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*Research article*

# Expressions of the Laguerre polynomial and some other special functions in terms of the generalized Meijer $G$-functions

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**Abstract:** In this paper, we investigate the relation of generalized Meijer $G$-functions with some other special functions. We prove the generalized form of Laguerre polynomials, product of Laguerre polynomials with exponential functions, logarithmic functions in terms of generalized Meijer $G$-functions. The generalized confluent hypergeometric functions and generalized tricomi confluent hypergeometric functions are also expressed in terms of the generalized Meijer $G$-functions.

**Keywords:** Meijer $G$-function; generalized Meijer $G$-function; Laguerre polynomial; hypergeometric function; confluent hypergeometric function
**Mathematics Subject Classification:** 33C60, 33C10, 45P05

## 1. Introduction

Special functions is a basic and important field of mathematics, because of applications in functional analysis, physics, statistics, mathematical analysis etc. The special functions can not be defined in formal way but all the functions which are important enough and assigned by their own names like exponential functions, logarithmic functions, trigonometric functions, Laguerre polynomials, Lagrange polynomials, Hermite polynomials, Mcrobert $E$-functions, Meijer $G$-functions, Fox $H$-functions etc. are considered to be the special functions. The Pochhammer symbol, gamma function, beta function and hypergeometric functions are fundamentals in the field of special functions.

The Pochhammer's symbol is defined [1] as

$$(a)_n = \begin{cases} a(a+1)(a+2)\cdots(a+n-1) & \text{for } n \geq 1 \\ 1 & \text{for } n = 0, a \neq 0, \end{cases} \tag{1.1}$$

where $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

The gamma function is defined [1] as

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt, \tag{1.2}$$

where $\Re(a) > 0$.

The relation of gamma function with Pochhammer symbol is given as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{1.3}$$

The beta function is defined [1] in the integral and gamma function forms as

$$\begin{aligned}\beta(a,b) &= \int_0^1 t^{a-1}(1-t)^{b-1} dt, \quad \Re(a) > 0, \Re(b) > 0 \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.\end{aligned} \tag{1.4}$$

The Laguerre polynomial is defined [1] as

$$L_v^\lambda(t) = \sum_{i=0}^{v} \frac{(-1)^i (1+\lambda)_v t^i}{i!(v-i)!(1+\lambda)_i}, \tag{1.5}$$

where $v$ is non-negative integer. The researchers [2–4] worked on the Laguerre polynomials and proved some important relations and applications in different fields.

The Gauss hypergeometric and confluent hypergeometric functions are defined [1] respectively as

$${}_2F_1\left[\begin{matrix} v, w \\ l \end{matrix}\middle| t\right] = \sum_{n=0}^{\infty} \frac{(v)_n (w)_n t^n}{(l)_n n!} \tag{1.6}$$

and

$${}_1F_1\left[\begin{matrix} v \\ l \end{matrix}\middle| t\right] = \sum_{n=0}^{\infty} \frac{(v)_n t^n}{(l)_n n!}, \tag{1.7}$$

where $v, w, l$ are parameters with $l$ not be zero or negative integer and $|t| < 1$.

Shen-Yang et al. [5] worked on the functional inequalities for the Gaussian hypergeometric function and generalized elliptic integral of the first kind. They discussed some applications of the Gauss hypergeometric functions in different areas especially in functional analysis. Almost all the special functions and elementary functions can be obtained from hypergeometric functions as special cases. Several applications of Gauss hypergeometric functions are discussed in [6–8]. Wang et al. [9] worked on the generalization of the inequalities and proved the generalization of Ramanujan's cubic transformation inequalities for zero balanced hypergeometric functions. Qiu et al. [10] investigated the generalization of the inequalities and proved sharp Lenden transformation inequalities for hypergeometric functions. They also proved some properties and gave some applications of the Ramanujan's constant.

The Meijer $G$-functions are important in various fields especially in applied mathematics and mathematical statistics. These functions naturally appear as a solution of differential equations. Meijer

$G$-functions are interesting because of greater flexibility in the orders $p, q, m, n$ and the parameters. By adopting different values of these parameters, we can obtain different known special functions. Because of the fact that Miejer $G$-functions can be defined in different ways, these functions are easy to compute. Almost all the elementary functions can be presented in the form of Meijer $G$-functions by choosing different parameters. Meijer $G$-functions are applicable in mathematical physics because these can be expressed in the form of hypergeometric function. So, almost all the special functions can be obtained as special cases of Meijer $G$-functions.

