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Research article

The limit of reciprocal sum of some subsequential Fibonacci numbers

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Abstract: This paper deals with the sum of reciprocal Fibonacci numbers. Let $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for any $n \in \mathbb{N}$. In this paper, we prove new estimates on $\displaystyle\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}$, where $m \in \mathbb{N}$ and $0 \le \ell \le m-1$. As a consequence of some inequalities, we prove

$$\lim_{n\to\infty} \left\{ \left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}} \right)^{-1} - (f_{mn-\ell} - f_{m(n-1)-\ell}) \right\} = 0.$$

And we also compute the explicit value of $\left\lfloor \left(\displaystyle\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}} \right)^{-1} \right\rfloor$. The interesting observation is that the value depends on $m(n+1)+\ell$.

Keywords: Fibonacci number; reciprocal sum; floor function; convergent series; Catalan's identity
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1. Introduction

Recently many interesting formulas of the floor function for tails of infinite convergent series have been obtained. The motivation of such research comes from the sum of reciprocal Fibonacci numbers. Let $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for all $n \in \mathbb{N}$. The *Fibonacci numbers* f_n can be written as the closed form by Binet's formula [9]

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

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where α and β are two solutions of $x^2-x-1=0$. No closed form for the *reciprocal Fibonacci constant*

$$\sum_{k=1}^{\infty}\frac{1}{f_k}\approx 3.359885666243\ldots$$

is known, but this number has been proved irrational by André-Jeannin [1] in 1989.
Problem A. Let $\sum_{k=1}^{\infty}a_k$ be a convergent series. Find the value of

$$\left\lfloor\left(\sum_{k=n}^{\infty}a_k\right)^{-1}\right\rfloor,$$

where $\lfloor x\rfloor$ denotes the greatest integer $\leq x$.

In 2008, Ohtsuka and Nakamura [7] proved

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{f_k}\right)^{-1}\right\rfloor=\begin{cases}f_{n-2}, & n\geq 2 \text{ is even;}\\ f_{n-2}-1, & n\geq 1 \text{ is odd.}\end{cases}\tag{1.1}$$

The similar results have been obtained for some generalized Fibonacci numbers in [3]. In [10, 12, 13], the reciprocal sums of some *linear subsequences* of Fibonacci numbers or Pell numbers have been obtained as following.

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{f_{2k}}\right)^{-1}\right\rfloor=f_{2n-1}-1,\quad \left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{f_{2k-1}}\right)^{-1}\right\rfloor=f_{2n-2},\tag{1.2}$$

and

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{f_{3k}}\right)^{-1}\right\rfloor=\begin{cases}f_{3n-1}+f_{3n-4}, & n\geq 2 \text{ is even;}\\ f_{3n-1}+f_{3n-4}-1, & n\geq 3 \text{ is odd.}\end{cases}\tag{1.3}$$

Also similar results for Fibonacci polynomials have been proved in [2, 11]. It is surprising that the floor function for tails of convergent series leads to such a simple formula. However, the above result does not give any information on the convergence of the decimal part of tails of convergent series.

Now we suggest the following new question on the convergence of the reciprocal sum.
Problem B.* Let $\sum_{k=1}^{\infty}a_k$ be a convergent series. Find a simple form of a function g_n (if it exists) such that

$$\lim_{n\to\infty}\left\{\left(\sum_{k=n}^{\infty}a_k\right)^{-1}-g_n\right\}=0.\tag{1.5}$$

*In [8], one can see the similar problem of finding g_n satisfying

$$\left(\sum_{k=n}^{\infty}a_k\right)^{-1}\sim g_n,\tag{1.4}$$

where $u_n\sim v_n$ means that u_n/v_n tends to 1 as $n\to\infty$. Since (1.5) implies (1.4) and the converse is not true, our main theorems are quite stronger results than Theorem 1 in [8].

For the proof of Problem B, we need sharper inequality than one in Problem A. Recently, there are some results [5, 6] to the above question for the Hurwitz function and the reciprocal sum of two products of Fibonacci numbers by the authors of this paper.

The main results of this paper are that we give the complete answers to Problem A and Problem B when $a_k = \frac{1}{f_{mk-\ell}}$ for any $m \in \mathbb{N}$ and $0 \le \ell \le m-1$, where a_k is the reciprocal of the most general form of the linear subsequences of Fibonacci numbers. More precisely, we prove that g_n has the form of

$$g_n := g_{n,m,\ell} = f_{mn-\ell} - f_{m(n-1)-\ell}.$$

In Section 2, we prove the inequalities sharper than in [7] and as a consequence, we obtain the following.

Theorem 1.1. *It holds that*

$$\lim_{n\to\infty}\left\{\left(\sum_{k=n}^{\infty}\frac{1}{f_k}\right)^{-1} - f_{n-2}\right\} = 0.$$

In fact, $(\sum_{k=n}^{\infty}\frac{1}{f_k})^{-1}$ is increasing to f_{n-2} when n is odd, and decreasing to f_{n-2} when n is even. See the inequalities in Proposition 2.4.

In Sections 3 and 4, we prove new inequalities for all linear subsequential Fibonacci numbers $(\sum_{k=n}^{\infty}\frac{1}{f_{mk-\ell}})^{-1}$. The estimates are sharp enough to solve Problems A and B. More precisely, we prove the following.

