(\nu_1, \nu_2) = \nu_1 - \nu_2$ in* (5.8)*, then we attain the following inequality*

$$
\left|\frac{\psi(\nu_1) + \psi(\frac{\nu_2}{c})}{2} - \frac{c}{(\nu_2 - c\nu_1)} \int_{\nu_1}^{\frac{\nu_2}{c}} \psi(x)dx\right| \leq \frac{(\nu_2 - c\nu_1)}{c} \left[\frac{1}{2^{\left(\frac{2q+1}{q}\right)}}\right] \left(|\psi'(\nu_1)|^q + |\psi'\left(\frac{\nu_2}{c}\right)|^q\right)^{1/q}.
$$

Theorem 5.6. *Let $\mathbb{X}^\circ \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \mathbb{X}^\circ \times \mathbb{X}^\circ \to \mathbb{R}$ and A $\nu_1, \nu_2 \in \mathbb{X}^\circ$ with $\nu_2 + \eta(\nu_1, \nu_2) \leq \nu_2$. Suppose $\psi : [\nu_2 + \eta(\nu_1, \nu_2), \nu_2]$ be a differentiable mapping on $\mathbb{X}^\circ$. If $|\psi'|^q$ is s-type preinvex on $(\nu_1, \nu_1 + \eta(\nu_2, \nu_1))$ for $p, q > 1$, $\frac{1}{q} + \frac{1}{p} = 1$ and $s \in [0, 1]$, then the following inequality holds:*

$$\left|\frac{\psi(\nu_1) + \psi\left(\nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)}{2} - \frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)} \psi(x)dx\right|$$
$$\leq \frac{\eta(\nu_2, c\nu_1)}{2c} \left(\frac{1}{2(p+1)}\right)^{1/p} \left[\left\{\frac{3-2s}{6}|\psi'(\nu_1)|^q + \frac{3-s}{6}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}\right.$$
$$\left. + \left\{\frac{3-s}{6}|\psi'(\nu_1)|^q + \frac{3-2s}{6}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}\right]. \tag{5.11}$$

Proof. Using Lemma 5.1 and Hölder-İşcan inequality, one has

$$\left|\frac{\psi(\nu_1) + \psi\left(\nu_1 + \eta(\frac{\nu_2}{c}, \nu_1)\right)}{2} - \frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)} \psi(x)dx\right|$$
$$\leq \frac{\eta(\nu_2, c\nu_1)}{2c} \left[\left(\int_0^1 (1-\kappa)|1-2\kappa|^p d\kappa\right)^{1/p} \left(\int_0^1 (1-\kappa)|\psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)|^q d\kappa\right)^{1/q}\right.$$
$$\left. + \left(\int_0^1 \kappa|1-2\kappa|^p d\kappa\right)^{1/p} \left(\int_0^1 \kappa|\psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)|^q d\kappa\right)^{1/q}\right]. \tag{5.12}$$

Since $|\psi'|^q$ is s-type preinvex on $(\nu_1, \nu_1 + \eta(\nu_2, \nu_1))$, we have

$$\int_0^1 |\psi'\left(\frac{\nu_2}{c} + \kappa\eta(\nu_1, \frac{\nu_2}{c})\right)|^q d\kappa = |\psi'(\nu_1)|^q \int_0^1 (1 - s(1-\kappa))d\kappa + |\psi'\left(\frac{\nu_2}{c}\right)|^q \int_0^1 (1 - s\kappa)d\kappa.$$

Now, Eq (5.12) becomes

$$\left|\frac{\psi(\nu_1) + \psi(\nu_1 + \eta(\frac{\nu_2}{c}, \nu_1))}{2} - \frac{c}{\eta(\nu_2, c\nu_1)} \int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)} \psi(x)dx\right|$$
$$\leq \frac{\eta(\nu_2, c\nu_1)}{2c} \left[\left(\int_0^1 (1-\kappa)|1-2\kappa|^p d\kappa\right)^{1/p} \left(|\psi'(\nu_1)|^q \int_0^1 (1-\kappa)(1-s(1-\kappa))d\kappa\right.\right.$$
$$\left. + |\psi'\left(\frac{\nu_2}{c}\right)|^q \int_0^1 (1-\kappa)(1-s\kappa)d\kappa\right)^{1/q}$$
$$\left. + \left(\int_0^1 \kappa|1-2\kappa|^p d\kappa\right)^{1/p} \left(|\psi'(\nu_1)|^q \int_0^1 \kappa(1-s(1-\kappa))d\kappa + |\psi'\left(\frac{\nu_2}{c}\right)|^q \int_0^1 \kappa(1-s\kappa)d\kappa\right)^{1/q}\right]. \tag{5.13}$$