The Meijer $G$-function is defined [11–15] in integral form as

$$G^{m,n}_{p,q}\left[\begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{matrix}\middle| t\right] = \frac{1}{2\pi i}\oint_L \frac{\Gamma_k(b_{1,m} - s)\Gamma_k(k - a_{1,n} + s)}{\Gamma_k(a_{n+1,p} - s)\Gamma_k(k - b_{m+1,q} + s)} t^{s/k} ds, \tag{1.8}$$

where $m, n, p, q$ are orders of $G$-function with the conditions $0 \le m \le q$ and $0 \le n \le p$, and $a_p, b_q$ are parameters. Also no poles of any $\Gamma(b_j - s), j = 1, 2, ..., m$ coincides with a pole of $\Gamma(1 - a_i + s), i = 1, 2, ..., n$.

The Meijer $G$-functions in term of hypergeometric functions are given as

$$\begin{aligned} G^{m,n}_{p,q}\left[\begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{matrix}\middle| t\right] &= \sum_{i=1}^{m} \frac{\prod_{h=1}^{m}\Gamma(b_h - b_i)^* \prod_{h=1}^{n}\Gamma(1 - a_h + b_i) t^{b_i}}{\prod_{h=m+1}^{q}\Gamma(1 - b_h + b_i)\prod_{h=n+1}^{p}\Gamma(a_h - b_i)} \\ &\times {}_pF_{q-1}\left[\begin{matrix} 1 - a_p + b_i \\ (1 - b_q + b_i)^* \end{matrix}\middle| (-1)^{p-m-n} t\right] \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} G^{m,n}_{p,q}\left[\begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{matrix}\middle| t\right] &= \sum_{i=1}^{m} \frac{\prod_{h=1}^{n}\Gamma(a_i - a_h)^* \prod_{h=1}^{m}\Gamma(1 - a_i + b_h) t^{a_i - 1}}{\prod_{h=n+1}^{p}\Gamma(1 - a_i + a_h)\prod_{h=m+1}^{q}\Gamma(a_i - b_h)} \\ &\times {}_qF_{p-1}\left[\begin{matrix} 1 - a_i + b_q \\ (1 - a_i + a_p)^* \end{matrix}\middle| (-1)^{q-m-n} t^{-1}\right], \end{aligned} \tag{1.10}$$

with the same conditions on $m, n, p, q$.

The researchers [16–27] worked on Meijer $G$-functions and proved immense amount of properties and applications.

In the last two decades, special functions have gained much attention of the researchers [28–38] after the development of $k$-symbol and $k$-functions by Diaz et al. [39, 40] in (2005) and (2007), which are considered to be the generalization of special functions.

Diaz and Pariguan [40], firstly defined generalization of gamma and beta functions in $k$-form, $k > 0$ as

$$\Gamma_k(a) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{a}{k}-1}}{(a)_{n,k}}, \qquad k > 0, a \in \mathbb{C}\backslash kZ^- \tag{1.11}$$

and

$$\beta_k(a, b) = \frac{1}{k}\int_0^1 t^{\frac{a}{k}-1}(1 - t)^{\frac{b}{k}-1} dt, \qquad \Re(a) > 0, \Re(b) > 0, \tag{1.12}$$

where $(a)_{n,k}$ is the Pochhammer $k$-symbol defined as

$$(a)_{n,k} = \begin{cases} a(a + k)(a + 2k)\cdots(a + (n - 1)k) & \text{for } n \ge 1 \\ 1 & \text{for } n = 0, a \ne 0. \end{cases} \tag{1.13}$$

Motivated by Prudnikov [25], our main aim is to obtain the generalization of some functions by keeping in view the idea of the extension and generalization of the special functions. We first define the generalized Meijer $G$-functions and then find some important relations between the generalized Meijer $G$-functions and some known generalized special functions. We investigate the relation of generalized Meijer $G$-functions with Laguerre polynomial, product of Laguerre polynomial with exponential functions, logarithmic functions and confluent hypergeometric functions. These relations will be helpful in different areas especially in mathematical physics and functional field of analysis.