Theorem 1.2. *For any $m \in \mathbb{N}$ and $0 \le \ell \le m-1$, we have*

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{f_{mk-\ell}}\right)^{-1}\right\rfloor = \begin{cases} f_{mn-\ell} - f_{m(n-1)-\ell} - 1, & m(n+1)+\ell \text{ is even},\\ f_{mn-\ell} - f_{m(n-1)-\ell}, & m(n+1)+\ell \text{ is odd}.\end{cases}$$

The formula in [3] does not imply Theorems 1.2. For the proof of Theorem 1.2, we prove some inequalities in Theorem 3.6 and 4.4.

The main contribution of this paper is the answer to Problem B in the case $a_k = 1/f_{mk-\ell}$.

Theorem 1.3. *For any $m \in \mathbb{N}$ and $0 \le \ell \le m-1$, we have*

$$\lim_{n\to\infty}\left\{\left(\sum_{k=n}^{\infty}\frac{1}{f_{mk-\ell}}\right)^{-1} - (f_{mn-\ell} - f_{m(n-1)-\ell})\right\} = 0. \tag{1.6}$$

It is interesting that the limit $f_{mn-\ell} - f_{m(n-1)-\ell}$ in (1.6) is an integer. In general, it is not guaranteed that the limit in (1.6) is an integer. In fact, in other case, the limit is not an integer in [5, 6].

Remark 1.4. It is hard to compute the form of $\sum_{k=n}^{\infty}\frac{1}{f_{mk-\ell}}$ using the Binet's formula. The advantage of this paper is to prove (1.6) without use of explicit forms. Moreover, our results imply all known formulas.

(i) If $m = 1$ and $\ell = 0$, then $g_n = f_n - f_{n-1} = f_{n-2}$ in (1.1).
(ii) If $m = 2$ and $\ell = 0$, then $g_n = f_{2n} - f_{2n-2} = f_{2n-1}$ in (1.2). If $m = 2$ and $\ell = 1$, then $g_n = f_{2n-1} - f_{2n-3} = f_{2n-2}$ in (1.2).

(iii) If $m = 3$ and $\ell = 0$, then $g_n = f_{3n} - f_{3n-3} = f_{3n-1} + f_{3n-4}$ in (1.3).

And also we can construct infinitely many different formulas of various types. For example, Theorem 1.2 provides the new formula

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{f_{8k-5}} \right)^{-1} \right\rfloor = f_{8n-5} - f_{8n-13}.$$

Remark 1.5. We need to explain the main contributions of this paper.

(i) For Problem A, there are many similar results for various series of Fibonacci numbers as (1.1)–(1.3). Theorem 1.2 covers all previously known results for all linear subsequences of Fibonacci numbers. Here the term $m(n + 1) + \ell$ is a new observation in Theorem 1.2.

(ii) Problem B is a quite new problem suggested by the authors of this paper. For the proof of Problem B, we need sharper inequality. (For example, see Propositions 2.3 and 2.4.) Thus Problem B is a harder task than Problem A. So far, there is no general theory for Problem B.

2. Proof of Theorem 1.1

The following lemma plays an important role in proving the essential inequalities.

Lemma 2.1. *[7] Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive real numbers with $\lim_{n\to\infty} a_n = 0$.*

(i) If $a_n < b_n + a_{n+1}$ holds for any $n \in \mathbb{N}$, then $a_n < \sum_{k=n}^{\infty} b_k$ holds for any $n \in \mathbb{N}$.

(ii) If $a_n < (b_n + b_{n+1}) + a_{n+2}$ holds for any even (or odd, respectively) integer n, then $a_n < \sum_{k=n}^{\infty} b_k$ holds for any even (or odd, respectively) integer n.

Proof. (i) If we apply the inequality repeatedly, we obtain

$$a_n < b_n + a_{n+1} < b_n + (b_{n+1} + a_{n+2}) < \cdots < \sum_{k=n}^{n+\ell} b_k + a_{n+\ell+1}$$

for any $\ell \in \mathbb{N}$. As $\ell \to \infty$, the proof is done. (ii) Use the inequality

$$a_n < \sum_{k=n}^{n+2\ell-1} b_k + a_{n+2\ell}$$

for any $\ell \in \mathbb{N}$. □

The following property is powerful for computing the Fibonacci numbers.

Lemma 2.2 (Catalan's identity)**.** *[9] For all $n, k \in \mathbb{N}$ with $n > k$, we have*

$$f_n^2 = f_{n-k} f_{n+k} + (-1)^{n+k} f_k^2.$$

In [7], Ohtsuka and Nakamura proved the following inequalities in order to obtain the formula for $\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{f_k} \right)^{-1} \right\rfloor$.

Proposition 2.3. *[7] For all $n \geq 1$, we have*

(i) $f_{n-2} - 1 < \left(\sum_{k=n}^{\infty} \frac{1}{f_k} \right)^{-1} < f_{n-2}$, *when n is odd;*

(ii) $f_{n-2} < \left(\sum_{k=n}^{\infty} \frac{1}{f_k} \right)^{-1} < f_{n-2} + 1$, *when n is even.*

In fact, one can see that the difference $\left| \left(\sum_{k=n}^{\infty} \frac{1}{f_k} \right)^{-1} - f_{n-2} \right|$ gets smaller as n is large (See Table 1).

