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Research article

Robust strong duality for nonconvex optimization problem under data uncertainty in constraint

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Abstract: This paper deals with the robust strong duality for nonconvex optimization problem with the data uncertainty in constraint. A new weak conjugate function which is abstract convex, is introduced and three kinds of robust dual problems are constructed to the primal optimization problem by employing this weak conjugate function: the robust augmented Lagrange dual, the robust weak Fenchel dual and the robust weak Fenchel-Lagrange dual problem. Characterizations of inequality (1.1) according to robust abstract perturbation weak conjugate duality are established by using the abstract convexity. The results are used to obtain robust strong duality between noncovex uncertain optimization problem and its robust dual problems mentioned above, the optimality conditions for this noncovex uncertain optimization problem are also investigated.

Keywords: weak conjugate function; the inequality; Robust abstract perturbational weak conjugate duality; Robust strong duality
Mathematics Subject Classification: 90C46, 90C48

1. Introduction

Robust optimization problems [4, 5, 7, 8, 22, 23, 29–31] and robust dual theory [3, 6, 10–13, 15, 16, 18, 19, 28] have attracted much attention of mathematical researchers. Many of the works in this area were considered convex robust optimization problems, in [6, 15] robust Lagrangian strong duality was established in convex optimization and in [16] robust Lagrangian strong duality theorem was given whenever the Lagrangian function is convex. Moreover, duality theory which is based on conjugate function plays an important role in optimization. In convex analysis, dual problem is constructed in terms of conjugate functions by using the well-known Legendre-Fenchel transform. Robust classical conjugate duality was presented for convex optimization problem in [18]. Furthermore, in [13], characterizations of inequality below

$$p(x) = \sup_{v\in V} F_v(x, 0_Y) \geq l(x) \tag{1.1}$$

in terms of robust abstract perturbational duality were established, where X, Y are locally convex Hausdorff topological vector spaces, $V \neq \emptyset$ is an uncertainty set, $F_V : X \times Y \to \bar{R} = R \cup \{\pm\infty\}$ for each $v \in V$, $l : X \to \bar{R}$ is a lower semicontinuous proper convex function. The results were then applied to robust DC and robust convex optimization problems, and strong Fenchel duality and strong Lagrangian duality for these classes of robust problems were also obtained.

It is well known that dual problems constructed by using general augmented Lagrangian functions or weak conjugate functions, and strong duality conditions for noncovex optimization problems were comprehensively studied by researchers [1, 2, 13, 14, 17, 20, 21, 25, 27, 33]. In particular, the conjugate function theory, developed by Azimov and Gasimov in [1], used superlinear functions of the form $\langle x^*, x\rangle - c\|x\|$ instead of linear functions $\langle x^*, x\rangle$ used in convex analysis. They extended the usual definition of the subdifferential, using this class of functions, and established duality relations in terms of so-called weak subdifferentiability of the perturbation function associated with the problem under consideration. By using weak conjugate function and weak subdifferential given in [1], Küçük ect. in [17] constructed weak Fenchel conjugate dual problem and weak Fenchel-Lagrange conjugate dual problem, presented necessary and sufficient conditions for the strong duality of the dual problems and nonconvex scalar optimization problem; In [33], the duality scheme and strong duality theorems for nonconvex optimization problem were presented, which are based on the weak conjugate function and the weak subdifferential concept given in [1].

Nevertheless, there are few duality results on nonconvex robust optimization problem in the literature, since it is not only very hard to verify the zero duality gap conditions formulated in terms of perturbation and/or dualizing parameterization functions, but also to derive the conditions formulated in terms of objective and constraint functions. Motivated by [13, 17, 33], the aim of this paper is to formulate robust dual problems by using the weak conjugate function we introduced (see Definition 2.1) and establish robust strong duality results for nonconvex uncertain optimization problem. Characterization of general inequality (1.1) above with uncertainty is established according to robust perturbational weak conjugate duality, where we only assume the right hand function l in (1.1) is abstract convex [9, 24], which covers very broad classes on nonconvex functions. Then the results are used as key tools to obtain the strong duality for the robust augmented Lagrange dual (RD_L), robust weak Fenchel dual (RD_F^w) and the robust weak Fenchel-Lagrange dual problems (RD_{LF}^w) which are all defined by using the weak conjugate function, and are also applied to investigate the optimality conditions for nonconvex robust optimization problem.

The paper is organized as follows. In section 2, we recall some notations and introduce some preliminary results which will be used in the rest of paper. In section 3, we construct three types of robust dual problems for the primal optimization problem by using the weak conjugate function and obtain the strong duality respectively by establishing the inequality (1.1) via robust perturbation weak conjugate duality. In section 4, we investigate the relations among the optimal objective values of (RD_L), (RD_F^w), (RD_{LF}^w) and the robust optimization (RP) of (UP). Finally, section 5, we present necessary and sufficient optimality conditions for (RD_L), (RD_F^w), $R(D_{LF}^w)$ and (RP).

2. Preliminary results

In this section, we introduce the definitions of weak conjugate, weak biconjugate function, weak subdifferentials and some basic theorems and lemmas about these notions.

Throughout this paper, let X, Y be two locally convex vector spaces with their topological dual spaces X^* and Y^*, endowed with the weak* topologies $W(X^*, X)$ and $W(Y^*, Y)$, respectively. Let $D \subset Y$ be a nonempty closed convex cone, the dual cone of D is defined by

$$D^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \ \forall\, y \in D\},$$

where we use the notation $\langle \cdot, \cdot\rangle$ for the value of the continuous linear function $y^* \in Y^*$ and $y \in Y$. We use the notation $R_+ = \{x \mid x \in R, \ x \geq 0\}$. We also recall the corresponding concepts and results on (extended) real-valued functions. Let $f : G \to \bar{R}$, $g : G \to \bar{R}$ be functions defined on a set $G \subseteq X$, then the inequality $f \leq g$ means that $f(x) \leq g(x)$ for all $x \in G$. The domain and the epigraph of f are

$$\mathrm{dom} f = \{x \in X : \ f(x) < +\infty\},$$

and

$$\mathrm{epi} f = \{(x, r) \in X \times R : \ x \in \mathrm{dom} f, \ f(x) \leq r\},$$

respectively. The strict epigraph of $f : X \to \bar{R}$ is the set

$$\mathrm{epi}_s f = \{(x, r) \in X \times R : \ x \in \mathrm{dom} f, \ f(x) < r\}.$$

The function f is said to be proper if $\mathrm{dom} f \neq \emptyset$ and $f(x) \neq -\infty$. Let H be a set of functions $h : \ G \to \bar{R}$. The set $\mathrm{supp}(f, H) = \{h \in H \mid h \leq f\}$ is called the support set of f with respect to H. The function $\mathrm{co}_H f : \ G \to \bar{R}$ defined by $\mathrm{co}_H f(x) = \sup\{h(x) \mid h \in \mathrm{supp}(f, H)\}$ is called the $H-$convex hull of f. A function $f : \ G \to \bar{R}$ is called abstract convex with respect to H (or $H-$convex) at a point $x \in G$ if there exist a set $U \subseteq \mathrm{supp}(f, H)$ such that $f(x) = \sup\{h(x) \mid h \in H\}$. It is clear that f is $H-$convex at x if and only if $f(x) = \mathrm{co}_H f(x)$. If f is $H-$convex at each point $x \in G$, then f is called $H-$convex on G.

