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Research article

Edge-fault-tolerant strong Menger edge connectivity of bubble-sort graphs

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Abstract: This paper studies the edge-fault-tolerant strong Menger edge connectivity of n-dimensional bubble-sort graph B_n. We give the values of faulty edges that B_n can tolerant when B_n is strongly Menger edge connected under two conditions. When there are $(n-3)$ faulty edges removed from B_n, the B_n network is still working and it is strongly Menger edge connected. When the condition of any vertex in B_n has at least two neighbors is imposed, the number of faulty edges that can removed from B_n is $(2n-6)$ when B_n is also strongly Menger edge connected. And two special cases are used to illustrate the correctness of the conclusions. The conclusions can help improve the reliability of the interconnection networks.

Keywords: edge-disjoint paths; fault-tolerant; strong Menger edge connectivity; bubble-sort graph
Mathematics Subject Classification: 05C40, 68M15

1. Introduction

Graph theory can be used to analyze the connections between things, arguably in the most general sense, because it is based on set theory. Social networks, traffic networks, power networks, distributed control systems, molecular structures, interconnection networks and so on, any system you can think of that involves connections between things can be modeled using graphs. Faults are inevitable in these application. Many researchers have done much works to improve the reliability of these systems, see [1–4]. The stability of systems is only under very strict conditions. Maintain systems stability in case of failure brings great challenges to theoretical research. And because of the complexity of many systems in engineering applications. Many systems are modeling of interconnected systems for security and reliability. Therefore, effective fault-tolerant control techniques are urgently needed. For this purpose, two methods are commonly used. One method is the algorithm in the proof, various algorithms are used in a lot of works, such as [5–11]. The other method is the discussion of whether the connectivity condition is satisfied after removing some vertices or edges in the topology structure of the interconnection network, this method is used by a large number of researchers, such as [12–15]. In this paper, we use the second method to discuss the application of graph theory to the interconnection network. An interconnection network can be modeled as an undirected graph G. The edge connectivity is one of the basic parameters to estimate the fault tolerance of the interconnection network. The edge-disjoint paths relate closely with the edge connectivity. In this paper, we discuss the edge-fault-tolerant strong Menger edge connectivity of a special graph.

Let $G = (V(G), E(G))$ be an undirected simple connected graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set. For any vertex $u \in V(G)$, $N_G(u)$ denotes the set of neighbors of u and $E_G(u)$ denotes the set of edges which are associated with u. $d_G(u) = |E_G(u)|$ denotes the degree of u and $\delta(G) = min\{d_G(u) : u \in V(G)\}$ denotes the minimum degree of vertices in G. G is called k-regular when $d_G(u) = k$ for each $u \in V(G)$. Let $F \subseteq V(G)$ be a vertex set, $F_e \subseteq E(G)$ be an edge set, and $E_G(F)$ be the set of edges with only one end in F. Let $G - F$ be the subgraph of G whose vertex set is $V(G) - F$ and edge set is $E(G) - \{uv \in E(G) : \{u, v\} \cap F \neq \emptyset\}$. Let $G - F_e$ be the subgraph of G whose vertex set is $V(G)$ and edge set is $E(G) - F_e$. When $G - F$ is disconnected or has only 1 vertex, F is called a vertex cut of G. When $G - F_e$ is disconnected, F_e is called an edge cut of G. The (vertex) connectivity $\kappa(G)$ of G is the minimal cardinality of F. The edge connectivity $\lambda(G)$ of G is the minimal cardinality of F_e. $P_k = ux_1x_2 \cdots x_{k-2}v$ denotes a (u, v)-path on k distinct vertices $u, x_1, x_2, \cdots, x_{k-2}, v$. Let $P^1 = ux_1x_2 \cdots x_kv$ and $P^2 = uy_1y_2 \cdots y_mv$ be two distinct (u, v)-paths of G. P^1 and P^2 are called disjoint (u, v)-paths when $V(P^1) \cap V(P^2) = \{u, v\}$. P^1 and P^2 are called edge-disjoint (u, v)-paths when $E(P^1) \cap E(P^2) = \emptyset$. If $G - F$ has no (u, v)-path, then $F \subseteq V(G) - \{u, v\}$ is a (u, v)-cut. If $G - F_e$ has no (u, v)-path, then $F_e \subseteq E(G)$ is a (u, v)-edge-cut. Much attention has been paid to the vertex connectivity and edge connectivity in recent years. Menger [16] proposed the local connectivity.

Theorem 1.1. *[16] Let $u, v \in V(G)$ be two distinct vertices. Then the minimal size of a (u, v)-edge-cut is equal to the maximum number of edge-disjoint (u, v)-paths.*

On the base of Menger's Theorem, Oh et al. [10] proposed the strong Menger connectivity (the maximal local-connectivity) and Qiao et al. [12] proposed the strong Menger edge connectivity.

Definition 1.2. *[12] Let $u, v \in V(G)$ be two distinct vertices. If there are $min\{d_G(u), d_G(v)\}$ edge-disjoint (u, v)-paths, then G is strongly Menger edge connected.*

Since the vertices and edges may fault, the discussion of fault-tolerance of interconnection networks is crucial. Oh et al. [10, 11] proposed the m-fault-tolerant strong Menger connectivity. Shih et al. [15] proposed the m-conditional fault-tolerant strong Menger connectivity.

Definition 1.3. *Let $m \geq 1$ be an integer, $F_e \subseteq E(G)$ be an edge set with $|F_e| \leq m$.*

(1) [17] If $G - F_e$ is strongly Menger edge connected, then G is m-edge-fault-tolerant strongly Menger edge connected.

(2) [12] If $G - F_e$ is strongly Menger edge connected for any edge set F_e with $\delta(G - F_e) \geq 2$, then G is m-conditional edge-fault-tolerant strongly Menger edge connected.

The fault-tolerant strong Menger edge connectivity is used in many topologies of interconnection networks, such as the n-dimensional star graph S_n, the hypercube-like network HL_n, the augmented cubes AQ_n, the bubble-sort star graph BS_n, the folded hypercube FQ_n, the hypercube Q_n and so on. Table 1 is a summary of some recent researches.

The n-dimensional bubble-sort graph B_n is an attractive topology structure of interconnection networks. It is regular and bipartite [23]. And it has been studied by many researchers, see [13, 14, 24–26]. From Table 2, we can see that S_n, B_n and BS_n have the same number of vertices, but B_n has the largest diameter. The fault-tolerant strong Menger edge connectivity of B_n has not been studied on B_n. In this paper, we discuss the number of faulty edges that B_n can tolerant without failure. The main contributions are listed in the following:

(1) Discussing the minimum number of vertices in the biggest connected component of B_n when at most $(2n-5)$ edges are removed from B_n.