Since,

$$\int_0^1 (1-\kappa)(1-s(1-\kappa))\, d\kappa = \int_0^1 \kappa(1-s\kappa)\, d\kappa = -\frac{2s-3}{6}$$

$$\int_0^1 \kappa\left(1-s\left(1-\kappa\right)\right)d\kappa=\int_0^1 \left(1-\kappa\right)\left(1-s\kappa\right)d\kappa=-\frac{s-3}{6}$$
$$\int_0^1 \kappa|1-2\kappa|^p d\kappa=\int_0^1 (1-\kappa)|1-2\kappa|^p d\kappa=\frac{1}{2(p+1)}.$$

The proof of the Theorem is completed by using these computations in (5.13). □

Corollary 11. *If we choose $s=1$ in (5.11), then we attain the following inequality*

$$\left|\frac{\psi(\nu_1)+\psi\left(\nu_1+\eta(\frac{\nu_2}{c},\nu_1)\right)}{2}-\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx\right|$$
$$\leq\frac{\eta(\nu_2,c\nu_1)}{2c}\left(\frac{1}{2(p+1)}\right)^{1/p}\left[\left\{\frac{1}{6}|\psi'(\nu_1)|^q+\frac{1}{3}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}+\left\{\frac{1}{3}|\psi'(\nu_1)|^q+\frac{1}{6}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}\right].$$

Corollary 12. *If we choose $\eta(\nu_1,\nu_2)=\nu_1-\nu_2$ in (5.11), then we attain the following inequality*

$$\left|\frac{\psi(\nu_1)+\psi(\frac{\nu_2}{c})}{2}-\frac{c}{(\nu_2-c\nu_1)}\int_{\nu_1}^{\frac{\nu_2}{c}}\psi(x)dx\right|\leq\frac{(\nu_2-c\nu_1)}{2c}\left(\frac{1}{2(p+1)}\right)^{1/p}\left[\left\{\frac{3-2s}{6}|\psi'(\nu_1)|^q+\frac{3-s}{6}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}\right.$$
$$\left.+\left\{\frac{3-s}{6}|\psi'(\nu_1)|^q+\frac{3-2s}{6}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}\right].$$

Corollary 13. *If we choose $s=1$ and $\eta(\nu_1,\nu_2)=\nu_1-\nu_2$ in (5.11), then we attain the following inequality*

$$\left|\frac{\psi(\nu_1)+\psi(\frac{\nu_2}{c})}{2}-\frac{c}{(\nu_2-c\nu_1)}\int_{\nu_1}^{\frac{\nu_2}{c}}\psi(x)dx\right|$$
$$\leq\frac{(\nu_2-c\nu_1)}{2c}\left(\frac{1}{2(p+1)}\right)^{1/p}\left[\left\{\frac{1}{6}|\psi'(\nu_1)|^q+\frac{1}{3}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}+\left\{\frac{1}{3}|\psi'(\nu_1)|^q+\frac{1}{6}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}\right].$$

Theorem 5.7. *Let $\mathbb{X}^\circ\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:\mathbb{X}^\circ\times\mathbb{X}^\circ\to\mathbb{R}$ and $\nu_1,\nu_2\in\mathbb{X}^\circ$ with $\nu_2+\eta(\nu_1,\nu_2)\leq\nu_2$. Suppose $\psi:[\nu_2+\eta(\nu_1,\nu_2),\nu_2]$ be a differentiable mapping on I°. If $|\psi'|^q$ is s-type preinvex on $(\nu_1,\nu_1+\eta(\nu_2,\nu_1))$ for $q\geq 1$ and $s\in[0,1]$, then the following inequality holds:*

$$\left|\frac{\psi(\nu_1)+\psi\left(\nu_1+\eta(\frac{\nu_2}{c},\nu_1)\right)}{2}-\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx\right|$$
$$\leq\frac{\eta(\nu_2,c\nu_1)}{2c}\left(\frac{1}{4}\right)^{1-1/q}\left[\left\{\frac{4-3s}{16}|\psi'(\nu_1)|^q+\frac{4-s}{16}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}+\left\{\frac{4-s}{16}|\psi'(\nu_1)|^q+\frac{4-3s}{16}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}\right].\tag{5.14}$$

