**Mathematics**

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(12): 13608–13621.
DOI:10.3934/math.2021791
Received: 30 June 2021
Accepted: 18 September 2021
Published: 24 September 2021

*Research article*

# Bohr-type inequalities for bounded analytic functions of Schwarz functions

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**Abstract:** In this paper, some new versions of Bohr-type inequalities for bounded analytic functions of Schwarz functions are established. Most of these inequalities are sharp. Some previous inequalities are generalized.

**Keywords:** Bohr radius; Bohr-Rogosinski radius; Bohr-type inequality; Bounded analytic functions; Schwarz functions
**Mathematics Subject Classification:** 30A10, 30C45

## 1. Introduction

Let $\mathcal{B}$ denote the class of analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ defined in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ such that $|f(z)| < 1$ for $z \in \mathbb{D}$.

We call

$$B^f(z) := \sum_{k=0}^{\infty} |a_k||z|^k$$

the Bohr sum of $f(z)$. The well known Bohr radius problem is to find $r_0$, such that

$$B^f(z) \le 1 \tag{1.1}$$

holds for $|z| \le r_0$. The constant $r_0 = 1/3$ is sharp, which is called the Bohr radius. The radius was originally obtained in 1914 by Bohr [13] with $1/6$. Later, Wiener, Riesz and Schur established the inequality (1.1) for $r = |z| \le 1/3$ and showed that the constant $1/3$ cannot be improved [22, 25, 26].

There are lots of works about the classical Bohr inequality and its generalized forms. Ali et al., [8] and Kayumov and Ponnusamy [15] considered the problem of Bohr radius for the classes of even and odd analytic functions and for alternating series, respectively. In [19], the authors generalized and improved several Bohr inequalities. In [21], several Bohr-type inequalities were obtained when the Taylor coefficients of classical Bohr inequality are partly replaced by higher order derivatives of $f$.

It is worth pointing out that Bohr's radius problem deal with analytic functions from unit disk $\mathbb{D}$ into $\mathbb{D}$ initially, but later it was generalized to mappings from $\mathbb{D}$ to punctured disk [4] or other domains [2]. For more discussion on the Bohr radius for analytic functions [3, 7, 9, 27].

Recently, Bohr's inequality has created enormous interest in various setting. For example, Bohr's idea is extended to functions of several complex variables and multi-dimensions [5, 6, 11, 12]. In addition, the authors study the Bohr radius for harmonic mappings [1, 10, 17, 20].

The Bohr-Rogosinski sum $R_N^f(z)$ of $f \in \mathcal{B}$ is defined by

$$R_N^f(z) := |f(z)| + \sum_{k=N}^{\infty} |a_k| r^k, \quad |z| = r.$$

Observe that if $N = 1$ and $f(z)$ is replaced by $f(0)$, then the Bohr-Rogosinski sum is the Bohr sum. The corresponding Bohr-Rogosinski radius problem is to find $R_N$, such that

$$R_N^f(z) \leq 1$$

holds for $|z| \leq R_N$. Recently, Kayumov and Ponnusamy [16] have given the Bohr-Rogosinski radius of $f$. In [21], the author also solved some problem of the Bohr-Rogosinski radius.

Let $S_N(z) = \sum_{k=0}^{N-1} a_k z^k$ denotes the partial sums of $f$. The corresponding Rogosinski radius is $|z| < 1/2$ for $|S_N(z)| < 1$ [18, 23, 24]. It is obvious that

$$|S_N(z)| = \left| f(z) - \sum_{k=N}^{\infty} a_k z^k \right| \leq R_N^f(z).$$

Hence, the Rogosinski radius is related to the Bohr-Rogosinski radius.

Let $\mathcal{B}_m = \{\omega \in \mathcal{B} : \omega(0) = \cdots = \omega^{(m-1)}(0) = 0, \omega^{(m)}(0) \neq 0\}$ be the classes of Schwarz functions, where $m \in \mathbb{N} = \{1, 2, \cdots\}$. Our aim of this article is to generalize or improve many versions of Bohr-type inequalities for bounded analytic functions of Schwarz functions.

The paper is organized as follows. In Section 2, we state some lemmas. In Section 3, we present many theorems which improve several versions of Bohr-Rogosinski inequalities and Bohr's type inequalities for bounded analytic functions. There are some corollaries and an open problem in Section 4.

## 2. Some Lemmas

In order to establish our main results, we need the following some lemmas which will play the key role in proving the main results of this paper.

