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Research article

Exponential attractors for nonclassical diffusion equations with arbitrary polynomial growth nonlinearity

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Abstract: In this paper, the dynamical behavior of the nonclassical diffusion equation is investigated. First, using the asymptotic regularity of the solution, we prove that the semigroup $\{S(t)\}_{t\geq 0}$ corresponding to this equation satisfies the global exponentially $\kappa-$dissipative. And then we estimate the upper bound of fractal dimension for the global attractors $\mathscr{A}$ for this equation and $\mathscr{A} \subset H_0^1(\Omega) \cap H^2(\Omega)$. Finally, we confirm the existence of exponential attractors $\mathscr{M}$ by validated differentiability of the semigroup $\{S(t)\}_{t\geq 0}$. It is worth mentioning that the nonlinearity f satisfies the polynomial growth of arbitrary order.

Keywords: nonclassical diffusion equation; exponential attractor; arbitrary polynomial growth; fractal dimension; global exponentially $\kappa-$dissipative
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1. Introduction

In this paper, we consider the following nonclassical diffusion equation on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$:

$$u_t - \nu\Delta u_t - \Delta u + f(u) = g, \quad \text{in } \Omega \times \mathbb{R}^+. \tag{1.1}$$

The problem is supplemented with initial data

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

and the boundary condition

$$u(x, t)|_{\partial\Omega} = 0, \quad \text{for all } t \in \mathbb{R}^+, \tag{1.3}$$

where ν is a positive constant and $g = g(x) \in L^2(\Omega)$. For nonlinearity, we always assume that

$$f(s) \in C^1(\mathbb{R}, \mathbb{R}), f(0) = 0, \tag{1.4}$$

and it satisfies the following conditions: For any $s \in \mathbb{R}$,

$$\alpha|s|^p - \beta \leq f(s)s \leq \gamma|s|^p + \delta, \qquad p \geq 2, \tag{1.5}$$

where α, γ, β and δ are positive constants given, and there is a positive constant l such that

$$f'(s) \geq -l. \tag{1.6}$$

This equation appears as an extension of the usual diffusion equation in fluid mechanics, solid mechanics and heat conduction theory (see e.g., [1–3]). The Eq (1.1) with a one-time derivative appearing in the highest order term is called pseudo-parabolic or Sobolev-Galpern equation [4]. The existence of global attractors or uniform attractors for this equation has been considered in many monographs and lectures (see e.g., [5–13] and the references therein). As nonlinearity satisfies arbitrary polynomial growth condition, the asymptotic behavior of the solution for the nonclassical diffusion equation, especially the existence of exponential attractors, has received considerably less attention in the literature. In some cases similar to Eq (1.1) for some recent results on this equation, the reader can refer to [14–17].

In recent years, the existence of exponential attractors for different types of evolution models has been studied by many works of literature (see, e.g., [18–21] and references therein). Generally, exponential attractors can be constructed for dissipative systems which possesses a certain kind of smoothing property. Actually, not only does the smoothing property provide us with an exponential attractive compact set $\mathscr{M}$(i.e. the exponential attractivity of the semigroup), but also it ensures the finite dimensionality of this set. In order to obtain the smoothing property of dissipative systems, we need split the solution of our problem into two parts, one part exponentially decay, and the other part is in some suitable phase spaces with higher regularity (for example, the domain of a suitable fractional power of the operator Δ). For our problem, we note that the two terms which are $\Delta\partial_t u$ and the nonlinearity f make problem (1.1) differ from usual reaction-diffusion equations or wave-type equations. For the Eq (1.1), if initial data belongs to $H_0^1(\Omega)$, then its solution is always in $H_0^1(\Omega)$, and has no higher regularity, that is similar as hyperbolic equations. Furthermore, when "$n > 4$" the imbedding $D(A) \hookrightarrow L^\infty(\Omega)$ is not true, so it's very difficult to obtain the squeezing property for the semigroup $\{S(t)\}_{t\geq 0}$ associated with this equations. These characters cause some difficulties in studying the existence and the regularity of exponential attractors for equation (1.1) when the nonlinearity f satisfies the polynomial growth of arbitrary order and $f \in C^1$. For the limit of our knowledge, the existence and the regularity of exponential attractors of equation (1.1) is still not confirmed when the nonlinearity f satisfies (1.4)–(1.6).

The main purpose in this paper is to consider the existence of exponential attractors for Eq (1.1). In particular, by verifying the asymptotic regularity of global weak solutions of problem (1.1), we also obtain the regularity of the exponential attractor $\mathscr{M}$, i.e. $\mathscr{M} \subset D(A)$ with $u_0 \in H_0^1(\Omega)$. First, to obtain the finite fractal dimension of global attractors in $H_0^1(\Omega)$, we verify the asymptotic regularity of the semigroup of solutions corresponding to problem (1.1) by using a new decomposition method (or technique) as in [22]. It is worth noticing that authors only proved the the existence of global attractors (autonomous) or uniform attractors (non-autonomous) under a polynomial growth nonlinearity in [22–24]. Second, to obtain an exponential attractor in $H_0^1(\Omega)$, we prove that the semigroup is Fréchet differentiable on $H_0^1(\Omega)$. Obviously, the result obtained in this paper essentially improve and complement earlier ones in [25, 26] with critical nonlinearity.

This paper is organized as follows. In section 2, we recall some basic concepts as to exponential attractors and useful results that will be used later. In section 3, by using the ideas in [22], we first verify the asymptotic regularity of the semigroup for problem (1.1). Then we obtain the existence and regularity of global attractors of problem (1.1). Finally, the existence and regularity of exponential attractor is proved by using the ideas in [27].

2. Preliminaries

For conveniences, hereafter let $|u|$ be the modular (or absolute value) of u, $|\Omega|$ be the measure of the bounded domain $\Omega \subset \mathbb{R}^n$, $|\cdot|_p$ be the norm of $L^p(\Omega)(1 \le p \le \infty)$ and $(\cdot, \cdot)$ be the inner product of $L^2(\Omega)$. C denotes any positive constant which may be different from line to line even in the same line. For the family of Hilbert spaces $D(A^{\frac{s+1}{2}})$, $s \ge 0$, their inner products and norms are respectively,

$$((\cdot,\ \cdot))_s = (A^{\frac{s+1}{2}}\cdot, A^{\frac{s+1}{2}}\cdot) \quad \text{and} \quad \|\cdot\|_s = |A^{\frac{s+1}{2}}\cdot|_2.$$

Let X be a complete metric space. A one-parameter family of (nonlinear) mappings $S(t) : X \to X(t \ge 0)$ is called semigroup provided that:

(1)$S(0) = I$;

(2)$S(t + s) = S(t)S(s)$ for all $t, s \ge 0$.