Proof. Using Lemma 5.1 and Improved power-mean inequality, one has

$$\left|\frac{\psi(\nu_1)+\psi\left(\nu_1+\eta(\frac{\nu_2}{c},\nu_1)\right)}{2}-\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx\right|$$

$$\leq\frac{\eta(\nu_2,c\nu_1)}{2c}\left[\left(\int_0^1(1-\kappa)|1-2\kappa|d\kappa\right)^{1-1/q}\left(\int_0^1(1-\kappa)|1-2\kappa||\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)|^q d\kappa\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1\kappa|1-2\kappa|d\kappa\right)^{1-1/q}\left(\int_0^1\kappa|1-2\kappa||\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)|^q d\kappa\right)^{1/q}\right]. \tag{5.15}$$

Since, $|\psi'|^q$ is s-type preinvex on $(\nu_1,\nu_1+\eta(\nu_2,\nu_1))$, we have

$$\int_0^1|\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)|^q d\kappa=|\psi'(\nu_1)|^q\int_0^1(1-s(1-\kappa))d\kappa+|\psi'\left(\frac{\nu_2}{c}\right)|^q\int_0^1(1-s\kappa)d\kappa.$$

Now, Eq (5.15) becomes

$$\left|\frac{\psi(\nu_1)+\psi\left(\nu_1+\eta(\frac{\nu_2}{c},\nu_1)\right)}{2}-\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx\right|$$
$$\leq\frac{\eta(\nu_2,c\nu_1)}{2c}\left[\left(\int_0^1(1-\kappa)|1-2\kappa|d\kappa\right)^{1-1/q}\left(|\psi'(\nu_1)|^q\int_0^1(1-\kappa)|1-2\kappa|(1-s(1-\kappa))d\kappa\right.\right. \tag{5.16}$$
$$\left.+|\psi'\left(\frac{\nu_2}{c}\right)|^q\int_0^1(1-\kappa)|1-2\kappa|(1-s\kappa)d\kappa\right)^{1/q}$$
$$\left.+\left(\int_0^1\kappa|1-2\kappa|d\kappa\right)^{1-1/q}\left(|\psi'(\nu_1)|^q\int_0^1\kappa|1-2\kappa|(1-s(1-\kappa))d\kappa+|\psi'\left(\frac{\nu_2}{c}\right)|^q\int_0^1\kappa|1-2\kappa|(1-s\kappa)d\kappa\right)^{1/q}\right]. \tag{5.17}$$

Since,

$$\int_0^1(1-\kappa)|1-2\kappa|\,(1-s\,(1-\kappa))\,d\kappa=\int_0^1\kappa|1-2\kappa|\,(1-s\kappa)\,d\kappa=-\frac{3s-4}{16},$$
$$\int_0^1\kappa|1-2\kappa|\,(1-s\,(1-\kappa))\,d\kappa=\int_0^1(1-\kappa)|1-2\kappa|\,(1-s\kappa)\,d\kappa=-\frac{s-4}{16},$$
$$\int_0^1\kappa|1-2\kappa|d\kappa=\int_0^1(1-\kappa)|1-2\kappa|d\kappa=\frac{1}{4}.$$

The proof of the Theorem is completed by using these computations in (5.17). □

Corollary 14. *If we choose* $s=1$ *in* (5.14)*, then we attain the following inequality*

$$\left|\frac{\psi(\nu_1)+\psi\left(\nu_1+\eta(\frac{\nu_2}{c},\nu_1)\right)}{2}-\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx\right|$$
$$\leq\frac{\eta(\nu_2,c\nu_1)}{2c}\left(\frac{1}{4}\right)^{1-1/q}\left[\left\{\frac{1}{16}|\psi'(\nu_1)|^q+\frac{3}{16}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}+\left\{\frac{3}{16}|\psi'(\nu_1)|^q+\frac{1}{16}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}\right].$$

Corollary 15. *If we choose* $\eta(\nu_1,\nu_2)=\nu_1-\nu_2$*, in* (5.14)*, then we attain the following inequality*

$$\left|\frac{\psi(\nu_1)+\psi(\frac{\nu_2}{c})}{2}-\frac{c}{(\nu_2-c\nu_1)}\int_{\nu_1}^{\frac{\nu_2}{c}}\psi(x)dx\right|$$
$$\leq\frac{(\nu_2-c\nu_1)}{2c}\left[\frac{1}{4}\right]^{1-1/q}\left[\left\{\frac{4-3s}{16}|\psi'(\nu_1)|^q+\frac{4-s}{16}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}+\left\{\frac{4-s}{16}|\psi'(\nu_1)|^q+\frac{4-3s}{16}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}\right].$$