**Lemma 2.1.** *(Schwarz-Pick lemma) Let $\phi(z)$ be analytic in the unit disk $\mathbb{D}$ and $|\phi(z)| < 1$. Then*

$$\frac{|\phi(z_1) - \phi(z_2)|}{|1 - \overline{\phi(z_1)}\phi(z_2)|} \leq \frac{|z_1 - z_2|}{|1 - \overline{z_1} z_2|} \quad for \quad z_1, z_2 \in \mathbb{D},$$

*and equality holds for distinct $z_1, z_2 \in \mathbb{D}$ if and only if $\phi$ is a Möbius transformation. In particular,*

$$|\phi'(z)| \leq \frac{1 - |\phi(z)^2|}{1 - |z^2|} \quad for \quad z \in \mathbb{D},$$

*and equality holds for some $z \in \mathbb{D}$ if and only if $\phi$ is a Möbius transformation.*

**Lemma 2.2.** ([14]) *Suppose $f(z)$ is analytic in the unit disk $\mathbb{D}$ and $|f(z)| \le 1$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $|a_n| \le 1 - |a_0|^2$ for all $n \in \mathbb{N}$.*

**Lemma 2.3.** *For $0 \le x \le x_0 \le 1$, it holds that*

$$\Phi(x) := x + A(1 - x^2) \le \Phi(x_0) \quad \textit{whenever} \quad 0 \le A \le 1/2.$$

The proof is simple, we omit it.

**Lemma 2.4.** *There is a unique root $\xi_m$ of the equation*

$$r^{2m} + 2r^m - 1 = 0, \tag{2.1}$$

*and a unique root $\alpha_{m,n}$ of the equation*

$$r^m(r^m + 2)(2r^{2n} - r^n + 1) + 2r^{2n} + r^n - 1 = 0 \tag{2.2}$$

*for $r \in (0, 1)$ and $m, n \in \mathbb{N}$, respectively. Furthermore, $\alpha_{m,n} \le \xi_m$ for $m \ge n$.*

*Proof.* Firstly, it is obvious that there is a unique root of Eq (2.1) on $(0, 1)$.

Secondly, we show that $\alpha_{m,n}$ is the unique root of Eq (2.2). Let

$$k(r) = r^m(r^m + 2)(2r^{2n} - r^n + 1) + 2r^{2n} + r^n - 1.$$

Then we have

$$\begin{aligned} k'(r) =& [4(m + n)r^{2m+2n-1} + 4(2n + m)r^{2n+m-1} + 4nr^{2n-1} - 2nr^{m+n-1}] \\ &+ (2mr^{2m-1} - 2mr^{2m+n-1}) + (nr^{n-1} - nr^{2m+n-1}) \\ &+ (2mr^{m-1} - 2mr^{m+n-1}) > 0, \quad for \quad m \ge n. \end{aligned}$$

Observe that $k(0)k(1) < 0$. Thus the monotonicity of $k(r)$ implies that there is an $\alpha_{m,n}$ that is the unique root of (2.2).

Finally, we need to show that $\alpha_{m,n} \le \xi_m$. Assuming that $\alpha_{m,n} > \xi_m$, then $\xi_m^{2m} + 2\xi_m^m - 1 = 0$ implies that

$$\begin{aligned} k(\xi_m) =& \xi_m^m(\xi_m^m + 2)(2\xi_m^{2n} - \xi_m^n + 1) + 2\xi_m^{2n} + \xi_m^n - 1 \\ =& (\xi_m^{2m} + 2\xi_m^m)(2\xi_m^{2n} - \xi_m^n + 1) + 2\xi_m^{2n} + \xi_m^n - 1 \\ =& (2\xi_m^{2n} - \xi_m^n + 1) + 2\xi_m^{2n} + \xi_m^n - 1 \\ =& 4\xi_m^{2n}, \quad for \quad \xi_m \in (0, 1). \end{aligned}$$

Then, $k(\xi_m) > 0 = k(\alpha_{m,n})$. This contradicts the monotonicity of $k$. □

## 3. Main results

**Theorem 3.1.** *Suppose that* $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$, $a := |a_0|$ *and* $\omega_m \in \mathcal{B}_m, \omega_n \in \mathcal{B}_n$ *for* $m, n \in \mathbb{N}$. *Then we have*

$$|f(\omega_m(z))| + \sum_{k=N}^{\infty} |a_k||\omega_n(z)|^k \leq 1 \quad for \quad |z| = r \leq R_{m,n,N}, \tag{3.1}$$

*where* $R_{m,n,N}$ *is the unique root in* $(0, 1)$ *of the equation*

$$2r^{nN}(1 + r^m) - (1 - r^n)(1 - r^m) = 0, \tag{3.2}$$

*and the radius* $R_{m,n,N}$ *cannot be improved.*

**Remark 3.1.** *1). If* $m = 1, n = 1$, *and* $\omega_m(z) = \omega_n(z) = z$, *then Theorem* 3.1 *reduces to the main result of Theorem* 1 *of* [16].
*2). If* $m \to \infty$ *in* (3.2), then $\lim_{m\to\infty} R_{m,n,N} = A_{n,N}$, where $A_{n,N}$ is the positive root of the equation $2r^{nN} = 1 - r^n$. Also, $A_{1,1} = 1/3$. It is the well-known classical Bohr radius.