Table 1. Difference between $\left(\sum_{k=n}^{\infty} \frac{1}{f_k} \right)^{-1}$ and f_{n-2}.

n	$\left(\sum_{k=n}^{\infty} \frac{1}{f_k} \right)^{-1}$	f_{n-2}	n	$\left(\sum_{k=n}^{\infty} \frac{1}{f_k} \right)^{-1}$	f_{n-2}
5	$1.8991\cdots$	2	6	$3.0622\cdots$	3
11	$33.9943\cdots$	34	12	$55.0034\cdots$	55
17	$609.9996\cdots$	610	18	$987.0001\cdots$	987
23	$10945.9999\cdots$	10946	24	$17711.0000\cdots$	17711

Motivated by the above investigation, we refine the inequalities in Proposition 2.3 as follows.

Proposition 2.4. *For any positive integers n, we have*

(i) $f_{n-2} - c_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_k} \right)^{-1} < f_{n-2}$, *when n is odd;*

(ii) $f_{n-2} < \left(\sum_{k=n}^{\infty} \frac{1}{f_k} \right)^{-1} < f_{n-2} + c_n$, *when n is even,*

where $c_n = \frac{1}{f_n}$ is the reciprocal of Fibonacci numbers.

Proof. At first we need to compare the values of

$$\frac{1}{f_{n-2} + (-1)^n c_n} - \frac{1}{f_n + (-1)^n c_{n+2}} \quad \text{and} \quad \frac{1}{f_n} + \frac{1}{f_{n+1}}.$$

Note that

$$
\begin{aligned}
&\frac{1}{f_{n-2}+(-1)^n c_n}-\frac{1}{f_n+(-1)^n c_{n+2}}-\left(\frac{1}{f_n}+\frac{1}{f_{n+1}}\right)\\
=&\frac{f_{n-1}-(-1)^n(c_n-c_{n+2})}{(f_{n-2}+(-1)^n c_n)(f_n+(-1)^n c_{n+2})}-\frac{f_{n+2}}{f_n f_{n+1}}\\
=&\frac{A}{(f_{n-2}+(-1)^n c_n)(f_n+(-1)^n c_{n+2})f_n f_{n+1}},
\end{aligned}
$$

where

$$
\begin{aligned}
A :=&f_n(f_{n-1}f_{n+1}-f_{n-2}f_{n+2})-c_n c_{n+2}f_{n+2}\\
&-(-1)^n\{(c_n-c_{n+2})f_n f_{n+1}+c_n f_n f_{n+2}+c_{n+2}f_{n+2}f_{n-2}\}.
\end{aligned}
$$

By Lemma 2.2, we have

$$
f_{n-1}f_{n+1}-f_{n-2}f_{n+2}=(f_n^2-(-1)^{n-1})-(f_n^2-(-1)^n)=2\cdot(-1)^n.
$$

It follows that

$$
\begin{aligned}
A =&(-1)^n\{2f_n-c_n f_n f_{n+2}-c_{n+2}f_{n+2}f_{n-2}-(c_n-c_{n+2})f_n f_{n+1}\}-c_n c_{n+2}f_{n+2}\\
=&(-1)^n\left\{2f_n-f_{n+2}-f_{n-2}-\left(\frac{1}{f_n}-\frac{1}{f_{n+2}}\right)f_n f_{n+1}\right\}-\frac{1}{f_n}.
\end{aligned}
$$

Since $2f_n-f_{n+2}-f_{n-2}=-f_n$, we have

$$
A=(-1)^n\left(-f_n-\frac{f_{n+1}^2}{f_{n+2}}\right)-\frac{1}{f_n}.
$$

Now we will show that $A>0$ if n is odd and $A<0$ if n is even.

(i) If n is odd, then

$$
A=f_n+\frac{f_{n+1}^2}{f_{n+2}}-\frac{1}{f_n}>f_n-\frac{1}{f_n}\geq 0.
$$

It follows that

$$
\frac{1}{f_{n-2}-c_n}>\left(\frac{1}{f_n}+\frac{1}{f_{n+1}}\right)+\frac{1}{f_n-c_{n+2}} \tag{2.1}
$$

for all odd integers n. If we apply Lemma 2.1 (ii) to the inequality (2.1), we obtain

$$
\frac{1}{f_{n-2}-c_n}>\sum_{k=n}^{\infty}\frac{1}{f_k},\ \text{so that}\ f_{n-2}-c_n<\left(\sum_{k=n}^{\infty}\frac{1}{f_k}\right)^{-1}.
$$

(ii) If n is even, then

$$
A=-f_n-\frac{f_{n+1}^2}{f_{n+2}}-\frac{1}{f_n}<0.
$$

Similarly as the proof of (i), we can prove (ii) using Lemma 2.1 (ii). □

Since $c_n = \frac{1}{f_n} \to 0$ as $n \to \infty$, Proposition 2.4 implies the following consequence.