Corollary 16. *If we choose $s = 1$ and $\eta(\nu_1,\nu_2) = \nu_1 - \nu_2$, in* (5.14)*, then we attain the following inequality*

$$\left|\frac{\psi(\nu_1)+\psi(\frac{\nu_2}{c})}{2}-\frac{c}{(\nu_2-c\nu_1)}\int_{\nu_1}^{\frac{\nu_2}{c}}\psi(x)dx\right|$$
$$\leq\frac{(\nu_2-c\nu_1)}{2c}\left(\frac{1}{4}\right)^{1-1/q}\left[\left\{\frac{1}{16}|\psi'(\nu_1)|^q+\frac{3}{16}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}+\left\{\frac{3}{16}|\psi'(\nu_1)|^q+\frac{1}{16}|\psi'\left(\frac{\nu_2}{c}\right)|^q\right\}^{1/q}\right].$$

Theorem 5.8. *Let $\mathbb{X}^\circ \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \mathbb{X}^\circ \times \mathbb{X}^\circ \to \mathbb{R}$ and $\nu_1,\nu_2 \in \mathbb{X}^\circ$ with $\nu_2 + \eta(\nu_1,\nu_2) \leq \nu_2$. Suppose $\psi : [\nu_2 + \eta(\nu_1,\nu_2), \nu_2]$ be a differentiable mapping on $\mathbb{X}^\circ$. If $|\psi'|^q$ is s-type preinvex on $(\nu_1, \nu_1 + \eta(\nu_2,\nu_1))$ for $q \geq 1$ and $s \in [0,1]$, then the following inequality holds:*

$$\left|\frac{\psi(\nu_1)+\psi\left(\nu_1+\eta(\frac{\nu_2}{c},\nu_1)\right)}{2}-\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx\right|$$
$$\leq\frac{\eta(\nu_2,c\nu_1)}{2c}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\frac{2-s}{4}\left[|\psi'(\nu_1)|^q+|\psi'\left(\frac{\nu_2}{c}\right)|^q\right]\right\}^{1/q}. \tag{5.18}$$

Proof. Using Lemma 5.1 and power-mean inequality, one has

$$\left|\frac{\psi(\nu_1)+\psi\left(\nu_1+\eta(\frac{\nu_2}{c},\nu_1)\right)}{2}-\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx\right|$$
$$\leq\frac{\eta(\nu_2,c\nu_1)}{2c}\int_0^1|1-2\kappa||\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)|d\kappa$$
$$\leq\frac{\eta(\nu_2,c\nu_1)}{2c}\left(\int_0^1|1-2\kappa|d\kappa\right)^{1-1/q}\left(\int_0^1|1-2\kappa||\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)|^qd\kappa\right)^{1/q}. \tag{5.19}$$

Since $|\psi'|^q$ is s-type preinvex on $(\nu_1, \nu_1 + \eta(\nu_2,\nu_1))$, we have

$$\int_0^1|\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)|^qd\kappa=|\psi'(\nu_1)|^q\int_0^1(1-s(1-\kappa))d\kappa+|\psi'\left(\frac{\nu_2}{c}\right)|^q\int_0^1(1-s\kappa)d\kappa.$$

Now, Eq (5.19) becomes

$$\left|\frac{\psi(\nu_1)+\psi\left(\nu_1+\eta(\frac{\nu_2}{c},\nu_1)\right)}{2}-\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx\right|$$
$$\leq\frac{\eta(\nu_2,c\nu_1)}{2c}\left(\int_0^1|1-2\kappa|d\kappa\right)^{1-1/q}\left(|\psi'(\nu_1)|^q\int_0^1|1-2\kappa|(1-s(1-\kappa))d\kappa\right.$$

$$+|\psi'\left(\frac{\nu_2}{c}\right)|^q \int_0^1 |1-2\kappa|(1-s\kappa)d\kappa\right)^{1/q}. \tag{5.20}$$