(2) Proving that when B_n has $(n-3)$ faulty edges, B_n is edge-fault-tolerant strongly Menger edge connected for $n \geq 3$.

(3) Using the inductive hypothesis to discuss the minimum number of vertices in the biggest connected component of B_n when at most $(3n-8)$ edges are removed from B_n.

(4) Proving under the condition that each vertex in B_n has at least two neighbors is imposed, when B_n has $(2n-6)$ faulty edges, B_n is $(2n-6)$-conditional edge-fault-tolerant strongly Menger edge connected for $n \geq 5$.

Table 1. The summary of recent researches.

Contributions	Reference	Topology structure
m-fault-tolerant strong Menger connectivity	[10, 11]	S_n
	[15]	HL_n
	[18]	AQ_n
m-fault-tolerant strong Menger edge connectivity	[19]	BS_n
m-conditional fault-tolerant strong Menger connectivity	[20]	FQ_n
	[15]	HL_n
m-conditional edge-fault-tolerant strong Menger edge connectivity	[12]	Q_n
	[12]	FQ_n
	[21]	HL_n
	[22]	BS_n

Table 2. The comparison of S_n, B_n and BS_n.

Topology structure	Regular degree	The number of vertices	Diameter	Connectivity
S_n	$n-1$	$n!$	$\lfloor \frac{3(n-1)}{2} \rfloor$	$n-1$
B_n	$n-1$	$n!$	$\frac{n(n-1)}{2}$	$n-1$
BS_n	$2n-3$	$n!$	$\lfloor \frac{3(n-1)}{2} \rfloor$	$2n-3$

2. The n-dimensional bubble-sort graph B_n

We will give the definition and propositions of the n-dimensional bubble-sort graph.

Let $[1, n] = \{1, 2, \cdots, n\}$. Let (ij) be a transposition with $i, j \in [1, n]$. We use $k_1 k_2 \cdots k_n$ to represent the permutation $\binom{1\ 2\ \cdots\ n}{k_1\ k_2\ \cdots\ k_n}$. Then $\binom{1\ 2\ \cdots\ i \cdots\ j \cdots\ n}{1\ 2\ \cdots\ i \cdots\ j \cdots\ n}(ij) = \binom{1\ 2\ \cdots\ i \cdots\ j \cdots\ n}{1\ 2\ \cdots\ j \cdots\ i \cdots\ n}$ $(*)$. In order to the convenience of discussion, the equation of $(*)$ is written as $(12 \cdots i \cdots j \cdots n)(ij) = (12 \cdots j \cdots i \cdots n)$.

Definition 2.1. *[23] The n-dimensional bubble-sort graph B_n has the vertex set of the entire $n!$ permutations of $[1, n]$ for $n \geq 2$. Any two distinct vertices $u, v \in V(B_n)$ are neighboring if and only if $u = v(i, i+1)$ for $1 \leq i \leq n-1$. Each vertex $u \in V(B_n)$ has the form of $u = u_1 u_2 \cdots u_n$.*

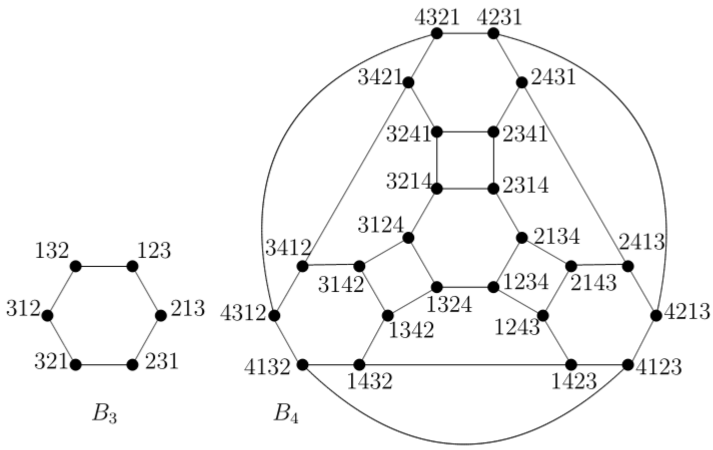

Figure 1. The illustration of B_3 and B_4.

By Definition 2.1, B_n has $n!$ vertices and it is $(n-1)$-regular. When $n = 3$ and $n = 4$, the graphs are given in Figure 1. Obviously, when the last position of each vertex $u = u_1u_2\cdots u_n \in V(B_n)$ has a fixed integer $i \in [1, n]$, B_n can be decomposed into n sub-bubble-sort graphs $B_n^1, B_n^2, \cdots, B_n^n$ along the last position. Then $B_n^i \cong B_{n-1}$ for $i \in [1, n]$. For distinct $i, j \in [1, n]$, when $u \in V(B_n^i)$, $u' = u(n-1, n) \in V(B_n^j)$ is the outside neighbor of u, $E_{ij}(B_n) = \{uu' \in E(B_n) : u \in V(B_n^i), u' \in V(B_n^j)\}$ is the cross-edge set of B_n. Let F_1 and F_2 be two distinct vertex sets of B_n, $E_{B_n}(F_1, F_2)$ be the set of edges between F_1 and F_2. Let $F_e \subseteq E(B_n)$, $F_e^i = F_e \cap E(B_n^i)$, $F_e^0 = F_e - \cup_{i=1}^n F_e^i$. Let $F_e^{[t,s]} = F_e^t \cup F_e^{t+1} \cdots \cup F_e^s$ be an edge set, and $B_n^{[t,s]} = B_n[V(B_n^t) \cup V(B_n^{t+1}) \cup \cdots \cup V(B_n^s)]$ be the subgraph of B_n induced by $B_n^t, B_n^{t+1}, \cdots, B_n^s$, $B_n^{[t,s]} - F_e^{[t,s]} = B_n[V(B_n^t - F_e^t) \cup V(B_n^{t+1} - F_e^{t+1}) \cup \cdots \cup V(B_n^s - F_e^s)]$ be the subgraph of B_n induced by $B_n^t - F_e^t, B_n^{t+1} - F_e^{t+1}, \cdots, B_n^s - F_e^s$ for $t, s \in [1, n]$ and $t < s$.

Some useful propositions are given in the following.