The generalized Meijer $G$-function in the integral form can be defined as

$$G^{m,n}_{k,p,q}\left[\begin{matrix}a_1,a_2,...,a_p\\ b_1,b_2,...,b_q\end{matrix}\middle| t\right] = \frac{1}{2\pi i}\oint_L \frac{\Gamma_k(b_{1,m}-s)\Gamma_k(k-a_{1,n}+s)}{\Gamma_k(a_{n+1,p}-s)\Gamma_k(k-b_{m+1,q}+s)} t^{s/k} ds, \tag{1.14}$$

where $m,n,p,q$ are orders of $G$-function with the conditions $0 \leq m \leq q$ and $0 \leq n \leq p$, and $a_p, b_q$ are parameters, $k > 0$. Also no poles of any $\Gamma_k(b_j-s)$, $j = 1,2,...,m$ coincides with a pole of $\Gamma_k(k-a_i+s)$, $i = 1,2,...,n$.

The generalized Meijer $G$-functions in terms of generalized hypergeometric functions are defined as respectively

$$\begin{aligned} G^{m,n}_{k,p,q}\left[\begin{matrix}a_1,a_2,...,a_p\\ b_1,b_2,...,b_q\end{matrix}\middle| t\right] &= \sum_{i=1}^{m} \frac{\prod_{h=1}^{m}\Gamma_k(b_h-b_i)^* \prod_{h=1}^{n}\Gamma_k(k-a_h+b_i) t^{\frac{b_i}{k}}}{\prod_{h=m+1}^{q}\Gamma(k-b_h+b_i)\prod_{h=n+1}^{p}\Gamma_k(a_h-b_i)} \\ &\quad \times {}_pF_{q-1,k}\left[\begin{matrix}k-a_p+b_i\\ (k-b_q+b_i)^*\end{matrix}\middle| (-1)^{p-m-n} t\right], \end{aligned} \tag{1.15}$$

for $p < q$ and

$$\begin{aligned} G^{m,n}_{k,p,q}\left[\begin{matrix}a_1,a_2,...,a_p\\ b_1,b_2,...,b_q\end{matrix}\middle| t\right] &= \sum_{i=1}^{m} \frac{\prod_{h=1}^{n}\Gamma_k(a_i-a_h)^* \prod_{h=1}^{m}\Gamma(k-a_i+b_h) t^{\frac{a_i}{k}-1}}{\prod_{h=n+1}^{p}\Gamma_k(k-a_i+a_h)\prod_{h=m+1}^{q}\Gamma_k(a_i-b_h)} \\ &\quad \times {}_qF_{p-1,k}\left[\begin{matrix}k-a_i+b_q\\ (k-a_i+a_p)^*\end{matrix}\middle| (-1)^{q-m-n} t^{-1}\right] \end{aligned} \tag{1.16}$$

for $p > q$ with the same conditions on $m,n,p,q$ as that of integral definition.

## 2. Laguerre polynomials in the form of generalized $G$-functions

In this section, we prove the relation of generalized Laguerre polynomials and product of Laguerre polynomial with exponential functions in the form of generalized Meijer $G$-functions.

**Theorem 2.1.** *Let $\lambda$ be an arbitrary real number and $v$ be a non-negative integer. Then for $k > 0$, the following holds*

$$L^{\lambda}_{v,k}(t) = \Gamma_k(k+\lambda+vk) G^{1,0}_{k,1,2}\left[\begin{matrix}vk+k\\ 0,-\lambda\end{matrix}\middle| t\right]. \tag{2.1}$$

*Proof.* Applying the definition (1.15) of the generalized Meijer $G$-function on the right hand side, we have

$$G^{1,0}_{k,1,2}\left[\begin{matrix} vk+k \\ 0,-\lambda \end{matrix}\middle| t\right] = \frac{1}{\Gamma_k(k+\lambda)\Gamma_k(vk+k)}\ {}_1F_{1,k}\left[\begin{matrix} -vk \\ k+\lambda \end{matrix}\middle| t\right]$$
$$= \frac{(k+\lambda)_{v,k}}{\Gamma_k(k+\lambda+vk)k^v v!}\ {}_1F_{1,k}\left[\begin{matrix} -vk \\ k+\lambda \end{matrix}\middle| t\right].$$

