AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(12): 13170–13186.
DOI: 10.3934/math.2021761
Received: 06 May 2021
Accepted: 23 August 2021
Published: 15 September 2021

Research article

Fixed point results of an implicit iterative scheme for fractal generations

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Abstract: In this paper, we derive the escape criteria for general complex polynomial $f(x) = \sum_{i=0}^{p} a_i x^i$ with $p \geq 2$, where $a_i \in \mathbb{C}$ for $i = 0, 1, 2, \ldots, p$ to generate the fractals. Moreover, we study the orbit of an implicit iteration (i.e., Jungck-Ishikawa iteration with s-convexity) and develop algorithms for Mandelbrot set and Multi-corn or Multi-edge set. Moreover, we draw some complex graphs and observe how the graph of Mandelbrot set and Multi-corn or Multi-edge set vary with the variation of a_i's.

Keywords: fixed point theory; fractals; Jungck-Ishikawa iteration
Mathematics Subject Classification: 37F45, 37F50, 47J25

1. Introduction

In recent years, fractal geometry plays an important role in software engineering. The word fractal was first time used by Mandelbrot in 1970, when he visualized the complex graph for a function $f(z) = z^2 + c$ [1]. The obtained image was self similar and he named it a fractal. Infact, he extended the work by G. Julia and examined the properties of Julia sets [2]. He exhibited that Julia sets have best extravagance of artistic patterns. After his work a progression of research have been done on various types of fractals. For example, the generalized Mandelbrot set was studied in [3]. Some rational, trigonometric, logarithmic and exponential functions were used to generate fractals in [4]. The quaternions, bi-complex and tri-complex function were used to generate fractals in [5,6] and in [7], the authors produced some generalized fractals (for example Julia and Mandelbrot sets).

The fixed point theory picked up the most noteworthy focus when Rani et al. in [8] and [9] utilized some fixed point iterative technique in the representation of fractals. They introduced some superior fractals and examined their properties. After their exploration the fixed point theory turned into a typical part of mathematics and software engineering. The fractals generated by Picard, Mann, Ishikawa, S, CR and SP were presented in [10–13] and [4]. The threshold escape radii for Jungck-Mann, Jungck-Ishikawa and Jungck-Noor with the blend of s-convexity in the second sense were demonstrated in [14,15]. Boundaries of Julia sets were presented in [16]. The organic took after pictures were shown in [17] and Modified outcomes for Julia sets and Mandelbrot sets were built up in [18].

This work present some fractals by implicit iterations. We derive the escape radius by extended Jungck-Ishikawa iteration with s-convex combination for general complex polynomial. For derived threshold escape radius, we establish algorithms to generate some kind of fractals in Jungck-Ishikawa orbit with s-convex combination (JIO) (i.e., Mandelbrot set and Multi-corn or Multi-edge set). We discuss the behavior of some complex polynomials in the form of some examples and demonstrate that the fractal image also depends upon a_i's. Furthermore, we show that for $p = 3$ the Mandelbrot set is not necessarily cubic, it may be quadratic also and same arguments for $p > 3$.

2. Preliminaries

Definition 2.1 (Julia set [19]). *Consider $f_c : \mathbb{C} \to \mathbb{C}$ a complex polynomial depends upon $c \in \mathbb{C}$. The filled Julia set, denoted by F_{f_c} for a function f_c can be defined by*

$$F_{f_c} = \{z \in \mathbb{C} : \left|f_c^p(z)\right| \nrightarrow \infty \textit{ as } p \to \infty\}, \tag{2.1}$$

where $f_c^p(z)$ is p-th iterate of function f_c. Julia set B_{f_c} for complex polynomial f_c can be defined as the boundary of filled Julia set F_{f_c}, i.e., $B_{f_c} = \partial F_{f_c}$. (The boundary of filled Julia set is called the Julia set.)

Definition 2.2 (Mandelbrot set [20]). *The mandelbrot set is defined as the collection of parameters c for which the filled Julia set of $f_c : \mathbb{C} \to \mathbb{C}$ is connected and the mandelbrot set is denoted by M. Mathematically,*

$$M = \{c \in \mathbb{C} : F_{f_c} \textit{ is connected}\}, \tag{2.2}$$

or mathematical definition of mandelbrot set can also be written as [21]:

$$M = \{c \in \mathbb{C} : \left|f_c^p(\theta)\right| \nrightarrow \infty \text{ as } p \to \infty\}, \tag{2.3}$$

f has only critical point θ (i.e., $f'(\theta) = 0$). So we choose θ as the initial point.

Definition 2.3 (Multi-corn or Multi-edge set [18]). *Let $A_c(z) = \bar{z}^p + c$, where $c \in \mathbb{C}$. The Multi-corn set $\mathcal{M}^*$ for A_c is defined as the collection of all $c \in \mathbb{C}$ for which the orbit of 0 under the action of A_c is bounded, i.e.,*

$$\mathcal{M}^* = \{c \in \mathbb{C} : |A_c^n(0)| \nrightarrow \infty \textit{ as } n \to \infty\} \tag{2.4}$$

Multi-corn or Multi-edge set for $p = 2$ is called the Tri-corn or Tri-edge set.