Let $\mathfrak{L}$ be a set of functions defined on a set G. Functions $h_{l,r}$ of the form $h_{l,r} = l(x) - r$, $x \in G$, with $l \in \mathfrak{L}$ and $r \in R$ are called $\mathfrak{L}-$affine. Denoted by $H_{\mathfrak{L}}$ the set of all $\mathfrak{L}-$affine functions. Denoted by Γ_X the union of the set of all functions $f : \ G \to R \cup \{+\infty\}$ and the function $-\infty$, where $-\infty(x) = -\infty$ for all $x \in G$.

We now introduce the definitions of new weak conjugate and weak biconjugate functions. First we need to have a function σ for the above definitions. It is assumed that $\sigma : \ Y \to R_+$ is continuous function with the following properties:

$$\sigma(0) = 0, \ \sigma(y) \neq 0 \text{ if } y \neq 0. \tag{2.1}$$

Definition 2.1. *(a) A function $f^w : \ X^* \times R_+ \to \bar{R}$ defined by*

$$f^w(x^*, c) = \sup_{x \in X}\{\langle x^*, x\rangle - c\sigma(x) - f(x)\}$$

is called the weak conjugate function of f. This function is H_X-convex;
(b) The function $f^{ww} : \ X \to \bar{R}$ defined by

$$f^{ww}(x) = \sup_{(x^*, c) \in X^* \times R_+}\{\langle x^*, x\rangle - c\sigma(x) - f^w(x^*, c)\}$$

is called the weak biconjugate function of f. This function is $H_{X^*\times R_+}-$convex. The classical result of abstract convex analysis states that $f \in H_X$ is abstract convex with respect to $H_{X^*\times R_+}$ at a point x if and only if $f(x) = f^{ww}(x)$.

Remark 2.1. *In the definition of weak biconjugate function if $r = 0$, then $f^w(x^*, 0) = f^*(x^*)$ where $f^*(x^*)$ is the classical conjugate function; if $\sigma(x) = \|x\|$, then $f^w(x^*, c)$ reduces to the $f^w(0, x^*, c)$ in [1].*

Definition 2.2. *Let X be a locally convex vector space. Let $f : X \to R$ be a single valued function and $x_0 \in X$ be a point with $f(x_0)$ is finite. A pair $(x^*, c) \in X^* \times R_+$ is called weak subgradient of f at x_0 if*

$$f(x) - f(x_0) \geq \langle x^*, x - x_0\rangle - c\sigma(x - x_0) \ \ for \ \ all \ x \in X.$$

The set $\partial^w f(x_0) = \{(x^*, c) \in X^* \times R_+ \mid f(x) - f(x_0) \geq \langle x^*, x - x_0\rangle - c\sigma(x - x_0),\ \forall\ x \in X\}$ of all weak subgradients of f at x_0 is called the weak subdifferential of f at x_0. If $\partial^w f(x_0) \neq \emptyset$, then f is called weakly subdifferentiable at x_0.

Remark 2.2. *If $\sigma(x) = \|x\|$, then the definition of 2.2 reduces to the corresponding definition in [1].*

Consider the following optimization problem with uncertain parameter in the constraint:

$$(UP) \qquad \inf_{x\in Q}\{f(x) \mid \ g(x, v) \in -D\},$$

where $f : X \to \bar{R}$ and $g : X \times Z \to Y$ are given functions, Z is another locally convex vector space, $Q \subset X$ is a nonempty closed set, v is uncertain parameter and belongs to $V \subseteq Z$.

For each $v \in V$, we denote

$$S_v = \{x \in Q \mid g(x, v) \in -D\}.$$

In this paper, robust optimization approach is applied to (UP). Now, we associate with (UP) its robust counterpart

$$(RP) \qquad \inf_{x\in Q}\{f(x) \mid \ g(x, v) \in -D,\ \forall\ v \in V\}.$$

We denote the feasible set of (RP) by

$$S = \{x \in Q \mid g(x, v) \in -D,\ \forall\ v \in V\} = \bigcap_{v\in V} S_v.$$

The problem (RP) is called the robust primal problem of (UP). The infimum for problem (RP) is denoted by inf (RP) and every element $x \in S$ such that $f(x) = \inf\ (RP)$ is called a robust solution of (UP) (or a solution of (RP)).

The Lagrange perturbation function of (UP) is $F : V \times X \times Y \to \bar{R}$ define as follows:

$$F_v(x, y) = \begin{cases} f(x),\ g(x, v) + y \in -D,\ x \in Q \\ +\infty,\ \text{otherwise}. \end{cases} \tag{2.2}$$

The weak conjugate function of F_v is $F_v^w :\ X^* \times R_+ \times Y^* \times R_+ \to \bar{R}$ given by

$$\begin{aligned} F_v^w(x^*, c, y^*, d) &= \sup_{(x,y)\in X\times Y}\{\langle x^*, x\rangle - c\sigma(x) + \langle y^*, y\rangle - d\sigma(y) - F_v(x, y)\} \\ &= \sup_{x\in Q}\ \sup_{y\in -D-g(x,v)}\{\langle x^*, x\rangle - c\sigma(x) + \langle y^*, y\rangle - d\sigma(y) - f(x)\} \end{aligned} \tag{2.3}$$

for all $(x^*, c, y^*, d) \in X^* \times R_+ \times Y^* \times R_+$.

Remark 2.3. *It follows immediately from the definition of weak biconjugate function, we have*

$$\begin{aligned} F_v^{ww}(x,0) &= \sup_{(x^*,c,y^*,d)} \{\langle x^*, x\rangle - c\sigma(x) + \langle y^*, 0\rangle - d\sigma(0) - F_v^w(x^*, c, y^*, d)\} \\ &= \sup_{(x^*,c,y^*,d)} \{\langle x^*, x\rangle - c\sigma(x) - \sup_{(x,y)}\{\langle x^*, x\rangle - c\sigma(x) + \langle y^*, y\rangle - d\sigma(y) - F_v(x,y)\}\} \\ &\leq \sup_{(x^*,c,y^*,d)} \{\langle x^*, x\rangle - c\sigma(x) - \langle x^*, x\rangle + c\sigma(x) + F_v(x,0)\} \\ &= F_v(x,0). \end{aligned}$$

Remark 2.4. *Considering (1.1) and the definition of $F_v(x,y)$, we can conclude that*

$$p(x) = \begin{cases} f(x), & x \in S, \\ +\infty, & \textit{otherwise}. \end{cases}$$

Let $q: X^* \times R_+ \to \bar{R}$ be the function defined by

$$q(x^*, c) = \inf_{v\in V} \inf_{(y^*,d)\in Y^*\times R_+} F_v^w(x^*, c, y^*, d), \quad \forall\, (x^*, c) \in X^* \times R_+.$$

Let the projection $\Pi: (x^*, c, y^*, d, r) \in X^* \times R_+ \times Y^* \times R_+ \times R \to (x^*, c, r) \in X^* \times R_+ \times R$, and let

$$\Lambda = \bigcup_{v\in V} \Pi(\mathrm{epi} F_v^w)$$

Lemma 2.1. *Let $p^w : X^* \times R_+ \to \bar{R}$ be a weak conjugate function, then p^w is lower semicontinuous and convex on $X^* \times R_+$.*

Proof. By the definition of weak conjugate function, we have

$$\begin{aligned} p^w(x^*, c) &= \sup_{x\in X}\{\langle x^*, x\rangle - c\sigma(x) - p(x)\} \\ &= \sup_{x\in X}\{\langle x^*, x\rangle - (p + c\sigma)(x)\} \\ &= (p + c\sigma)^*(x^*), \end{aligned}$$

where $(\rho + c\sigma)(x) = \rho(x) + c\sigma(x)$, so p^w is lower semicontinuous on $X^* \times R_+$. Since x^* is linear function and c is a constant, so convexity is easy to obtain. The proof is complete. □

The following lemmas generalize [[13], Lemmas 2.1 and 2.2].

