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Research article

Oscillation theorems of solution of second-order neutral differential equations

Ali Muhib[1,2], **Hammad Alotaibi**[3,*], **Omar Bazighifan**[4] **and Kamsing Nonlaopon**[5,*]

[1] Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt
[2] Department of Mathematics, Faculty of Education – Al-Nadirah, Ibb University, Ibb, Yemen
[3] Department of Mathematics, Faculty of Science, Taif University, P.O.Box 11099, Taif 21944, Saudi Arabia
[4] Department of Mathematics, Faculty of Science, Hadhramout University, Seiyun, Yemen
[5] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

* **Correspondence:** Email: hm.alotaibi@tu.edu.sa, nkamsi@kku.ac.th; Tel: +660866421582.

Abstract: In this paper, we aim to explore the oscillation of solutions for a class of second-order neutral functional differential equations. We propose new criteria to ensure that all obtained solutions are oscillatory. The obtained results can be used to develop and provide theoretical support for and further develop the oscillation study for a class of second-order neutral differential equations. Finally, an illustrated example is given to demonstrate the effectiveness of our new criteria.

Keywords: neutral differential equation; oscillation criteria; second-order neutral differential equation
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

In this work, we are concerned with the oscillation of solutions of the second-order neutral differential equation

$$\left(r(l)\left(u(l)+p(l)\,u(\tau(l))\right)'\right)'+\int_a^b q(l,s)\,f(u(\eta(l,s)))\,\mathrm{d}s=0,\qquad l\geq l_0. \tag{1.1}$$

Throughout this work, we assume:

(S_1) $p,\ r\in C^1([l_0,\infty)),\ r(l)>0,\ r'(l)\geq 0,\ 0\leq p(l)<1$ and $\inf_{l\geq l_0} p(l)\neq 0,\ p,q$ do not vanish identically on any half-line of form $[l_0,\infty)$;

(S_2) $q \in C([l_0,\infty)\times(a,b),[0,\infty)), q(l,s)\geq 0$ and

$$\int_{l_0}^{\infty}\frac{1}{r(s)}\mathrm{d}s<\infty; \tag{1.2}$$

(S_3) $f \in C(\mathbb{R},\mathbb{R})$, and there exists a positive constant k such that $f(l,u)/u^{\alpha}\geq k$ for $u\neq 0$, α is a quotient of odd positive integers and $\alpha>1$;

(S_4) $\tau \in C([l_0,\infty),(0,\infty))$, $\tau(l)\leq l$ and $\lim_{l\to\infty}\tau(l)=\infty$;

(S_5) $\eta \in C([l_0,\infty)\times(a,b),(0,\infty)), \eta(l,s)\geq l, \eta$ has nonnegative partial derivatives and $\lim_{l\to\infty}\eta(l,s)=\infty$.

By a solution of (1.1) we mean a function $u\in C^1[l_u,\infty)$, $l_u\geq l_0$, which has the property $r(l)(z'(l))\in C^1[l_u,\infty)$, and satisfies (1.1) on $[l_u,\infty)$. We consider only those solutions u of (1.1) which satisfy $\alpha\sup\{|u(l)|: l\geq L\}>0$ for all $L\geq l_u$. A solution u of (1.1) is said to be non-oscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory. Equation (1.1) itself is called oscillatory if all its solutions are oscillatory.

The neutral differential equations find a wide range of applications in certain high-tech fields, such as control theory, mechanical engineering, physics, population dynamics, economics, and so on, see [1, 2]. Thus, we can see that investigating the oscillatory and asymptotic behavior of solutions of neutral differential equations is of great importance. During the past period, many papers appeared on the oscillatory behavior of differential equations of neutral and delay type, see [3–16], and the references mentioned therein.

Second-order differential equations are derived from nuclear physics, fluid mechanics, gas dynamics, and astrophysics applications. Wong [17] established the oscillation criteria for equation

$$u''(l)+r(l)|u(l)|^{\alpha}\, sgnu(l)=0$$

in the super-linear case. After that, many researchers developed several criteria for oscillation for second-order differential equations, see for example [18–33] and the references mentioned therein.