So,

$$\Gamma_k(k+\lambda+vk)G^{1,0}_{k,1,2}\left[\begin{matrix} vk+k \\ 0,-\lambda \end{matrix}\middle| t\right] = \frac{(k+\lambda)_{v,k}}{k^v v!}\ {}_1F_{1,k}\left[\begin{matrix} -vk \\ k+\lambda \end{matrix}\middle| t\right]. \tag{2.2}$$

The generalized form of Laguerre polynomial (1.5) for $k > 0$, can be defined as

$$L^{\lambda}_{v,k}(t) = \frac{(k+\lambda)_{v,k}}{k^v v!}\ {}_1F_{1,k}\left[\begin{matrix} -vk \\ k+\lambda \end{matrix}\middle| t\right]. \tag{2.3}$$

Thus from (2.2) and (2.3), we obtain the desired result. □

**Corollary 2.1.** *Let $\lambda$ be an arbitrary real number and $v$ be a non-negative integer. Then for $k > 0$, the following holds*

$$L^{\lambda}_{v,k}(\frac{1}{t}) = \Gamma_k(k+\lambda+vk)G^{0,1}_{k,2,1}\left[\begin{matrix} k,\lambda+k \\ -vk \end{matrix}\middle| t\right]. \tag{2.4}$$

*Proof.* By using the definition (1.16) of the generalized Meijer $G$-function, we have

$$G^{0,1}_{k,2,1}\left[\begin{matrix} k,\lambda+k \\ -vk \end{matrix}\middle| t\right] = \frac{(k+\lambda)_{v,k}}{\Gamma_k(k+\lambda+vk)k^v v!}\ {}_1F_{1,k}\left[\begin{matrix} -vk \\ k+\lambda \end{matrix}\middle| \frac{1}{t}\right]. \tag{2.5}$$

As the Laguerre polynomial for $k > 0$, is defined as

$$L^{\lambda}_{v,k}(\frac{1}{t}) = \frac{(k+\lambda)_{v,k}}{k^v v!}\ {}_1F_{1,k}\left[\begin{matrix} -vk \\ k+\lambda \end{matrix}\middle| \frac{1}{t}\right]. \tag{2.6}$$

Thus from (2.5) and (2.6), we obtain the desired result. □

Now, we are considering the product of exponential function with Laguerre polynomial.

**Theorem 2.2.** *Let $\lambda$ be an arbitrary real number and $v$ be a non-negative integer. Then for $k > 0$, the following holds*

$$e^{-t}L^{\lambda}_{v,k}(t) = \frac{1}{k^v v!}G^{1,1}_{k,1,2}\left[\begin{matrix} -vk-\lambda \\ 0,-\lambda \end{matrix}\middle| t\right]. \tag{2.7}$$

*Proof.* By considering the series of exponential and Laguerre polynomial, and taking their product, we have

$$e^{-t}L_{v,k}^{\lambda}(t) = [\sum_{n=0}^{\infty} \frac{(-t)^n}{n!}][\frac{(k+\lambda)_{v,k}}{k^v v!} \sum_{m=0}^{\infty} \frac{(-vk)_{m,k}t^m}{(k+\lambda)_{m,k}m!}]$$
$$= \frac{(k+\lambda)_{v,k}}{k^v v!} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^n(-vk)_{m,k}t^{m+n}}{(k+\lambda)_{m,k}m!n!}$$
$$= \frac{(k+\lambda)_{v,k}}{k^v v!} \sum_{n=0}^{\infty} \frac{(-1)^n t^n (k+\lambda+vk)_{n,k}}{(k+\lambda)_{n,k}n!}. \tag{2.8}$$

By using (1.15), the generalized Meijer $G$-function on the right hand side of (2.7) will take the following form

$$G_{k,1,2}^{1,1}\left[\begin{matrix}-vk-\lambda\\ 0,-\lambda\end{matrix}\middle| t\right] = (k+\lambda)_{v,k} \sum_{n=0}^{\infty} \frac{(-1)^n t^n (k+\lambda+vk)_{n,k}}{(k+\lambda)_{n,k}n!}. \tag{2.9}$$