Furthermore, we say that semigroup $\{S(t)\}_{t\ge 0}$ is a C_0 semigroup or continuous semigroup if $S(t)x_0$ is continuous in $x_0 \in X$ and $t \in \mathbb{R}$.

The pair $(S(t), X)$ is usually referred to as a dynamical system. A set $\mathscr{A} \subset X$ is called the global attractor for $\{S(t)\}_{t\ge 0}$ in X if

$(i)\mathscr{A}$ is compact in X,

$(ii)S(t)\mathscr{A} = \mathscr{A}$ for all $t \ge 0$, and

(iii) for any bounded set $B \subset X$, $dist(S(t)B, \mathscr{A}) \to 0$ as $t \to 0$, where

$$dist(B, \mathscr{A}) = \sup_{b\in B} \inf_{a\in\mathscr{A}} \|b - a\|_X.$$

A set B_0 is called a bounded absorbing set for $(S(t), X)$ if for any bounded set $B \subset X$, there exists $t_0 = t_0(B)$ such that $S(t)B \subset B_0$ for all $t \ge t_0$. A set E is called positively invariant w.r.t. $S(t)$ if for all $t \ge 0, S(t)E \subset E$.

Lemma 2.1. *[19, 20, 28] A continuous semigroup $\{S(t)\}_{t\ge 0}$ has a global attractor $\mathscr{A}$ if and only if $S(t)$ has a bounded absorbing set B and for an arbitrary sequence of points $x_n \in B(n = 1, 2, \cdots .)$, the sequence $\{S(t_n)x_n\}_{n=1}^{\infty}$ has a convergence subsequence in B.*

In fact, we know that

$$\mathscr{A} = \bigcap_{t\ge 0} \overline{\bigcup_{s\ge t} S(s)B}.$$

Now, we briefly review the basic concept of the Kuratowski measure of noncompactness and restate its basic property, which will be used to characterize the existence of exponential attractors for the dynamical system $(S(t), X)$ (the readers refer to [29, 30] for more details).

Let X be a Banach space and B be a bounded subset of X. The Kuratowski measure of noncompactness $\kappa(B)$ of B is defined by

$$\kappa(B) = \inf\{\delta > 0 | B \text{ admits a finite cover by sets of diameter} \leq \delta\}.$$

Let $B, B_1, B_2 \subset X$. Then

(1)$\kappa(B) = 0$ if and only if B is compact;

(2)$\kappa(B_1 + B_2) \leq \kappa(B_1) + \kappa(B_2)$;

(3)$\kappa(B_1) \leq \kappa(B_2)$, for $B_1 \subset B_2$;

(4)$\kappa(B_1 \cup B_2) \leq \max\{\kappa(B_1), \kappa(B_2)\}$;

(5)$if F_1 \supset F_2 \cdots$ are non-empty closed sets in X such that $\kappa(F_n) \to 0$ as $n \to \infty$, then $F = \bigcap_{n=1}^{\infty} F_n$ is nonempty and compact. In addition, let X be an infinite dimensional Banach space with a decomposition $X = X_1 \oplus X_2$ and let $P : X \to X_1, Q : X \to X_2$ be projectors with $\dim X_1 < \infty$. Then

(6)$\kappa(B(\varepsilon)) = 2\varepsilon$, where $B(\varepsilon)$ is a ball of radius ε;

(7)$\kappa(B) < \varepsilon$, for any bounded subset $B \subset X$ for which the diameter of QB is less than ε.

Definition 2.2. *[19, 28] A semigroup $S(t)$ is called ω-limit compact if for every bounded subset B of X and for any $\varepsilon > 0$, there exists a $t_0 > 0$ such that*

$$\kappa\left(\bigcup_{t \geq t_0} S(t)B\right) \leq \varepsilon.$$

Definition 2.3. *[19, 20] Let $n(\mathscr{M}, \varepsilon), \varepsilon > 0$ denote the minimum number of balls of X of radius ε which are needed to cover $\mathscr{M}$. The fractal dimension of $\mathscr{M}$, which is also called the capacity of $\mathscr{M}$, is the number*

$$\dim_f \mathscr{M} = \overline{\lim_{\varepsilon \to 0}} \frac{\ln n(\mathscr{M}, \varepsilon)}{\ln \frac{1}{\varepsilon}}.$$

Definition 2.4. *[18, 20, 31] Let $\{S(t)\}_{t \geq 0}$ be a semigroup on complete metric space X. A set $\mathscr{M} \subset X$ is called an exponential attractor for $S(t)$, if the following properties hold*

(1) The set $\mathscr{M}$ is compact in X and has finite fractal dimension;

(2) The set $\mathscr{M}$ is positively invariant, i.e., $S(t)\mathscr{M} \subset \mathscr{M}$ for any $t > 0$;

(3) The set $\mathscr{M}$ is an exponentially attracting set for the semigroup $S(t)$, i.e., there exist two positive constants $l, k = k(B)$, such that for every bounded subset $B \subset X$, it follows that

$$dist(S(t)B, \mathscr{M}) \leq ke^{-lt}.$$

Definition 2.5. *[27] Let $(X; d)$ be a complete metric space. A continuous semigroup $\{S(t)\}_{t \geq 0}$ on X is called global exponentially κ-dissipative if for each bounded subset $B \subset X$, there exist positive constants k and l such that*

$$\kappa\left(\overline{\bigcup_{s \geq t} S(s)B}\right) \leq ke^{-lt}, \quad \forall t \geq 0,$$

where κ is the Kuratowski measure of noncompactness.

Let X be a Banach space with the following decomposition

$$X = X_1 \oplus X_2, \quad \dim X_1 < \infty,$$

and denote projections by $P : X \to X_1$ and $(I - P) : X \to X_2$. In addition, let $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on X. Using the idea in [29], the concepts "Condition (C^*)" is introduced in [27].

Condition (C^*)**:** For any bounded set $B \subset X$, there exist positive constants t_0, k and l, such that for any $\varepsilon > 0$, there is a finite dimensional subspace $X_1 \subset X$ satisfying

(I) $\{\|PS(t)B\|_X\}_{t\geq 0}$ is bounded, and

(II) $\|(I - P)S(t)B\|_X < ke^{-lt} + \varepsilon$, for $t \geq t_0$,

where $P : X \to X_1$ is a bounded projection.