Theorem 2.5 (Theorem 1.1, again)**.** *It holds that*

$$\lim_{n\to\infty} \left\{ \left(\sum_{k=n}^{\infty} \frac{1}{f_k} \right)^{-1} - f_{n-2} \right\} = 0.$$

3. Estimates of enne $\left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}} \right)^{-1}$ when m is even

Throughout this paper, we write

$$c_n = \frac{1}{f_n}, \quad g_n = f_{mn-\ell} - f_{m(n-1)-\ell}, \quad 0 \le \ell \le m-1.$$

For simplicity, we assume that $n \ge 2$. Then g_n is well-defined, since $m(n-1) - \ell \ge m - \ell \ge 1$. In fact, it is enough to consider the case when n is sufficiently large.

We begin this section with the elementary and important properties of Fibonacci numbers.

Lemma 3.1. *[4, 9] For any positive integers m, n, we have*

(i) $f_{m+n} = f_{m-1}f_n + f_m f_{n+1}$,
(ii) $f_{m+n} \ge f_{m+1}f_n$ *and equality holds only if* $n = 1$.

Proof. We only prove (ii). By (i), we have

$$f_{m+n} = f_{m-1}f_n + f_m f_{n+1} \ge f_{m-1}f_n + f_m f_n = f_{m+1}f_n$$

for any positive integers m, n. □

Lemma 3.2. *Define $g_n := f_{mn-\ell} - f_{m(n-1)-\ell}$. Then for all $m \in \mathbb{N}$, we have*

$$(g_{n+1} - g_n) f_{mn-\ell} - g_n g_{n+1} = (-1)^{m(n-1)-\ell+1} f_m^2.$$

Proof. By Lemma 2.2, we have

$$\begin{aligned}(g_{n+1} - g_n) f_{mn-\ell} - g_n g_{n+1} =& (f_{m(n+1)-\ell} - 2f_{mn-\ell} + f_{m(n-1)-\ell}) f_{mn-\ell} \\ & - (f_{mn-\ell} - f_{m(n-1)-\ell})(f_{m(n+1)-\ell} - f_{mn-\ell}) \\ =& f_{m(n-1)-\ell} f_{m(n+1)-\ell} - f_{mn-\ell}^2 \\ =& (-1)^{m(n-1)-\ell+1} f_m^2.\end{aligned}$$

□

Proposition 3.3. *Let $m \in \mathbb{N}$ be even. Then for any $n \in \mathbb{N}$, we have*

(i) $\left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}} \right)^{-1} < g_n$, *when ℓ is even;*

(ii) $\left(\sum_{k=n}^{\infty}\frac{1}{f_{mk-\ell}}\right)^{-1} > g_n$*, when* ℓ *is odd.*

Proof. By Lemma 3.2, we have

$$\begin{aligned}\frac{1}{g_n}-\frac{1}{g_{n+1}}-\frac{1}{f_{mn-\ell}} &=\frac{g_{n+1}-g_n}{g_ng_{n+1}}-\frac{1}{f_{mn-\ell}}\\&=\frac{(g_{n+1}-g_n)f_{mn-\ell}-g_ng_{n+1}}{g_ng_{n+1}f_{mn-\ell}}\\&=\frac{(-1)^{\ell+1}f_m^2}{g_ng_{n+1}f_{mn-\ell}},\end{aligned}$$

since $(-1)^{m(n-1)-\ell+1}=(-1)^{\ell+1}$ for any even positive integer m.

(i) If ℓ is even, then

$$\frac{1}{g_n}<\frac{1}{f_{mn-\ell}}+\frac{1}{g_{n+1}}.$$

By Lemma 2.1 (i), it completes the proof.

(ii) If ℓ is odd, then we can prove (ii) in a similar way. □

Lemma 3.4. *For any positive integers* m,n*, we have*

$$f_{m+2n}-f_m=\sum_{k=1}^{n}f_{m+2k-1}.$$

Proof. It is easily checked that

$$\begin{aligned}&f_m+(f_{m+1}+f_{m+3}+\cdots+f_{m+2n-1})\\=&f_{m+2}+(f_{m+3}+f_{m+5}+\cdots+f_{m+2n-1})\\=&\cdots\\=&f_{m+2n-2}+f_{m+2n-1}\\=&f_{m+2n}.\end{aligned}$$

□

Proposition 3.5. *Let* $m\in\mathbb{N}$ *be even and let* $c_n=\frac{1}{f_n}$*. Then for any* $n\geq 2$*, we have*

(i) $\left(\sum_{k=n}^{\infty}\frac{1}{f_{mk-\ell}}\right)^{-1} > g_n-c_n$*, when* ℓ *is even;*

(ii) $\left(\sum_{k=n}^{\infty}\frac{1}{f_{mk-\ell}}\right)^{-1} < g_n+c_n$*, when* ℓ *is odd.*

Proof. (i) If ℓ is even, then we will prove that

$$\frac{1}{g_n - c_n} > \frac{1}{f_{mn-\ell}} + \frac{1}{g_{n+1} - c_{n+1}}. \tag{3.1}$$