So, from (2.8) and (2.9), we obtain the desired result. □

**Corollary 2.2.** *Let $\lambda$ be an arbitrary real number and $v$ be a non-negative integer. Then for $k > 0$, the following holds*

$$e^{-\frac{1}{t}}L_{v,k}^{\lambda}(\frac{1}{t}) = \frac{1}{k^v v!}G_{k,2,1}^{1,1}\left[\begin{matrix}k,\lambda+k\\ \lambda+vk+k\end{matrix}\middle| t\right]. \tag{2.10}$$

*Proof.* By considering the series of exponential and Laguerre polynomial on the lefty hand side and taking their product, we have

$$e^{-\frac{1}{t}}L_{v,k}^{\lambda}(t) = \frac{(k+\lambda)_{v,k}}{k^v v!} \sum_{n=0}^{\infty} \frac{(-1)^n t^n (k+\lambda+vk)_{n,k}}{(k+\lambda)_{n,k}n!}. \tag{2.11}$$

By using (1.16), the generalized Meijer $G$-function on the right hand side of (2.10) will take the form

$$G_{k,2,1}^{1,1}\left[\begin{matrix}k,\lambda+k\\ \lambda+vk+k\end{matrix}\middle| t\right] = (k+\lambda)_{v,k} \sum_{n=0}^{\infty} \frac{(-1)^n (k+\lambda+vk)_{n,k}}{(k+\lambda)_{n,k}t^n n!}. \tag{2.12}$$

So, from (2.11) and (2.12), we obtain the desired result. □

**Proposition 2.1.** *Let $v$ be a non-negative integer and $\lambda$ be zero in theorem (2.2). Then for $k > 0$, the following holds*

$$e^{-t}L_{v,k}(t) = \frac{1}{\Gamma_k(vk+k)}G_{k,1,2}^{1,1}\left[\begin{matrix}-vk\\ 0,0\end{matrix}\middle| t\right]. \tag{2.13}$$

*Proof.* By considering the series of exponential and Laguerre polynomial, and taking their product, we have

$$e^{-t}L_{v,k}(t) = [\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}][\sum_{m=0}^{\infty}\frac{(-vk)_{m,k}t^m}{(k)_{m,k}m!}]$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n(-vk)_{m,k}t^{m+n}}{(k)_{m,k}m!n!}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n t^n(k+vk)_{n,k}}{\Gamma_k(k+nk)n!}. \tag{2.14}$$

By using (1.15), the generalized Meijer $G$-function on the right hand side of (2.13) will take the form

$$G^{1,1}_{k,1,2}\left[\begin{matrix}-vk\\0,0\end{matrix}\middle|t\right] = \Gamma_k(k+vk)\sum_{n=0}^{\infty}\frac{(-1)^n t^n(k+vk)_{n,k}}{(k)_{n,k}n!}. \tag{2.15}$$

So, from (2.14) and (2.15), we obtain the required result. □

**Proposition 2.2.** *Let $v$ be a non-negative integer and $\lambda$ be zero in corollary (2.2). Then for $k > 0$, the following holds*

$$e^{-\frac{1}{t}}L_{v,k}(\frac{1}{t}) = \frac{1}{\Gamma_k(k+vk)}G^{1,1}_{k,2,1}\left[\begin{matrix}k,k\\vk+k\end{matrix}\middle|t\right]. \tag{2.16}$$

*Proof.* By adopting the same process as that of corollary (2.2), one can prove easily. □

## 3. Relations of the generalized Meijer $G$-functions with some other known functions

In this section, we prove the relations of generalized Meijer $G$-functions with generalized form of some known functions as logarithmic functions and confluent hypergeometric functions.