Lemma 2.6. *[27] Let X be a Banach space and $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on X for which Condition (C^*) holds. Then $\{S(t)\}_{t\geq 0}$ is a global exponentially $\kappa-$dissipative semigroup.*

Lemma 2.7. *[27] Let X be a Banach space, and $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on X. If $\{S(t)\}_{t\geq 0}$ is global exponentially $\kappa-$dissipative and has a bounded absorbing subset $B_0 \subset X$, then there exists a compact subset $\mathscr{M}$ such that*

(i) $\mathscr{M}$ is positive invariant,

(ii) $\mathscr{M}$ exponentially attracts any bounded subset $B \subset X$.

Theorem 2.8. *[27] Let $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on a Banach space X, if $\{S(t)\}_{t\geq 0}$ is a global exponential $\kappa-$dissipative semigroup and satisfies*

(i) the fractal dimension of global attractor $\mathscr{A}$ is finite, i.e., $\dim_f(\mathscr{A}) < \infty$,

(ii) there exists a constant $\varepsilon > 0$, such that for any $T^* > 0$, $S = S(T^*) : \mathscr{N}^{\varepsilon}(\mathscr{A}) \to \mathscr{N}^{\varepsilon}(\mathscr{A})$ is a C^1 map.

Then there exists a exponential attractor $\mathscr{M}$ with the finite fractal dimension.

3. Exponential attractors in $H_0^1(\Omega)$

3.1. Priori estimates

The following general existence and uniqueness of solutions for the nonclassical diffusion equations can be obtained by the *Galerkin* approximation methods, here we only formulate the results:

Lemma 3.1. *[22, 32] Assume that $g \in L^2(\Omega)$, and f satisfies (1.4)–(1.6). Then for any initial data $u_0 \in H_0^1(\Omega)$ and any $T > 0$, there exists a unique solution u for the problem (1.1)–(1.2) which satisfies*

$$u \in C^1(0, T; H_0^1(\Omega)) \cap L^p(0, T; L^p(\Omega)).$$

Moreover, we have the following Lipschitz continuity: For any $u_0^i(u_0^i \in H_0^1(\Omega)$, denote by $u_i(i = 1, 2)$ the corresponding solutions of Eq (1.1), then for all $t \geq T$

$$\|u_1(t) - u_2(t)\|_0^2 \leq Q\left(\|u_0^1\|_0, \|u_0^2\|_0, T\right)\left(\|u_0^1 - u_0^2\|_0^2\right), \tag{3.1}$$

where $Q(\cdot)$ is a monotonically increasing function.

By *Lemma* 3.1, we can define a semigroup $\{S(t)\}_{t\geq 0}$ in $H_0^1(\Omega)$ as the following:

$$S(t):\mathbb{R}^+\times H_0^1(\Omega)\to H_0^1(\Omega),$$

$$u_0\to u(t)=S(t)u_0,$$

and $\{S(t)\}_{t\geq 0}$ is a continuous semigroup on $H_0^1(\Omega)$.

Lemma 3.2. *[22,23] Let* (1.4)–(1.6) *hold, and* $g\in L^2(\Omega)$*. Then for any bounded subset* $B\subset H_0^1(\Omega)$*, there exist positive* ρ_0 *and* $T_0=T_0(\|B\|_0)$ *such that*

$$|S(t)u_0|_2^2+\|S(t)u_0\|_0^2\leqslant\rho_0,\quad \textit{for all } t\geqslant T_0 \textit{ and all } u_0\in B, \tag{3.2}$$

where ρ_0 *depends only on* $|g|_2$ *and is independent of the initial value* u_0 *and time* t*.*

Combining with (3.1), we know that $S(t)$ maps the bounded set of $H_0^1(\Omega)$ into a bounded set for all $t\geqslant 0$, that is

Corollary 3.3. *Let* (1.4)–(1.6) *hold and* $g\in L^2(\Omega)$*, then for any bounded (in* $H_0^1(\Omega)$*) subset* B*, there is an* $M_B=M(\|B\|_0,|g|_2)$ *such that*

$$|S(t)u_0|_2^2+\|S(t)u_0\|_0^2\leqslant M_B\quad \textit{for all } t\geqslant 0 \textit{ and all } u_0\in B. \tag{3.3}$$

Lemma 3.4. *[23] Let* (1.4)–(1.6) *hold,* $g\in L^2(\Omega)$ *and* B *be any bounded subset* $H_0^1(\Omega)$*, then there exists a positive constant* χ *which depends only on* $|g|_2$ *and* ρ_0*, such that for any* $u_0\in B$*, the following estimate*

$$|u(t)|_p^p\leq\chi,$$

holds for any $t\geq T_0$ *(from Lemma 3.2).*

For brevity, in the sequel, let B_0 be the bounded absorbing set obtained in *Lemma* 3.2 and let $\rho_1=\rho_0+\chi$, i.e.,

$$B_0=\left\{u\in H_0^1(\Omega)\cap L^p(\Omega):\ |u|_2^2+\|u\|_0^2+|u|_p^p\leq\rho_1\right\}. \tag{3.4}$$

Lemma 3.5. *Let* (1.4)–(1.6) *hold,* $g\in L^2(\Omega)$ *and* B *be any bounded subset of* $H_0^1(\Omega)$*, for any* $u_0\in B$*, then there exist positive constants* C *which depends on* $\|B\|_0$*, such that*

$$|u_t(s)|_2^2+\|u_t(s)\|_0^2\leq C$$

holds for any $t\geq 0$*.*

Proof. Multiplying (1.1) by u, and then integrating in Ω, it now follows that

$$\frac{1}{2}\frac{d}{dt}\left(|u|_2^2+\nu|\nabla u|_2^2\right)+|\nabla u|_2^2=-<f(u),u>+(g,u). \tag{3.5}$$

Using the assumptions (1.4)–(1.6), we have

$$-<f(u),u>\leq\beta|\Omega|-\alpha\int_\Omega|u|^p. \tag{3.6}$$

By the *Hölder inequality*, combining with (3.5) and (3.6), it follows that

$$\frac{1}{2}\frac{d}{dt}\Big(|u|_2^2 + \nu\|u\|_0^2\Big) + \alpha|u|_p^p \le \beta|\Omega| + \frac{|g|_2^2}{2\lambda_1}, \tag{3.7}$$

where λ_1 is the first eigenvalue of $-\Delta$ with the boundary condition (1.3).

Taking $t \ge 0$ and integrating (3.7) over $[t, t+1]$, we obtain

$$\int_t^{t+1} |u(s)|_p^p ds \le M_0, \tag{3.8}$$

where $M_0 = \frac{1}{\alpha}\left(2\beta|\Omega| + (1+\nu)M_B + \frac{|g|_2^2}{\lambda_1}\right)$.