Since,

$$\int_0^1 |1-2\kappa|\,(1-s\,(1-\kappa))\,d\kappa = \int_0^1 |1-2\kappa|\,(1-s\kappa)\,d\kappa = -\frac{s-2}{4},$$
$$\int_0^1 |1-2\kappa| d\kappa = \frac{1}{2}.$$

The proof of the Theorem gets completed by using these computations in (5.20). □

Corollary 17. *If we choose* $s = 1$ *in* (5.18)*, then we attain the following inequality*

$$\left|\frac{\psi(\nu_1)+\psi\left(\nu_1+\eta(\frac{\nu_2}{c},\nu_1)\right)}{2} - \frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)} \psi(x)dx\right| \le \frac{\eta(\nu_2,c\nu_1)}{2^{\frac{2q+1}{q}}c}\left\{\left[|\psi'(\nu_1)|^q + |\psi'\left(\frac{\nu_2}{c}\right)|^q\right]\right\}^{1/q}.$$

Corollary 18. *If we choose* $\eta(\nu_1,\nu_2) = \nu_1-\nu_2$ *in* (5.18)*, then we attain the following inequality*

$$\left|\frac{\psi(\nu_1)+\psi(\frac{\nu_2}{c})}{2} - \frac{c}{(\nu_2-c\nu_1)}\int_{\nu_1}^{\frac{\nu_2}{c}} \psi(x)dx\right| \le \frac{(\nu_2-c\nu_1)}{2c}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\frac{2-s}{4}\left[|\psi'(\nu_1)|^q + |\psi'\left(\frac{\nu_2}{c}\right)|^q\right]\right\}^{1/q}.$$

Corollary 19. *If we choose* $s = 1$ *and* $\eta(\nu_1,\nu_2) = \nu_1-\nu_2$ *in* (5.18)*, then we attain the following inequality*

$$\left|\frac{\psi(\nu_1)+\psi(\frac{\nu_2}{c})}{2} - \frac{c}{(\nu_2-c\nu_1)}\int_{\nu_1}^{\frac{\nu_2}{c}} \psi(x)dx\right| \le \frac{(\nu_2-c\nu_1)}{2^{\frac{2q+1}{q}}c}\left\{\left[|\psi'(\nu_1)|^q + |\psi'\left(\frac{\nu_2}{c}\right)|^q\right]\right\}^{1/q}.$$

Theorem 5.9. *Let* $\mathbb{X}^\circ \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\eta : \mathbb{X}^\circ \times \mathbb{X}^\circ \to \mathbb{R}$ *and* $\nu_1,\nu_2 \in \mathbb{X}^\circ$ *with* $\nu_2+\eta(\nu_1,\nu_2) \le \nu_2$*. Suppose* $\psi : [\nu_2+\eta(\nu_1,\nu_2),\nu_2]$ *be a differentiable mapping on* $\mathbb{X}^\circ$*. If* $|\psi'|^q$ *is s-type preinvex on* $(\nu_1,\nu_1+\eta(\nu_2,\nu_1))$ *for* $p,q>1$, $\frac{1}{q}+\frac{1}{p}=1$ *and* $s\in[0,1]$*, then the following inequality holds:*

$$\left|\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)} \psi(x)dx - \psi\left(\frac{2\nu_1+\eta(\nu_2,\nu_1)}{2c}\right)\right|$$
$$\le \frac{\eta(\nu_2,c\nu_1)}{c}\left[\left(\frac{1}{p+1}\right)^{1/p}\left\{\frac{2-s}{2}[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q]\right\}^{1/q} + \left\{\frac{4-3s}{8}[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q]\right\}^{1/q}\right]. \tag{5.21}$$

Proof. Using Lemma 5.2 and Hölder's inequality, one has

$$\left|\frac{c}{\eta(\nu_2,c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)} \psi(x)dx - \psi\left(\frac{2\nu_1+\eta(\nu_2,\nu_1)}{2c}\right)\right|$$
$$=\frac{\eta(\nu_2,c\nu_1)}{c}\left\{\int_0^1 \kappa\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)d\kappa - \int_{1/2}^1 \psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)d\kappa\right\}$$

$$\leq \frac{\eta(\nu_2, c\nu_1)}{c}\left[\left(\int_0^1 \kappa^p d\kappa\right)^{1/p}\left\{\int_0^1 |\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)|^q d\kappa\right\}^{1/q}+\left\{\int_{1/2}^1 |\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)|^q d\kappa\right\}^{1/q}\right]$$