Wang et al. [21] considered the oscillation behavior of solutions of the second-order neutral differential equations of the form

$$(r(l)(u(l)+p(l)u(l-\tau))')'+\int_a^b q(l,s)u^{\alpha}(\eta(l,s))\,\mathrm{d}\sigma(s)=0, \tag{1.3}$$

where $\alpha=1$.

Baculikova and Dzurina [34] studied the asymptotic and oscillation behavior of the solutions of second-order delay differential equation

$$(r(l)(u'(l))^{\alpha})'+p(l)u^{\alpha}(\tau(l))=0,$$

where

$$\int_{l_0}^{l}r^{-1/\alpha}(s)\,\mathrm{d}s\to\infty \quad \text{as } l\to\infty.$$

The main purpose of this work is to establish new criteria for oscillation of (1.1). New criteria ensure that all solutions are oscillatory, which is an extension and expansion of previous results. An example was provided to illustrate the results.

2. Main results

In this section, we will mention our main results.

To prove our results, we will need the following notation:

$$z(l) := u(l) + p(l)\,u(\tau(l)),$$
$$\pi(l) = \int_l^\infty \frac{1}{r(s)}\mathrm{d}s.$$

Lemma 2.1. *Assume that* (1.2) *holds and*

$$\int_{l_2}^\infty \int_a^b q(v,s)\,\mathrm{d}s\mathrm{d}v = \infty. \tag{2.1}$$

Moreover, assume that (1.1) *has a positive solution* $u(l)$ *on* $[l_1,\infty)$*, where* $l_1 \in [l_0,\infty)$ *is sufficiently large. Then,*

$$z(l) > 0,\ z'(l) < 0 \ \text{ and } \ (r(l)z'(l))' \le 0, \quad l \ge l_1. \tag{2.2}$$

Furthermore,

$$\left(\frac{z(l)}{\pi(l)}\right)' \ge 0, \quad l \ge l_1. \tag{2.3}$$

Proof. Suppose that $u(l)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $u(l) > 0, u(\tau(l)) > 0$ and $u(\eta(l,s)) > 0$ for $l \ge l_1 \ge l_0$. From (1.1), we have

$$\begin{aligned}(r(l)z'(l))' &= -\int_a^b q(l,s)\,f(u(\eta(l,s)))\,\mathrm{d}s\\ &\le -k\int_a^b q(l,s)\,u^\alpha(\eta(l,s))\,\mathrm{d}s \le 0,\ l \ge l_1.\end{aligned} \tag{2.4}$$

Therefore, $z'(l)$ is either eventually negative or eventually positive. Assume on the contrary that there exists $l_2 \ge l_1$ such that $z'(l) > 0$. Then,

$$u(l) = z(l) - p(l)\,u(\tau(l)) \ge z(l) - p(l)\,z(\tau(l)) \ge z(l)(1-p(l)), \quad l \ge l_2,$$

and so,

$$u(\eta(l,s)) \ge z(\eta(l,s))(1-p(\eta(l,s))), \quad l \ge l_2, \tag{2.5}$$

which together with (2.4) implies that

$$(r(l)z'(l))' \le -k\int_a^b q(l,s)\,z^\alpha(\eta(l,s))(1-p(\eta(l,s)))^\alpha\,\mathrm{d}s.$$

Since $\eta(l,s)$ is nondecreasing with respect to s, we get $\eta(l,s) \ge \eta(l,a)$ for $s \in (a,b)$, and so

$$(r(l)z'(l))' \le -kz^\alpha(\eta(l,a))\int_a^b q(l,s)(1-p(\eta(l,s)))^\alpha\,\mathrm{d}s. \tag{2.6}$$