Lemma 2.2. *One has*
(i) $q^w = \sup_{v\in V} F_v^{ww}(x, 0_Y) \leq p$;
(ii) $p^w \leq q^{ww} \leq q$;
(iii) $epi_s p = \bigcup_{v\in V} \Pi(epi_s F_v^w)$;
(iv) $\Lambda \subset epiq \subset epip^w$ *and* $\overline{co}\Lambda \subset epip^w$.

Proof. For any $x \in X$, from the definition of q^w one has

$$\begin{aligned}
q^w(x) &= \sup_{(x^*,c)\in X^*\times R_+}\{\langle x^*,x\rangle - c\sigma(x) - q(x^*,c)\} \\
&= \sup_{(x^*,c)\in X^*\times R_+}\{\langle x^*,x\rangle - c\sigma(x) - \inf_{v\in V}\inf_{(y^*,d)\in Y^*\times R_+} F_v^w(x^*,c,y^*,d)\} \\
&= \sup_{v\in V}\sup_{\substack{(x^*,c)\in X^*\times R_+\\(y^*,d)\in Y^*\times R_+}}\{\langle x^*,x\rangle - c\sigma(x) + \langle y^*,0\rangle - d\sigma(0) - F_v^w(x^*,c,y^*,d)\} \\
&= \sup_{v\in V} F_v^{ww}(x,0_Y). \qquad (2.4)
\end{aligned}$$

Since $F_v^{ww}(x,0_Y) \le F_v(x,0_Y) \le p(x)$ for all $v \in V$ and $x \in X$, (2.4) yields $q^w(x) \le p(x)$ and (i) holds, while (ii) follows from (i).

Proof of (iii). Take $(x^*,c,r) \in \mathrm{epi}_s q$. Then

$$\inf_{\substack{v\in V\\(y^*,d)\in Y^*\times R_+}} F_v^w(x^*,c,y^*,d) < r,$$

which implies there exist $\bar{v} \in V$ and $(\bar{y}^*,\bar{d}) \in Y^* \times R_+$ such that

$$F_{\bar{v}}^w(x^*,c,\bar{y}^*,\bar{d}) < r,$$

so $(x^*,c,\bar{y}^*,\bar{d},r) \in \mathrm{epi}_s F_{\bar{v}}^w$ and $(x^*,c,r) = \Pi(\mathrm{epi}_s F_{\bar{v}}^w) \subset \bigcup_{v\in V}\Pi(\mathrm{epi}_s F_{\bar{v}}^w)$, which mean $\mathrm{epi}_s q \subset \bigcup_{v\in V}\Pi(\mathrm{epi}_s F_v^w)$.

On the other hand, take $(x^*,c,r) \in \bigcup_{v\in V}\Pi(\mathrm{epi}_s F_v^w)$, then there exist $\bar{v} \in V$ such that $(x^*,c,r) \in \Pi(\mathrm{epi}_s F_{\bar{v}}^w)$. Since Π is surjective, there is $(\bar{y}^*,\bar{d}) \in Y^* \times R_+$ such that $(x^*,c,\bar{y}^*,\bar{d},r) \in \mathrm{epi}_s F_{\bar{v}}^w$, and so $(x^*,c,r) \in \mathrm{epi}_s q$ as

$$q(x^*,c) = \inf_{v\in V}\inf_{(y^*,d)\in Y^*\times R_+} F_v^w(x^*,c,y^*,d) \le F_{\bar{v}}^w(x^*,c,\bar{y}^*,\bar{d}) < r,$$

for all $(x^*,c) \in X^* \times R_+$. Thus, $\mathrm{epi}_s q \supset \bigcup_{v\in V}\Pi(\mathrm{epi}_s F_v^w)$ which, together with the inclusion above, proves that (iii) holds.

Proof of (iv). Since Π is surjective and

$$q(x^*,c) = \inf_{v\in V}\inf_{(y^*,d)\in Y^*\times R_+} F_v^w(x^*,c,y^*,d) \le F_v^w(x^*,c,\bar{y}^*,\bar{d})$$

for all $v \in V$, $(\bar{y}^*,\bar{d}) \in Y^* \times R_+$ and $(x^*,c) \in X^* \times R_+$, it follows that $\Lambda \subset \mathrm{epi}q$. By (ii), $\mathrm{epi}q \subset \mathrm{epi}p^w$, so $\overline{co}\Lambda \subset \mathrm{epi}p^w$. □

Lemma 2.3. *Assume that there exists $\bar{x} \in X$ such that $\sup_{v\in V} F_v^{ww}(x,0_Y) < +\infty$. Then one has $epiq^{ww} = \overline{co}\Lambda$. Moreover, the following statements are equivalent:*
(i) $p^{ww} = \sup_{v\in V} F_v^{ww}(\cdot,0_Y)$;
(ii) $epip^w = \overline{co}\Lambda$.

Proof. Observe that $q^w = \sup_{v\in V} F_v^{ww}(\cdot, 0_Y)$, and so by assumption, one obtains dom$q^w \neq \emptyset$. According to [34], epi$q^{ww} = \overline{co}(\text{epi}q)$ which, together with Lemma 2.1 (iii), implies

$$\overline{co}(\text{epi}q) = \overline{co}(\text{epi}_s q) = \overline{co}(\bigcup_{v\in V} \Pi\text{epi}_s F_v^w) = \overline{co}(\bigcup_{v\in V} \Pi\text{epi} F_v^w) = \overline{co}\Lambda.$$

For the equivalence of (i) and (ii), note that in light of Lemma 2.1, (i) is equivalent to $p^{ww} = q^w$, which means also that $q^{ww} = p^w$. The last equality and epi$q^{ww} = \overline{co}\Lambda$ show epip^w = epi$q^{ww} = \overline{co}\Lambda$, which is (ii). The proof is complete. □

3. Robust strong duality for nonconvex uncertain optimization problem

The aim of this section is to construct three types of robust dual problems for (UP) by using weak conjugate function: the robust augmented Lagrange dual, the robust weak Fenchel dual and the robust weak Fenchel-Lagrange dual problem, to establish characterization of inequality (1.1) according to robust abstract perturbational weak conjugate duality, and finally, by employing these results to obtain robust strong duality results for (UP).

3.1. Robust augmented Lagrange duality

To define an augmented Lagrange function for (UP), we need augmented function σ to be a continuous function with the properties (2.1). For each fixed $v \in V$, the uncertain augmented Lagrange function associated with (UP) is given by

$$\begin{aligned} L_v(x, y^*, d) &= \inf_{y\in Y}\{F_v(x, y) - \langle y^*, y\rangle + d\sigma(y)\} \\ &= \inf_{y\in Y} \begin{cases} f(x) - \langle y^*, y\rangle + d\sigma(y), & g(x, v) + y \in -D,\ x \in Q \\ +\infty, & \text{otherwise}, \end{cases} \end{aligned}$$

for $x \in X$, $y \in Y$, $y^* \in Y^*$ and $d \in R_+$, where function $F_v(x, y)$ is defined in (2.2). By using the definition of $F_v(x, y)$, we can concretize the augmented Lagrange associated with (UP)

$$L_v(x, y^*, d) = \inf_{y\in -D-g(x,v)}\{f(x) - \langle y^*, y\rangle + d\sigma(y)\},$$

for $x \in Q$, $y^* \in Y^*$ and $d \in R_+$.