In past years researchers utilized various ways to deal with produce Julia sets. Some famous algorithms to envision the Julia sets are, distance estimator, escape time and potential function calculations. To create filled Julia sets, just Julia sets and Fatou spaces, we use escape time calculations. The escape time calculation repeat the function upto the longing number of iterations. The algorithm create two sets, one is comprises of focuses for which the JIO doesn't disappear to boundlessness (for example filled Julia set or limit of Julia set) and the subsequent set comprises of focuses for which the JIO break to boundlessness (for example Fatou areas).

Definition 2.4 (Jungck iteration [22]). *Let $P, Q : X \to X$ be two maps such that P is one to one and Q is differentiable of degree greater than and equal to 2. For any $x_0 \in X$ the Jungck iteration is defined in the following way*

$$P(x_{k+1}) = Q(x_k), \tag{2.5}$$

where $k = 0, 1, \ldots$.

Definition 2.5 (Jungck-Mann iteration [14]). *Let $P, Q : \mathbb{C} \to \mathbb{C}$ be two complex maps such that Q is a complex polynomial of degree greater than and equal to 2, also differentiable and P is injective. For any $x_0 \in \mathbb{C}$ the Jungck-Mann iteration defined as:*

$$P(x_{k+1}) = (1 - a)^s P(x_k) + a^s Q(x_k), \tag{2.6}$$

where $a, s \in (0, 1]$, $n = 0, 1, 2, \ldots$.

Definition 2.6 (Jungck-Mann iteration with s-convex combination in second sense [14]). *Consider $P, Q : \mathbb{C} \to \mathbb{C}$ are two complex valued mapping where Q is a complex differentiable polynomial having degree more than or equal to 2 and P is an injective map. For any $x_0 \in \mathbb{C}$ the Jungck-Mann iteration with s-convex combination in second sense can be defined as:*

$$P(x_{k+1}) = (1 - a)^s Q(x_k) + a^s P(x_k), \tag{2.7}$$

where $a, s \in (0, 1]$, $k = 0, 1, 2, \ldots$.

Remark 2.7. *One can observe that iteration (2.6) becomes:*

- *Picard orbit when $P(x) = x$ and $a, s = 1$,*
- *Mann orbit when $P(x) = x$ and $s = 1$,*
- *Jungck Mann orbit when $s = 1$.*

Definition 2.8 (Jungck-Ishikawa iteration [14]). *Consider $P, Q : \mathbb{C} \to \mathbb{C}$ are two complex valued mapping where Q is a complex differentiable polynomial having degree more than or equal to 2 and P is an injective map. For any $x_0 \in \mathbb{C}$ the Jungck-Ishikawa iteration is defined in the following way*

$$\begin{cases} P(x_{k+1}) = (1 - a)P(x_k) + aQ(y_k), \\ P(y_k) = (1 - b)P(x_k) + bQ(x_k), \end{cases} \tag{2.8}$$

where $a, b \in (0, 1]$ and $k = 0, 1, 2, \ldots$.

Along these lines, in proposed iteration we manage two distinct mappings, we break f into two mappings P and Q so that $f = Q - P$ and P is injective. This kind of arrangement of f restriction us to receive P as injective mapping and Q as analytical mapping. In this manner we infer new threshold escape radius and execute in our algorithms to imagine a fractals.

3. Escape criteria for general complex polynomial via Jungck-Ishikawa iteration with s-convexity (JIO)

Right now, we demonstrate the threshold escape radius for Jungck-Ishikawa iteration with s-convex mix in second sense for general complex polynomial. In this section we prove the threshold escape radius for Jungck-Ishikawa iteration with s-convex combination in second sense for general complex polynomial.

Definition 3.1 (Jungck-Ishikawa iteration with s-convex combination [14])**.** *Consider $P, Q : \mathbb{C} \to \mathbb{C}$ are two complex valued mapping where Q is a complex differentiable polynomial having degree more than or equal to 2 and P is an injective map. For any $x_0 \in \mathbb{C}$ the Jungck-Ishikawa iteration with s-convex combination in second sense is defined in the following way*

$$\begin{cases} P(x_{k+1}) = (1-a)^s P(x_k) + a^s Q(y_k), \\ P(y_k) = (1-b)^s P(x_k) + b^s Q(x_k), \end{cases} \tag{3.1}$$

where $s, a, b \in (0, 1]$ and $k = 0, 1, 2, \ldots$.