$$\leq \frac{\eta(\nu_2, c\nu_1)}{c}\left[\left(\frac{1}{p+1}\right)^{1/p}\left\{\int_0^1 [1-s(1-\kappa)]|\psi'(\nu_1)|^q d\kappa+\int_0^1 [1-s\kappa]|\psi'\left(\frac{\nu_2}{c}\right)|^q d\kappa\right\}^{1/q}\right.$$

$$\left.+\left\{\int_{1/2}^1 [1-s(1-\kappa)]|\psi'(\nu_1)|^q d\kappa+\int_{1/2}^1 [1-s\kappa]|\psi'\left(\frac{\nu_2}{c}\right)|^q d\kappa\right\}^{1/q}\right]$$

$$=\frac{\eta(\nu_2, c\nu_1)}{c}\left[\left(\frac{1}{p+1}\right)^{1/p}\left\{\frac{2-s}{2}[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q]\right\}^{1/q}+\left\{\frac{4-3s}{8}[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q]\right\}^{1/q}\right].$$

□

Corollary 20. *If we choose $s = 1$ in* (5.21)*, then we attain the following inequality*

$$\left|\frac{c}{\eta(\nu_2, c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx-\psi\left(\frac{2\nu_1+\eta(\nu_2,\nu_1)}{2c}\right)\right|$$
$$\leq \frac{\eta(\nu_2, c\nu_1)}{c}\left[\left(\frac{1}{p+1}\right)^{1/p}\left\{\frac{1}{2}\left[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q\right]\right\}^{1/q}+\left\{\frac{1}{8}\left[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q\right]\right\}^{1/q}\right].$$

Corollary 21. *If we choose $\eta(\nu_1, \nu_2) = \nu_1 - \nu_2$ in* (5.21)*, then we attain the following inequality*

$$\left|\frac{c}{(\nu_2 - c\nu_1)}\int_{\nu_1}^{\frac{\nu_2}{c}}\psi(x)dx-\psi\left(\frac{\nu_1+\nu_2)}{2c}\right)\right|$$
$$\leq \frac{\eta(\nu_2, c\nu_1)}{c}\left[\left(\frac{1}{p+1}\right)^{1/p}\left\{\frac{2-s}{2}\left[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q\right]\right\}^{1/q}+\left\{\frac{4-3s}{8}\left[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q\right]\right\}^{1/q}\right].$$

Corollary 22. *If we choose $s = 1$ and $\eta(\nu_1, \nu_2) = \nu_1 - \nu_2$ in* (5.21)*, then we attain the following inequality*

$$\left|\frac{c}{(\nu_2 - c\nu_1)}\int_{\nu_1}^{\frac{\nu_2}{c}}\psi(x)dx-\psi\left(\frac{\nu_1+\nu_2)}{2c}\right)\right|$$
$$\leq \frac{(\nu_2 - c\nu_1)}{c}\left[\left(\frac{1}{p+1}\right)^{1/p}\left\{\frac{1}{2}\left[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q\right]\right\}^{1/q}+\left\{\frac{1}{8}\left[|\psi'(\nu_1)|q+|\psi'(\frac{\nu_2}{c})|^q\right]\right\}^{1/q}\right].$$

Theorem 5.10. *Let $\mathbb{X}^\circ \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \mathbb{X}^\circ \times \mathbb{X}^\circ \to \mathbb{R}$ and $\nu_1, \nu_2 \in \mathbb{X}^\circ$ with $\nu_2 + \eta(\nu_1, \nu_2) \leq \nu_2$. Suppose $\psi : [\nu_2 + \eta(\nu_1, \nu_2), \nu_2]$ be a differentiable mapping on $\mathbb{X}^\circ$. If $|\psi'|^q$ is s-type preinvex on $(\nu_1, \nu_1 + \eta(\nu_2, \nu_1))$ for $q \geq 1$, and $s \in [0, 1]$, then the following inequality holds:*

$$\left|\frac{c}{\eta(\nu_2, c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx-\psi\left(\frac{2\nu_1+\eta(\nu_2,\nu_1)}{2c}\right)\right|$$
$$\leq \frac{\eta(\nu_2, c\nu_1)}{c}\left[\left(\frac{1}{2}\right)^{1-1/q}\left\{\frac{3-s}{6}|\psi'(\nu_1)|^q+\frac{3-2s}{6}|\psi'(\frac{\nu_2}{c})|^q\right\}^{1/q}+\left\{\frac{4-3s}{8}\left[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q\right]\right\}^{1/q}\right]. \tag{5.22}$$