*Proof.* Since $f \in \mathcal{B}$, $a := |a_0|$ and $\omega_m \in \mathcal{B}_m, \omega_n \in \mathcal{B}_n$, by the Schwarz lemma and the Schwarz-Pick lemma, respectively, we obtain

$$|\omega_m(z)| \leq |z|^m, \quad |\omega_n(z)| \leq |z|^n, \quad and \quad |f(z)| \leq \frac{|z| + a}{1 + a|z|}$$

for $z \in \mathbb{D}$. It follows that

$$|f(\omega_m(z))| \leq \frac{|\omega_m(z)| + a}{1 + a|\omega_m(z)|} \leq \frac{r^m + a}{1 + ar^m}, \quad |z| = r < 1. \tag{3.3}$$

By using inequality (3.3) and Lemma 2.2, we have

$$|f(\omega_m(z))| + \sum_{k=N}^{\infty} |a_k||\omega_n(z)|^k \leq \frac{r^m + a}{1 + ar^m} + (1 - a^2)\frac{r^{nN}}{1 - r^n} := u_{m,n,N}(r).$$

Now, we need to show that $u_{m,n,N}(r) \leq 1$ holds for $r \leq R_{m,n,N}$. It is equivalent to showing $v_{m,n,N}(r) \leq 0$, where

$$\begin{aligned}
v_{m,n,N}(r) =& [u_{m,n,N}(r) - 1](1 + ar^m)(1 - r^n) \\
=& (r^m + a)(1 - r^n) + (1 - a^2)r^{nN}(1 + ar^m) - (1 + ar^m)(1 - r^n) \\
=& (1 - a)(r^n - r^{m+n} + r^{nN} + r^m + ar^{nN} + ar^{m+nN} + a^2 r^{m+nN} - 1) \\
\leq& (1 - a)(r^n - r^{m+n} + r^{nN} + r^m + r^{nN} + r^{m+nN} + r^{m+nN} - 1) \\
=& (1 - a)[2r^{nN}(1 + r^m) - (1 - r^n)(1 - r^m)].
\end{aligned}$$

Obviously, it is enough to show that $2r^{nN}(1 + r^m) - (1 - r^n)(1 - r^m) \leq 0$ holds for $r \leq R_{m,n,N}$. Let $g(r) = 2r^{nN}(1 + r^m) - (1 - r^n)(1 - r^m)$. Then it is easy to verify that $g(0)g(1) < 0$ and $g(r)$ is a continuous and increasing function of $r \in [0, 1]$. Thus $R_{m,n,N}$ is the unique root of $g(r)$ and $g(r) \leq 0$ holds for $r \leq R_{m,n,N}$.

Next we show the radius $R_{m,n,N}$ is sharp. For $a \in [0, 1)$, let

$$\omega_m(z) = z^m, \quad \omega_n(z) = z^n \quad \textit{and} \quad f(z) = \frac{a+z}{1+az} = a + (1-a^2)\sum_{k=1}^{\infty}(-a)^{k-1}z^k, \quad z \in \mathbb{D}. \tag{3.4}$$

Taking $z = r$, substituting (3.4) into the left side of inequality (3.1), then we have

$$|f(r^m)| + \sum_{k=N}^{\infty}|a_k|r^{nk} = \frac{r^m + a}{1 + ar^m} + \sum_{k=N}^{\infty}(1-a^2)a^{k-1}r^{nk} = \frac{r^m + a}{1 + ar^m} + (1-a^2)\frac{a^{N-1}r^{nN}}{1 - ar^n}. \tag{3.5}$$

Now we just need to show that if $r > R_{m,n,N}$, then there exists an $a$, such that the right side of (3.5) is greater than 1. That is

$$\frac{r^m + a}{1 + ar^m} + (1-a^2)\frac{a^{N-1}r^{nN}}{1 - ar^n} > 1.$$

Namely, we need to prove that

$$(1-a)[r^{m+nN}a^{N+1} + (r^{m+nN} + r^{nN})a^N + r^{nN}a^{N-1} + (r^n - r^{m+n})a + r^m - 1] > 0. \tag{3.6}$$

Let

$$A_1(a, r) = r^{m+nN}a^{N+1} + (r^{m+nN} + r^{nN})a^N + r^{nN}a^{N-1} + (r^n - r^{m+n})a + r^m - 1.$$

Observe that $A_1(a, r)$ is a continuous and increasing function of $a \in [0, 1)$. It holds that $A_1(a, r) \leq A_1(1, r) = 2r^{nN}(1 + r^m) - (1 - r^n)(1 - r^m) = g(r)$ for $r \in (0, 1)$. Furthermore, by the monotonicity of $g(r)$, if $r > R_{m,n,N}$, then $A_1(1, r) > 0$. Hence, by the continuity of $A_1(a, r)$, if $r > R_{m,n,N}$, we have

$$\lim_{a\to 1^-} A_1(a, r) = A_1(1, r) > 0.$$

Therefore, if $r > R_{m,n,N}$, then there exists an $a$, such that inequality (3.6) holds. □