Remark 3.2. *We observed that the Jungck-Ishikawa Orbit with s-convexity change into:*

- *Picard orbit when $P(x) = x, b = 0$ and $a, s = 1$,*
- *Mann orbit when $P(x) = x, b = 0$ and $s = 1$,*
- *Ishikawa orbit when $P(x) = x$ and $s = 1$,*
- *Jungck-Ishikawa orbit when $s = 1$.*

We utilize Jungck-Ishikawa iteration with s-convex combination in second sense for proving that the polynomial $f(x) = \sum_{i=0}^{p} a_i x^i$ where $p \geq 2$, $a_i \in \mathbb{C}$ for $i = 0, 1, 2, \ldots, p$ and $\left|a_p\right| > \sum_{i=2}^{p-1} |a_i|$ with choice $Q(z) = \sum_{i=2}^{p} a_i x^i + a_0$ and $P(z) = a_1 x$ to generate some kind of fractals:

Theorem 3.3. *Suppose that $|x| \geq |a_0| > \eta_1 = \left(\frac{2(1+|a_1|)}{sa(\alpha-\beta)}\right)^{\frac{1}{p-1}}$ and $|x| \geq |a_0| > \eta_2 = \left(\frac{2(1+|a_1|)}{sb(\alpha-\beta)}\right)^{\frac{1}{p-1}}$ where $\alpha = \left|a_p\right|, \beta = \sum_{2}^{p-1} |a_i|$ also $a, b, s \in (0, 1]$, then the sequence $\{x_k\}_{k\in\mathbb{N}}$ define as follows:*

$$\begin{cases} P(x_{k+1}) = (1-a)^s P(x_k) + a^s Q(y_k), \\ P(y_k) = (1-b)^s P(x_k) + b^s Q(x_k), \end{cases} \tag{3.2}$$

where $s, a, b \in (0, 1]$ and $k = 0, 1, 2, \ldots$. Then $|x_k| \to \infty$ as $k \to \infty$.

Proof. Since $f(x) = \sum_{i=0}^{p} a_i x^i$, where $a_i \in \mathbb{C}$ for $i = 0, 1, 2, \ldots, p$, $x_0 = x$ and $y_0 = y$. Handling f as $f = Q - P$ with choice $Q(x) = \sum_{i=2}^{p} a_i x^i + a_0$ and $P(x) = a_1 x$, then

$$\begin{aligned} |P(y_0)| &= |(1-b)^s P(x) + b^s Q(x)| \\ &= \left|(1-b)^s a_1 x + (1-(1-b))^s \left(\sum_{i=2}^{p} a_i x^i + a_0\right)\right|. \end{aligned}$$

Now, using the fact that $s \leq 1$ and expansion to degree 1 of b and $1 - b$, we arrive at

$$\begin{aligned} |a_1 y_0| &\geq (1 - s(1-b))|\sum_{i=2}^{p} a_i x^i + a_0| - (1 - sb)|a_1 x| \\ &\geq |(s - s(1-b))(\sum_{i=2}^{p} a_i x^i + a_0)| - |(1 - sb) a_1 x|. \end{aligned}$$

Since $|x| \geq |a_0|$ and $sb < 1$ we have

$$|a_1 y_0| \geq sb \left|\sum_{i=2}^{p} a_i x^i\right| - sb\,|a_0| - (1 - sb)\,|a_1 x|$$

$$
\begin{aligned}
&= sb\left|\sum_{i=2}^{p} a_i x^i\right| - sb\,|a_0| - |a_1 x| + sb\,|a_1 x| \\
&\geq sb\left|\sum_{i=2}^{p} a_i x^i\right| - |x| - |a_1|\,|x| \\
&= |x|\,(sb\left|\sum_{i=2}^{p} a_i x^{i-1}\right| - (1 + |a_1|)).
\end{aligned}
$$

This provides

$$
\begin{aligned}
|y_0| &\geq |x|\left(\frac{sb\left|\sum_{i=2}^{p} a_i x^{i-1}\right|}{1+|a_1|} - 1\right) \\
&\geq |x|\left(\frac{sb\left|x^{p-1}\right|\left(\left|a_p\right| - \sum_{i=2}^{p-1}|a_i|\right)}{1+|a_1|} - 1\right) \\
&= |x|\left(\frac{sb\left|x^{p-1}\right|(\alpha-\beta)}{1+|a_1|} - 1\right) \\
|y_0| &\geq sb\,|x|\,.
\end{aligned}
$$

Because $|x| \geq |a_0| > \left(\frac{2(1+|a_1|)}{sb(\alpha-\beta)}\right)^{\frac{1}{p-1}}$ where $\alpha = \left|a_p\right|, \beta = \sum_{2}^{p-1}|a_i|$, this produced the situation $|x|\left(\frac{|x|^{p-1}(sb(\alpha-\beta))}{1+|a_1|} - 1\right) > |x| \geq sb|x|$.