We can infer from (3.8) that for any $\tau > 0$, there exists $\tau_0 \in (0, \tau]$ such that

$$|u(\tau_0)|_p^p \le M_0. \tag{3.9}$$

Multiplying (1.1) by $u_t(t)$ and integrating in Ω, we have

$$|u_t|_2^2 + \|u_t\|_0^2 + \frac{d}{dt}\Big(\frac{1}{2}\|u(t)\|_0^2 + \int_\Omega F(u(t)) - (g, u(t))\Big) \le 0, \tag{3.10}$$

whence

$$\int_\Omega F(u(t)) \le \frac{1}{\lambda_1}|g|_2^2 + M_B + \int_\Omega F(u(\tau_0)), \tag{3.11}$$

and

$$F(s) = \int_0^s f(\upsilon)d\upsilon.$$

From assumptions (1.4) – (1.6), then there are positive constants $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, such that

$$\tilde{\alpha}|s|^p - \tilde{\beta} \le F(s) \le \tilde{\gamma}|s|^p + \tilde{\delta}, \tag{3.12}$$

holds for any $s \in \mathbb{R}$.

Plugging (3.12) and (3.14) into (3.11), then there exists a positive constant $M_2 = M_2\Big(M_B, |g|_2^2\Big)$ (of course M_2 also depends on these coefficients, e.g., $\tilde{\alpha}, \tilde{\gamma}$ etc.) such that

$$|u(t)|_p^p \le M_2 \tag{3.13}$$

holds for all $t \ge 0$.

From (3.10), there exists a positive constant $M_3 = M_3\Big(M_B, M_2, |g|_2^2\Big)$ such that

$$\int_0^t \Big(|u_t(s)|_2^2 + \|u_t(s)\|_0^2\Big)ds \le M_3.$$

Similarly, for any $\tau > 0$, there exists $\tau_1 \in (0, \tau]$ such that

$$|u_t(\tau_1)|_2^2 + \|u_t(\tau_1)\|_0^2 \le M_3. \tag{3.14}$$

In order to obtain the estimate about u_t, differentiate the first equation of (1.1) with respect to t and let $z = \partial_t u$, then z satisfies the following equality

$$z_t(t) - \nu\Delta z_t(t) - \Delta z(t) + f'(u(t))z = 0. \tag{3.15}$$

Multiplying (3.15) by $z(t)$, and integrating in Ω, we have

$$\frac{1}{2}\frac{d}{dt}\Big(|z|_2^2 + \nu\|z\|_0^2\Big) + \|z\|_0^2 \leq l|z|_2^2. \tag{3.16}$$

Taking $t \geq \tau_1$ and integrating (3.16) over $[\tau_1, t]$. Thus, we obtain

$$|z(t)|_2^2 + \nu\|z(t)\|_0^2 \leq |u_t(\tau_1)|_2^2 + \nu\|u_t(\tau_1)\|_0^2 + l\int_0^t |u_t(s)|_2^2 ds.$$

Let $C = \frac{M_3 l \max\{1,\nu\}}{\min\{1,\nu\}}$, then the proof is completed.

3.2. Exponential attractors

In the following, we will prove the asymptotic regularity of solutions for the Eq (1.1) with initial-boundary conditions (1.2)–(1.3) in $H_0^1(\Omega)$ by using a new decomposition method (or technique).

In order to obtain the regularity estimates later, we decompose the solution $S(t)u_0 = u(t)$ into the sum:

$$S(t)u_0 = S_1(t)u_0 + S_2(t)u_0, \tag{3.17}$$

where $S_1(t)u_0 = v(t)$ and $S_2(t)u_0 = \omega(t)$ solve the following equations respectively,

$$\begin{cases} v_t - \Delta v - \nu\Delta v_t + f(u) - f(\omega) + \mu v = 0, \\ v(0) = u_0, \\ v|_{\partial\Omega} = 0, \end{cases} \tag{3.18}$$

and

$$\begin{cases} \omega_t - \Delta\omega - \nu\Delta\omega_t + f(\omega) - \mu v = g, \\ \omega(0) = 0, \\ \omega|_{\partial\Omega} = 0, \end{cases} \tag{3.19}$$

where the constant $\mu > 2l\max\{\beta, 1\}$ given, l is from (1.6).

Remark 3.6. *It is easy to verify the existence and uniqueness of the decomposition (3.17) corresponding to (3.18) and (3.19).*

In fact, we can rewrite (3.19) as the following

$$\begin{cases} \omega_t - \Delta\omega - \nu\Delta\omega_t + f(\omega) + \mu\omega = g + \mu u, \\ \omega(0) = 0, \\ \omega|_{\partial\Omega} = 0, \end{cases} \tag{3.20}$$

where u is the unique solution of Eq (1.1) with (1.2), so $g + \mu u \in L^2_{loc}(\mathbb{R}^+, L^2(\Omega))$ is known. The existence and uniqueness of solutions ω corresponding to Eq (3.20) can be obtained by the *Galerkin* approximation methods(see e.g., [19]). By the superposition principle of solutions of partial differential equations, the existence and uniqueness of solutions v for Eq (3.18) can be proved.

We will establish some priori estimates about the solutions of Eqs (3.18) and (3.19), which are the basis of our analysis. The proof is similar to [24]. We also note that this proof was mentioned in [22].

Lemma 3.7. *Let f satisfy* (1.4)–(1.6) *and B be any bounded set of $H_0^1(\Omega)$. Assume that $S_1(t)u_0 = v(t)$ is the solutions of (3.18) with initial data $v(0) = u_0 \in B$. Then, there exists a positive constant d_0 which only depend on l, μ and ν such that*

$$|S_1(t)u_0|_2^2 + \|S_1(t)u_0\|_0^2 \leq k_0 e^{-d_0 t}. \tag{3.21}$$

for every $t \geq 0$ holds, where $k_0 = k_0(\|u\|_0) > 0$ is a monotonically increasing continuous function about $\|u\|_0$.