Define the Riccati function by
$$\omega(l)=\frac{r(l)\,z'(l)}{z^{\alpha}(\eta(l,a))}>0.$$
Differentiating and using (1.1) and (2.5), we arrive at
$$\begin{aligned}\omega'(l)&=\frac{(r(l)\,z'(l))'}{z^{\alpha}(\eta(l,a))}-\frac{\alpha r(l)\,z'(l)\,z'(\eta(l,a))\,\eta'(l,a)}{z^{\alpha+1}(\eta(l,a))}\\&\le -k\int_a^b q(l,s)\,(1-p(\eta(l,s)))^{\alpha}\,\mathrm{d}s-\frac{\alpha r(l)\,z'(l)\,z'(\eta(l,a))\,\eta'(l,a)}{z^{\alpha+1}(\eta(l,a))}\\&\le -k\int_a^b q(l,s)\,(1-p(\eta(l,s)))^{\alpha}\,\mathrm{d}s-\frac{\alpha z'(\eta(l,a))\,\eta'(l,a)}{z(\eta(l,a))}\omega(l)\\&\le -k\int_a^b q(l,s)\,(1-p(\eta(l,s)))^{\alpha}\,\mathrm{d}s.\end{aligned}\tag{2.7}$$
Integrating (2.7) from l_2 to l, we obtain
$$\begin{aligned}\omega(l)&\le\omega(l_2)-k\int_{l_2}^{l}\int_a^b q(v,s)\,(1-p(\eta(v,s)))^{\alpha}\,\mathrm{d}s\mathrm{d}v\\&\le\omega(l_2)-k\inf_{l\ge l_2}(1-p(\eta(l,b)))^{\alpha}\int_{l_2}^{l}\int_a^b q(v,s)\,\mathrm{d}s\mathrm{d}v.\end{aligned}\tag{2.8}$$
From (2.1), we find that (2.8) comes to contradiction with the positivity of $\omega(l)$. Hence, the case $z'(l)>0$ is impossible. Thus, $z(l)$ satisfies (2.2) for $l\ge l_1$. Now, it follows from the monotonicity of $r(l)\,z'(l)$ that
$$z(l)\ge-\int_l^{\infty}\frac{r(s)\,z'(s)}{r(s)}\mathrm{d}s\ge -r(l)\,z'(l)\,\pi(l),\tag{2.9}$$
that is
$$z(l)+r(l)\,z'(l)\,\pi(l)\ge 0,\quad l\ge l_1.\tag{2.10}$$
Now
$$\left(\frac{z(l)}{\pi(l)}\right)'=\frac{\pi(l)\,z'(l)-z(l)\,\pi'(l)}{\pi^2(l)}.\tag{2.11}$$
Using (2.10) and (2.11), we conclude that
$$\left(\frac{z(l)}{\pi(l)}\right)'=\frac{r(l)\,\pi(l)\,z'(l)+z(l)}{r(l)\,\pi^2(l)}\ge 0,\quad l\ge l_1.$$
This completes the proof. □

Theorem 2.1. *Assume that* (1.2) *holds. If*
$$0<1-p(l)\frac{\pi(\tau(l))}{\pi(l)}<1,\ \ \inf_{l\ge l_1}\left(1-p(l)\frac{\pi(\tau(l))}{\pi(l)}\right)>0\tag{2.12}$$
and
$$\int_{l_0}^{\infty}\frac{1}{r(\kappa)}\int_{l_0}^{\kappa}\int_a^b q(v,s)\,\pi^{\alpha}(\eta(v,s))\,\mathrm{d}s\mathrm{d}v\mathrm{d}\kappa=\infty,\tag{2.13}$$
then, (1.1) *is oscillatory.*