Proposition 2.2. *[25] Any vertex $u \in V(B_n^i)$ has only one outside neighbor for all $i \in [1, n]$.*

Proposition 2.3. *[25] $|E_{ij}(B_n)| = (n-2)!$ for distinct $i, j \in [1, n]$.*

Proposition 2.4. *[25] Any two distinct vertices $u, v \in V(B_n^i)$ have different outside neighbors for all $i \in [1, n]$.*

Proposition 2.5. *[27] $\kappa(B_n) = \lambda(B_n) = n - 1$ for $n \geq 2$.*

3. The edge-fault-tolerant strong Menger edge connectivity of B_n

In this part, we will discuss the edge-fault-tolerant strong Menger edge connectivity of B_n, firstly, one necessary conclusion is proved.

Lemma 3.1. *Let $F_e \subseteq E(B_n)$ with $|F_e| \leq 2n-5$ for $n \geq 3$. There is a connected component C of $B_n - F_e$ with $|V(C)| \geq n! - 1$.*

Proof. We prove this conclusion by induction on n. Let C be the largest connected component of $B_n - F_e$.

When $n = 3$, $|F_e| \leq 2\times 3 - 5 = 1$, see from Figure 1, $B_3 - F_e$ is connected. Then there is a connected component C of $B_3 - F_e$ with $|V(C)| = |V(B_3)| = 3! > 3! - 1 = 5$. When $n = 4$, $|F_e| \leq 2 \times 4 - 5 = 3$, see from Figure 1, there is at most 1 vertex can be isolated. Then there is a connected component C of $B_4 - F_e$ with $|V(C)| \geq 4! - 1 = 23$. Assume that the conclusion holds for $n-1$, $n \geq 5$, we will prove that the conclusion holds for n. Without loss of generality, let $|F_e^1| = max\{|F_e^i| : 1 \leq i \leq n\}$. Let C_i be the largest connected component of $B_n^i - F_e^i$ for all $i \in [1, n]$.

Claim 1. For distinct $i, j \in [1, n]$, if $|F_e^i| \leq n-3$ and $|F_e^j| \leq n-3$, then $B_n[V(B_n^i - F_e^i) \cup V(B_n^j - F_e^j)]$ is connected.

By Proposition 2.5, $\lambda(B_n^i) = n-2$. Since $|F_e^i| \leq n-3$ and $|F_e^j| \leq n-3$, both $B_n^i - F_e^i$ and $B_n^j - F_e^j$ are connected. Note that $|F_e^0| \leq |F_e| \leq 2n-5 < (n-2)! = |E_{ij}(B_n)|$ for $n \geq 5$, $B_n[V(B_n^i - F_e^i) \cup V(B_n^j - F_e^j)]$ is connected by Proposition 2.3. The proof of Claim 1 is complete.

Case 1. $|F_e^1| \leq n-3$.

Since $|F_e^1| = max\{|F_e^i| : 1 \leq i \leq n\}$, $|F_e^i| \leq n-3$ and $|F_e^0| \leq |F_e| \leq 2n-5$ for all $i \in [1, n]$. By Claim 1, $B_n - F_e$ is connected. So $|V(C)| = |V(B_n)| = n! > n! - 1$.

Case 2. $n-2 \leq |F_e^1| \leq 2n-7$.

Since $|F_e| \leq 2n-5$, $|F_e^{[2,n]}| \leq |F_e| - |F_e^1| \leq (2n-5)-(n-2) = n-3$ and $|F_e^0| \leq |F_e| - |F_e^1| \leq n-3$. So $|F_e^j| \leq n-3 < n-2 = \lambda(B_{n-1})$ for all $j \in [2, n]$. By Claim 1, $B_n^{[2,n]} - F_e^{[2,n]}$ is connected. By the assumption, there is an isolated vertex and a connected component C_1 in $B_n^1 - F_e^1$ with $|V(C_1)| \geq (n-1)! - 1$. Note that $|E_{B_n}(V(C_1), V(B_n^2)) - F_e| \geq |E_{12}(B_n)| - |V(B_n^1) - V(C_1)| - |F_e^0| \geq (n-2)! - [(n-1)! - ((n-1)! - 1)] - (n-3) \geq (n-2)! - n + 2 > 0$ for $n \geq 5$. By Proposition 2.2, $B_n[V(C_1) \cup V(B_n^{[2,n]} - F_e^{[2,n]})]$ is connected and it is C. So $|V(C)| \geq |V(C_1)| + |V(B_n^{[2,n]} - F_e^{[2,n]})| \geq n! - 1$.

Case 3. $2n-6 \leq |F_e^1| \leq 2n-5$.

In this case, $|F_e^{[2,n]}| \leq |F_e| - |F_e^1| \leq (2n-5) - (2n-6) = 1$. $|F_e^0| \leq |F_e| - |F_e^1| \leq 1$. So $|F_e^j| \leq 1 < n-2 = \lambda(B_{n-1})$ for all $j \in [2, n]$ and $n \geq 5$. By Claim 1, $B_n^{[2,n]} - F_e^{[2,n]}$ is connected. By Proposition 2.2, each vertex $u \in V(B_n^1)$ has 1 outside neighbor in $B_n^{[2,n]}$, combining this with $|F_e^0| \leq 1$, there is at most one vertex in B_n^1 can be isolated from the component C. So $|V(C)| \geq n! - 1$.

By Case 1–3, When $B_n - F_e$ is disconnected, there is a largest connected component C of $B_n - F_e$ with $|V(C)| \geq n! - 1$ when $|F_e| \leq 2n-5$. □

Remark 3.2. *The conclusion of Lemma 3.1 is optimal in that there is an edge set $F_e \subseteq E(B_n)$ with $|F_e| = 2n-4$ such that $B_n - F_e$ has a connected component C with $|V(C)| \leq n! - 2$ for $n \geq 4$.*

Let $F = \{u, v\}$ be a vertex set of B_n with $uv \in E(B_n)$. Then $|E_{B_n}(F)| = 2n-4$ by Proposition 2.5. Let $F_e = E_{B_n}(F) \subseteq E(B_n)$. Then uv is isolated in $B_n - F$. So $|V(C)| \leq n! - 2$.