**Theorem 3.2.** *Suppose that* $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$, $a := |a_0|$ *and* $\omega_m \in \mathcal{B}_m, \omega_n \in \mathcal{B}_n$ *with* $m, n \in \mathbb{N}$ *and* $m \geq n$. *Then we have*

$$|f(\omega_m(z))| + |f'(\omega_m(z))||\omega_m(z)| + \sum_{k=2}^{\infty}|a_k||\omega_n(z)|^k \leq 1 \quad for \quad |z| = r \leq \alpha_{m,n}, \tag{3.7}$$

*where* $\alpha_{m,n}$ *is the unique root in* $(0, 1)$ *of the equation*

$$r^m(r^m + 2)(2r^{2n} - r^n + 1) + 2r^{2n} + r^n - 1 = 0.$$

*The radius* $\alpha_{m,n}$ *cannot be improved.*

**Remark 3.2.** *If* $m = 1, n = 1$, *and* $\omega_m(z) = \omega_n(z) = z$ *in Theorem 3.2, then it reduces to Theorem 2.1 of* [21].

*Proof.* By the hypothesis, inequality (3.3) still holds. Then by Schwarz-Pick lemma, Lemma 2.2 and Lemma 2.3, respectively, we obtain

$$|f(\omega_m(z))| + |f'(\omega_m(z))||\omega_m(z)| + \sum_{k=2}^{\infty}|a_k||\omega_n(z)|^k$$

$$
\begin{aligned}
&\leq |f(\omega_m(z))| + \frac{1-|f(\omega_m(z))|^2}{1-|\omega_m(z)^2|}|\omega_m(z)| + (1-a^2)\sum_{k=2}^{\infty}|\omega_n(z)|^k \\
&\leq |f(\omega_m(z))| + \frac{r^m}{1-r^{2m}}(1-|f(\omega_m(z))|^2) + (1-a^2)\sum_{k=2}^{\infty}|\omega_n(z)|^k \\
&\leq \frac{r^m+a}{1+ar^m} + \left[1-\left(\frac{r^m+a}{1+ar^m}\right)^2\right]\frac{r^m}{1-r^{2m}} + (1-a^2)\frac{r^{2n}}{1-r^n} \\
&= \frac{r^m+a}{1+ar^m} + \frac{(1-a^2)r^m}{(1+ar^m)^2} + (1-a^2)\frac{r^{2n}}{1-r^n} \\
&= 1 + \frac{(1-a)\Phi_{m,n}(a,r)}{(1+ar^m)^2(1-r^n)}, \quad for \quad r \leq \xi_m, \qquad (3.8)
\end{aligned}
$$

where

$$\Phi_{m,n}(a,r) = r^{2m+2n}a^3 + r^{m+2n}(r^m+2)a^2 + [r^{2m}(1-r^n) + (2r^m+1)r^{2n}]a + 2r^m(1-r^n) + r^{2n} + r^n - 1,$$

and $\xi_m$ is the unique root in $(0,1)$ of the equation $r^{2m} + 2r^m - 1 = 0$.

Let

$$\Phi_{m,n}(1,r) = r^m(r^m+2)(2r^{2n} - r^n + 1) + 2r^{2n} + r^n - 1.$$

It is obvious that $\Phi_{m,n}(1,r)$ is monotonically increasing function of $r \in [0,1)$. By the hypothesis or Lemma 2.4, we have $\Phi_{m,n}(1,\alpha_{m,n}) = 0$. Thus the monotonicity of $\Phi_{m,n}(1,r)$ implies that

$$\Phi_{m,n}(1,r) \leq 0 \quad for \quad r \leq \alpha_{m,n}.$$

Furthermore, observe that $\Phi_{m,n}(a,r)$ is a monotonically increasing function of $a \in [0,1]$ for each fixed $r \in [0,1)$. Thus

$$\Phi_{m,n}(a,r) \leq \Phi_{m,n}(1,r) \leq 0 \quad for \quad r \leq \alpha_{m,n}. \qquad (3.9)$$

Therefore, by inequalities (3.8) and (3.9), we obtain inequality (3.7).