Proof. Suppose that $u(l)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $u(l) > 0, u(\tau(l)) > 0$ and $u(\eta(l,s)) > 0$ for $l \geq l_1 \geq l_0$. By comparing (2.1) with (2.13), we find that (2.1) is necessary for (2.13) to be correct. In fact, since the function

$$\int_{l_0}^{\infty}\int_a^b q(v,s)\pi^{\alpha}(\eta(v,s))\,\mathrm{d}s\mathrm{d}v$$

is unbounded due to (1.2) and $\pi'(l) < 0$, (2.1) must hold. Thus, by Lemma 2.1, $z(l)$ satisfies (2.2). It follows from (2.3) that there is $c > 0$ such that

$$\frac{z(l)}{\pi(l)} \geq c \tag{2.14}$$

and

$$\begin{aligned} u(l) &= z(l) - p(l)u(\tau(l)) \geq z(l) - p(l)z(\tau(l)) \\ &\geq z(l) - p(l)\frac{\pi(\tau(l))z(l)}{\pi(l)} = z(l)\left(1 - p(l)\frac{\pi(\tau(l))}{\pi(l)}\right). \end{aligned} \tag{2.15}$$

Using (2.14) and (2.15) in (1.1), we have

$$(r(l)z'(l))' \leq -k\int_a^b q(l,s)z^{\alpha}(\eta(l,s))\left(1 - p(\eta(l,s))\frac{\pi(\tau(\eta(l,s)))}{\pi(\eta(l,s))}\right)^{\alpha}\mathrm{d}s \tag{2.16}$$

$$\leq -k\int_a^b q(l,s)\left(1 - p(\eta(l,s))\frac{\pi(\tau(\eta(l,s)))}{\pi(\eta(l,s))}\right)^{\alpha}c^{\alpha}\pi^{\alpha}(\eta(l,s))\,\mathrm{d}s. \tag{2.17}$$

Integrating (2.17) from l_1 to l, we have

$$r(l)z'(l) - r(l_1)z'(l_1) \leq -kc^{\alpha}\int_{l_1}^{l}\int_a^b q(v,s)\left(1 - p(\eta(v,s))\frac{\pi(\tau(\eta(v,s)))}{\pi(\eta(v,s))}\right)^{\alpha}\pi^{\alpha}(\eta(v,s))\,\mathrm{d}s\mathrm{d}v,$$

that is,

$$z'(l) \leq -\frac{kc^{\alpha}}{r(l)}\int_{l_1}^{l}\int_a^b q(v,s)\left(1 - p(\eta(v,s))\frac{\pi(\tau(\eta(v,s)))}{\pi(\eta(v,s))}\right)^{\alpha}\pi^{\alpha}(\eta(v,s))\,\mathrm{d}s\mathrm{d}v. \tag{2.18}$$

Integrating (2.18) from l_1 to l and taking (2.12) and (2.13) into account, we get

$$\begin{aligned} z(l) &\leq z(l_1) - \int_{l_1}^{l}\frac{kc^{\alpha}}{r(\kappa)}\int_{l_1}^{\kappa}\int_a^b q(v,s)\left(1 - p(\eta(v,s))\frac{\pi(\tau(\eta(v,s)))}{\pi(\eta(v,s))}\right)^{\alpha}\pi^{\alpha}(\eta(v,s))\,\mathrm{d}s\mathrm{d}v\mathrm{d}\kappa \\ &\leq z(l_1) - kc^{\alpha}\inf_{l\geq l_1}\left(1 - p(\eta(l,b))\frac{\pi(\tau(\eta(l,b)))}{\pi(\eta(l,b))}\right)^{\alpha}\int_{l_1}^{l}\frac{1}{r(\kappa)}\int_{l_1}^{\kappa}\int_a^b q(v,s)\pi^{\alpha}(\eta(v,s))\,\mathrm{d}s\mathrm{d}v\mathrm{d}\kappa, \end{aligned}$$

which is a contradiction. This completes the proof. □

Theorem 2.2. *Assume that* (1.2) *and* (2.1) *hold. If*

$$0 < 1 - p(l)\frac{\pi(\tau(l))}{\pi(l)} < 1 \tag{2.19}$$

and

$$W'(l) \geq \left(k \int_a^b q(l,s) M^\alpha(\eta(l,s)) \left(1 - p(\eta(l,s)) \frac{\pi(\tau(\eta(l,s)))}{\pi(\eta(l,s))}\right)^\alpha \mathrm{d}s\right) W^{\alpha^2}(\eta(l,a)) \tag{2.20}$$