Theorem 3.3. *B_n is $(n-3)$-edge-fault-tolerant strongly Menger edge connected for $n \geq 3$.*

Proof. Let $S_e \subseteq E(B_n)$ be a faulty edge set with $|S_e| \leq n-3$. By Proposition 2.5, we have $\lambda(B_n) = n-1$ for $n \geq 2$. Since $n-3 < n-1$, $B_n - S_e$ is connected. Let two distinct vertices $u, v \in V(B_n)$, $a = min\{d_{B_n - S_e}(u), d_{B_n - S_e}(v)\}$ and $E_f \subseteq E(B_n) - S_e$ with $|E_f| \leq a-1$. By Definition 1.3, if B_n is $(n-3)$-edge-fault-tolerant strongly Menger edge connected, then there should be a edge-disjoint fault-free (u, v)-paths connect u and v. We prove this by contradiction, suppose that u and v are disconnected in $B_n - S_e$ by removing an edge set E_f. That means u and v are disconnected in $B_n - S_e - E_f$. Since $a = min\{d_{B_n - S_e}(u), d_{B_n - S_e}(v)\} \leq \kappa(B_n) = n-1$ and $|E_f| \leq a-1$, we have $|E_f| \leq n-2$. Note that $d_{B_n - S_e}(u) \leq n-1$, $d_{B_n - S_e}(v) \leq n-1$ and $|E_f| \leq n-2$. Then $|S_e \cup E_f| \leq (n-3) + (n-2) = 2n-5$. By Lemma 3.1, $B_n - S_e - E_f$ has a connected component C with $|V(C)| \geq n! - 1$. Since $|V(B_n)| = n!$, combining this with u and v are disconnected in $B_n - S_e - E_f$, we have $|V(C)| = n! - 1$ and only one vertex can be isolated. Without loss of generality, we suppose that v is the isolated vertex. Then $u \in V(C)$ and

$v \notin V(C)$. Moreover, the edges which are attached to v belonging to E_f, that is $E_{B_n}(\{v\}, N_{B_n - S_e}(v)) \subseteq E_f$. This means $|E_f| \geq d_{B_n - S_e}(v)$, which contradicts with $|E_f| \leq a - 1 \leq d_{B_n - S_e}(v) - 1$. Hence B_n is $(n-3)$-edge-fault-tolerant strongly Menger edge connected. □

4. The m-conditional edge-fault-tolerant strong Menger edge connectivity of B_n

In this part, we will discuss the m-conditional edge-fault-tolerant strong Menger edge connectivity of B_n, firstly, two necessary conclusions are proved.

Lemma 4.1. *Let $F_e \subseteq E(B_5)$ with $|F_e| \leq 7$. There is a connected component C of $B_5 - F_e$ with $|V(C)| \geq 5! - 2$.*

Proof. Let $|F_e^1| = max\{|F_e^i| : 1 \leq i \leq 5\}$. Let C_i be the largest connected component of $B_5^i - F_e^i$ for all $i \in [1, 5]$.

Claim 1. If $|F_e^i| \leq 2$, $|F_e^j| \leq 2$ and $|E_{ij}(B_5)| > |F_e^0|$, then $B_5[V(B_5^i - F_e^i) \cup V(B_5^j - F_e^j)]$ is connected for distinct $i, j \in [1, 5]$.

By Proposition 2.5, $\lambda(B_5) = 3$, then both $B_5^i - F_e^i$ and $B_5^j - F_e^j$ are connected for distinct $i, j \in [1, 5]$. Since $|E_{ij}(B_5)| = (5-2)! = 6 > |F_e^0|$ for distinct $i, j \in [1, 5]$, $B_5[V(B_5^i - F_e^i) \cup V(B_5^j - F_e^j)]$ is connected by Proposition 2.3. The proof of claim 1 is complete.

Case 1. $|F_e^1| \leq 2$.

In this case, $|F_e^0| \leq |F_e| - |F_e^1| \leq 7$ and $|F_e^i| \leq 7$ for all $i \in [1, 5]$. By Proposition 2.5, $B_5^i - F_e^i$ is connected for $i \in [1, 5]$. Since $|E_{ij}(B_5)| = (5-2)! = 6$ by Proposition 2.3 for distinct $i, j \in [1, 5]$ and $|F_e^0| \leq 7$, at most one $E_{ij}(B_5) - F_e = \emptyset$ for distinct $i, j \in [1, 5]$. Without loss of generality, we suppose that $E_{12}(B_5) - F_e = \emptyset$. Then only $B_5^1 - F_e^1$ is not connected to $B_5^2 - F_e^2$, but both $B_5^1 - F_e^1$ and $B_5^2 - F_e^2$ are connected to $B_5^s - F_e^s$ for all $s \in [3, 5]$. So $B_5 - F_e$ is connected and $|V(C)| = |V(B_5)| = 5! > 5! - 2$.

Case 2. $3 \leq |F_e^1| \leq 7$.

In this case, $|F_e^0| \leq |F_e| - |F_e^1| \leq 7 - 3 = 4$ and $|F_e^{[2,5]}| \leq |F_e| - |F_e^1| \leq 4$.

Subcase 2.1. $|F_e^j| \leq 2$, for all $j \in [2, 5]$.

In this subcase, $B_5^j - F_e^j$ is connected by Proposition 2.5 for all $j \in [2, 5]$. Since $|E_{st}(B_5)| = (5-2)! = 6 > 4 \geq |F_e^0|$ for distinct $s, t \in [2, 5]$, $B_5^{[2,5]} - F_e^{[2,5]}$ is connected by Claim 1. If $B_5^1 - F_e^1$ is connected, this is similar to Case 1, we have $B_5 - F_e$ is connected and $|V(C)| = |V(B_5)| = 5! > 5! - 2$. If $B_5^1 - F_e^1$ is disconnected, we assume on the contrary that $|V(C)| \leq 5! - 3$, then there are at least three distinct vertices $u_1, u_2, u_3 \in V(B_5^1 - F_e^1)$ but $u_1, u_2, u_3 \notin V(C)$. By Propositions 2.2 and 2.4, there are three distinct vertices $u_1', u_2', u_3' \in V(B_5^{[2,5]}) \subseteq V(C)$ and $\{u_1u_1', u_2u_2', u_3u_3'\} \subseteq E(B_5)$. Furthermore, $\{u_1u_1', u_2u_2', u_3u_3'\} \subseteq F_e^0$. At this moment, $|F_e^0| \geq 3$, $|F_e^1| \leq |F_e| - |F_e^0| \leq 7 - 3 = 4$. Combining this with $|F_e^0| \leq 4$, we have $3 \leq |F_e^0| \leq 4$ and $3 \leq |F_e^1| \leq 4$. When $|F_e^1| = 3$, there is only one vertex can be isolated in $B_5^1 - F_e^1$ (Figure 1), which contradicts with the assumption. When $|F_e^1| = 4$, we have $|F_e^0| = 3$, a K_2 or a 4-cycle can be isolated in $B_5^1 - F_e^1$ (Figure 1.), but only the 4-cycle can satisfy the assumption. By Propositions 2.2 and 2.4, each vertex of the 4-cycle has an outside neighbor in $B_5^{[2,5]}$ and the outside neighbors are distinct, then $|F_e^0| \geq 4$, which contradicts with $|F_e^0| = 3$. So the assumption does not hold. Therefore, $|V(C)| \geq 5! - 2$.