Now, in next iteration, we arrive at

$$
\begin{aligned}
|P(x_1)| &= |(1-a)^s P(x_0) + a^s Q(y_0)| \\
|a_1 x_1| &= \left|(1-a)^s a_1 x + a^s\left(\sum_{i=2}^{p} a_i y^i + a_0\right)\right| \\
&\geq \left|(1-sa)a_1 x + (1-s(1-a)\left(\sum_{i=2}^{p} a_i y^i + a_0\right)\right| \\
&\geq \left|(1-s(1-a)\left(\sum_{i=2}^{p} a_i y^i + a_0\right)\right| - |(1-sa)a_1 x| \\
&\geq sa\left|\sum_{i=2}^{p} a_i y^i\right| - (1+|a_1|)\,|x| \\
&\geq |x|\left(s^2 ab\left|\sum_{i=2}^{p} a_i x^i\right| - (1+|a_1|)\right). \\
&\geq |x|\left(s^2 ab\left|x^{p-1}\right|\left(\left|a_p\right| - \sum_{i=2}^{p-1}|a_i|\right) - (1+|a_1|)\right).
\end{aligned}
$$

Thus

$$
|x_1| \geq |x|\left(\frac{s^2 ab\left|x^{p-1}\right|(\alpha-\beta)}{1+|a_1|} - 1\right). \tag{3.3}
$$

Since $|x| > \left(\frac{2(1+|a_1|)}{sa(\alpha-\beta)}\right)^{\frac{1}{p-1}}$ and $|x| > \left(\frac{2(1+|a_1|)}{sb(\alpha-\beta)}\right)^{\frac{1}{p-1}}$, then $|x|^{p-1} > \left(\frac{2(1+|a_1|)}{s^2ab(\alpha-\beta)}\right)$ and this implies $\frac{s^2ab(\alpha-\beta)|x|^{p-1}}{1+|a_1|} - 1 > 1$. Therefore there exists $\lambda > 0$ such that $\frac{s^2ab(\alpha-\beta)|x|^{p-1}}{1+|a_1|} - 1 > 1 + \lambda$. Consequently $|x_1| > (1 + \lambda)|x|$. In particular $|x_1| > |x|$. So we may iterate to find $|x_k| > (1 + \lambda)^k|x|$. Hence, the orbit of z tends to infinity and this completes the proof. □

Corollary 3.4. *Suppose that*

$$|a_0| > \eta_1 \textit{and } |a_0| > \eta_2,$$

then the Jungck-Ishikawa orbit with s-convexity escapes to infinity.

Corollary 3.5. *Suppose that* $a, b, s \in (0, 1]$ *and*

$$|x| > \max\{|a_0|, \eta_1, \eta_2\},$$

therefore there exists $\lambda > 0$ *such that* $|x_k| > (1 + \lambda)^k|x|$ *and* $|x_k| \to \infty$ *as* $k \to \infty$.

Corollary 3.6. *Suppose that*

$$|x_m| > \max\{|a_0|, \eta_1, \eta_2\},$$

for some $m \geq 0$. *Therefore, we have some* $\lambda > 0$ *s.t* $|x_{m+k}| > (1 + \lambda)^k|x_m|$ *and* $|x_k| \to \infty$ *as* $k \to \infty$.

Now we prove the converse of Theorem 3.3.

Theorem 3.7. *Suppose that* $\{x_k\}_{k\in\mathbb{N}}$ *be the sequence of points in Jungck-Ishikawa orbit with s-convexity for complex polynomial* $f(x) = \sum_{i=0}^{p} a_i x^i$ *with* $p \geq 2$, *where* $a_i \in \mathbb{C}$ *for* $i = 0, 1, 2, \ldots, p$ *such that* $|x_k| \to \infty$ *as* $k \to \infty$, *then* $|x| \geq |a_0| > \eta_1 = \left(\frac{2(1+|a_1|)}{sa(\alpha-\beta)}\right)^{\frac{1}{p-1}}$ *and* $|x| \geq |a_0| > \eta_2 = \left(\frac{2(1+|a_1|)}{sb(\alpha-\beta)}\right)^{\frac{1}{p-1}}$ *where* $\alpha = \left|a_p\right|, \beta = \sum\limits_{2}^{p-1} |a_i|$ *and* $a, b, s \in (0, 1]$.

Proof. Since $\{x_k\}_{k\in\mathbb{N}}$ is the sequence of points in Jungck-Ishikawa orbit with s-convexity for complex polynomial $f(x) = \sum_{i=0}^{p} a_i x^i$ with $p \geq 2$ such that $|x_k| \to \infty$ as $k \to \infty$, therefore there exists $\lambda > 0$ such that

$$|x_k| > (1 + \lambda)^k|x|.$$

For $k = 1$, we get

$$|x_1| \geq (1 + \lambda)|x|. \tag{3.4}$$

Since $f(x) = \sum_{i=0}^{p} a_i x^i$, where $a_i \in \mathbb{C}$ for $i = 0, 1, 2, \ldots, p$, $x_0 = x$ and $y_0 = y$. We break down the function f in such a way that: $Q(x) = \sum_{i=2}^{p} a_i x^i + a_0$ and $P(x) = a_1 x$, then

$$\begin{aligned}
|P(y_0)| &= |(1 - b)^s P(x) + b^s Q(x)| \\
&= \left|(1 - b)^s a_1 x + (1 - (1 - b))^s \left(\sum_{i=2}^{p} a_i x^i + a_0\right)\right|.
\end{aligned}$$