is oscillatory, where

$$M(l) = k \int_l^\infty \pi(v) \int_a^b q(v,s) \pi^\alpha(\eta(v,s)) \left(1 - p(\eta(v,s)) \frac{\pi(\tau(\eta(v,s)))}{\pi(\eta(v,s))}\right)^\alpha \mathrm{d}s\mathrm{d}v,$$

then, (1.1) *is oscillatory.*

Proof. Suppose that $u(l)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $u(l) > 0, u(\tau(l)) > 0$ and $u(\eta(l,s)) > 0$ for $l \geq l_1 \geq l_0$. Since (2.1) holds, we can conclude that $z(l)$ satisfies (2.2) for $l \geq l_1$.

Considering the following

$$(z(l) + r(l) z'(l) \pi(l))' = z'(l) + (r(l) z'(l))' \pi(l) + r(l) z'(l) \pi'(l) = (r(l) z'(l))' \pi(l), \tag{2.21}$$

from (1.1) and (2.15), (2.21) becomes

$$\begin{aligned}(z(l) + r(l) z'(l) \pi(l))' &\leq -k\pi(l) \int_a^b q(l,s) u^\alpha(\eta(l,s)) \mathrm{d}s \\ &\leq -k\pi(l) \int_a^b q(l,s) z^\alpha(\eta(l,s)) \left(1 - p(\eta(l,s)) \frac{\pi(\tau(\eta(l,s)))}{\pi(\eta(l,s))}\right)^\alpha \mathrm{d}s \leq 0. \end{aligned} \tag{2.22}$$

Thus, we find $\Theta(l) = z(l) + r(l) z'(l) \pi(l) \geq 0$ is nonincreasing. Integrating (2.22) from l to ∞ and using (2.10), we have

$$\begin{aligned}\Theta(l) &\geq \Theta(\infty) + \int_l^\infty k\pi(v) \int_a^b q(v,s) z^\alpha(\eta(v,s)) \left(1 - p(\eta(v,s)) \frac{\pi(\tau(\eta(v,s)))}{\pi(\eta(v,s))}\right)^\alpha \mathrm{d}s\mathrm{d}v \\ &\geq k \int_l^\infty \pi(v) \int_a^b q(v,s) (-r(\eta(v,s)) z'(\eta(v,s)))^\alpha \pi^\alpha(\eta(v,s)) \left(1 - p(\eta(v,s)) \frac{\pi(\tau(\eta(v,s)))}{\pi(\eta(v,s))}\right)^\alpha \mathrm{d}s\mathrm{d}v \\ &\geq k \int_l^\infty \pi(v) \int_a^b q(v,s) (-r(v) z'(v))^\alpha \pi^\alpha(\eta(v,s)) \left(1 - p(\eta(v,s)) \frac{\pi(\tau(\eta(v,s)))}{\pi(\eta(v,s))}\right)^\alpha \mathrm{d}s\mathrm{d}v \\ &\geq (-r(l) z'(l))^\alpha k \int_l^\infty \pi(v) \int_a^b q(v,s) \pi^\alpha(\eta(v,s)) \left(1 - p(\eta(v,s)) \frac{\pi(\tau(\eta(v,s)))}{\pi(\eta(v,s))}\right)^\alpha \mathrm{d}s\mathrm{d}v,\end{aligned}$$

since $r(l) z'(l) \pi(l) < 0$, we get

$$\begin{aligned}z(l) &\geq (-r(l) z'(l))^\alpha k \int_l^\infty \pi(v) \int_a^b q(v,s) \pi^\alpha(\eta(v,s)) \left(1 - p(\eta(v,s)) \frac{\pi(\tau(\eta(v,s)))}{\pi(\eta(v,s))}\right)^\alpha \mathrm{d}s\mathrm{d}v \\ &\geq M(l) (-r(l) z'(l))^\alpha. \end{aligned} \tag{2.23}$$