Subcase 2.2. $|F_e^j| \geq 3$, for some $j \in [2, 5]$.

Note that $|F_e| \leq 7$ and $|F_e^1| \geq 3$. There is only one F_e^j can satisfy $|F_e^j| \geq 3$ for all $j \in [2, 5]$. Otherwise, if there are two F_e^j's satisfy $|F_e^j| \geq 3$, then $|F_e| \geq 3 \times 3 = 9 > 7$, a contradiction. Without loss of generality, we suppose that $|F_e^2| \geq 3$. Since $|F_e^1| = max\{|F_e^i| : 1 \leq i \leq 5\}$, we have $|F_e^2| = 3$. Otherwise, if $|F_e^2| = 4$, then $|F_e| \geq 2 \times 4 = 8 > 7$, a contradiction. Then $3 \leq |F_e^1| \leq 4$, $|F_e^{[3,5]}| \leq 1$ and $|F_e^0| \leq 1$. Note that $|F_e^s| \leq 1$ and $|E_{st}(B_5)| = (5-2)! = 6 > 1 \geq |F_e^0|$ for distinct $s, t \in [3,5]$, $B_5^{[3,5]} - F_e^{[3,5]}$ is connected by Claim 1.

Subcase 2.2.1. $B_5^i - F_e^i$ is connected for all $i \in [1,2]$.

In this subcase, both $B_5^1 - F_e^1$ and $B_5^2 - F_e^2$ are connected. Since $|E_{i3}(B_5)| = 6 > 1 \geq |F_e^0|$ for all $i \in [1,2]$, $B_5 - F_e$ is connected. So $|V(C)| = |V(B_5)| = 5! > 5! - 2$.

Subcase 2.2.2. Only $B_5^1 - F_e^1$ is disconnected.

In this subcase, $B_5^2 - F_e^2$ is connected. Since $|E_{23}(B_5)| = 6 > 1 \geq |F_e^0|$, $B_5^{[2,5]} - F_e^{[2,5]}$ is connected by Proposition 2.3. This is similar to Case 2.1, we have $|V(C)| \geq 5! - 2$.

Subcase 2.2.3. Only $B_5^2 - F_e^2$ is disconnected.

In this subcase, $B_5^1 - F_e^1$ is connected. Note that $|F_e^2| = 3 = 2 \times 4 - 5$. There is a connected component C_2 of $B_5^2 - F_e^2$ with $|V(C_2)| \geq 4! - 1$ by Lemma 3.1. Since $|E_{13}(B_5)| = 6 > 1 \geq |F_e^0|$, $B_5[V(B_5^1 - F_e^1) \cup V(B_5^{[3,5]} - F_e^{[3,5]})]$ is connected by Proposition 2.3. Since $|E_{B_5}(V(C_2), V(B_5^3)) - F_e| \geq |E_{23}(B_5)| - |V(B_5^2) - V(C_2)| - |F_e^0| \geq (5-2)! - [4! - (4! - 1)] - 1 = 4 > 0$, $B_5[V(C_2) \cup V(B_5^1 - F_e^1) \cup V(B_5^{[3,5]} - F_e^{[3,5]})]$ is connected and it is C. So $|V(C)| \geq 5! - 1 > 5! - 2$.

Subcase 2.2.4. $B_5^i - F_e^i$ is disconnected for all $i \in [1,2]$.

In this subcase, $B_5^1 - F_e^1$ is disconnected and $B_5^2 - F_e^2$ is disconnected. Since $|V(C_2)| \geq 4! - 1$ by Lemma 3.1, there is at most one isolated vertex $u_2 \in V(B_5^2 - F_e^2 - V(C_2))$. Note that $3 \leq |F_e^1| \leq 4$ and $B_5^1 - F_e^1$ is disconnected. Then $B_5^1 - F_e^1 - V(C_1)$ can only be one isolated vertex u_1 with $3 \leq |F_e^1| \leq 4$, a $K_2 = u_1v_1$ with $|F_e^1| = 4$ or a 4-cycle with $|F_e^1| = 4$. When $B_5^1 - F_e^1 - V(C_1)$ is a K_2 or a 4-cycle, $|F_e^1| = 4$, this moment, $|F_e^0| = |F_e| - |F_e^1| - |F_e^2| = 7 - 4 - 3 = 0$. $|E_{B_5}(V(C_i), V(B_5^3)) - F_e| \geq |E_{13}(B_5)| - |V(B_5^i) - V(C_i)| - |F_e^0| \geq (5-2)! - [4! - (4! - 4)] - 1 = 1 > 0$ for all $i \in [1,2]$, $B_5[V(C_i) \cup V(B_5^{[3,5]} - F_e^{[3,5]})]$ is connected by Proposition 2.3. Then $B_5[V(C_1) \cup V(C_2) \cup V(B_5^{[3,5]} - F_e^{[3,5])})]$ is connected and it is C. By Proposition 2.2 and Proposition 2.4, each vertex in $B_5^1 - F_e^1 - V(C_1)$ has an outside neighbor and the outside neighbors are distinct, combining this with $B_5^2 - F_e^2 - V(C_2)$ has only 1 isolated vertex u_2, $B_5 - F_e$ is connected. So $|V(C)| = |V(B_5)| = 5! > 5! - 2$. When $B_5^1 - F_e^1 - V(C_1)$ is an isolated vertex u_1, $|V(C_1)| \geq 4! - 1$. Since $|E_{B_5}(V(C_i), V(B_5^3)) - F_e| \geq |E_{13}(B_5)| - |V(B_5^i) - V(C_i)| - |F_e^0| \geq (5-2)! - [4! - (4! - 1)] - 1 = 4 > 0$ for all $i \in [1,2]$, $B_5[V(C_i) \cup V(B_5^{[3,5]} - F_e^{[3,5]})]$ is connected by Proposition 2.3. Then $B_5[V(C_1) \cup V(C_2) \cup V(B_5^{[3,5]} - F_e^{[3,5])})]$ is connected and it is C. There is an isolated vertex in $B_5^i - F_e^i - V(C_i)$ for all $i \in [1,2]$, so $|V(C)| \geq 5! - 2$. □

Lemma 4.2. *Let $F_e \subseteq E(B_n)$ with $|F_e| \leq 3n-8$ for $n \geq 5$. There is a connected component C of $B_n - F_e$ with $|V(C)| \geq n! - 2$.*

Proof. We prove this conclusion by induction on n. When $n = 5$, the conclusion holds by Lemma 4.1. Assume that the conclusion holds for $n-1$ with $n \geq 6$, we will prove that the conclusion holds for n. Let $|F_e^1| = max\{|F_e^i| : 1 \leq i \leq n\}$. Let C_i be the largest connected component of $B_n^i - F_e^i$ for all $i \in [1,n]$.