Using the fact that $s \leq 1$ and expansion upto degree 1 of b and $1-b$, we get

$$
\begin{aligned}
|a_1 y_0| &\geq (1-s(1-b))|\sum_{i=2}^{p} a_i x^i + a_0| - (1-sb)|a_1 x| \\
&\geq |(s-s(1-b))(\sum_{i=2}^{p} a_i x^i + a_0)| - |(1-sb)a_1 x|.
\end{aligned}
$$

Since for the generation of Mandelbrot sets it must be true $|x| \geq |a_0|$ and $sb < 1$ we have

$$
\begin{aligned}
|a_1 y_0| &\geq sb\left|\sum_{i=2}^{p} a_i x^i\right| - sb\,|a_0| - (1-sb)\,|a_1 x| \\
&= sb\left|\sum_{i=2}^{p} a_i x^i\right| - sb\,|a_0| - |a_1 x| + sb\,|a_1 x| \\
&\geq sb\left|\sum_{i=2}^{p} a_i x^i\right| - |x| - |a_1|\,|x| \\
&= |x|\,(sb\left|\sum_{i=2}^{p} a_i x^{i-1}\right| - (1+|a_1|)).
\end{aligned}
$$

This provides

$$
\begin{aligned}
|y_0| &\geq |x|\left(\frac{sb\left|\sum_{i=2}^{p} a_i x^{i-1}\right|}{1+|a_1|} - 1\right) \\
&\geq |x|\left(\frac{sb\left|x^{p-1}\right|\left(\left|a_p\right| - \sum_{i=2}^{p-1}|a_i|\right)}{1+|a_1|} - 1\right) \\
&= |x|\left(\frac{sb\left|x^{p-1}\right|(\alpha-\beta)}{1+|a_1|} - 1\right) \\
|y_0| &\geq sb\,|x|.
\end{aligned}
$$

Because the Mandelbrot set is bounded therefore $|x|\left(\frac{sb\left|x^{p-1}\right|(\alpha-\beta)}{1+|a_1|} - 1\right) \geq 1$.

In next step of iteration we have

$$
\begin{aligned}
|P(x_1)| &= |(1-a)^s P(x_0) + a^s Q(y_0)| \\
|a_1 x_1| &= \left|(1-a)^s a_1 x + a^s\left(\sum_{i=2}^{p} a_i y^i + a_0\right)\right| \\
&\geq \left|(1-sa)a_1 x + (1-s(1-a)\left(\sum_{i=2}^{p} a_i y^i + a_0\right)\right| \\
&\geq \left|(1-s(1-a)\left(\sum_{i=2}^{p} a_i y^i + a_0\right)\right| - |(1-sa)a_1 x|
\end{aligned}
$$

$$\geq sa\left|\sum_{i=2}^{p} a_i y^i\right| - (1+|a_1|)\,|x|$$
$$\geq |x|\left(s^2ab\left|\sum_{i=2}^{p} a_i x^i\right| - (1+|a_1|)\right).$$
$$\geq |x|\left(s^2ab\left|x^{p-1}\right|\left(\left|a_p\right| - \sum_{i=2}^{p-1}|a_i|\right) - (1+|a_1|)\right).$$

Thus

$$|x_1| \geq |x|\left(\frac{s^2ab\left|x^{p-1}\right|(\alpha-\beta)}{1+|a_1|} - 1\right). \tag{3.5}$$

Comparing (3.4) and (3.5), we have

$$\frac{s^2ab\,(\alpha-\beta)\left|x^{p-1}\right|}{1+|a_1|} - 1 = 1+\lambda$$
$$\frac{s^2ab\,(\alpha-\beta)\left|x^{p-1}\right|}{1+|a_1|} - 1 > 1,$$

because $\lambda > 0$. This implies

$$|x| > \left(\frac{2(1+|a_1|)}{s^2ab\,(\alpha-\beta)}\right)^{\frac{1}{p-1}}.$$

As a result, we obtain $|x| > \left(\frac{2(1+|a_1|)}{sa(\alpha-\beta)}\right)^{\frac{1}{p-1}}$ and $|x| > \left(\frac{2(1+|a_1|)}{sb(\alpha-\beta)}\right)^{\frac{1}{p-1}}$ where $p \geq 2$ and $a, b, s \in (0,1]$. To visualize complex fractal $|x| \geq |a_0|$ must exist, because for any given point $|x| < |a_0|$, we have to compute the Jungck-Ishikawa orbit with s-convexity of x. If for some k, $|x_k|$ lies outside the circle of radius $\max\{|a_0|, \eta_1, \eta_2\}$, we observed that Jungck-Ishikawa orbit with s-convexity escapes. Hence, x is not in the Julia sets and also, is not in Mandelbrot sets. But if the sequence $\{x_k\}_{k\in\mathbb{N}}$ is bounded to obey $|x| \geq |a_0|$, then by definition of complex fractals, the sequence $\{x_k\}_{k\in\mathbb{N}}$ lies in Jungck-Ishikawa orbit with s-convexity. Hence the result. □

4. Applications of fractals via proposed iteration

This section consists of two subsection. In first subsection we demonstrate some graphical examples of quadratic, cubic and quadric Mandelbrot sets in JIO and in second we present some graphs of Multicorn or Multi-edge sets in JIO.