Substituting (2.23) in (2.16), we see that $W(l) = -r(l) z'(l)$ is a positive solution of the following inequality

$$W'(l) \geq k \int_a^b q(l,s) M^\alpha(\eta(l,s)) \left(1 - p(\eta(l,s)) \frac{\pi(\tau(\eta(l,s)))}{\pi(\eta(l,s))}\right)^\alpha W^{\alpha^2}(\eta(l,s)) \mathrm{d}s.$$

Using increasing property of $W(l)$, we get

$$W'(l) \geq kW^{\alpha^2}(\eta(l,a)) \int_a^b q(l,s) M^{\alpha}(\eta(l,s)) \left(1 - p(\eta(l,s)) \frac{\pi(\tau(\eta(l,s)))}{\pi(\eta(l,s))}\right)^{\alpha} \mathrm{d}s,$$

that is,

$$W'(l) \geq \left(k \int_a^b q(l,s) M^{\alpha}(\eta(l,s)) \left(1 - p(\eta(l,s)) \frac{\pi(\tau(\eta(l,s)))}{\pi(\eta(l,s))}\right)^{\alpha} \mathrm{d}s\right) W^{\alpha^2}(\eta(l,a)),$$

which is a contradiction. This completes the proof. □

3. Example

In this section, we will present example to illustrate the results.

Example 3.1. *Consider the second-order neutral differential equation*

$$\left(l^4\left(u(l) + \left(\frac{l^{-3}}{3(l-1)^{-3}}\right)u(l-1)\right)'\right)' + \int_{1/2}^{1} 3^{\alpha} 4(l+s+1)^{3\alpha} l^3 u^{\alpha}(l+s+1)\,\mathrm{d}s = 0, \qquad (3.1)$$

where $\alpha > 1,\ l_0 = 0$. *Note that* $r(l) = l^4$, $p(l) = l^{-3}/3(l-1)^{-3}$, $q(l,s) = 3^{\alpha}4(l+s+1)^{3\alpha} l^3$, $\tau(l) = l-1,\ \eta(l,s) = l+s+1,\ a = 1/2,\ b = 1$ *and* $f(u) = u^{\alpha}$.

Now, we find

$$\pi(l) = \int_l^{\infty} \frac{1}{r(s)} ds = \frac{1}{3l^3}.$$

By using Theorem 2.1, we see that

$$0 < 1 - p(l)\frac{\pi(\tau(l))}{\pi(l)} = \frac{2}{3} < 1$$

and the condition (2.13) *is satisfied, where*

$$\int_{l_0}^{\infty} \frac{1}{r(\kappa)} \int_{l_0}^{\kappa} \int_a^b q(v,s)\pi^{\alpha}(\eta(v,s))\,\mathrm{d}s\mathrm{d}v\mathrm{d}\kappa = \int_0^{\infty} \frac{1}{2}\mathrm{d}\kappa = \infty.$$

Then (3.1) *is oscillatory.*

4. Conclusions

This article presents an interesting outcome by studying a class of second-order neutral functional differential equations. By explicitly taking advantage of proposing new criteria, we ensure that all solutions are oscillatory. The obtained results can provide theoretical support and empower the oscillation study for a class of second-order neutral differential equations. An illustrated example is presented to verify our results. For researchers interested in this field, and as part of our future research, we seek to find new results for the oscillation of the differential equation

$$\left(r(l)(u(l) + p(l)u(\tau(l)))^{(n-1)}\right)' + \int_a^b q(l,s) f(u(\eta(l,s)))\,\mathrm{d}s = 0$$

under condition

$$\int_{l_0}^{\infty} \frac{1}{r(s)}\mathrm{d}s < \infty.$$

Conflict of interest

The author declares that there is no competing interest.

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