Note that

$$\begin{aligned}
\frac{1}{g_n - c_n} - \frac{1}{g_{n+1} - c_{n+1}} - \frac{1}{f_{mn-\ell}} &= \frac{g_{n+1} - g_n + (c_n - c_{n+1})}{(g_n - c_n)(g_{n+1} - c_{n+1})} - \frac{1}{f_{mn-\ell}} \\
&> \frac{g_{n+1} - g_n}{(g_n - c_n)(g_{n+1} - c_{n+1})} - \frac{1}{f_{mn-\ell}} \\
&= \frac{f_{mn-\ell}(g_{n+1} - g_n) - (g_n - c_n)(g_{n+1} - c_{n+1})}{(g_n - c_n)(g_{n+1} - c_{n+1})f_{mn-\ell}}.
\end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned}
&f_{mn-\ell}(g_{n+1} - g_n) - (g_n - c_n)(g_{n+1} - c_{n+1}) \\
=&\{f_{mn-\ell}(g_{n+1} - g_n) - g_n g_{n+1}\} + c_n g_{n+1} + c_{n+1} g_n - c_n c_{n+1} \\
>& - f_m^2 + c_n g_{n+1} - c_n c_{n+1} \\
>& - f_m^2 + c_n g_{n+1} - 1.
\end{aligned} \tag{3.2}$$

Note that for any $n \geq 2$, by Lemmas 3.1 (i) and 3.4,

$$\begin{aligned}
c_n g_{n+1} &= \frac{f_{n(m+1)-\ell} - f_{mn-\ell}}{f_n} \\
&= \frac{f_{n(m+1)-\ell-1} + f_{n(m+1)-\ell-3} + \cdots + f_{mn-\ell+1}}{f_n} \\
&> \frac{f_{n(m+1)-\ell-1} + f_{n(m+1)-\ell-3}}{f_n} \\
&> f_{m(n+1)-\ell-n} + f_{m(n+1)-\ell-n-2} \\
&> f_m f_{mn-\ell-n+1} + 1 \\
&\geq f_m^2 f_{mn-\ell-n-m+2} + 1.
\end{aligned}$$

Since $mn - \ell - n - m + 2 \geq (m-1)(n-2) + 1 \geq 1$ for $n \geq 2$ and $\ell \leq m - 1$, we have

$$c_n g_{n+1} \geq f_m^2 + 1. \tag{3.3}$$

By (3.2) and (3.3), we obtain the inequality

$$f_{mn-\ell}(g_{n+1} - g_n) - (g_n - c_n)(g_{n+1} - c_{n+1}) > 0,$$

which implies (3.1). By Lemma 2.1 (i), it completes the proof.

(ii) If ℓ is odd, then we will show that

$$\frac{1}{g_n + c_n} < \frac{1}{f_{mn-\ell}} + \frac{1}{g_{n+1} + c_{n+1}}. \tag{3.4}$$

Note that

$$\begin{aligned}\frac{1}{g_n+c_n}-\frac{1}{g_{n+1}+c_{n+1}}-\frac{1}{f_{mn-\ell}} &= \frac{g_{n+1}-g_n-(c_n-c_{n+1})}{(g_n+c_n)(g_{n+1}+c_{n+1})}-\frac{1}{f_{mn-\ell}}\\ &< \frac{g_{n+1}-g_n}{(g_n+c_n)(g_{n+1}+c_{n+1})}-\frac{1}{f_{mn-\ell}}\\ &= \frac{f_{mn-\ell}(g_{n+1}-g_n)-(g_n-c_n)(g_{n+1}-c_{n+1})}{(g_n+c_n)(g_{n+1}+c_{n+1})f_{mn-\ell}}.\end{aligned}$$

By Lemma 3.2 and (3.3), we have

$$\begin{aligned}f_{mn-\ell}(g_{n+1}-g_n)-(g_n-c_n)(g_{n+1}-c_{n+1}) =& f_m^2-c_ng_{n+1}-c_{n+1}g_n-c_nc_{n+1}\\ <& f_m^2-c_ng_{n+1}<0.\end{aligned}$$

By Lemma 2.1 (i), it completes the proof. □

Combining Propositions 3.3 and 3.5, we obtain the following.

Theorem 3.6. *Let $c_n=\frac{1}{f_n}$. If m is even, then*

(i) $g_n-c_n<\left(\sum_{k=n}^{\infty}\frac{1}{f_{mk-\ell}}\right)^{-1}<g_n$, *when ℓ is even;*

(ii) $g_n<\left(\sum_{k=n}^{\infty}\frac{1}{f_{mk-\ell}}\right)^{-1}<g_n+c_n$, *when ℓ is odd.*

Example 3.7. *Table 2 shows some examples for $m=4,6$.*

Table 2. Examples for Theorem 3.6.

n	$\left(\sum_{k=n}^{\infty}\frac{1}{f_{4k-3}}\right)^{-1}$	$f_{4n-3}-f_{4n-7}$	n	$\left(\sum_{k=n}^{\infty}\frac{1}{f_{6k-2}}\right)^{-1}$	$f_{6n-2}-f_{6n-8}$
5	$1364.0008\cdots$	1364	3	$931.9963\cdots$	932
6	$9349.0001\cdots$	9349	4	$16723.9997\cdots$	16724

4. Estimates of $\left(\sum_{k=n}^{\infty}\frac{1}{f_{mk-\ell}}\right)^{-1}$ when m is odd

Recall that $g_n=f_{mn-\ell}-f_{m(n-1)-\ell}$ for $n\geq 2$ and $0\leq\ell\leq m-1$. We already proved when $m=1$ in Proposition 2.4. Throughout this section, we assume that $m\geq 3$ is an odd integer.