Proof. Multiplying (3.18) by $v(t)$, and integrating in Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(|v|_2^2 + \nu|\nabla v|_2^2\Big) + |\nabla v|_2^2 + \int_\Omega \Big(f(u) - f(\omega)\Big)v + \mu|v|_2^2 = 0. \tag{3.22}$$

By assumptions (1.4)–(1.6), we have

$$\int_\Omega \Big(f(u) - f(\omega)\Big)v = \int_\Omega \Big(f(u) - f(\omega)\Big)(u - \omega) \geq -l|v|_2^2,$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\Big(|v|_2^2 + \nu|\nabla v|_2^2\Big) + \Big(\mu - l\Big)|v|_2^2 + |\nabla v|_2^2 \leq 0.$$

By the definition of μ, then $\mu - l \geq l > 0$. Let

$$d_0 = 2\min\{\mu - l, \frac{1}{\nu}\},$$

then we have

$$\frac{d}{dt}\Big(|v|_2^2 + \nu|\nabla v|_2^2\Big) + d_0\Big(|v|_2^2 + \nu|\nabla v|_2^2\Big) \leq 0.$$

By the *Gronwall Lemma*, for all $t \geq 0$, we have the following estimation

$$|v|_2^2 + \nu|\nabla v|_2^2 \leq (|u_0|_2^2 + \nu|\nabla u_0|_2^2)e^{-d_0 t}.$$

Taking

$$k_0 = \frac{\max\{1, \nu\}}{\min\{1, \nu\}}(|u_0|_2^2 + \|u_0\|_0^2),$$

then for all $t \geq 0$, we have

$$|S_1(t)u_0|_2^2 + \|S_1(t)u_0\|_0^2 \leq k_0 e^{-d_0 t}.$$

This proof is completed.

Next, we will consider the asymptotic regularity of the solution $u(t)$ for (1.1), that is to verify the regularity of the solution $\omega(t)$ for Eq (3.19). Concerning the solution ω to Eq (3.19), we have the following result, which shows asymptotic regularity of the solution u to Eq (1.1) with the initial-boundary conditions (1.2)–(1.3).

Lemma 3.8. *Let f satisfy* (1.4)–(1.6)*, $\omega(t)$ be the solutions of the Eq (3.19). Then the solution satisfies the following estimate: there is a positive constant ρ_2 such that*

$$\|\omega\|_1^2 \leq \rho_2 \tag{3.23}$$

for every $t \geq T_1$, where $T_1 = T_1(T_0) \geq T_0$ is a constant.

Proof. Multiplying the first equation of (3.19) by $\omega(t)$, and integrating in Ω, we have

$$\frac{1}{2}\frac{d}{dt}\Big(|\omega|_2^2 + \nu|\nabla\omega|_2^2\Big) + \mu|\omega|_2^2 + |\nabla\omega|_2^2 + \int_\Omega f(\omega)\omega = (g,\omega) + \mu(u,\omega). \tag{3.24}$$

Using the assumptions (1.4)–(1.6), we have

$$\int_\Omega f(\omega)\omega \geq -\beta|\Omega| + \alpha\int_\Omega |\omega|^p. \tag{3.25}$$

Combining with (3.24) and (3.25), we obtain that

$$\frac{1}{2}\frac{d}{dt}\Big(|\omega|_2^2 + \nu\|\omega\|_0^2\Big) + \frac{\mu}{2}|\omega|_2^2 + \|\omega\|_0^2 + \alpha|\omega|_p^p \leq \beta|\Omega| + \frac{1}{\mu}|g|_2^2 + \mu|u|_2^2. \tag{3.26}$$

Therefore, for all $t \geq T_0$, we have

$$\frac{d}{dt}\Big(|\omega|_2^2 + \nu\|\omega\|_0^2\Big) + \mu|\omega|_2^2 + 2\|\omega\|_0^2 \leq 2\beta|\Omega| + \frac{2}{\mu}|g|_2^2 + 2\mu|u|_2^2.$$

Let $d_1 = \min\{\mu, \frac{2}{\nu}\} \geq d_0$, then

$$\begin{aligned}
|\omega|_2^2 + \|\omega\|_0^2 &\leq \frac{2}{\min\{1,\nu\}d_1}\left(\beta|\Omega| + \frac{1}{\mu}|g|_2^2\right) + \frac{2\mu}{\min\{1,\nu\}}e^{-d_1 t}\int_0^t e^{d_1 s}|u(s|_2^2 ds \\
&\leq \frac{2}{\min\{1,\nu\}d_1}\left(\beta|\Omega| + \frac{1}{\mu}|g|_2^2 + \mu\rho_0\right) + \frac{2\mu M_B}{d_1\min\{1,\nu\}}e^{-d_1(t-T_0)}
\end{aligned} \tag{3.27}$$

where M_B from *Corollary* 3.3.

Furthermore, by (3.26), we obtain also

$$\frac{d}{dt}\Big(|\omega|_2^2 + \nu|\nabla\omega|_2^2\Big) + 2\alpha|\omega|_p^p \leq 2\beta|\Omega| + \frac{2}{\mu}|g|_2^2 + 2\mu|u|_2^2.$$

For any $t \geq T_0$, it follows that

$$\begin{aligned}
\int_t^{t+1}|\omega(s)|_p^p ds \leq &\frac{1}{\alpha}\left(\beta|\Omega| + \frac{1}{\mu}|g|_2^2\right) + \frac{\mu}{\alpha}\int_t^{t+1}|u(s|_2^2 ds \\
&+ \frac{\max\{1,\nu\}}{\alpha}(|\omega(t)|_2^2 + \|\omega(t)\|_0^2).
\end{aligned} \tag{3.28}$$

Taking

$$\begin{aligned}
M =& \frac{2}{\min\{1,\nu\}d_1}\left(\mu\rho_0 + \beta|\Omega| + \frac{1}{\mu}|g|_2^2\right), \\
M_1 =& \frac{M}{\alpha}\left(\frac{\min\{1,\nu\}d_1}{2} + \max\{1,\nu\}\right), \\
K_0 =& \frac{2\mu M_B}{d_1\min\{1,\nu\}},
\end{aligned}$$

and

$$T_0^* = \max\{T_0, T_0 + \frac{1}{d_1}\ln\frac{2K_0}{M}\},$$

then for all $t \geq T_0^*$, we have that

$$|\nabla\omega(t)|_2^2 \leq 2M, \quad \int_t^{t+1} |\omega(s)|_p^p ds \leq M_1. \tag{3.29}$$

Multiplying (3.19) by $-\Delta\omega(t)$ and integrating in Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(|\nabla\omega|_2^2 + \nu|\Delta\omega|_2^2\Big) + |\Delta\omega|_2^2 = \int_\Omega f(\omega)\Delta\omega - \mu(v, \Delta\omega) - (g, \Delta\omega). \tag{3.30}$$

By the Hölder inequality and assumptions (1.4)–(1.6), we have

$$\int_\Omega f(\omega)\Delta\omega \leq l|\nabla\omega|_2^2, \tag{3.31}$$

$$-\mu(v, \Delta\omega) \leq \mu^2|v|_2^2 + \frac{1}{4}|\Delta\omega|_2^2, \tag{3.32}$$

$$-(g, \Delta\omega) \leq |g|_2^2 + \frac{1}{4}|\Delta\omega|_2^2. \tag{3.33}$$

Plugging (3.31)–(3.33) into (3.30) , it follows that

$$\frac{d}{dt}\Big(|\nabla\omega|_2^2 + \nu|\Delta\omega|_2^2\Big) + l|\nabla\omega|_2^2 + |\Delta\omega|_2^2 \leq 3l|\nabla\omega|_2^2 + 2\mu^2|v|_2^2 + 2|g|_2.$$