Proof. Using Lemma 5.2 and power-mean inequality, one has

$$\begin{aligned}
&\left|\frac{c}{\eta(\nu_2, c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx-\psi\left(\frac{2\nu_1+\eta(\nu_2,\nu_1)}{2c}\right)\right|\\
=&\frac{\eta(\nu_2, c\nu_1)}{c}\left\{\int_0^1\kappa\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)d\kappa-\int_{1/2}^1\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)d\kappa\right\}\\
\leq&\frac{\eta(\nu_2, c\nu_1)}{c}\left[\left(\int_0^1\kappa d\kappa\right)^{1-1/q}\left\{\int_0^1\kappa|\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)|^q d\kappa\right\}^{1/q}+\left\{\int_{1/2}^1|\psi'\left(\frac{\nu_2}{c}+\kappa\eta(\nu_1,\frac{\nu_2}{c})\right)|^q d\kappa\right\}^{1/q}\right]\\
\leq&\frac{\eta(\nu_2, c\nu_1)}{c}\left[\left(\frac{1}{2}\right)^{1-1/q}\left\{\int_0^1\kappa[1-s(1-\kappa)]|\psi'(\nu_1)|^q d\kappa+\int_0^1\kappa[1-s\kappa]|\psi'\left(\frac{\nu_2}{c}\right)|^q d\kappa\right\}^{1/q}\right.\\
&\left.+\left\{\int_{1/2}^1[1-s(1-\kappa)]|\psi'(\nu_1)|^q d\kappa+\int_{1/2}^1[1-s\kappa]|\psi'\left(\frac{\nu_2}{c}\right)|^q d\kappa\right\}^{1/q}\right]\\
=&\frac{\eta(\nu_2, c\nu_1)}{c}\left[\left(\frac{1}{2}\right)^{1-1/q}\left\{\frac{3-s}{6}|\psi'(\nu_1)|^q+\frac{3-2s}{6}|\psi'(\frac{\nu_2}{c})|^q\right\}^{1/q}+\left\{\frac{4-3s}{8}\left[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q\right]\right\}^{1/q}\right].
\end{aligned}$$

□

Corollary 23. *If we choose $s = 1$ in* (5.22)*, then we attain the following inequality*

$$\begin{aligned}
&\left|\frac{c}{\eta(\nu_2, c\nu_1)}\int_{\nu_1}^{\nu_1+\eta(\frac{\nu_2}{c},\nu_1)}\psi(x)dx-\psi\left(\frac{2\nu_1+\eta(\nu_2,\nu_1)}{2c}\right)\right|\\
&\qquad\leq\frac{\eta(\nu_2, c\nu_1)}{c}\left[\left(\frac{1}{2}\right)^{1-1/q}\left\{\frac{1}{3}|\psi'(\nu_1)|^q+\frac{1}{6}|\psi'(\frac{\nu_2}{c})|^q\right\}^{1/q}+\left\{\frac{1}{8}\left[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q\right]\right\}^{1/q}\right].
\end{aligned}$$

Corollary 24. *If we choose $\eta(\nu_1,\nu_2)=\nu_1-\nu_2$ in* (5.22)*, then we attain the following inequality*

$$\begin{aligned}
&\left|\frac{c}{(\nu_2-c\nu_1)}\int_{\nu_1}^{\frac{\nu_2}{c}}\psi(x)dx-\psi\left(\frac{\nu_1+\nu_2)}{2c}\right)\right|\\
&\leq\frac{(\nu_2-c\nu_1)}{c}\left[\left(\frac{1}{2}\right)^{1-1/q}\left\{\frac{3-s}{6}|\psi'(\nu_1)|^q+\frac{3-2s}{6}|\psi'(\frac{\nu_2}{c})|^q\right\}^{1/q}+\left\{\frac{4-3s}{8}\left[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q\right]\right\}^{1/q}\right].
\end{aligned}$$