Claim 1. If $|F_e^i| \leq n-3$, $|F_e^j| \leq n-3$ and $|E_{ij}(B_n)| > |F_e^0|$, then $B_n[V(B_n^i - F_e^i) \cup V(B_n^j - F_e^j)]$ is connected for distinct $i, j \in [1,n]$ and $n \geq 6$.

By Proposition 2.5, $\lambda(B_n^i) = n-2$. Since $|F_e^i| \leq n-3$ and $|F_e^j| \leq n-3$, both $B_n^i - F_e^i$ and $B_n^j - F_e^j$ are connected for distinct $i, j \in [1,n]$. Since $|E_{ij}(B_n)| = (n-2)! > |F_e^0|$ for distinct $i, j \in [1,n]$ and $n \geq 6$, $B_n[V(B_n^i - F_e^i) \cup V(B_n^j - F_e^j)]$ is connected by Proposition 2.3. The proof of claim 1 is complete.

Case 1. $|F_e^1| \leq n-3$.

Since $|F_e^1| \leq n-3$, $|F_e^i| \leq n-3$ for all $i \in [1,n]$ and $|F_e^0| \leq |F_e| - |F_e^1| \leq 3n-8$. Note that $|E_{ij}(B_n)| = (n-2)! > 3n-8 \geq |F_e^0|$ for $n \geq 6$. We have $B_n - F_e$ is connected by Claim 1. Thus, $|V(C)| = |V(B_n)| = n! > n! - 2$.

Case 2. $n-2 \leq |F_e^1| \leq 2n-7$.

In this case, $|F_e^0| \leq |F_e| - |F_e^1| \leq (3n-8) - (n-2) = 2n-6$ and $|F_e^{[2,n]}| \leq |F_e| - |F_e^1| \leq 2n-6$.

Subcase 2.1. $|F_e^j| \leq n-3$, for all $j \in [2,n]$.

Since $|E_{st}(B_n)| = (n-2)! > 2n-6 \geq |F_e^0|$ for distinct $s,t \in [2,n]$ and $n \geq 6$, $B_n[V(B_n^s - F_e^s) \cup V(B_n^t - F_e^t)]$ is connected by Claim 1. Then $B_n^{[2,n]} - F_e^{[2,n]}$ is connected and it is a subgraph of C. Since $n-2 \leq |F_e^1| \leq 2n-7$, we have $|V(C_1)| \geq (n-1)! - 1$ by Lemma 3.1. Since $|E_{B_n}(V(C_1), V(B_n^{[2,3]})) - F_e| \geq |E_{12}(B_n)| + |E_{13}(B_n)| - 2|V(B_n^1) - V(C_1)| - |F_e^0| \geq 2(n-2)! - 2[(n-1)! - ((n-1)! - 1)] - (2n-6) = 2(n-2)! - 2n + 4 > 0$ for $n \geq 6$, we have $B_n[V(C_1) \cup V(B_n^{[2,n]} - F_e^{[2,n]})]$ is connected and it is C. So $|V(C)| \geq n! - 1 > n! - 2$.

Subcase 2.2. $n-2 \leq |F_e^j| \leq 2n-7$, for some $j \in [2,n]$.

Since $|F_e| \leq 3n-8$, there is at most one F_e^j can satisfy $n-2 \leq |F_e^j| \leq 2n-7$ for some $j \in [2,n]$. Otherwise, if there are two F_e^j's satisfy $n-2 \leq |F_e^j| \leq 2n-7$ for some $j \in [2,n]$, then $|F_e| \geq 3(n-2) = 3n-6 > 3n-8$, a contradiction. Without loss of generality, we suppose that $n-2 \leq |F_e^2| \leq 2n-7$, then $|F_e^0| \leq |F_e| - 2|F_e^1| \leq (3n-8) - 2(n-2) = n-4$, $|F_e^s| \leq |F_e| - 2|F_e^1| \leq n-4 < n-3$ and $|F_e^t| \leq n-3$ for distinct $s,t \in [3,n]$ and $n \geq 6$. Since $|E_{st}(B_n)| = (n-2)! > n-4 \geq |F_e^0|$, we have $B_n[V(B_n^s - F_e^s) \cup V(B_n^t - F_e^t)]$ is connected by Claim 1 for distinct $s,t \in [3,n]$ and $n \geq 6$. Then $B_n^{[3,n]} - F_e^{[3,n]}$ is connected and it is a subgraph of C. There is only 1 isolated vertex $u_i \in V(B_n^i - F_e^i - V(C_i))$ and $|V(C_i)| \geq (n-1)! - 1$ when $B_n^i - F_e^i$ is disconnected by Lemma 3.1 for all $i \in [1,2]$. Since $|E_{B_n}(V(C_i), V(B_n^3)) - F_e| \geq |E_{i3}(B_n)| - |V(B_n^i) - V(C_i)| - |F_e^0| \geq (n-2)! - [(n-1)! - ((n-1)! - 1)] - (n-4) = (n-2)! - n + 3 > 0$ for $n \geq 6$, $B_n[V(C_i) \cup V(B_n^{[3,n]} - F_e^{[3,n]})]$ is connected by Proposition 2.3 for all $i \in [1,2]$. Then $B_n[V(C_1) \cup V(C_2) \cup V(B_n^{[3,n]} - F_e^{[3,n]})]$ is connected.

Subcase 2.2.1. $B_n^i - F_e^i$ is connected for all $i \in [1,2]$.

In this subcase, both $B_n^1 - F_e^1$ and $B_n^2 - F_e^2$ are connected. since $|F_e^0| \leq n-4 < (n-2)! = |E_{i3}(B_n)|$ for all $i \in [1,2]$, $B_n - F_e$ is connected by Proposition 2.3, $|V(C)| = |V(B_n)| = n! > n! - 2$.

Subcase 2.2.2. Only one of $B_n^1 - F_e^1$ and $B_n^2 - F_e^2$ is disconnected.