The uncertain dual function of (UP) is

$$\phi_v(y^*, d) = \inf_{x\in Q} L_v(x, y^*, d),\ \text{for } (Y^*, d) \in Y^* \times R_+.$$

Then uncertain augmented Lagrange dual problem of (UP) is defined as

$$(UD_L) \qquad \sup_{(y^*,d)\in Y^*\times R^+} \phi_v(\lambda, r).$$

The optimistic counterpart of the uncertain augmented Lagrange dual (UD_L) is a deterministic maximization problem given by

$$(RD_L) \quad \sup_{(v,y^*,d)\in V\times Y^*\times R^+} \phi_v(y^*,d) = \sup_{(v,y^*,d)\in V\times Y^*\times R^+} \inf_{x\in Q} L_v(x,y^*,d)$$
$$= \sup_{(v,y^*,d)\in V\times Y^*\times R^+} \inf_{x\in Q} \inf_{y\in -D-g(x,v)} \{f(x) - \langle y^*, y\rangle + d\sigma(y)\}.$$

Now, when $x^* = 0_{X^*}$ and $c = 0$ in (2.3), the value of the function $F_v^w(0,0,y^*,d)$ simply denoted by

$$F_v^w(0,0,y^*,d) = \sup_{x\in Q} \sup_{y\in -D-g(x,v)} \{\langle y^*, y\rangle - d\sigma(y) - f(x)\}.$$

Hence,

$$-F_v^w(0,0,y^*,d) = \inf_{x\in Q} \inf_{y\in -D-g(x,v)} \{f(x) - \langle y^*, y\rangle + d\sigma(y)\}.$$

As a result, robust augmented Lagrange dual problem for (UP) with respect to F_v can be given by

$$(RD_L) \quad \sup_{(y^*,d)\in Y^*\times R_+} \sup_{v\in V}\{-F_v^w(0,0,y^*,d)\}.$$

The supremum for problem (RD_L) is denoted by $\sup(RD_L)$ and any element $(v,y^*,d) \in V\times Y^*\times R_+$ such that $-F_v^w(0,0,y^*,d) = \sup(RD_L)$ is termed as a solution of (RD_L).

Theorem 3.1. *(Weak duality)* $\sup(RD_L) \le \inf(RP)$.

Proof. For arbitrary $(v,y^*,d) \in V\times Y^*\times R_+$,

$$\begin{aligned} -F_v^w(0,0,y^*,d) &= -\sup_{x\in Q} \sup_{y\in -D-g(x,v)} \{\langle y^*, y\rangle - d\sigma(y) - f(x)\} \\ &= \inf_{x\in Q} \inf_{y\in -D-g(x,v)} \{f(x) - \langle y^*, y\rangle + d\sigma(y)\} \\ &\le \inf_{x\in S_V} f(x) \\ &\le \inf_{x\in S} f(x) = \inf(RP), \end{aligned}$$

so we conclude that $\sup(RD_L) \le \inf(RP)$ □

In the following sections we always assume Γ is a set of functions defined on X, H is a set of functions and $H \ne \emptyset$, define $\Gamma_H(X) = \{l \in \Gamma(x) \mid l(x) \text{ is } H\text{-convex on } X\}$.

Theorem 3.2. *(Robust abstract perturbational weak conjugate duality) Consider the following statements:*

(a_1) $epip^w = \Lambda$;

(b_1) For any $l \in \Gamma_H(X)$, the following assertions are equivalent:

(b_{1_1}) $p(x) \ge l(x)$;

(b_{1_2}) for all $(x^*,c) \in X^*\times R_+$,

there exist $(v,y^*,d) \in V\times Y^*\times R_+$ such that $F_v^w(x^*,c,y^*,d) \le l^w(x^*,c)$.

One has $(a_1) \Leftrightarrow (b_1)$.

Proof. $(a_1) \Rightarrow (b_1)$ Assume that (a_1) holds. Let $l \in \Gamma_H(x)$. If (b_1) holds, i.e., $p \geq l$, then $p^w \leq l^w$, and hence for all $(x^*, c) \in$ dom l^w, $(x^*, c, l^w(x^*, c)) \in$ epip^w. Since (a_1) holds, there exist $v \in V$ such that $(x^*, c, l^w(x^*, c)) \in \Pi(\text{epi}F_v^w)$, and so there exist $(y^*, d) \in Y^* \times R_+$ such that $F_v^w(x^*, c, y^*, d) \leq l^w(x^*, c)$, which implies that (b_{1_2}) holds.

Conversely, if (b_{1_2}) holds, then for any $(x^*, c) \in$ dom l^w, there exist $(v, y^*, d) \in V \times Y^* \times R_+$ such that $F_v^w(x^*, c, y^*, d) \leq l^w(x^*, c)$. Taking Lemma 2.1 (ii) into account, we get $p^w(x^*, c) \leq q(x^*, c) \leq F_v^w(x^*, c, y^*, d) \leq l^w(x^*, c)$, and hence $l = l^{ww} \leq p^{ww} \leq p$, the " = " above is from l being H-convex.

$(b_1) \Rightarrow (a_1)$. Assume that (b_1) holds, we will show that (a_1) holds. To this aim, considering Lemma 2.1 (iv) into account, it is sufficient to prove that

$$\text{epi}p^w \subset \Lambda.$$

Take every $(x^*, c, r) \in$ epip^w. Then $\langle x^*, x\rangle - c\sigma(x) - p(x) \leq p^w(x^*, c) \leq r$ for all $x \in X$. Now set $l(x) = \langle x^*, x\rangle - c\sigma(x) - r$, then $l \in \Gamma_H(X)$ and $l \leq p$, so by (b_1), there exist $(v, y^*, d) \in V \times Y^* \times R_+$ such that $F_v^w(x^*, c, y^*, d) \leq l^w(x^*, c) \leq r$. This shows that $(x^*, c, r) \in \Pi(\text{epi}F_v^w)$ and hence $(x^*, c, r) \in \Lambda$. □

Remark 3.1. *Theorem 3.2 generalizes [[13], Theorem 3.1]. In [13] the authors used the classical conjugate function and assumed the right-side function $l(x)$ of inequality (1.1) is convex lower semicontinuous, whereas Theorem 3.2 in this paper, we employ the weak conjugate function and only assume $l(x)$ is abstract convex, which covers very broad classes on nonconvex function.*

Theorem 3.3. *(Strong duality) Let epi$p^w = \Lambda$, then there exist $(v_0, y_0^*, d_0) \in V \times Y^* \times R_+$ such that (v_0, y_0^*, d_0) is a solution of (RD_L) and* $\sup(RD_L) = \inf(RP)$.

Proof. Let $l(x) = \inf_{x\in X} p(x)$, then $l(x) = \inf(RP)$ for all $x \in S$. From assumption epi$p^w = \Lambda$ and considering $x^* = 0_{X^*}$, $c = 0$ in Theorem 3.2, then there exist $(v_0, y_0^*, d_0) \in V \times Y^* \times R_+$ such that

$$F_{v_0}^w(0, 0, y_0^*, d_0) \leq l^w(0, 0) = -\inf(RP),$$

which is equivalent to

$$-F_{v_0}^w(0, 0, y_0^*, d_0) \geq \inf(RP),$$

considering weak duality of Theorem 3.1, we have

$$\inf(RP) \geq \sup(RD_L) \geq -F_{v_0}^w(0, 0, y_0^*, d_0) \geq \inf(RP),$$

this yields (v_0, y_0^*, d_0) is a solution of (RD_L) and $\sup(RD_L) = \inf(RP)$. □

Theorem 3.4. *Assume that dom$p \neq \emptyset$ and $p^{ww} = \sup_{v\in V} F_v^{ww}(\cdot, 0_Y)$. Then the following statements are equivalent:*

(f_1) $\Lambda = \overline{co}\Lambda$;

(b_1) For any $\Gamma_H(X)$, the following assertions are equivalent:

(b_{1_1}) $p(x) \geq l(x)$;

(b_{1_2}) for all $(x^*, c) \in X^* \times R_+$, there exist $(v, y^*, d) \in V \times Y^* \times R_+$ such that $F_v^w(x^*, c, y^*, d) \leq l^w(x^*, c)$.