4.1. Generation of Mandelbrot sets

Now we present some examples of Mandelbrot sets in Jungck-Ishikawa orbit with s-convex combination (JIO). In each example we set the maximum of iteration at $K = 25$, $a_0 = x$, $a = 0.9, b = 0.5$ and $s = 0.5$. The algorithms run in Mathematica at Dell machine with spec. Intel(R) Core(TM)i5-3320M CPU @ 2.60 GHz and 4GB RAM to visualize the Mandelbrot sets.

Algorithm 1: Mandelbrot set generation

Input: $f_c = \sum_{i=0}^{p} a_i x^i$ with $p = 2$ a complex polynomial, A-area for image, K-fixed number of iterations, $colourscale[0..h-1]$ colour scale with h colours.

Output: Complex graph of Mandelbrot set in area A.

for $a_0 \in A$ **do**
  $R = \max\{|a_0|, \eta_1, \eta_2\}$-threshold escape radius
  $k = 0$
  $z_0 = a_0$-initial guess for f_c
  while $k \leq K$ **do**
    $x_{k+1} = f_c(x_k)$
    if $|x_{k+1}| > R$ **then**
      break
    $k = k + 1$
  $i = \lfloor (h-1)\frac{k}{K} \rfloor$
  colour a_0 with $colourmap[i]$

In first example we generate the Mandelbrot sets of complex polynomial $f(x) = \sum_{i=0}^{p} a_i x^i$ with $p = 2$ at different values of a_1 and a_2 and observe that the image changes with the change of a_1 and a_2. The visualized images shown in Figures 1 and 2. The area occupied by images and values of a_1 and a_2 were given in Table 1.

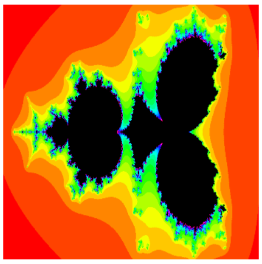

Figure 1. Quadratic Mandelbrot set in JIO with escape time 30.201 sec.

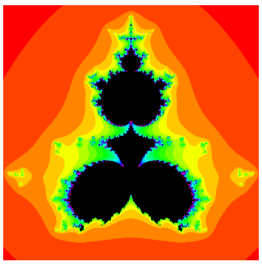

Figure 2. Quadratic Mandelbrot set in JIO with escape time 30.639 sec.

Table 1. Input parameters for quadratic Mandelbrot sets.

Figure	a_1	a_2	Area
1	2	0.5	$[-20, 8.5] \times [-9, 9]$
2	2	$2\mathbf{i}$	$[-3.5, 3.5] \times [-2, 5]$

In second example we generate the Mandelbrot sets of complex polynomial $f(x) = \sum_{i=0}^{p} a_i x^i$ with $p = 3$ at different values of a_1, a_2 and a_3. From Figures 3–5, we notice that the image of cubic Mandelbrot set changes with the change of a_1, a_2 and a_3. At some values of a_1, a_2 and a_3 the images of cubic Mandelbrot set resembled with quadratic Mandelbrot set. The area occupied by images and values of a_1, a_2 and a_3 were given in Table 2.

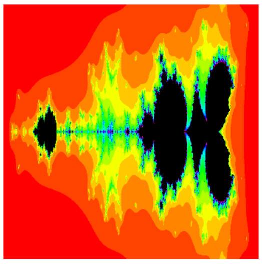

Figure 3. Cubic Mandelbrot set in JIO with escape time 47.97 sec.

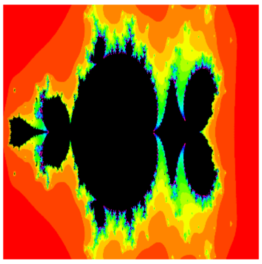

Figure 4. Cubic Mandelbrot set in JIO with escape time 57.096 sec.

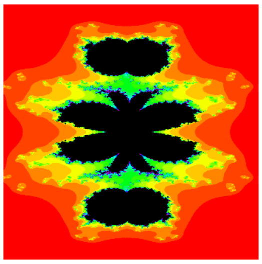

Figure 5. Cubic Mandelbrot set in JIO with escape time 46.457 sec.

Table 2. Input parameters for cubic Mandelbrot sets.

Figure	a_1	a_2	a_3	Area
3	2	20	30	$[-1.5, 0.3] \times [-0.3, 0.3]$
4	$\frac{1}{20}$	$\frac{1}{2}$	1	$[-0.028, 0.009] \times [-0.008, 0.008]$
5	$\frac{1}{2}$	$\frac{1}{2}$	50	$[-0.05, 0.05] \times [-0.09, 0.09]$

In third example we generate the Mandelbrot sets of complex polynomial $f(x) = \sum_{i=0}^{p} a_i x^i$ with $p = 4$ at different values of a_1, a_2, a_3 and a_4. From Figures 6–9, we notice that the image of quadric Mandelbrot set also changes with the change of a_1, a_2, a_3 and a_4. At some values of a_1, a_2, a_3 and a_4 the images of quadric Mandelbrot set resembled with quadratic and cubic Mandelbrot sets. The area occupied by images and the values of a_1, a_2, a_3 and a_4 were given in Table 3.