Lemma 4.1. *If $m\in\mathbb{N}$ is odd, then we have*

(i) $(g_{n+2}-g_n)f_{mn-\ell}f_{m(n+1)-\ell}-g_ng_{n+2}(f_{mn-\ell}+f_{m(n+1)-\ell})=(-1)^{n+\ell}f_m^2(f_{m(n+2)-\ell}-f_{m(n-1)-\ell})$,

(ii) $g_{n+2}(f_{mn-\ell}+f_{m(n+1)-\ell})=-(-1)^{n+\ell}f_m^2+f_{m(n+1)-\ell}(f_{m(n+2)-\ell}-f_{mn-\ell})$,

(iii) $g_n(f_{mn-\ell} + f_{m(n+1)-\ell}) = -(-1)^{n+\ell} f_m^2 + f_{mn-\ell}(f_{m(n+1)-\ell} - f_{m(n-1)-\ell})$.

Proof. Note that

$$\begin{aligned}
&(g_{n+2} - g_n) f_{mn-\ell} f_{m(n+1)-\ell} - g_n g_{n+2}(f_{mn-\ell} + f_{m(n+1)-\ell}) \\
=&(f_{m(n+2)-\ell} - f_{m(n+1)-\ell} - f_{mn-\ell} + f_{m(n-1)-\ell}) f_{mn-\ell} f_{m(n+1)-\ell} \\
&\quad - (f_{mn-\ell} - f_{m(n-1)-\ell})(f_{m(n+2)-\ell} - f_{m(n+1)-\ell})(f_{mn-\ell} + f_{m(n+1)-\ell}) \\
=& - f_{m(n+2)-\ell}(f_{mn-\ell}^2 - f_{m(n-1)-\ell} f_{m(n+1)-\ell}) \\
&\quad - f_{m(n-1)-\ell}(f_{m(n+1)-\ell}^2 - f_{mn-\ell} f_{m(n+2)-\ell}).
\end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}
&(g_{n+2} - g_n) f_{mn-\ell} f_{m(n+1)-\ell} - g_n g_{n+2}(f_{mn-\ell} + f_{m(n+1)-\ell}) \\
&\qquad = -(-1)^{m(n+1)-\ell} f_m^2 f_{m(n+2)-\ell} - (-1)^{mn-\ell} f_m^2 f_{m(n-1)-\ell}.
\end{aligned}$$

If m is odd, then $(-1)^{m(n+1)-\ell} = -(-1)^{n+\ell}$. It completes the proof of (i). Note that

$$\begin{aligned}
&g_{n+2}(f_{mn-\ell} + f_{m(n+1)-\ell}) \\
=&(f_{m(n+2)-\ell} - f_{m(n+1)-\ell})(f_{mn-\ell} + f_{m(n+1)-\ell}) \\
=& - (f_{m(n+1)-\ell}^2 - f_{mn-\ell} f_{m(n+2)-\ell}) + f_{m(n+1)-\ell}(f_{m(n+2)-\ell} - f_{mn-\ell}).
\end{aligned}$$

By Lemma 2.2 (i), we obtain (ii). Similarly, we also obtain (iii). □

The previous lemma shows that we need to split the cases when $n + \ell$ is odd or even.

Proposition 4.2. *Let $m \in \mathbb{N}$ be odd. Then for any $n \geq 2$, we have*

(i) $\left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} < g_n$*, when $n + \ell$ is odd;*

(ii) $\left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} > g_n$*, when $n + \ell$ is even.*

Proof. Since m is odd, by Lemma 4.1 (i), we have

$$\begin{aligned}
\frac{1}{g_n} - \frac{1}{g_{n+2}} - \left(\frac{1}{f_{mn-\ell}} + \frac{1}{f_{m(n+1)-\ell}}\right) =& \frac{g_{n+2} - g_n}{g_n g_{n+2}} - \frac{f_{mn-\ell} + f_{m(n+1)-\ell}}{f_{mn-\ell} f_{m(n+1)-\ell}} \\
=& \frac{(-1)^{n+\ell} f_m^2 (f_{m(n+2)-\ell} - f_{m(n-1)-\ell})}{g_n g_{n+2} f_{mn-\ell} f_{m(n+1)-\ell}}.
\end{aligned}$$

(i) If $n + \ell$ is odd, then

$$\frac{1}{g_n} < \frac{1}{f_{mn-\ell}} + \frac{1}{f_{m(n+1)-\ell}} + \frac{1}{g_{n+2}}. \tag{4.1}$$

By Lemma 2.1 (ii), it completes the proof.