Proof. Since $p^{ww} = \sup_{v\in V} F_v^{ww}(\cdot, 0_Y)$, Lemma 2.3 and f_1 give epi$p^w = \overline{co}\Lambda = \Lambda$. The conclusion follows from Theorem 3.2. □

Remark 3.2. *Theorem 3.4 gives an equivalent condition that the set Λ is closed convex set.*

Next we give a sufficient condition for $p^{ww} = \sup_{v\in V} F_v^{ww}(\cdot, 0)$.

Proposition 3.1. *Assume that for any $(x^*, c) \in X^* \times R_+$, there exists $\bar{v} \in V$ such that $\partial_y F_{\bar{v}}(\cdot, 0_Y) \neq \emptyset$ on $S_{\bar{v}}$ and $\sup_{x\in S}\{\langle x^*, x\rangle - c\sigma(x) - f(x)\} = \sup_{x\in S_{\bar{v}}}\{\langle x^*, x\rangle - c\sigma(x) - f(x)\}$, then $p^{ww} = \sup_{v\in V} F_v^{ww}(\cdot, 0)$.*

Proof. Since for any $(x^*, c) \in X^* \times R_+$, there exists $\bar{v} \in V$ such that $\partial_y F_{\bar{v}}(\cdot, 0_Y) \neq \emptyset$, so there exist $(y^*, d) \in Y^* \times R_+$ such that $(y^*, d) \in \partial_y F_{\bar{v}}(\cdot, 0_Y)$, which implies

$$-F_{\bar{v}}(x, 0) \geq -F_{\bar{v}}(x, y) + \langle y^*, y\rangle - d\sigma(y),\ x \in S_{\bar{v}},\ y \in Y,$$

that is

$$-F_{\bar{v}}(x, 0) \geq \langle y^*, y\rangle - d\sigma(y) - F_{\bar{v}}(x, y),\ x \in S_{\bar{v}},\ y \in Y. \tag{3.1}$$

Following from (3.1), we obtain

$$\begin{aligned} F_{\bar{v}}^{w}(x^*, c, y^*, d) &= \sup_{(x,y)\in X\times Y} \{\langle x^*, x\rangle - c\sigma(x) + \langle y^*, y\rangle - d\sigma(y) - F_{\bar{v}}(x, y)\} \qquad (3.2)\\ &\leq \sup_{x\in X}\{\langle x^*, x\rangle - c\sigma(x) - F_{\bar{v}}(x, 0)\}\\ &= \sup_{x\in S_{\bar{v}}}\{\langle x^*, x\rangle - c\sigma(x) - f(x)\} \end{aligned}$$

Since $\sup_{x\in S}\{\langle x^*, x\rangle - c\sigma(x) - f(x)\} = \sup_{x\in S_{\bar{v}}}\{\langle x^*, x\rangle - c\sigma(x) - f(x)\}$, together with (3.2), we get

$$\begin{aligned} F_{\bar{v}}^{w}(x^*, c, y^*, d) &\leq \sup_{x\in S}\{\langle x^*, x\rangle - c\sigma(x) - f(x)\}\\ &= \sup_{x\in X}\{\langle x^*, x\rangle - c\sigma(x) - P(x)\}\\ &= p^{w}(x^*, c). \end{aligned} \tag{3.3}$$

The first equality above follows from Remark 2.1. Taking account into (3.3) and the definition of $p^{ww}(x)$, one has

$$\begin{aligned} p^{ww}(x) &= \sup_{(x^*,c)\in X^*\times R_+} \{\langle x^*, x\rangle - c\sigma(x) - p^{w}(x^*, c)\}\\ &\leq \sup_{(x^*,c)\in X^*\times R_+} \{\langle x^*, x\rangle - c\sigma(x) - F_{\bar{v}}^{w}(x^*, c, y^*, d)\}\\ &\leq \sup_{v\in V} \sup_{(x^*,c,y^*,d)\in X^*\times R_+\times Y^*\times R_+} \{\langle x^*, x\rangle - c\sigma(x) - F_v^{w}(x^*, c, y^*, d)\}\\ &= \sup_{v\in V} F_v^{ww}(x, 0)\\ &\leq p^{ww}(x). \end{aligned}$$

Which implies $p^{ww} = \sup_{v\in V} F_v^{ww}(\cdot, 0)$. The proof is complete. □

Remark 3.3. *We note that $\partial_y F_{\bar{v}}(\cdot, 0_Y) \neq \emptyset$ on $S_{\bar{v}}$ is a general assumption. In fact, we just need to assume augmented function $\sigma(y)$ satisfy $\sigma(y) \geq \|y\|$ for all $y \in Y$ and take $(y^*, d) \in Y^* \times R_+$ such that $\|y^*\| \leq d$, then (3.1) holds from the definition of $F_{\bar{v}}(x, 0)$ and $F_{\bar{v}}(x, y)$ (see (2.2)), so we can conclude that $(y^*, d) \in \partial_y F_{\bar{v}}(\cdot, 0_Y)$, which implies $\partial_y F_{\bar{v}}(\cdot, 0_Y) \neq \emptyset$.*

Remark 3.4. *In the proof of the strong duality theorem 3.3, our sufficient condition epip*$^w = \Lambda$ *is different from the existed conditions. Duality Theorem 11.59 in [26], the condition was supposed dualizing parameterization* $\phi(x, y)$ *is level-bounded in x locally uniformly in y, duality theorems in [1, 32], conditions were assumed perturbation function* $h = \inf_x \phi(x, y)$ *is proper and weakly subdifferential at the origin* $0 \in Y$*. All these conditions were formulated in terms of dualization parameterization or perturbation functions associated with the given problem.*

We recall the assumption epip$^w = \Lambda$ *in theorem 3.3, which employ the epigraph of function* F_v^w *defined by (2.3), it is also related to dualization parameterization* $F_v(x, y)$*, and we know that epip*$^w = \overline{co}\Lambda$ *is easy to satisfy from Lemma 2.3, Proposition 3.1 and Remark 3.1. Moreover, Theorem 3.4 gives an equivalent condition that the set* Λ *is closed convex set, so it is worth further exploration to find the sufficient conditions that can ensure the set* Λ *is not only a closed convex set but also only related to the objective function and the constraint function.*

3.2. Robust weak Fenchel conjugate duality

The Fenchel perturbation function of (UP) is $F : V \times X \times X \to \bar{R}$ defined as

$$F_v(x, u) = \begin{cases} f(x + u), & g(x, v) \in -D,\ x \in Q, \\ +\infty, & \text{otherwise.} \end{cases}$$

The weak conjugate function of F_v is $F_v^w : X^* \times R_+ \times X^* \times R \to R_+$ defined as

$$\begin{aligned} F_v^w(x^*, c, u^*, d) &= \sup_{(x,u)\in X\times X} \{\langle x^*, x\rangle - c\sigma(x) + \langle u^*, u\rangle - d\sigma(u) - F_v(x, u)\} \\ &= \sup_{x\in S_v}\sup_{u\in X}\{\langle x^*, x\rangle - c\sigma(x) + \langle u^*, u\rangle - d\sigma(u) - f(x + u)\} \\ &= \sup_{x\in S_v}\sup_{\gamma\in X}\{\langle x^*, x\rangle - c\sigma(x) + \langle u^*, \gamma - x\rangle - d\sigma(\gamma - x) - f(\gamma)\}, \end{aligned}$$

for $x^*,\ u^* \in X^*$ and $c,\ d \in R_+$, where $\gamma = x + u$. By choosing $x^* = 0_{X^*}$, $c = 0$, we have

$$F_v^w(0, 0, u^*, d) = \sup_{x\in S_v}\sup_{\gamma\in X}\{\langle u^*, \gamma - x\rangle - d\sigma(\gamma - x) - f(\gamma)\}.$$