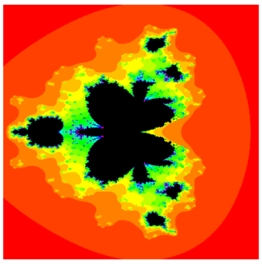

Figure 6. Quadric Mandelbrot set in JIO with escape time 67.408 sec.

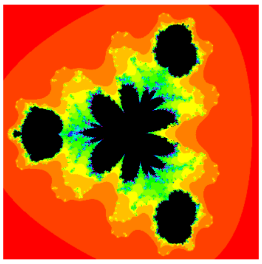

Figure 7. Quadric Mandelbrot set in JIO with escape time 85.816 sec.

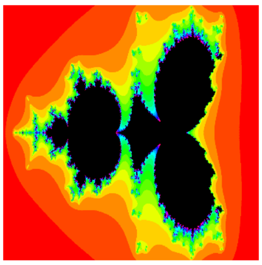

Figure 8. Quadric Mandelbrot set in JIO with escape time 104.458 sec.

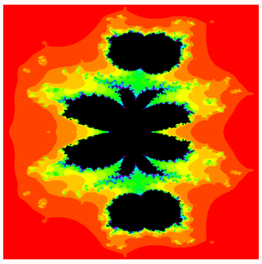

Figure 9. Quadric Mandelbrot set in JIO with escape time 59.093 sec.

Table 3. Input parameters for quadric Mandelbrot sets.

Figure	a_1	a_2	a_3	a_4	Area
6	$\frac{1}{2}$	1	1	5	$[-0.3, 0.3] \times [-0.4, 0.4]$
7	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{i}$	50	$[-0.4, 0.4] \times [-0.15, 0.15]$
8	$\frac{1}{10}$	28	30	50	$[-0.0009, 0.0004] \times [-0.0004, 0.0004]$
9	1	0	30	35	$[-0.2, 0.2] \times [-0.3, 0.3]$

4.2. Generation of Multi-corn or Multi-edge sets

Here we present some examples of Multi-corn or Multi-edge sets in Jungck-Ishikawa orbit with s-convex combination (JIO) for conjugate complex polynomial $f(x) = \sum_{i=0}^{p} a_i x^i$ with $p = 2$. In each example we set the maximum of iteration at $K = 25$, $a_0 = x$, $a = 0.9, b = 0.5$ and $s = 0.5$ as we set in previous subsection. The algorithms run in Mathematica at same machine we used for Mandelbrot sets.

Algorithm 2: Multi-corn set generation

Input: $f_c = \sum_{i=0}^{p} a_i x^i$ with $p = 2$ a conjugate complex polynomial, A area for image, K fixed number of iterations, $colourscale[0..h-1]$ colourscale with h colours.

Output: Complex graph of Multi-corn or Multi-edge set in area A.

for $a_0 \in A$ **do**
  $R = \max\{|a_0|, \eta_1, \eta_2\}$ threshold escape radius
  $k = 0$
  $x_0 = a_0$ initial guess for f_c
  while $k \leq K$ **do**
    $x_{k+1} = f_c(x_k)$
    if $|x_{k+1}| > R$ **then**
      break
    $k = k + 1$
  $i = \lfloor (h-1)\frac{k}{K} \rfloor$
  colour a_0 with $colourmap[i]$

In this example we visualize the Multi-corn sets for complex polynomial $f(x) = \sum_{i=0}^{p} a_i x^i$ with $p = 2$ at different values of a_1 and a_2 these Multi-corn sets are actually the Tri-corn sets. Moreover we observe that the image of Multi-corn set also changes with the change of a_1 and a_2. The generated images shown in Figures 10 and 11. The area occupied by images and values of a_1 and a_2 were given in Table 4.

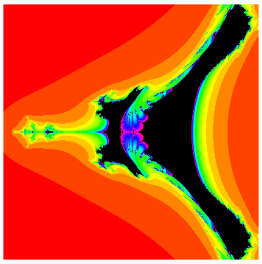

Figure 10. Tri-corn set in JIO with escape time 47.16 sec.

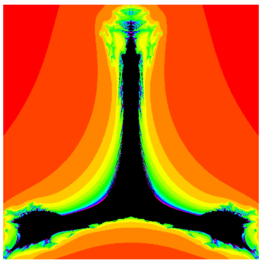

Figure 11. Tri-corn set in JIO with escape time 51.09 sec.

Table 4. Input parameters for Tri-corn sets.

Figure	a_1	a_2	Area
10	2	0.5	$[-20, 8.5] \times [-9, 9]$
11	2	$2\mathbf{i}$	$[-3.5, 3.5] \times [-2, 5]$

In second last example we generate some cubic Multi-corn sets for complex polynomial $f(x) = \sum_{i=0}^{p} a_i x^i$ with $p = 3$ at different values of a_1, a_2 and a_3. The Figures 12–14 show that some images of quadractic and cubic Multi-corn sets resembled with each other. Also we note that the Multi-corn set changes with the change of a_1, a_2 and a_3. The area occupied by images and values of a_1 and a_2 were given in Table 5.