(ii) If $n + \ell$ is even, the proof is similar. □

Proposition 4.3. *Let $m \in \mathbb{N}$ be odd and let $c_n = \frac{1}{f_n}$. Then for any $n \geq 4$, we have*

(i) $\left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} > g_n - c_n$, *when $n + \ell$ is odd;*

(ii) $\left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} < g_n + c_n$, *when $n + \ell$ is even.*

Proof. Note that

$$\begin{aligned}
&\frac{1}{g_n + (-1)^{n+\ell} c_n} - \frac{1}{g_{n+2} + (-1)^{n+\ell} c_{n+2}} - \left(\frac{1}{f_{mn-\ell}} + \frac{1}{f_{m(n+1)-\ell}}\right) \\
=&\frac{g_{n+2} - g_n - (-1)^{n+\ell}(c_n - c_{n+2})}{(g_n + (-1)^{n+\ell} c_n)(g_{n+2} + (-1)^{n+\ell} c_{n+2})} - \frac{f_{mn-\ell} + f_{m(n+1)-\ell}}{f_{mn-\ell} f_{m(n+1)-\ell}} \\
=&\frac{A}{(g_n + (-1)^{n+\ell} c_n)(g_{n+2} + (-1)^{n+\ell} c_{n+2}) f_{mn-\ell} f_{m(n+1)-\ell}},
\end{aligned}$$

where

$$\begin{aligned}
A =&\{g_{n+2} - g_n - (-1)^{n+\ell}(c_n - c_{n+2})\} f_{mn-\ell} f_{m(n+1)-\ell} \\
&- (g_n + (-1)^{n+\ell} c_n)(g_{n+2} + (-1)^{n+\ell} c_{n+2})(f_{mn-\ell} + f_{m(n+1)-\ell}).
\end{aligned}$$

By Lemma 4.1, we have

$$\begin{aligned}
A =&\{(g_{n+2} - g_n) f_{mn-\ell} f_{m(n+1)-\ell} - g_n g_{n+2}(f_{mn-\ell} + f_{m(n+1)-\ell})\} + \cdots \\
=&(-1)^{n+\ell} f_m^2 (f_{m(n+2)-\ell} - f_{m(n-1)-\ell}) + \cdots,
\end{aligned}$$

where $\cdots$ means the terms which contain c_n or c_{n+2}. Thus we have

$$\begin{aligned}
A =&(-1)^{n+\ell} f_m^2 (f_{m(n+2)-\ell} - f_{m(n-1)-\ell}) \\
&- (-1)^{n+\ell}\{c_n g_{n+2}(f_{mn-\ell} + f_{m(n+1)-\ell}) + c_{n+2} g_n (f_{mn-\ell} + f_{m(n+1)-\ell})\} \\
&- (-1)^{n+\ell}(c_n - c_{n+2}) f_{mn-\ell} f_{m(n+1)-\ell} \\
&- c_n c_{n+2}(f_{mn-\ell} + f_{m(n+1)-\ell}).
\end{aligned}$$

For simplicity, we write

$$A_1 = f_m^2 (f_{m(n+2)-\ell} - f_{m(n-1)-\ell}).$$

By Lemma 4.1 (ii) and (iii), we write

$$\begin{aligned}
A_2 :=&c_n g_{n+2}(f_{mn-\ell} + f_{m(n+1)-\ell}) \\
=&-(-1)^{n+\ell} c_n f_m^2 + c_n f_{m(n+1)-\ell}(f_{m(n+2)-\ell} - f_{mn-\ell}), \\
A_3 :=&c_{n+2} g_n (f_{mn-\ell} + f_{m(n+1)-\ell}) \\
=&-(-1)^{n+\ell} c_{n+2} f_m^2 + c_{n+2} f_{mn-\ell}(f_{m(n+1)-\ell} - f_{m(n-1)-\ell})
\end{aligned}$$

and

$$A_4 := (c_n - c_{n+2}) f_{mn-\ell} f_{m(n+1)-\ell},$$
$$A_5 := c_n c_{n+2} (f_{mn-\ell} + f_{m(n+1)-\ell}).$$

It follows that

$$A = (-1)^{n+\ell}(A_1 - A_2 - A_3 - A_4) - A_5.$$

Note that by Lemma 3.1 (ii), we have

$$c_n f_{m(n+1)-\ell} = \frac{f_{m(n+1)-\ell}}{f_n} > f_{mn+m-\ell-n+1} > f_m^2 f_{mn-m-n-\ell+3} \geq f_m^2,$$

since $mn - m - n - \ell + 3 \geq (m-1)(n-2) + 2 \geq 2$ for any $n \geq 2$. Similarly we have

$$c_{n+2} f_{mn-\ell} = \frac{f_{mn-\ell}}{f_{n+2}} > f_{mn-\ell-n-1} > f_m^2 f_{mn-\ell-n-2m+1} \geq f_m^2,$$

since $mn - \ell - n - 2m + 1 \geq (m-1)(n-3) - 1 \geq 1$ for any $n \geq 4$. Thus we have

$$A_2 > -(-1)^{n+\ell} c_n f_m^2 + f_m^2 (f_{m(n+2)-\ell} - f_{mn-\ell}),$$
$$A_3 > -(-1)^{n+\ell} c_{n+2} f_m^2 + f_m^2 (f_{m(n+1)-\ell} - f_{m(n-1)-\ell}).$$

It follows that

$$-A_1 + A_2 + A_3 > f_m^2 (f_{m(n+1)-\ell} - f_{mn-\ell}) - (-1)^{n+\ell}(c_n + c_{n+2}) f_m^2. \tag{4.2}$$

Now we must show that $A > 0$ if $n + \ell$ is odd and $A < 0$ if $n + \ell$ is even.