Hence, the robust Fenchel dual problem (RD_F^w) with respect to F_v is defined as

$$\begin{aligned} (RD_F^w) \qquad & \sup_{(u^*,d)\in X^*\times R_+}\ \sup_{v\in V}\{-F_v^w(0, 0, u^*, d)\} \\ = \ & \sup_{(u^*,d)\in X^*\times R_+}\ \sup_{v\in V}\inf_{x\in S_v}\inf_{\gamma\in X}\{f(\gamma) - \langle u^*, \gamma - x\rangle + d\sigma(\gamma - x)\} \end{aligned}$$

The supremum for problem (RD_F) is denoted by $\sup(RD_F)$ and any element $(v, u^*, d) \in V \times X^* \times R_+$ such that $-F_v^w(0, 0, u^*, d) = \sup(RD_F)$ is termed as a solution of (RD_F).

Remark 3.5. $\sup(RD_F^w) \leq \inf(RP)$ *follows immediately from the definition of* $F_v^w(0, 0, u^*, d)$.

Let the projection $\Pi_1 : (x^*, c, u^*, d, r) \in X^* \times R_+ \times X^* \times R_+ \times R \to (x^*, c, r) \in X^* \times R_+ \times R$ and let

$$\Lambda_1 = \bigcup_{v\in V} \Pi_1(\text{epi}F_v^w)$$

Theorem 3.5. *(Strong duality) Let epip*$^w = \Lambda_1$*, then there exists* $(v_0, u_0^*, d_0) \in V \times X^* \times R_+$ *such that* (v_0, u_0^*, d_0) *is a solution of* (RD_F) *and* $\sup(RD_F) = \inf(RP)$

Proof. The proof is similar to that of Theorem 3.3. □

3.3. Fenchel-Lagrange weak conjugate duality

The Fenchel-Lagrange perturbation function of (UP) is $F : V \times X \times X \times Y \to \bar{R}$ defined as

$$F_v(x,u,y) = \begin{cases} f(x+u), & g(x,v) + y \in -D,\ x \in Q, \\ +\infty, & \text{otherwise.} \end{cases}$$

The weak conjugate function of $F_v(x,u,y)$ is defined as $F_v^w : X^* \times R_+ \times X^* \times R_+ \times Y^* \times R_+ \to R_+$:

$$\begin{aligned} F_v^w(x^*,c,u^*,d,y^*,e) &= \sup_{(x,u,y)\in X\times X\times Y} \{\langle x^*,x\rangle - c\sigma(x) + \langle u^*,u\rangle - d\sigma(u) + \langle y^*,y\rangle - e\sigma(y) - F_v(x,u,y)\} \\ &= \sup_{\substack{x\in Q\\ u\in X}} \sup_{y\in -D-g(x,v)} \{\langle x^*,x\rangle - c\sigma(x) + \langle u^*,u\rangle - d\sigma(u) + \langle y^*,y\rangle - e\sigma(y) - f(x+u)\} \\ &= \sup_{\substack{x\in Q\\ \gamma\in X}} \sup_{y\in -D-g(x,v)} \{\langle x^*,x\rangle - c\sigma(x) + \langle u^*,\gamma - x\rangle - d\sigma(\gamma - x) + \langle y^*,y\rangle - e\sigma(y) - f(\gamma)\}, \end{aligned}$$

where $\gamma = x + u$. By choosing $x^* = 0_{X^*}$, $c = 0$ and $d = e$, we have

$$F_v^w(0,0,u^*,d,y^*,d) = \sup_{\substack{x\in Q\\ \gamma\in X}} \sup_{y\in -D-g(x,v)} \{\langle u^*,\gamma - x\rangle - d\sigma(\gamma - x) + \langle y^*,y\rangle - d\sigma(y) - f(\gamma)\}.$$

Hence, the robust Fenchel-Lagrange dual problem (RD_{FL}^w) with respect to F_v is defined as

$$\begin{aligned} (RD_{FL}^w) \qquad & \sup_{\substack{(u^*,d)\in X^*\times R_+\\ (y^*,d)\in Y^*\times R_+}} \sup_{v\in V} \inf_{\substack{x\in Q\\ \gamma\in X}} \inf_{y\in -D-g(x,v)} \{-F_v^w(0,0,u^*,d,y^*,d)\} \\ = \ & \sup_{\substack{(u^*,d)\in X^*\times R_+\\ (y^*,d)\in Y^*\times R_+}} \sup_{v\in V} \inf_{\substack{x\in Q\\ \gamma\in X}} \inf_{y\in -D-g(x,v)} \{f(\gamma) - \langle u^*,\gamma - x\rangle - \langle y^*,y\rangle + d\sigma(\gamma - x) + d\sigma(y)\} \end{aligned}$$

The supremum for problem (RD_{FL}^w) is denoted by $\sup(RD_{FL}^w)$ and any element $(v,u^*,d,y^*,d) \in V \times X^* \times R_+ \times X^* \times R_+$ such that $-F_v^w(0,0,u^*,d) = \sup(RD_{FL}^w)$ is termed as a solution of (RD_{FL}^w).

Theorem 3.6. *(Weak duality)* $\sup(RD_{FL}^w) \le \inf(RP)$.

Proof. For any $(v,u^*,d,y^*,d) \in V \times X^* \times R_+ \times Y^* \times R_+$,

$$\begin{aligned} -F_v^w(0,0,(v,u^*,d,y^*,d)) &= \inf_{\substack{x\in Q\\ \gamma\in X}} \inf_{y\in -D-g(x,v)} \{f(\gamma) - \langle u^*,\gamma - x\rangle - \langle y^*,y\rangle + d\sigma(\gamma - x) + d\sigma(y)\} \\ &\le \inf_{x\in Q} \inf_{y\in -D-g(x,v)} \{f(x) - \langle y^*,y\rangle + d\sigma(y)\} \\ &\le \inf_{x\in S_V} f(x) \\ &\le \inf_{x\in S} f(x) = \inf(RP), \end{aligned}$$

so we conclude that $\sup(RD_L) \le \inf(RP)$ □

Let the projection $\Pi_2 : (x^*,c,u^*,d,y^*,e,r) \in X^*\times R_+\times X^*\times R_+\times Y^*\times R_+\times R \to (x^*,c,r) \in X^*\times R_+\times R$ and let

$$\Lambda_2 = \bigcup_{v\in V} \Pi_2(\text{epi}F_v^w).$$

Theorem 3.7. *(Strong duality) Let* $epip^w = \Lambda_2$*, then there exists* $(v,u^*,d,y^*,d) \in V\times X^*\times R_+\times Y^*\times R_+$ *such that* (v,u^*,d,y^*,d) *is a solution of* (RD_{FL}^w) *and* $\inf(RP) = \sup(RD_{FL}^w)$.

Proof. The proof is similar to that of Theorem 3.3. □

4. Relationships among the objective values of dual problems (RD_L), (RD_F^w) and (RD_{FL}^w)

In this section, we examine the relations among the objective values of dual problems (RD_L), (RD_F^w) and (RD_{FL}^w).