Corollary 25. *If we choose $s = 1$ and $\eta(\nu_1,\nu_2)=\nu_1-\nu_2$ in* (5.22)*, then we attain the following inequality*

$$\begin{aligned}
&\left|\frac{c}{(\nu_2-c\nu_1)}\int_{\nu_1}^{\frac{\nu_2}{c}}\psi(x)dx-\psi\left(\frac{\nu_1+\nu_2)}{2c}\right)\right|\\
&\qquad\leq\frac{(\nu_2-c\nu_1)}{c}\left[\left(\frac{1}{2}\right)^{1-1/q}\left\{\frac{1}{3}|\psi'(\nu_1)|^q+\frac{1}{6}|\psi'(\frac{\nu_2}{c})|^q\right\}^{1/q}+\left\{\frac{1}{8}\left[|\psi'(\nu_1)|^q+|\psi'(\frac{\nu_2}{c})|^q\right]\right\}^{1/q}\right].
\end{aligned}$$

6. Applications

In this section, we recall the following special means of two positive number ν_1, ν_2 with $\nu_1 < \nu_2$:
(1) The arithmetic mean

$$A = A(\nu_1, \nu_2) = \frac{\nu_1 + \nu_2}{2}.$$

(2) The geometric mean

$$G = G(\nu_1, \nu_2) = \sqrt{\nu_1 \nu_2}.$$

(3) The harmonic mean

$$H = H(\nu_1, \nu_2) = \frac{2\nu_1 \nu_2}{\nu_1 + \nu_2}.$$

The following relationship are well-known in the literature.

$$H(\nu_1, \nu_2) \leq G(\nu_1, \nu_2) \leq A(\nu_1, \nu_2).$$

Proposition 3. *Let* $0 < \nu_1 < \nu_2$ *and* $s \in [0, 1]$. *then*

$$\frac{1}{2-s}\left(\nu_2 + \frac{1}{2}\eta(\nu_1, \nu_2)\right) \leq \frac{\eta(\nu_1, \nu_2) + 2\nu_2)}{2} \leq A(\nu_1, \nu_2)(2-s). \tag{6.1}$$

Proof. We attained the above inequality (6.1), if we put $\psi(\nu) = \nu$ for $\nu > 0$ in Theorem 4.1. □

Proposition 4. *Let* $\nu_1, \nu_2 \in (0, 1]$ *with* $\nu_1 < \nu_2$ *and* $s \in [0, 1]$, *then*

$$\frac{1}{2-s}\ln G(\nu_1, \nu_2) \leq \frac{1}{\nu_2(\nu_2 + \eta(\nu_1, \nu_2))} \leq \ln\left(\nu_2 + \frac{1}{2}\eta(\nu_1, \nu_2)\right)\frac{2-s}{2}. \tag{6.2}$$

Proof. We attained the above inequality (6.2), if we put $\psi(\nu) = -\ln \nu$ for $\nu \in (0, 1]$ in Theorem 4.1. □

Proposition 5. *Let* $\nu_1, \nu_2 \in (0, \infty)$ *with* $\nu_1 < \nu_2$ *and* $s \in [0, 1]$, *then*

$$\frac{1}{(2-s)\left[1 - \left(\frac{s}{2}\right)^j\right]\left(\nu_2 + \frac{1}{2}\eta(\nu_1, \nu_2)\right)} \leq \frac{\ln\left(1 + \frac{\eta(\nu_1, \nu_2)}{\nu_2}\right)}{\eta(\nu_1, \nu_2)} \leq \frac{2-s}{H(\nu_1, \nu_2)}. \tag{6.3}$$

Proof. We attained the above inequality (6.3), if we put $\psi(\nu) = \frac{1}{\nu}$ for $\nu > 0$ in Theorem 4.1. □

7. Conclusions

In this article, we addressed a novel idea of generalized preinvex function namely s-type preinvex function. Some algebraic properties were examined of the proposed definition. In the manner of the newly proposed definition, we described some novel versions of Hermite-Hadamard type inequality. Further, we established two new identities. Our presented results employing the new identities can be considered as refinements and remarkable extensions to the new family of preinvex functions. Our novel results can be deduced from the previously known results. Applications to special means were considered. The above estimations on the mentioned lemmas need an interesting and amazing comparison. Using Lemma 5.1, we examined three Theorems 5.4–5.6, in which we used the Hölder and Hölder-İşcan inequality. On the comparison, the Theorem 5.6 gives the better result as compared to the other Theorems 5.4 and 5.5. Similarly, employing Lemma 5.1, we examined two Theorems 5.7 and 5.8, in which we used power mean and improved power mean inequality. On the comparison, the Theorem 5.7 gives the better result as compared to the other Theorem 5.8. We hope the consequences and techniques of this article will energize and inspire the researchers to explore a more interesting sequel in this area.

Conflicts of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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