**Theorem 3.1.** *Let $p = 3$, $q = 3$, $m = 1$, $n = 2$ and $a_1 = k$, $a_2 = k$, $a_3 = \frac{k}{2}$, $b_1 = k, b_2 = 0$ and $b_3 = \frac{k}{2}$, with $k > 0$ in (1.15). Then for $|t| < 1$, the generalized Meijer $G$-function gives*

$$G^{1,2}_{k,3,3}\left[\begin{matrix}k,k,\frac{k}{2}\\k,0,\frac{k}{2}\end{matrix}\middle|t\right] = \frac{1}{k^2\pi}ln(1-kt). \tag{3.1}$$

*Proof.* By using the definition of generalized Meijer $G$-function (1.15) for given parameters, we have

$$G^{1,2}_{k,3,3}\left[\begin{matrix}k,k,\frac{k}{2}\\k,0,\frac{k}{2}\end{matrix}\middle|t\right] = \frac{1}{\Gamma_k(2k)\Gamma_k(\frac{3k}{2})\Gamma_k(\frac{-k}{2})}\sum_{n=0}^{\infty}\frac{(k)_{n,k}k^n t^{n+1}}{(2k)_{n,k}}$$
$$= \frac{-1}{k\pi}\sum_{n=0}^{\infty}\frac{(k)_{n,k}k^n t^{n+1}}{(2k)_{n,k}}. \tag{3.2}$$

Since, the Maclaurin series of logarithmic function is

$$ln(1-kt) = -kt - \frac{(kt)^2}{2} - \frac{(kt)^3}{3} - \cdots. \tag{3.3}$$

So, from (3.2) and (3.3), we obtain the required result. □

**Corollary 3.1.** *Let $p = 2$, $q = 2$, $m = 2$, $n = 1$ and $a_1 = 0$, $a_2 = k$, $b_1 = 0, b_2 = 0$ with $k > 0$ in (1.15). Then for $|t| < 1$ the generalized Meijer G-function gives*

$$G^{2,1}_{k,2,2}\left[\begin{matrix}0,k\\0,0\end{matrix}\middle|t\right] = \frac{1}{k^2}ln(1+\frac{k}{t}). \tag{3.4}$$

*Proof.* By using the definition of generalized Meijer $G$-function (1.15) for given parameters, we have

$$\begin{aligned}G^{2,1}_{k,2,2}\left[\begin{matrix}0,k\\0,0\end{matrix}\middle|t\right] &= \frac{1}{\Gamma_k(2k)}\sum_{n=0}^{\infty}\frac{(k)_{n,k}k^n(-1)^n}{(2k)_{n,k}t^{n+1}}\\ &= \frac{1}{k}\sum_{n=0}^{\infty}\frac{(k)_{n,k}k^n(-1)^n}{t^{n+1}(2k)_{n,k}}.\end{aligned} \tag{3.5}$$

So, we have

$$G^{2,1}_{k,2,2}\left[\begin{matrix}0,k\\0,0\end{matrix}\middle|t\right] = \frac{1}{k^2}ln(1+\frac{k}{t}).$$

□

The confluent hypergeometric functions can be expressed in the form of generalized Meijer $G$-functions as follows:

**Corollary 3.2.** *Let $p = 1, q = 2, m = 1, n = 1$ and $a_1 = k - a, b_1 = 0, b_2 = k - b$ for $k > 0$. Then from (1.15), the following relation holds*

$$G^{1,1}_{k,1,2}\left[\begin{matrix}k-a\\0,k-b\end{matrix}\middle|t\right] = \frac{\Gamma_k(a)}{\Gamma_k(b)}\ {}_1F_{1,k}\left[\begin{matrix}a\\0\end{matrix}\middle|-t\right]. \tag{3.6}$$

*Proof.* By using the definition of generalized Meijer $G$-function (1.15) for given parameters we can prove easily. □

**Proposition 3.1.** *Let $p = 1, q = 2, m = 2, n = 1$ and $a_1 = k - a, b_1 = 0, b_2 = k - b$ for $k > 0$. Then from (1.15), the generalized Meijer G-function can be expressed in the form of generalized confluent hypergeometric function of tricomi as*

$$G^{2,1}_{k,1,2}\left[\begin{matrix}k-a\\0,k-b\end{matrix}\middle|t\right] = \Gamma_k(a)\Gamma_k(k+a-b)\o_k(a;b;t), \tag{3.7}$$