Note that $n-2 \leq |F_e^1| \leq 2n-7$ and $n-2 \leq |F_e^2| \leq 2n-7$. Without loss of generality, we suppose that $B_n^1 - F_e^1$ is disconnected and $B_n^2 - F_e^2$ is connected. Then there is an isolated vertex in $B_n^1 - F_e^1 - V(C_1)$ and $B_n[V(C_1) \cup V(B_n^{[2,n]} - F_e^{[2,n]})]$ is connected by the above. So $|V(C)| \geq |V(B_n)| - 1 \geq n! - 1 > n! - 2$.

Subcase 2.2.3. $B_n^i - F_e^i$ is disconnected for all $i \in [1,2]$.

In this subcase, $B_n^1 - F_e^1$ is disconnected and $B_n^2 - F_e^2$ is disconnected, $B_n[V(C_1) \cup V(C_2) \cup V(B_n^{[3,n]} - F_e^{[3,n]})]$ is connected by the above, there are at most two vertices can not be contained in C. So $|V(C)| \geq |V(B_n)| - 2 \geq n! - 2$.

Case 3. $2n-6 \leq |F_e^1| \leq 3n-11$.

In this case, $|F_e^0| \leq |F_e| - |F_e^1| \leq (3n-8) - (2n-6) \leq n-2$ and $|F_e^{[2,n]}| \leq |F_e| - |F_e^1| \leq n-2$.

Subcase 3.1. $|F_e^j| \leq n-3$, for all $j \in [2,n]$.

Since $|F_e^s| \leq n-3$, $|F_e^t| \leq n-3$, $|E_{st}(B_n)| = (n-2)! > n-2 \geq |F_e^0|$ for distinct $s,t \in [2,n]$ and $n \geq 6$, we have $B_n^{[2,n]} - F_e^{[2,n]}$ is connected by Claim 1. Then $B_n^{[2,n]} - F_e^{[2,n]}$ is a subgraph of C. Note that $2n-6 \leq |F_e^1| \leq 3n-11$ and $B_n \cong B_{n-1}$ for $n \geq 6$. By the assumption, there is a connected component of $B_n^1 - F_e^1$ with $|V(C_1)| \geq (n-1)! - 2$. Since $|E_{B_n}(V(C_1), V(B_n^2)) - F_e| \geq |E_{12}(B_n)| - |V(B_n^1) - V(C_1)| - |F_e^0| \geq (n-2)! - [(n-1)! - ((n-1)! - 2)] - (n-2) = (n-2)! - n > 0$ for $n \geq 6$, $B_n[V(C_1) \cup V(B_n^{[2,n]} - F_e^{[2,n]})]$ is connected and it is C. So $|V(C)| \geq n! - 2$.

Subcase 3.2. $n-2 \le |F_e^j| \le 2n-7$, for some $j \in [2,n]$.

Note that $|F_e| \le 3n-8$ and $|F_e^{[2,n]}| \le n-2$. There is only one F_e^j can satisfy $|F_e^j| = n-2$ for all $j \in [2,n]$. Without loss of generality, we suppose that $|F_e^2| = n-2$. This moment, $|F_e^1| = |F_e| - |F_e^2| = (3n-8)-(n-2) = 2n-6$. Then $|F_e^s| = 0$ for all $s \in [3,n]$ and $|F_e^0| = 0$. $B_n[V(B_n^s - F_e^s) \cup V(B_n^t - F_e^t)]$ is connected by Claim 1. Then $B_n^{[3,n]} - F_e^{[3,n]}$ is connected and it is a subgraph of C.

Subcase 3.2.1. $B_n^i - F_e^i$ is connected for all $i \in [1,2]$.

In this subcase, both $B_n^1 - F_e^1$ and $B_n^2 - F_e^2$ are connected. Since $|F_e^0| = 0$, we have $B_n - F_e$ is connected by Proposition 2.3, $|V(C)| = |V(B_n)| = n! > n! - 2$.

Subcase 3.2.2. Only $B_n^1 - F_e^1$ is disconnected.

In this subcase, $B_n^2 - F_e^2$ is connected. Note that $|F_e^1| = 2n-6 < 2n-5$. There is a connected component C_1 of $B_n^1 - F_e^1$ with $|V(C_1)| \ge n! - 1$ by Lemma 3.1. $|E_{23}(B_n)| = (n-2)! > 0 = |F_e^0|$, we have $B_n^{[2,n]} - F_e^{[2,n]}$ is connected. Since $|E_{B_n}(V(C_1), V(B_n^3)) - F_e| \ge |E_{13}(B_n)| - |V(B_n^1) - V(C_1)| - |F_e^0| \ge (n-2)! - [(n-1)! - ((n-1)!-1)] - 0 = (n-2)! - 1 > 0$ for $n \ge 6$, we have $B_n[V(C_1) \cup V(B_n^{[2,n]} - F_e^{[2,n]})]$ is connected and it is C. So $|V(C)| \ge n! - 1 > n! - 2$.

Subcase 3.2.3. Only $B_n^2 - F_e^2$ is disconnected.

In this subcase, $B_n^1 - F_e^1$ is connected. Since $|F_e^2| = n-2 < 2n-5$ for $n \ge 6$, there is a connected component C_2 of $B_n^2 - F_e^2$ with $|V(C_2)| \ge n! - 1$ by Lemma 3.1. By Proposition 2.3 and $|F_e^0| = 0$, $B_n[V(B_n^1 - F_e^1) \cup V(B_n^{[3,n]} - F_e^{[3,n]})]$ is connected. Since $|E_{B_n}(V(C_2), V(B_n^3)) - F_e| \ge |E_{23}(B_n)| - |V(B_n^2) - V(C_2)| - |F_e^0| \ge (n-2)! - [(n-1)! - ((n-1)!-1)] - 0 = (n-2)! - 1 > 0$ for $n \ge 6$, we have $B_n[V(B_n^1 - F_e^1) \cup V(C_2) \cup V(B_n^{[3,n]} - F_e^{[3,n]})]$ is connected and it is C. So $|V(C)| \ge n! - 1 > n! - 2$.

Subcase 3.2.4. $B_n^i - F_e^i$ is disconnected for all $i \in [1,2]$.

In this subcase, $B_n^1 - F_e^1$ is disconnected and $B_n^2 - F_e^2$ is disconnected. By Subcase 3.2.2 and Subcase 3.2.3, we have $|V(C_1)| \ge n!-1$, $|V(C_2)| \ge n!-1$. Moreover, we have $B_n[V(C_1) \cup V(C_2) \cup V(B_n^{[3,n]} - F_e^{[3,n]})]$ is connected and it is C. So $|V(C)| \ge n! - 2$.

Case 4. $3n-10 \le |F_e^1| \le 3n-8$.