To show that the radius $\alpha_{m,n}$ is best possible, we consider the functions $\omega_m(z)$, $\omega_n(z)$ and $f(z)$ as in (3.4). Taking $z = r$, the left side of inequality (3.7) reduces to

$$|f(r^m)| + |f'(r^m)|r^m + \sum_{k=2}^{\infty}|a_k|r^{nk} = \frac{r^m+a}{1+ar^m} + \frac{(1-a^2)r^m}{(1+ar^m)^2} + (1-a^2)\frac{ar^{2n}}{1-ar^n}. \qquad (3.10)$$

We need to show that if $r > \alpha_{m,n}$, then there exists an $a$, such that the right side of (3.10) is larger than 1. That is

$$
\begin{aligned}
&(1-a)[r^{2m+2n}a^4 + (r^{2m+2n} + 2r^{m+2n})a^3 + (2r^{m+2n} - r^{n+2m} + r^{2n})a^2 \\
&+ (r^{2m} + r^{2n} + r^n - 2r^{m+n})a + 2r^m - 1] > 0.
\end{aligned}
$$

Namely, we need to show that

$$r^{2m+2n}a^4 + (r^{2m+2n} + 2r^{m+2n})a^3 + (2r^{m+2n} - r^{n+2m} + r^{2n})a^2 +$$

$$(r^{2m}+r^{2n}+r^{n}-2r^{m+n})a+2r^{m}-1>0. \tag{3.11}$$

For $m \geq n$, let

$$\begin{aligned} A_2(a,r) =& r^{2m+2n}a^4+(r^{2m+2n}+2r^{m+2n})a^3+(2r^{m+2n}-r^{n+2m}+r^{2n})a^2 \\ &+(r^{2m}+r^{2n}+r^{n}-2r^{m+n})a+2r^{m}-1. \end{aligned}$$

Observe that $A_2(a,r)$ is a continuous and increasing function for $a \in [0,1)$. It holds that $A_2(a,r) \leq A_2(1,r) = r^m(r^m+2)(2r^{2n}-r^n+1)+2r^{2n}+r^n-1 = \Phi_{m,n}(1,r)$ holds for $r \in (0,1)$. Furthermore, the monotonicity of $\Phi_{m,n}(1,r)$ leads to that if $r > \alpha_{m,n}$, then $A_2(1,r) > 0$. Hence, by the continuity of $A_2(a,r)$, if $r > \alpha_{m,n}$, we have

$$\lim_{a\to 1^-} A_2(a,r) = A_2(1,r) > 0.$$

Therefore, if $r > \alpha_{m,n}$, then there exists an $a$, such that inequality (3.11) holds. □

**Theorem 3.3.** *Suppose that* $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$*,* $a := |a_0|$ *and* $\omega_m(z) \in \mathcal{B}_m$*,* $\omega_n(z) \in \mathcal{B}_n$ *for* $m,n \in \mathbb{N}$*. Then we have*

$$|f(\omega_m(z))| + \sum_{k=1}^{\infty} |a_{sk}||\omega_n(z)|^{sk} \leq 1 \quad for \quad |z| = r \leq \beta_{m,n,s}, \tag{3.12}$$

*where* $s \in \mathbb{N}$*,* $\beta_{m,n,s}$ *is the unique root in* $(0,1)$ *of the equation*

$$r^m + 3r^{ns} + r^{m+ns} - 1 = 0. \tag{3.13}$$

*The radius* $\beta_{m,n,s}$ *cannot be improved.*

**Remark 3.3.** *1). If* $m = 1, n = 1$ *and* $\omega_m(z) = \omega_n(z) = z$ *in Theorem 3.3, it reduces to Theorem 2.5 of* [21].
2). If $m \to \infty$ in (3.13), then $\lim_{m\to\infty}\beta_{m,n,s} = A_{n,s}$, where $A_{n,s}$ is the positive root of the equation $3r^{ns}-1=0$. Also $A_{1,1} = \frac{1}{3}$ is the well-known classical Bohr radius.

*Proof.* Using inequality (3.3) and Lemma 2.2, we have

$$|f(\omega_m(z))| + \sum_{k=1}^{\infty} |a_{sk}||\omega_n(z)|^{sk} \leq \frac{r^m+a}{1+ar^m} + (1-a^2)\frac{r^{sn}}{1-r^{sn}}. \tag{3.14}$$

It is sufficient for us to prove that the right side of (3.14) is less than or equals to 1 for $r \leq \beta_{m,n,s}$. Actually, we just need to prove $\phi(r) \leq 0$ for $r \leq \beta_{m,n,s}$, where $\phi(r) = (r^m+a)(1-r^{sn}) + (1-a^2)r^{sn}(1+ar^m) - (1+ar^m)(1-r^{sn})$. Observe that

$$\begin{aligned} \phi(r) =& (1-a)[r^m+2r^{sn}-r^{m+sn}+(r^{sn}+r^{m+sn})a+r^{m+sn}a^2-1] \\ \leq& (1-a)(r^m+3r^{sn}+r^{m+sn}-1). \end{aligned}$$

Let $h(r) = r^m+3r^{sn}+r^{m+sn}-1$. Then it is easy to verify that $h(0)h(1) < 0$, $h(r)$ is a continuous and increasing function of $r \in [0,1]$. Thus $\beta_{m,n,s}$ is unique root of $h(r)$ and $h(r) \leq 0$ holds for $r \leq \beta_{m,n,s}$. Thus $\phi(r) \leq 0$ for $r \leq \beta_{m,n,s}$.