Proposition 4.1. *The inequality* $\sup(D_{FL}^w) \leq \sup(RD_L)$ *holds.*

Proof. Let (v, u^*, y^*, d) be an arbitrary element of $V \times X^* \times Y^* \times R_+$. It is known that

$$\begin{aligned}
& \inf_{x\in Q}\inf_{\gamma\in X}\inf_{y\in -D-g(x,v)}\{f(\gamma) - \langle u^*, \gamma - x\rangle - \langle y^*, y\rangle + d\sigma(\gamma - x) + d\sigma(y)\} \\
\leq\ & \inf_{x\in Q}\inf_{y\in -D-g(x,v)}\{f(x) - \langle y^*, y\rangle + d\sigma(y)\} \\
\leq\ & \sup_{(v,y^*,d)\in V\times Y^*\times R^+}\inf_{x\in Q}\inf_{y\in -D-g(x,v)}\{f(x) - \langle y^*, y\rangle + d\sigma(y)\} \\
=\ & \sup(RD_L).
\end{aligned}$$

As $(v, u^*, y^*, d) \in V \times X^* \times Y^* \times R_+$ is arbitrary element, we get

$$\begin{aligned}
\sup(D_{FL}^w) &= \sup_{(v,u^*,y^*,d)}\inf_{x\in Q}\inf_{\gamma\in X}\inf_{y\in -D-g(x,v)}\{f(\gamma) - \langle u^*, \gamma - x\rangle - \langle y^*, y\rangle + d\sigma(\gamma - x) + d\sigma(y)\} \\
&\leq \sup(RD_L).
\end{aligned}$$

□

Proposition 4.2. *The inequality* $\sup(RD_{FL}^w) \leq \sup(RD_F^w)$ *holds.*

Proof. Let (v, u^*, y^*, d) be an arbitrary element of $V \times X^* \times Y^* \times R_+$. It is known that $0 \in -D - g(x, v)$ for all $x \in S_v$, so we have

$$\begin{aligned}
& \inf_{\substack{x\in Q\\ \gamma\in X}}\inf_{y\in -D-g(x,v)}\{f(\gamma) - \langle u^*, \gamma - x\rangle - \langle y^*, y\rangle + d\sigma(\gamma - x) + d\sigma(y)\} \\
\leq\ & \inf_{\substack{x\in S_v\\ \gamma\in X}}\inf_{y\in -D-g(x,v)}\{f(\gamma) - \langle u^*, \gamma - x\rangle - \langle y^*, y\rangle + d\sigma(\gamma - x) + d\sigma(y)\} \\
\leq\ & \inf_{\substack{x\in S_v\\ \gamma\in X}}\{f(\gamma) - \langle u^*, \gamma - x\rangle + d\sigma(\gamma - x)\} \\
\leq\ & \sup_{(v,u^*,d)\in V\times X^*\times R_+}\inf_{\substack{x\in S_v\\ \gamma\in X}}\{f(\gamma) - \langle u^*, \gamma - x\rangle + d\sigma(\gamma - x)\} \\
=\ & \sup(RD_F^w).
\end{aligned}$$

Hence, taking the supremum in both sides over $(v, u^*, y^*, d) \in V \times X^* \times Y^* \times R_+$, we get

$$\sup(RD_{FL}^w) \leq \sup(RD_F^w).$$

This completes the proof. □

Proposition 4.3. *Let* $\Lambda_2 \bigcap(\{0_{X^*}\}\times\{0\}\times R_+) = epip^w \bigcap(\{0_{X^*}\}\times\{0\}\times R_+)$*, then* $\sup(RD_{FL}^w) = \sup(RD_F^w) = \sup(RD_L) = \inf(RP)$.

Proof. Under these assumptions and considering Theorem 3.7, it is known that $\sup(RD^w_{FL}) = \inf(RP)$. By Propositions 4.1 and 4.2, we obtain

$$\sup(RD^w_{FL}) = \sup(RD^w_F) = \sup(RD_L) = \inf(RP).$$

□

Now, we present an example of robust optimization problem which prove the relationships between the optimal values of the three proposed dual problems.

Example 1. Consider the following one-dimensional optimization with data uncertainty in constraint:

$$(UP) \qquad \inf_{x\in Q}\{f(x) \mid \ g(x,v) = vx \le 0\},$$

where $f: \ R \to R$ is defined as

$$f(x) = \begin{cases} -|x|, & x \le 1 \\ -1 \quad , & x > 1. \end{cases}$$

for all $x \in R$ and $Q = R$, the data $v \in [-1, 1]$ is uncertain.

In this example, we always assume function defined in (2.1) is $\sigma(x) = |x|$. Let us calculate the weak conjugate function of Lagrange perturbation $F_v(x,u)$ with $x^* = c = 0$. If $0 < v \le 1$, then $F^w_v(0,0,y^*,d) = +\infty$, if $-1 \le v \le 0$,

$$F^w_v(0,0,y^*,d) = \begin{cases} +\infty, & z^* - d > 0 \text{ or } z^* + d < 0, \text{ or } z^* + d > 0 \text{ and } 1 + (z^* + d)v > 0 \\ u^* - c + 1, & u^* - c \ge 0. \end{cases}$$

then we have $\sup(RD_L) = -1$.

Let us calculate the weak conjugate function of Fenchel perturbation $F_v(x,u)$ with $x^* = c = 0$. If $0 < v \le 1$, then $F^w_v(0,0,u^*,d) = +\infty$, otherwise if $-1 \le v \le 0$,

$$F^w_v(0,0,u^*,d) = \begin{cases} +\infty, & u^* + c + 1 < 0 \text{ or } u^* > c \\ u^* - c + 1, & u^* - c \ge 0. \end{cases}$$

Then we obtain $\sup(RD^w_F) = -1$.

We also get $\sup(RD^w_{FL}) = -\infty$. So we obtain $-\infty = \sup(RD^w_{FL}) < \sup(RD^w_F) = \sup(RD_L) = -1$.

Remark 4.1. *Consider Example 1, let $v \le 0$, from [[17], Example1], we know the classical conjugate function of Fenchel perturbation $F_v(x,y)$ with $x^* = 0$ is $F^*_v(0,y^*) = +\infty$, so we can conclude that the optimal value of robust Fenchel dual problem in classical sense (denoted by* $\sup(RD^*_F)$*) is*

$$\sup(RD^*_F) = \sup_{v\in[-1,0]} \sup_{y^*\in Y^*} -F^*_v(0,y^*) = -\infty < \sup(RD^w_F) = -1,$$

which shows that weak conjugate function is more likely to guarantee zero dual gaps than classical conjugate function.

5. Optimality conditions

In this section, we give optimality conditions for (RD_F^w), (D_{FL}^w) and (RP).

Theorem 5.1. *Let* $\Lambda_1 \bigcap(\{0_{X^*}\} \times \{0\} \times R_+) = epip^w \bigcap(\{0_{X^*}\} \times \{0\} \times R_+)$.
(a) If $\bar{x}$ *is a solution of* (RP)*, then there exists a solution* $(v_0, u_0^*, d_0) \in V \times X^* \times R_+$ *such that*

(i) $f(\bar{x}) = \inf_{x \in S_{v_0}} \inf_{\gamma \in X} \{f(\gamma) - \langle u_0^*, \gamma - x\rangle + d_0\sigma(\gamma - x)\}$,

(ii) $(u_0^*, d_0) \in \partial^w f(\bar{x})$.

(b) Conversely, if $\bar{x} \in S$ *and* $(v_0, u_0^*, d_0) \in V \times X^* \times R_+$ *satisfies conditions (i) and (ii), then* $\bar{x}$ *is a solution of (RP) and* (v_0, u_0^*, d_0) *is a solution of* (RD_F^w).