In this case, $|F_e^0| \le |F_e| - |F_e^1| \le (3n-8)-(3n-10) = 2$ and $|F_e^{[2,n]}| \le |F_e| - |F_e^1| = 2$. Since $|E_{ij}(B_n)| = (n-2)! > 2 \ge |F_e^0|$ for distinct $i, j \in [2,n]$ and $n \ge 6$, $B_n^{[2,n]} - F_e^{[2,n]}$ is connected by Claim 1. If $B_n^1 - F_e^1$ is connected, then $B_n - F_e$ is connected by Proposition 2.3 and $|V(C)| = |V(B_n)| = n! > n! - 2$. If $B_n^1 - F_e^1$ is disconnected, note that $|F_e^0| \le 2$, then at most two vertices in B_n^1 cannot be contained in C, thus, $|V(C)| \ge n! - 2$.

By Case 1-4, When $B_n - F_e$ is disconnected, there is a largest connected component C of $B_n - F_e$ with $|V(C)| \ge n! - 2$ when $|F_e| \le 3n-8$. □

Remark 4.3. *Lemma 4.2 does not hold for $n = 4$. When $n = 4$, $|F_e| = 3 \times 4 - 8 = 4$. See from Figure 1, $B_4 - F_e$ has a 4-cycle as its component except the largest connected component C. Then $|V(C)| = 4! - 4$.*

Remark 4.4. *The conclusion of Lemma 4.2 is optimal in that there is an edge set $F_e \subseteq E(B_n)$ with $|F_e| = 3n-7$ such that $B_n - F_e$ has a connected component C with $|V(C)| \le n! - 3$ for $n \ge 6$.*

Let a 2-path be uvw with the vertices in this order and $F = \{u, v, w\}$ be the vertex set of the 2-path. Then $|E_{B_n}(F)| = 3n-7$ by Proposition 2.5. Let $F_e = E_{B_n}(F) \subseteq E(B_n)$. Then uvw is a component of $B_n - F_e$ except C. So $|V(C)| \le n! - 3$.

Theorem 4.5. *B_n is $(2n-6)$-conditional edge-fault-tolerant strongly Menger edge connected for $n \ge 5$.*

Proof. Let $S_e \subseteq E(B_n)$ be an faulty edge set with $|S_e| \le 2n-6$ and $\delta(B_n - S_e) \ge 2$. Let two distinct vertices $u, v \in V(B_n)$, $a = min\{d_{B_n-S_e}(u), d_{B_n-S_e}(v)\}$ and $E_f \subseteq E(B_n) - S_e$ with $|E_f| \leq a-1$. By Theorem 1.1, Definition 1.2 and Definition 1.3, it is sufficient to prove that u is connected to v in $B_n - S_e - E_f$. We prove this by contradiction, assume that u and v are disconnected in $B_n - S_e - E_f$ for $E_f \subseteq E(B_n) - S_e$ with $|E_f| \leq a-1$. Note that $d_{B_n-S_e}(u) \leq n-1$, $d_{B_n-S_e}(v) \leq n-1$ and $|E_f| \leq n-2$ by Proposition 2.5. Then $|F_e| = |S_e \cup E_f| \leq (2n-6)+(n-2) = 3n-8$. By Lemma 4.2, $B_n - S_e - E_f$ has a connected component C with $|V(C)| \geq n!-2$. Note that u and v are disconnected in $B_n - S_e - E_f$, $|\{u, v\} \cap V(C)| \leq 1$. Without loss of generality, we suppose that $u \in V(C)$ and $v \notin V(C)$.

Case 1. $|V(C)| = n! - 1$.

In this case, v is the isolated vertex of $B_n - S_e - E_f$. Then $E_{B_n}(\{v\}, N_{B_n-S_e}(v)) \subseteq E_f$. This means $|E_f| \geq d_{B_n-S_e}(v)$, which contradicts with $|E_f| \leq a - 1 \leq d_{B_n-S_e}(v) - 1$. The assumption does not hold.

Case 2. $|V(C)| = n! - 2$.

In this case, there is another vertex $w \in V(B_n - S_e - E_f)$ and $w \notin V(C)$. If v and w are connected, then $K_2 = vw$ and $E_{B_n-S_e}(\{vw\}) \subseteq E_f$. Since $\delta(B_n - S_e) \geq 2$, v has at least one degree in E_f. Then $|E_f| \geq d_{B_n-S_e}(v) - 1 + 1 = d_{B_n-S_e}(v)$, which contradicts with $|E_f| \leq a - 1 \leq d_{B_n-S_e}(v) - 1$. If v and w are disconnected, then $E_{B_n}(\{v\}, N_{B_n-S_e}(v)) \subseteq E_f$. This means $|E_f| \geq d_{B_n-S_e}(v)$, which contradicts with $|E_f| \leq a - 1 \leq d_{B_n-S_e}(v) - 1$. The assumption does not hold.

Hence, B_n is $(2n-6)$-conditional edge-fault-tolerant strongly Menger edge connected for $n \geq 5$. □

5. Results discussion

The edge connectivity is one of the basic parameters to estimate the fault tolerance of the interconnection network. We show that B_n is $(n - 3)$-edge-fault-tolerant strongly Menger edge connected for $n \geq 3$ and $(2n - 6)$-conditional edge-fault-tolerant strongly Menger edge connected for $n \geq 5$. There are some problems need further discussion.

(1) The fault tolerance of an interconnected network is the maximum number of vertices or edges allowed to fail without affecting the communication of other fault-free vertices or edges. Let F_e be the set of faulty edges in B_n. We show that when B_n has $(n - 3)$ ($|F_e| = n - 3$) edges fault, there are $min\{d_{B_n-F_e}(u), d_{B_n-F_e}(v)\}$ edge-disjoint paths that can connect any two distinct vertices in B_n. This conclusion can ensure that B_n can work normally even with $(n - 3)$ faulty edge and improve the reliability of the system. When the condition of any vertex in B_n has at least two neighbors is imposed, the conclusion can ensure that B_n can work normally even with $(2n - 6)$ faulty edges and improve the reliability of the system.

(2) In fact, we don't have a counter example to show that B_n is not edge-fault-tolerant strongly Menger edge connected for larger $|F_e|$, the researchers can explore the more general $|F_e|$.

(3) In this paper, we discuss under the condition that each vertex or edge in B_n has a uniform and independent failure probability. But actually, vertex or edge may have different failure probabilities. In addition, vertices or edges may be related and fail at the same time, so that vertices or edges failures may not be independent. These need to be discussed furthermore.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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