To show the radius $\beta_{m,n,s}$ is sharp, we consider the functions $\omega_m(z)$, $\omega_n(z)$ and $f(z)$ is the same as (3.4). Taking $z = r$, the left side of inequality (3.12) reduces to

$$|f(r^m)| + \sum_{k=1}^{\infty} |a_{sk}| r^{nsk} = \frac{r^m + a}{1 + ar^m} + (1 - a^2)\frac{a^{s-1} r^{ns}}{1 - a^s r^{ns}}. \tag{3.15}$$

Next, we need to show that if $r > \beta_{m,n,s}$, then there exists an $a$, such that the right side of (3.15) is bigger than 1. That is

$$r^{m+ns}a^{s+1} + 2r^{ns}a^s + r^{ns}a^{s-1} + r^m - 1 > 0. \tag{3.16}$$

Let

$$A_3(a, r) = r^{m+ns}a^{s+1} + 2r^{ns}a^s + r^{ns}a^{s-1} + r^m - 1.$$

Observe that $A_3(a, r)$ is a continuous and increasing function for $a \in [0, 1)$. It follows that $A_3(a, r) \leq A_3(1, r) = r^{m+ns} + 3r^{ns} + r^m - 1 = h(r)$ holds for $r \in (0, 1)$. Furthermore, the monotonicity of $h(r)$ leads to that if $r > \beta_{m,n,s}$, then $A_3(1, r) > 0$. Hence, by the continuity of $A_3(a, r)$, if $r > \beta_{m,n,s}$, we have

$$\lim_{a \to 1^-} A_3(a, r) = A_3(1, r) > 0.$$

Therefore, if $r > \beta_{m,n,s}$, then there exists an $a$, such that inequality (3.16) holds. □

**Theorem 3.4.** *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$, $a := |a_0|$ and $\omega_m(z) \in \mathcal{B}_m$ , $\omega_n(z) \in \mathcal{B}_n$, where $m, n \in \mathbb{N}$ and $n \geq m$. Then we have*

$$\left| |f(\omega_m(z))| + \sum_{k=1}^{\infty} (-1)^k |a_k| |\omega_n(z)|^k \right| \leq 1 \quad for \quad |z| = r \leq \gamma_{m,n},$$

*where $\gamma_{m,n}$ is the unique root in $(0, 1)$ of the equation*

$$r^{m+2n} + r^m + 3r^{2n} - 1 = 0.$$

**Remark 3.4.** *If $m = 1, n = 1$, and $\omega_m(z) = \omega_n(z) = z$ in Theorem 3.4, then it reduces to Theorem 2.9 of* [21].

*Proof.* On the one hand, by the assumption, we have

$$\begin{aligned}
|f(\omega_m(z))| + \sum_{k=1}^{\infty} (-1)^k |a_k| |\omega_n(z)|^k \leq & \frac{r^m + a}{1 + ar^m} + \sum_{k=1}^{\infty} |a_{2k}| |\omega_n(z)|^{2k} - \sum_{k=1}^{\infty} |a_{2k-1}| |\omega_n(z)|^{2k-1} \\
\leq & \frac{r^m + a}{1 + ar^m} + \sum_{k=1}^{\infty} |a_{2k}| r^{2kn} \\
\leq & \frac{r^m + a}{1 + ar^m} + (1 - a^2)\frac{r^{2n}}{1 - r^{2n}}.
\end{aligned}$$

Now we need to show that above inequality is smaller than or equal to 1. It is sufficient for us to prove $\psi(r) \leq 0$, where $\psi(r) = (r^m + a)(1 - r^{2n}) + (1 - a^2)r^{2n}(1 + ar^m) - (1 + ar^m)(1 - r^{2n})$. Observe that

$$\psi(r) = (1 - a)[r^m + 2r^{2n} - r^{m+2n} + (r^{2n} + r^{m+2n})a + r^{m+2n}a^2 - 1]$$

$$\leq (1-a)(r^m + 3r^{2n} + r^{m+2n} - 1).$$

Furthermore, it is easy to verify that $r^m + 3r^{2n} + r^{m+2n} - 1$ is increasing on $r \in [0, 1)$ and have a unique zero $\gamma_{m,n}$. Therefore, we have $\psi(r) \leq 0$ for $r \leq \gamma_{m,n}$.