Proof. (a) Let $\Lambda_1 \bigcap(\{0_{X^*}\} \times \{0\} \times R_+) = \text{epi}p^w \bigcap(\{0_{X^*}\} \times \{0\} \times R_+)$. Theorem 3.5 ensures the existence of an optimal solution $(v_0, u_0^*, d_0) \in V \times X^* \times R_+$ of (RD_F^w) and $\sup(D_F^w) = \inf(RP)$. Since $\bar{x}$ is a solution of (RP), so we get

$$f(\bar{x}) = \inf_{x \in S_{v_0}} \inf_{\gamma \in X} \{f(\gamma) - \langle u_0^*, \gamma - x\rangle + d_0\sigma(\gamma - x)\},$$

which means condition (i) is satisfied.

From (i) we have

$$\begin{aligned} f(\bar{x}) &= \inf_{x \in S_{v_0}} \inf_{\gamma \in X} \{f(\gamma) - \langle u_0^*, \gamma - x\rangle + d_0\sigma(\gamma - x)\} \\ &\leq f(\gamma) - \langle u_0^*, \gamma - \bar{x}\rangle + d_0\sigma(\gamma - \bar{x}), \ \forall\, \gamma \in X \end{aligned}$$

which implies $(u_0^*, d_0) \in \partial^w f(\bar{x})$. Therefore, (ii) is satisfied.
(b) Let (i) be satisfied. Then

$$\begin{aligned} \inf(RP) &\leq f(\bar{x}) \\ &= \inf_{x \in S_{v_0}} \inf_{\gamma \in X} \{f(\gamma) - \langle u_0^*, \gamma - x\rangle + d_0\sigma(\gamma - x)\} \\ &\leq \sup_{(v,u^*,d) \in V \times X^* \times R_+} \inf_{x \in S_{v_0}} \inf_{\gamma \in X} \{f(\gamma) - \langle u_0^*, \gamma - x\rangle + d_0\sigma(\gamma - x)\} \\ &= \sup(RD_F^w). \end{aligned}$$

Considering the above formula and weak duality, it is easy to get the following,

$$\inf(RP) = \sup(D_F^w) = f(\bar{x}) = \inf_{x \in S_{v_0}} \inf_{\gamma \in X} \{f(r) - \langle u_0^*, \gamma - x\rangle + d_0\sigma(\gamma - x)\},$$

that is, $\bar{x}$ and (v_0, u_0^*, d_0) are solutions of (RP) and (RD_F^w) respectively. □

Theorem 5.2. *Let* $\Lambda_2 \bigcap(\{0_{X_*}\} \times \{0\} \times R_+) = epip^w \bigcap(\{0_{X_*}\} \times \{0\} \times R_+)$.
(a) If $\bar{x}$ *is a solution of* (RP)*, then there exist a solution* $(v_0, u_0^*, d_0, y_0^*, d_0) \in V \times X^* \times R_+ \times Y^* \times R_+$ *such that*

(i) $f(\bar{x}) = \inf_{\substack{x \in S_{v_0} \\ \gamma \in X}} \inf_{y \in -D-g(x,v_0)} \{f(\gamma) - \langle u_0^*, \gamma - x\rangle - \langle y_0^*, y\rangle + d_0\sigma(\gamma - x) + d_0\sigma(y)\}$*;*

(ii) $(u_0^*, d_0) \in \partial^w f(\bar{x})$*;*

(iii) $\inf_{y\in -D-g(\bar{x},v_0)}\{d_0\sigma(y)-\langle y_0^*,y\rangle\}=0.$

(b) Conversely, if $\bar{x}\in S$ *and* $(v_0,u_0^*,d_0,y_0^*,d_0)\in V\times X^*\times R_+\times Y^*\times R_+$ *satisfy conditions (i)–(iii), then* $\bar{x}$ *is solution of of (RP) and* $(v_0,u_0^*,d_0,y_0^*,d_0)$ *is solution of* (D_{FL}^w).

Proof. (a) Let $\Lambda_2\bigcap(\{0_{X^*}\}\bigcap\{0\}\times R_+)=\mathrm{epi}p^w\bigcap(\{0_{X^*}\}\times\{0\}\times R_+)$. Theorem 3.7 guarantees the existence of an optimal solution $(v_0,u_0^*,d_0,y_0^*,d_0)\in V\times X^*\times R_+\times Y^*\times R_+$ of (RD_{FL}^w) and $\sup(D_{FL}^w)=\inf(RP)$. Since $\bar{x}$ is a solution of (RP), then

$$f(\bar{x})=\inf_{\substack{x\in S_{v_0}\\ \gamma\in X}}\inf_{y\in -D-g(x,v_0)}\{f(\gamma)-\langle u_0^*,\gamma-x\rangle-\langle y_0^*,y\rangle+d_0\sigma(\gamma-x)+d_0\sigma(y)\}.$$

Therefore, (i) is satisfied.

As $x\in S_{v_0}$ implies $0_Y\in -D-g(x,v_0)$, so $\inf_{y\in -D-\bigcup_{x\in S_{v_0}}g(x,v_0)}\{d_0\sigma(y)-\langle y_0^*,y\rangle\}\le 0$, which means

$$\begin{aligned}f(\bar{x}) &= \inf_{\substack{x\in S_{v_0}\\ \gamma\in X}}\inf_{y\in -D-g(x,v_0)}\{f(\gamma)-\langle u_0^*,\gamma-x\rangle-\langle y_0^*,y\rangle+d_0\sigma(\gamma-x)+d\sigma(y)\}\\ &\le f(\gamma)-\langle u_0^*,\gamma-\bar{x}\rangle+d_0\sigma(\gamma-\bar{x}),\ \forall\,\gamma\in X,\end{aligned}$$

this implies $(u_0^*,d_0)\in\partial^w f(\bar{x})$, that is (ii) holds.

Considering (i) we also have

$$\begin{aligned}f(\bar{x}) &= \inf_{\substack{x\in S_{v_0}\\ \gamma\in X}}\inf_{y\in -D-g(x,v_0)}\{f(\gamma)-\langle u_0^*,\gamma-x\rangle-\langle y_0^*,y\rangle+d_0\sigma(\gamma-x)+d_0\sigma(y)\}\\ &\le f(\bar{x})+\inf_{y\in -D-g(\bar{x},v_0)}\{d_0\sigma(y)-\langle y_0^*,y\rangle\},\end{aligned}$$

which implies $\inf_{y\in -D-g(\bar{x},v_0)}\{d_0\sigma(y)-\langle y_0^*,y\rangle\}\ge 0$. It is obvious that $\inf_{y\in -D-g(\bar{x},v_0)}\{d_0\sigma(y)-\langle y_0^*,y\rangle\}\le 0$ for $\bar{x}\in S_{v_0}$, so we get $\inf_{y\in -D-g(\bar{x},v_0)}\{d_0\sigma(y)-\langle y_0^*,y\rangle\}=0$, the relation (iii) is proved.

The proof of (b) is similar to that of Theorem 5.1 (b). □

6. Conclusions

This paper deals with the robust strong duality for nonconvex optimization problem with the data uncertainty in constraint. We introduce a weak conjugate function and construct three kinds of robust dual problems for primal problem by employing this weak conjugate function, then we establish the robust strong duality between noncovex uncertain optimization problem and its robust dual problems. In particular, we note that the optimal value of robust conjugate dual problem in classical sense is less than that in weak conjugate sense (see Remark 4.1), which shows that weak conjugate function is more likely to guarantee zero dual gaps than classical conjugate function.

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Conflict of interest

The author declares no conflict of interest.

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