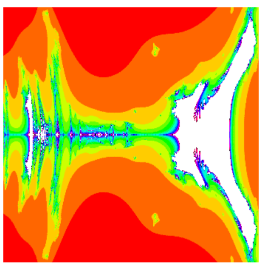

Figure 12. Multi-corn set in JIO with escape time 229.399 sec.

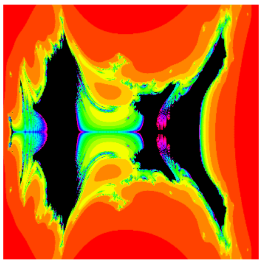

Figure 13. Multi-corn set in JIO with escape time 103.881 sec.

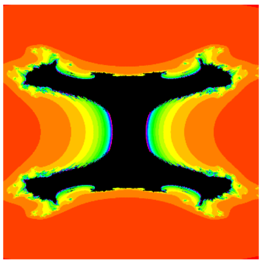

Figure 14. Multi-corn set in JIO with escape time 57.216 sec.

Table 5. Input parameters for Multi-corn sets.

Figure	a_1	a_2	a_3	Area
12	2	20	30	$[-1.3, 0.19] \times [-0.28, 0.28]$
13	$\frac{1}{20}$	$\frac{1}{2}$	1	$[-0.028, 0.009] \times [-0.008, 0.008]$
14	$\frac{1}{2}$	$\frac{1}{2}$	50	$[-0.05, 0.05] \times [-0.09, 0.09]$

In last example we present some Multi-corn sets for complex polynomial $f(x) = \sum_{i=0}^{p} a_i x^i$ with $p = 4$ at different values of a_1, a_2, a_3 and a_4. The resulting Figures 15–18 demonstrate the image of quadric Multi-corn set also changes with the change of a_1, a_2, a_3 and a_4. At some values of a_1, a_2, a_3 and a_4 the images resembled with quadratic and cubic Multi-corn sets. The area occupied by images and values of a_1 and a_2 were given in Table 6.

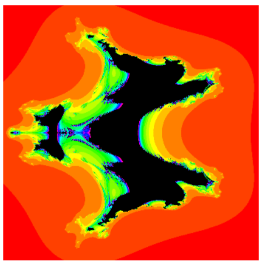

Figure 15. Multi-corn set in JIOwith escape time 93.086 sec.

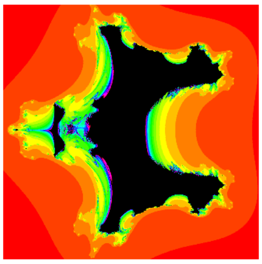

Figure 16. Multi-corn set in JIO with escape time 97.376 sec.

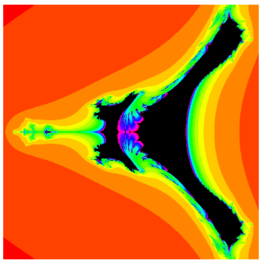

Figure 17. Multi-corn set in JIO with escape time 59.093 sec.

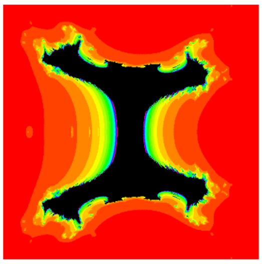

Figure 18. Multi-corn set in JIO with escape time 53.805 sec.

Table 6. Input parameters for Multi-corn sets.

Figure	a_1	a_2	a_3	a_4	Area
15	$\frac{1}{2}$	1	1	5	$[-0.3, 0.3] \times [-0.4, 0.4]$
16	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{i}$	50	$[-0.4, 0.4] \times [-0.15, 0.15]$
17	$\frac{1}{10}$	28	30	50	$[-0.0009, 0.0004] \times [-0.0004, 0.0004]$
18	1	0	30	35	$[-0.28, 0.28] \times [-0.3, 0.3]$

5. Conclusions

We studied implicit iteration as an application of fractal geometry. We derived the threshold radius of Jungck-Ishikawa with s-convexity for general complex polynomial $f(x) = \sum_{i=0}^{p} a_i x^i$ with $p \geq 2$, where $a_i \in \mathbb{C}$ for $i = 0, 1, 2, \ldots, p$ instead of $f(x) = x^p - ax + c$ to generate the fractals. We used the established radius in algorithms to visualize Mandelbrot set and Multi-corn or Multi-edge set. We showed in examples that the images of Mandelbrot sets and Multi-corn or Multi-edge sets vary with the variation in a_i's. For different values of a_i's in quadratic, cubic and quadric complex polynomials some resembled and inspiring images obtained. Our next work will demonstrate the derivations of threshold radii's for general complex polynomial via all other Jungck type iterations with s-convex combination in the first and second sense.

Conflict of interest

The authors declare that they do not have any conflict of interests.

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