Let $d_2 = \min\{l, \frac{1}{\nu}\} < d_0$ and $\varrho = \max\{3l, 2\mu^2\}$ then

$$\frac{d}{dt}\Big(|\nabla\omega|_2^2 + \nu|\Delta\omega|_2^2\Big) + d_2\Big(|\nabla\omega|_2^2 + \nu|\Delta\omega|_2^2\Big) \leq \varrho\left(|\nabla\omega|_2^2 + |v|_2^2\right) + 2|g|_2.$$

Combined with (3.29) and *Lemma* 3.7 , by the *Gronwall Lemma* we have

$$\begin{aligned}
|\nabla\omega|_2^2 + \nu|\Delta\omega|_2^2 \leq & \varrho e^{-d_2 t}\int_0^t e^{d_2 s}(|\nabla\omega(s)|_2^2 + |v(s)|_2^2)ds + \frac{2}{d_2}|g|_2 \\
\leq & \varrho e^{-d_2 t}\left(\int_0^t e^{d_2 s}(|\nabla\omega(s)|_2^2)ds + \int_0^t e^{d_2 s}|v(s)|_2^2 ds\right) + \frac{2}{d_2}|g|_2 \\
\leq & \frac{2}{d_2}|g|_2 + \frac{\varrho(M + K_0 e^{d_2 T_0})}{d_2}e^{-d_2(t-T_0^*)} + \frac{k_0}{d_0 - d_2}e^{-d_2 t} \\
& + \varrho e^{-d_2 t}\int_{T_0^*}^t e^{d_2 s}|\nabla\omega(s)|_2^2 ds \\
\leq & \frac{2}{d_2}|g|_2 + \left(\frac{\varrho(M + K_0 e^{d_2 T_0})}{d_2}e^{d_2 T_0^*} + \frac{k_0}{d_0 - d_2}\right)e^{-d_2 t} + \frac{2\varrho(d_2 + 1)M}{d_2}
\end{aligned}$$

Let

$$T_1 = \max\left\{T_0^*, \frac{1}{d_2}\ln\frac{d_2\left(\frac{\varrho(M+K_0 e^{d_2 T_0})}{d_2}e^{d_2 T_0^*} + \frac{k_0}{d_0 - d_2}\right)}{2\Big(|g|_2 + \varrho M(d_2 + 1)\Big)}\right\},$$

then it follows that

$$|\nabla\omega|_2^2 + |\Delta\omega|_2^2 \leq \frac{4}{d_2 \min\{1,\nu\}}\Big(2|g|_2 + \varrho M(d_2+1)\Big)$$

holds for any $t \geq T_1$. Let $\rho_2 = \rho_0 + \frac{4}{d_2 \min\{1,\nu\}}\Big(2|g|_2 + \varrho M(d_2+1)\Big)$, by *Lemma* 3.2, we get

$$|\omega|_2^2 + |\nabla\omega|_2^2 + |\Delta\omega|_2^2 \leq \rho_2.$$

This proof is completed.

Remark 3.9. *By the proof of the Lemma 3.7 and the Lemma 3.8, we find that the existence and regularity of global attractor $\mathscr{A}$ also can be proved under $g \in H^{-1}(\Omega)$.*

In order to obtain the exponential attractor, we verify that the semigroup $\{S(t)\}_{t\geq 0}$ satisfies globally exponential κ−dissipative. Let $\lambda_k (k = 1, 2, \cdots)$ be the eigenvalues of $-\Delta$ in $D(A)$ and $w_k (k = 1, 2, \cdots)$ be the eigenvectors correspondingly, $\{w_k\}_{k=1}^{\infty}$ is an orthonormal basis in $L^2(\Omega)$. Then we have

$$(w_i, w_j) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j = 1, 2, \cdots.$$

Further more, $\{w_k\}_{k=1}^{\infty}$ also form an orthogonal basis for $H_0^1(\Omega)$ and $D(A)$ and satisfies

$$\begin{aligned} &-\Delta w_k = \lambda_k w_k, \;\; k = 1, 2, \cdots, \\ &0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots, \quad \text{and } \lim_{k\to\infty} \lambda_k = +\infty. \\ &((w_i, w_j))_0 = \lambda_j \delta_{ij} = \lambda_i \delta_{ij}, \;\; i, j = 1, 2, \cdots. \end{aligned}$$

If we take an element of $L^2(\Omega)$ and project it onto the space spanned by the first m eigenfunctions $\{w_1, w_2, \cdots, w_m\}$, we get

$$P_m u = \sum_{j=1}^{m} (u, w_j) w_j = \sum_{j=1}^{m} u_j w_j.$$

We also define the projection orthogonal of P_m, $Q_m = I - P_m$,

$$Q_m u = \sum_{j=m+1}^{\infty} (u, w_j) w_j.$$

Let $u_1 = P_m u, u_2 = Q_m u = (I - P_m)u$, then $u = u_1 + u_2$ and it follows that

$$|u_2|_2^2 = \sum_{j=m+1}^{\infty} |(u, w_j)|^2 \leq |u|_2^2, \;\; \text{for any } u \in L^2(\Omega); \tag{3.34}$$

$$\|u_2\|_0^2 = ((u_2, u_2))_0 = \sum_{j=m+1}^{\infty} \lambda_j |(u, w_j)|^2 \geq \lambda_m |u_2|_2^2, \;\; \text{for any } u \in H_0^1(\Omega). \tag{3.35}$$

$$\|u_2\|_1^2 = ((u_2, u_2))_1 \geq \lambda_m \|u_2\|_0^2, \;\; \text{for any } u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{3.36}$$

Lemma 3.10. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, f satisfies* (1.4)–(1.6). *Assume further that $g \in L^2(\Omega)$, and the semigroup $\{S(t)\}_{t\geq 0}$ associated with the Eq (1.1) satisfies Condition(C^*) , that is, for any bounded subset $B \subset H_0^1(\Omega)$, there exist $k, l, T_2 > 0$ and $k(m)$, such that*

$$\|(I - P_m)\bigcup_{s\geq t} S(s)u_0\|_0^2 \leq ke^{-lt} + k(m), \text{ for any } u_0 \in B,$$

and

$$\lim_{m\to\infty} k(m) = 0$$

hold provided that $t \geq T_2$.