On the other hand, we have

$$\begin{aligned}
|f(\omega_m(z))| + \sum_{k=1}^{\infty}(-1)^k|a_k||\omega_n(z)|^k =& |f(\omega_m(z))| + \sum_{k=1}^{\infty}|a_{2k}||\omega_n(z)|^{2k} - \sum_{k=1}^{\infty}|a_{2k-1}||\omega_n(z)|^{2k-1} \\
\geq& -\sum_{k=1}^{\infty}|a_{2k-1}|r^{(2k-1)n} = -(|a_1|r^n + \sum_{k=1}^{\infty}|a_{2k+1}|r^{(2k+1)n}) \\
\geq& -\left((1-a^2)r^n + \sum_{k=1}^{\infty}|a_{2k+1}|r^{(2k)n}\right) \\
\geq& -\left(\frac{r^m + a}{1 + ar^m} + \sum_{k=1}^{\infty}|a_{2k+1}|r^{(2k)n}\right) \\
\geq& -\left(\frac{r^m + a}{1 + ar^m} + (1-a^2)\frac{r^{2n}}{1 - r^{2n}}\right).
\end{aligned}$$

It is obvious that the last item of above is greater than or equals $-1$ for all $r \leq \gamma_{m,n}$. We complete the proof. □

## 4. Some Corollaries

In Theorem 3.1, by setting with the combination of $\omega_n(z) = z$, $\omega_m(z) = z$, and $\omega_m(z) = \omega_n(z) = \omega(z)$, we get Corollaries 4.1–4.3, respectively.

**Corollary 4.1.** *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$, $a := |a_0|$ and $\omega \in \mathcal{B}_m$ for $m \in \mathbb{N}$. Then we have*

$$|f(\omega(z))| + \sum_{k=N}^{\infty}|a_k||z|^k \leq 1 \quad for \quad |z| = r \leq R_{m,1,N},$$

*where $R_{m,1,N}$ is the unique root in $(0, 1)$ of the equation*

$$2r^N(1 + r^m) - (1 - r)(1 - r^m) = 0,$$

*and the radius $R_{m,1,N}$ cannot be improved.*

**Corollary 4.2.** *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$, $a := |a_0|$ and $\omega \in \mathcal{B}_n$ for $n \in \mathbb{N}$. Then we have*

$$|f(z)| + \sum_{k=N}^{\infty}|a_k||\omega(z)|^k \leq 1 \quad for \quad |z| = r \leq R_{1,n,N},$$

*where $R_{1,n,N}$ is the unique root in $(0, 1)$ of the equation*

$$2r^{nN}(1 + r) - (1 - r^n)(1 - r) = 0,$$

*and the radius $R_{1,n,N}$ cannot be improved.*

**Corollary 4.3.** *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$, $a := |a_0|$ and $\omega \in \mathcal{B}_m$ for $m \in \mathbb{N}$. Then we have*

$$|f(\omega(z))| + \sum_{k=N}^{\infty} |a_k||\omega(z)|^k \leq 1 \quad for \quad |z| = r \leq R_{m,m,N},$$

*where $R_{m,m,N}$ is the positive root in $(0,1)$ of the equation*

$$2r^{mN}(1 + r^m) - (1 - r^m)^2 = 0,$$

*and the radius $R_{m,m,N}$ cannot be improved.*

In Theorem 3.2, setting $\omega_n(z) = z$ and $\omega_m(z) = \omega_n(z) = \omega(z)$, we have Corollaries 4.4 and 4.5, respectively.

**Corollary 4.4.** *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$, $a := |a_0|$ and $\omega \in \mathcal{B}_m$ for $m \in \mathbb{N}$. Then we have*

$$|f(\omega(z))| + |f'(\omega(z))||\omega(z)| + \sum_{k=2}^{\infty} |a_k||z|^k \leq 1 \quad for \quad |z| = r \leq \alpha_{m,1},$$

*where $\alpha_{m,1}$ is the unique root in $(0,1)$ of the equation*

$$r^m(r^m + 2)(2r^2 - r + 1) + 2r^2 + r - 1 = 0.$$

*The radius $\alpha_{m,1}$ cannot be improved.*

**Corollary 4.5.** *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$, $a := |a_0|$ and $\omega \in \mathcal{B}_m$ for $m \in \mathbb{N}$. Then we have*

$$|f(\omega(z))| + |f'(\omega(z))||\omega(z)| + \sum_{k=2}^{\infty} |a_k||\omega(z)|^k \leq 1 \quad for \quad |z| = r \leq \alpha_{m,m},$$

*where $\alpha_{m,m}$ is the unique root in $(0,1)$ of the equation*

$$2r^{4m} + 3r^{3m} + r^{2m} + 3r^m - 1 = 0.$$

*The radius $\alpha_{m,m}$ cannot be improved.*

In Theorem 3.3, setting $\omega_n(z) = z$, $\omega_m(z) = z$, and $\omega_m(z) = \omega_n(z) = \omega(z)$, we obtain Corollaries 4.6–4.8, respectively.