*where the generalized confluent hypergeometric function of tricomi is defined as*

$$\o_k(a;b;t) = \frac{\Gamma_k(b-k)}{\Gamma_k(a-b+k)}\ {}_1F_{1,k}\left[\begin{matrix}a\\b\end{matrix}\middle|t\right] + \frac{\Gamma_k(k-b)t^{1-\frac{b}{k}}}{\Gamma_k(a)}\ {}_1F_{1,k}\left[\begin{matrix}a-b+k\\2k-b\end{matrix}\middle|t\right].$$

*Proof.* By using the definition of generalized Meijer $G$-function (1.15), we can obtain

$$G^{2,1}_{k,1,2}\left[\begin{matrix}k-a\\0,k-b\end{matrix}\middle|t\right] = \Gamma_k(b-k)\Gamma_k(a)\ {}_1F_{1,k}\left[\begin{matrix}a\\b\end{matrix}\middle|t\right] + \Gamma_k(k-b)t^{1-\frac{b}{k}}\Gamma_k(k+a-b)\ {}_1F_{1,k}\left[\begin{matrix}a-b+k\\2k-b\end{matrix}\middle|t\right], \tag{3.8}$$

from which we get the required result. □

**Corollary 3.3.** *Let $p = 1, q = 2, m = 1, n = 1$ and $a_1 = k, a_2 = b, b_1 = a$ for $k > 0$. Then from (1.15), the following relation holds*

$$G^{1,1}_{k,1,2}\left[\begin{matrix}k,b\\a\end{matrix}\middle| t\right] = \frac{\Gamma_k(a)}{\Gamma_k(b)}\ {}_1F_{1,k}\left[\begin{matrix}a\\b\end{matrix}\middle| t\right]. \tag{3.9}$$

*Proof.* By using the definition of generalized Meijer $G$-function (1.15) for given parameters, we can prove easily. □

**Proposition 3.2.** *Let $p = 2, q = 1, m = 1, n = 2$ and $a_1 = k, a_2 = b, b_1 = a$ for $k > 0$. Then from (1.16), the generalized Meijer G-function can be expressed in the form of generalized confluent hypergeometric function of tricomi as*

$$G^{1,2}_{k,2,1}\left[\begin{matrix}k,b\\a\end{matrix}\middle| t\right] = \Gamma_k(a)\Gamma_k(k+a-b)\o_k(a;b;\frac{1}{t}), \tag{3.10}$$

*where the generalized confluent hypergeometric function of tricomi is defined as*

$$\o_k(a;b;\frac{1}{t}) = \frac{\Gamma_k(k-b)}{\Gamma_k(a-b+k)}\ {}_1F_{1,k}\left[\begin{matrix}a\\b\end{matrix}\middle| \frac{1}{t}\right] + \frac{\Gamma_k(b-k)}{\Gamma_k(a)t^{1-\frac{b}{k}}}\ {}_1F_{1,k}\left[\begin{matrix}a-b+k\\2k-b\end{matrix}\middle| \frac{1}{t}\right].$$

*Proof.* By using the definition of generalized Meijer $G$-function (1.15), we have

$$G^{1,2}_{k,2,1}\left[\begin{matrix}k,b\\a\end{matrix}\middle| t\right] = \Gamma_k(k-b)\Gamma_k(a)\ {}_1F_{1,k}\left[\begin{matrix}a\\b\end{matrix}\middle| \frac{1}{t}\right] + \Gamma_k(b-k)\Gamma_k(k+a-b)t^{\frac{b}{k}-1}\ {}_1F_{1,k}\left[\begin{matrix}k+a-b\\2k-b\end{matrix}\middle| \frac{1}{t}\right], \tag{3.11}$$

from which we can obtain the desired result. □

## 4. Conclusions

In this article, we gave the relations of some special functions with the generalized Meijer $G$-functions. The main focus was on Laguerre polynomials, product of Laguerre polynomials and exponential functions, logarithmic functions, confluent hypergeometric functions. We derived the generalized form of the said functions from generalized Meijer $G$-functions by choosing different parameters. There are several similar type of functions which can also be expressed in terms of the generalized Meijer $G$-functions.

## Conflict of interest

The authors declare that they have no competing interests.

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