Proof. For any $t > 0$, the solutions $u = S(t)u_0$, corresponding to (1.1)–(1.3), can be decomposed in the form

$$S(t)u_0 = S_1(t)u_0 + S_2(t)u_0,$$

where $S_1(t)u_0$ and $S_2(t)u_0$ are the solutions of system (3.18) and system (3.19)respectively. Then we have

$$\|(I - P_m)\bigcup_{s\geq t} S(s)u_0\|_0 \leq \|(I - P_m)\bigcup_{s\geq t} S_1(s)u_0\|_0 + \|(I - P_m)\bigcup_{s\geq t} S_2(s)u_0\|_0. \tag{3.37}$$

By *Lemma* 3.7, we have

$$\|v_2\|_0^2 \leq \|v\|_0^2 \leq 2k_0e^{-d_0t}. \tag{3.38}$$

For any $t \geq T_1$

$$\|\omega_2\|_0^2 \leq \frac{1}{\lambda_m}\|\omega_2\|_1^2 \leq \frac{1}{\lambda_m}\|\omega\|_1^2 \leq \frac{\rho_1}{\lambda_m}. \tag{3.39}$$

Let $k = \sqrt{2k_0}$, $l = \frac{1}{2}d_0$ and $k(m) = \sqrt{\frac{\rho_1}{\lambda_m}}$, then we have

$$\|(I - P_m)\bigcup_{s\geq t} S(s)u_0\|_0 \leq ke^{-lt} + k(m), \text{ for any } u_0 \in B,$$

and

$$\lim_{m\to\infty} k(m) = 0$$

provided that $t \geq T_2$. This proof is completed.

It follows from the conclusions in *Lemma* 2.7 and *Theorem* 3.10 that the semigroup $\{S(t)\}_{t\geq 0}$ associated with the Eq (1.1) satisfies the globally exponential κ-dissipative, then the semigroup has a compact and positive invariant set $\mathscr{M}$, which attracts any bounded subset $B \subset H_0^1(\Omega)$ exponentially. Next, we are aiming to prove that the fractal dimension of the set $\mathscr{M}$ is finite, to this end, the following result is necessary.

Lemma 3.11. *Assume further that f satisfies* (1.4)–(1.6) *and $g \in L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t\geq 0}$, corresponding to* (1.1)–(1.3), *possesses a global attractor $\mathscr{A}$ in $H_0^1(\Omega)$. Moreover, this attractor $\mathscr{A}$ is bounded in $D(A)$ and the fractal dimension of global attractor $\mathscr{A}$ is finite, i.e.,* $\dim_f(\mathscr{A}) < \infty$.

Proof. By *Lemma* 3.8, we just need to verify that the fractal dimension of global attractor $\mathscr{A}$ is finite. It's obvious $\mathscr{A} \subset D(A)$ for all $t \geq 0$. Take $T > 0$ fixing and let $S^n = S(nT)$, obviously S^n is a discrete dynamical system. The measure of non-compactness is exponentially decaying for S^n. Let $\theta = e^{-lT}$ and $r = k\theta$ (l, k from *Lemma* 3.10). Since $\mathscr{A}$ is compact, for r there exist $x_1, x_2, \cdots, x_N$ such that

$$\mathscr{A} \subset \bigcup_{i=1}^{N} B(x_i, k\theta) \subset \bigcup_{i=1}^{N} \{x_i + k\theta B(0,1)\}, \quad \mathscr{A} = \bigcup_{i=1}^{N} \left(\{x_i + k\theta B(0,1)\} \cap \mathscr{A} \right).$$

Because $\{x_i + k\theta B(0,1)\} \bigcap \mathscr{A}$ is precompact, so there exists a precompact set $B_i \subset B(0,1)$ such that $\{x_i + k\theta B(0,1)\} \bigcap \mathscr{A} = \{x_i + k\theta B_i\}$. For this θ, there exists $q \in \mathbb{N}$ such that $B_i \subset \bigcup_{j=1}^{q} B(y_{ij}, \theta)$, so we have

$$\mathscr{A} = \left(\bigcup_{i=1}^{N} \bigcup_{j=1}^{q} \{x_i + k\theta B(y_{ij}, \theta)\} \right) \bigcap \mathscr{A} = \left(\bigcup_{i=1}^{N} \bigcup_{j=1}^{q} \{x_i + k\theta y_{ij} + k\theta^2 B(0,1)\} \right) \bigcap \mathscr{A}.$$

Then there exist Nq open balls with radius $k\theta^2$ in $H_0^1(\Omega)$ covering $\mathscr{A}$. For any $n \in \mathbb{N}$, after iterations, we obtain that there exist at most Nq^{n-1} balls with radius $k\theta^n$ in $H_0^1(\Omega)$ covering $\mathscr{A}$. So for all $\varepsilon > 0$, let $n \geq [\frac{\ln k - \ln \varepsilon}{lT}] + 1$, then $k\theta^n = ke^{-nlT} < \varepsilon$. We get

$$\begin{aligned}
\dim_f(\mathscr{A}) &\leq \overline{\lim_{\varepsilon \to 0}} \frac{\ln Nq^{n-1}}{\ln \varepsilon^{-1}} \\
&\leq \overline{\lim_{\varepsilon \to 0}} \frac{\ln N + (n-1)\ln q}{\ln \varepsilon^{-1}} \\
&\leq \lim_{\varepsilon \to 0} \frac{lT \ln N + (\ln k - \ln \varepsilon)\ln q}{-lT \ln \varepsilon} \\
&\leq \frac{\ln q}{lT}.
\end{aligned}$$

This proof is completed.

Lemma 3.12. *For any $t > 0$, the semigroup $\{S(t)\}_{t \geq 0}$ is Fréchet differentiable on $H_0^1(\Omega)$.*

Proof. Let $S(t^*)(u_0 + hv_0) = v(t^*)$ and $S(t^*)(u_0) = u(t^*)$ be the solutions at the time t^* for the following equations respectively,

$$\begin{cases} v_t - \Delta v - \nu \Delta v_t + f(v) = g, \\ v(0) = u_0 + hv_0, \\ v|_{\partial\Omega} = 0, \end{cases} \tag{3.40}$$

and

$$\begin{cases} u_t - \Delta u - \nu \Delta u_t + f(u) = g, \\ u(0) = u_0, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{3.41}$$

And then, setting $\omega_h = \frac{v-u}{h} = \frac{S(t^*)\left(u_0 + hv_0\right) - S(t^*)\left(u_0\right)}{h}$, which clearly satisfies the following equation

$$\begin{cases} \frac{\partial}{\partial t}\omega_h - \Delta\omega_h - \nu \frac{\partial}{\partial t}\Delta\omega_h + f'(u + \theta(v-u))\omega_h = 0, \\ \omega_h(0) = v_0, \\ \omega_h|_{\partial\Omega} = 0, \end{cases} \tag{3.42}$$

where $f'(u+\theta(v-u)) = \frac{f(v)-f(u)}{h}$ and $0<\theta<1$.