**Corollary 4.6.** *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$, $a := |a_0|$ and $\omega \in \mathcal{B}_m$ for $m \in \mathbb{N}$. Then we have*

$$|f(\omega(z))| + \sum_{k=1}^{\infty} |a_{sk}||z|^{sk} \leq 1 \quad for \quad |z| = r \leq \beta_{m,1,s},$$

*where $\beta_{m,1,s}$ is the unique root in $(0,1)$ of the equation*

$$r^{m+s} + r^m + 3r^s - 1 = 0.$$

*The radius $\beta_{m,1,s}$ cannot be improved.*

**Corollary 4.7.** *Suppose that* $f(z)=\sum_{k=0}^{\infty}a_kz^k\in\mathcal{B}$, $a:=|a_0|$ *and* $\omega\in\mathcal{B}_n$ *for* $m\in\mathbb{N}$*. Then we have*

$$|f(z)|+\sum_{k=1}^{\infty}|a_{sk}||\omega(z)|^{sk}\leq 1 \quad for \quad |z|=r\leq\beta_{1,n,s},$$

*where* $\beta_{1,n,s}$ *is the unique root in* $(0,1)$ *of the equation*

$$r^{ns+1}+3r^{ns}+r-1=0.$$

*The radius* $\beta_{1,n,s}$ *cannot be improved.*

**Corollary 4.8.** *Suppose that* $f(z)=\sum_{k=0}^{\infty}a_kz^k\in\mathcal{B}$, $a:=|a_0|$ *and* $\omega\in\mathcal{B}_m$ *for* $m\in\mathbb{N}$*. Then we have*

$$|f(\omega(z))|+\sum_{k=1}^{\infty}|a_{sk}||\omega(z)|^{sk}\leq 1 \quad for \quad |z|=r\leq\beta_{m,m,s},$$

*where* $\beta_{m,m,s}$ *is the unique root in* $(0,1)$ *of the equation*

$$r^{ms+m}+3r^{ms}+r^m-1=0.$$

*The radius* $\beta_{m,m,s}$ *cannot be improved.*

In Theorem 3.3, setting $s=2$, we get Corollary 4.9.

**Corollary 4.9.** *Suppose that* $f(z)=\sum_{k=0}^{\infty}a_kz^k\in\mathcal{B}$, $a:=|a_0|$ *and* $\omega_m\in\mathcal{B}_m$, $\omega_n(z)\in\mathcal{B}_n$ *for* $m,n\in\mathbb{N}$*. Then we have*

$$|f(\omega_m(z))|+\sum_{k=1}^{\infty}|a_{2k}||\omega_n(z)|^{2k}\leq 1 \quad for \quad |z|=r\leq\beta_{m,n,2},$$

*where* $\beta_{m,n,2}$ *is the unique root in* $(0,1)$ *of the equation*

$$r^{m+2n}+r^m+3r^{2n}-1=0.$$

*The radius* $\beta_{m,n,2}$ *cannot be improved.*

In Theorem 3.4, setting $\omega_m(z)=z$, and $\omega_m(z)=\omega_n(z)=\omega(z)$, we give Corollaries 4.10 and 4.11, respectively.

**Corollary 4.10.** *Suppose that* $f(z)=\sum_{k=0}^{\infty}a_kz^k\in\mathcal{B}$, $a:=|a_0|$ *and* $\omega\in\mathcal{B}_n$ *for* $n\in\mathbb{N}$*. Then we have*

$$\left||f(z)|+\sum_{k=1}^{\infty}(-1)^k|a_k||\omega(z)|^k\right|\leq 1 \quad for \quad |z|=r\leq\gamma_{1,n},$$

*where* $\gamma_{1,n}$ *is the unique root in* $(0,1)$ *of the equation*

$$r^{2n+1}+3r^{2n}+r-1=0.$$

**Corollary 4.11.** *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$, $a := |a_0|$ and $\omega \in \mathcal{B}_m$ for $m \in \mathbb{N}$. Then we have*

$$\left| |f(\omega(z))| + \sum_{k=1}^{\infty} (-1)^k |a_k||\omega(z)|^k \right| \leq 1 \quad for \quad |z| = r \leq \gamma_{m,m},$$

*where $\gamma_{m,m}$ is the unique root in $(0, 1)$ of the equation*

$$r^{3m} + 3r^{2m} + r^m - 1 = 0.$$

It is worth pointing out that we have not proved radius $\gamma_{m,n}$ is sharp in Theorem 3.4. Therefore, the following problem is open.

**Open problem**: Find the largest radius $r_0$ for $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$, and $\omega_m(z) \in \mathcal{B}_m$, $\omega_n(z) \in \mathcal{B}_n$ with $m, n \in \mathbb{N}$, such that

$$\left| |f(\omega_m(z))| + \sum_{k=1}^{\infty} (-1)^k |a_k||\omega_n(z)|^k \right| \leq 1 \quad for \quad r \leq r_0.$$

## 5. Conclusions

We obtain some new versions of Bohr-type inequalities for bounded analytic functions of Schwarz functions by replacing the variable $z$ by Schwarz functions in function's power series expansions. we conclude that most of the corresponding Bohr radii are exact. These inequalities generalize the classical Bohr inequality and some earlier results on the Bohr inequality.

## Acknowledgments

This work is supported by the Foundation of Anhui Educational Committee (KJ2020A0002) and Natural Science Foundation of Anhui Province(1908085MA18), China.

## Conflict of interest

The authors declare no conflict of interest in this paper.

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