Case (1) : Assume that $n + \ell$ is odd. Then

$$\begin{aligned} A &= -A_1 + A_2 + A_3 + A_4 - A_5 \\ &> (-A_1 + A_2 + A_3) - A_5 \\ &> f_m^2 (f_{m(n+1)-\ell} - f_{mn-\ell}) - c_n c_{n+2} (f_{mn-\ell} + f_{m(n+1)-\ell}) \\ &\geq 4(f_{m(n+1)-\ell} - f_{mn-\ell}) - \frac{1}{2}(f_{mn-\ell} + f_{m(n+1)-\ell}) \\ &= \frac{1}{2}\{7 f_{m(n+1)-\ell} - 9 f_{mn-\ell}\} \\ &> \frac{1}{2}\{7 f_m f_{mn-\ell+1} - 9 f_{mn-\ell}\} > 0. \end{aligned}$$

Case (2) : Assume that $n + \ell$ is even. Then

$$\begin{aligned} A &= A_1 - A_2 - A_3 - A_4 - A_5 \\ &< A_1 - A_2 - A_3 \\ &< (c_n + c_{n+2}) f_m^2 - (f_{m(n+1)-\ell} - f_{mn-\ell}) f_m^2 \\ &< -(f_{m(n+1)-\ell} - f_{mn-\ell} - 2) f_m^2 < 0, \end{aligned}$$

since $f_{m(n+1)-\ell} - f_{mn-\ell} \geq f_{m+1} - f_m \geq f_4 - f_1 \geq 2$ for $m \geq 3$ and $n \geq 1$. □

Combining Propositions 4.2 and 4.3, we obtain the following.

Theorem 4.4. *Let $c_n = \frac{1}{f_n}$. If m is odd, then*

(i) $g_n - c_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} < g_n$, *when $n + \ell$ is odd;*

(ii) $g_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} < g_n + c_n$, *when $n + \ell$ is even.*

Example 4.5. *Table 3 shows some examples for $m = 3, 5$.*

Table 3. Examples for Theorem 4.4.

n	$\left(\sum_{k=n}^{\infty} \frac{1}{f_{3k-2}}\right)^{-1}$	$f_{3n-2} - f_{3n-5}$	n	$\left(\sum_{k=n}^{\infty} \frac{1}{f_{5k-3}}\right)^{-1}$	$f_{5n-3} - f_{5n-8}$
5	$177.9959\cdots$	178	4	$1452.9985\cdots$	1453
6	$754.0009\cdots$	754	5	$16114.0001\cdots$	16114

5. Proof of main theorems and concluding remarks

At first, we summarize the inequalities proved in Sections 3 and 4.

Theorem 5.1 (Theorem 3.6, again)**.** *Let $c_n = \frac{1}{f_n}$. If m is even, then*

(i) $g_n - c_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} < g_n$, *when ℓ is even;*

(ii) $g_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} < g_n + c_n$, *when ℓ is odd.*

Theorem 5.2 (Theorem 4.4, again)**.** *Let $c_n = \frac{1}{f_n}$. If m is odd, then*

(i) $g_n - c_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} < g_n$, *when $n + \ell$ is odd;*

(ii) $g_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} < g_n + c_n$, *when $n + \ell$ is even.*

Combining the above two theorems, we obtain the following.

Theorem 5.3. *Let $c_n = \frac{1}{f_n}$. For any $m \in \mathbb{N}$ and $0 \le \ell \le m - 1$, we have*

(i) $g_n - c_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} < g_n$, *when $m(n + 1) + \ell$ is even;*

(ii) $g_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} < g_n + c_n$, *when $m(n + 1) + \ell$ is odd.*

Now we can prove Theorems 1.2 and 1.3 simultaneously from Theorem 5.3. Since $g_n \in \mathbb{N}$ and $0 < c_n \le 1$ for all $n \in \mathbb{N}$, we can give an answer to the Problem A as following.

Theorem 5.4 (Theorem 1.2, again)**.** *For any* $m \in \mathbb{N}$ *and* $0 \le \ell \le m-1$*, we have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}} \right)^{-1} \right\rfloor = \begin{cases} f_{mn-\ell} - f_{m(n-1)-\ell} - 1, & m(n+1)+\ell \text{ is even}; \\ f_{mn-\ell} - f_{m(n-1)-\ell}, & m(n+1)+\ell \text{ is odd}. \end{cases}$$

Since $c_n = \frac{1}{f_n} \to 0$ as $n \to \infty$, by Theorem 5.3, we finally obtain the following theorem. This is the answer to the Problem B.

Theorem 5.5 (Theorem 1.3, again)**.** *For any* $m \in \mathbb{N}$ *and* $0 \le \ell \le m-1$*, we have*

$$\lim_{n\to\infty} \left\{ \left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}} \right)^{-1} - (f_{mn-\ell} - f_{m(n-1)-\ell}) \right\} = 0.$$

6. Conclusions

So far, the researches on the reciprocal sum have been concentrated to Problem A, which is the question on finding the floor function. Problem A does not give any information on the decimal or the convergence of the reciprocal sum. Thus Problem B is more natural than Problem A. But, so far there is no general theory for Problem B. We found the behavior of the tails of the reciprocal sums for the linear subsequential Fibonacci numbers. Our theorem covers all previous results and we can obtain many formulas of various types.

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Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

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