Multiplying Eq (3.42) by ω_h and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\Big(|\omega_h|_2^2 + \nu\|\omega_h\|_0^2\Big) + \|\omega_h\|_0^2 \le l|\omega_h|_2^2.$$

Then $\omega_h(t^*) \in H_0^1(\Omega)$ and

$$|\omega_h|_2^2 + \|\omega_h\|_0^2 \le Ce^{lt}(|v_0|_2^2 + \|v_0\|_0^2), \tag{3.43}$$

where C is a constant independent of h. On the other hand, we denote $W = W(x,t)$ which satisfies the following equation

$$\begin{cases} \frac{\partial}{\partial t}W - \Delta W - \nu\frac{\partial}{\partial t}\Delta W + f'(u)W = 0, \\ W(0) = v_0, \\ W|_{\partial\Omega} = 0. \end{cases} \tag{3.44}$$

Multiplying Eq (3.44) by W and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\Big(|W|_2^2 + \nu\|W\|_0^2\Big) + \|W\|_0^2 \le l|W|_2^2.$$

Then

$$|W|_2^2 + \|W\|_0^2 \le Ce^{lt}(|v_0|_2^2 + \|v_0\|_0^2), \tag{3.45}$$

where C is a constant independent of h.

Obviously, $W(t^*) \in H_0^1(\Omega)$ and the linear version $S'(t^*)$(if existed)

$$S'(t^*) = L : v_0(\in T_{u_0}(H_0^1(\Omega))) \mapsto W(t^*)(\in T_{S(t^*)u_0}(H_0^1(\Omega))),$$

where $T_{u_0}(X)$ denotes the tangent space at the point u_0 in Banach space X.

From (3.42) and (3.42), the difference $U_h = \omega_h - W$ satisfies

$$\begin{cases} \frac{\partial}{\partial t}U_h - \Delta U_h - \nu\frac{\partial}{\partial t}\Delta U_h + g_h\omega_h + f'(u)U_h + lU_h = lU_h, \\ U_h(0) = 0, \\ U_h|_{\partial\Omega} = 0, \end{cases} \tag{3.46}$$

where $g_h = f'(u+\theta(v-u)) - f'(u)$ and l from (1.6).

The homogenization of the above Eq (3.46) gives

$$\begin{cases} \frac{\partial}{\partial t}U - \Delta U - \nu\frac{\partial}{\partial t}\Delta U + f'(u)U + lU = lU, \\ U(0) = 0, \\ U|_{\partial\Omega} = 0. \end{cases} \tag{3.47}$$

It is obvious $U \equiv 0$ for the homogenization Eq (3.47).

Next, we consider the non-homogeneous Eq (3.46). It is obvious that $g_h\omega_h \in H^{-1}(\Omega)$, $U_h \in H_0^1(\Omega)$ and $g_h\omega_hU_h \in L^1(\Omega)$. Multiplying Eq (3.46) by U_h and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\Big(|U_h|_2^2 + \nu\|U_h\|_0^2\Big) + \|U_h\|_0^2 \le l|U_h|_2^2 + \Big|\int_\Omega g_h\omega_hU_h\Big|.$$

Combining with (3.28), (3.43) and (3.45), we get $g_h\omega_h U_h$ is uniform (w.r.t h) bounded, for a.e. $x \in \Omega$. Note that the non-homogeneous term $g_h\omega_h U_h \to 0$, a.e. $x \in \Omega$. And applying the Lebesgue dominated convergence theorem, one can deduce

$$\lim_{h\to 0}\int_\Omega |g_h\omega_h U_h| = 0.$$

So we obtain that

$$\lim_{h\to 0}\Big|\int_\Omega g_h\omega_h U_h\Big| = 0.$$

for any $t \in [0, +\infty)$. By the standard theory of ordinary differential equation, one see that $U_h \to 0$ in $H_0^1(\Omega)$ as $h \to 0$, that is, $\omega_h \to W$ in $H_0^1(\Omega)$ as $h \to 0$. It implies that the semigroup $\{S(t^*)\}_{t\geq 0}$ is Fréchet differentiable on $H_0^1(\Omega)$. This proof is completed.

Lemma 3.13. *There exists a constant L, such that for any $u_0 \in B_0$, the solution u(t) of the equation (1.1) with the initial-boundary conditions (1.2) and (1.3) satisfies*

$$|u(t_1) - u(t_2)|_2^2 + \|u(t_1) - u(t_2)\|_0^2 \leq L|t_1 - t_2|^2,$$

for any $t_1, t_2 \geq 0$.

Proof. Recalling *Lemma* 3.5, for any $t \geq 0$, there exists a constant C such that

$$|u_t(t)|_2^2 + \|u_t(t)\|_0^2 \leq C, \text{ and } |u(t)|_2^2 + \|u(t)\|_0^2 \leq C$$

It follows that

$$|u(t_1) - u(t_2)|_2^2 + \|u(t_1) - u(t_2)\|_0^2 \leq C|t_1 - t_2|^2.$$

This implies for any $T > 0$, the semigroup $\{S(t)\}_{t\geq 0}$ is uniformly Hölder continuous w.r.t t on $[0, T]$. This proof is completed.

Theorem 3.14 (Exponential attractor). *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and f satisfies* (1.4)–(1.6). *Then the semigroup $\{S(t)\}_{t\geq 0}$, corresponding to* (1.1)–(1.3)*, possesses a exponential attractor $\mathscr{M}$ in $H_0^1(\Omega)$.*

Proof. Combining with *Lemma* 3.2, *Lemma* 3.10, *Lemma* 3.12 and *Lemma* 3.13. as a direct application of the abstract theorem 2.8, we obtain the existence of a exponential attractor $\mathscr{M}$ in $H_0^1(\Omega)$. The proof is completed.

4. Conclusions

This paper mainly investigate the long-time behavior for nonclassical diffusion equations with arbitrary polynomial growth nonlinearity, including the following three results: (i) the existence and regularity of global attractors is obtained, it is worth noting that a new operator decomposition method is proposed; (ii) the global attractors have finite fractal dimension by combining with asymptotic regularity of solutions; (iii) we confirm the existence of exponential attractors by verifying Fréchet differentiability of semigroup. The above conclusions are more general, and essentially improve existing some results, it should be pointed out that these methods in this paper can also be used for other evolution equations.

Conflict of interest

The authors declares no conflict of interest in this paper.

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