Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(10): 11346–11379.
DOI: 10.3934/math.2021659
Received: 28 May 2021
Accepted: 19 July 2021
Published: 05 August 2021

Research article

Novel concepts of *m*-polar spherical fuzzy sets and new correlation measures with application to pattern recognition and medical diagnosis

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Abstract: In this paper, we introduce the notion of m-polar spherical fuzzy set (m-PSFS) which is a hybrid notion of m-polar fuzzy set (m-PFS) and spherical fuzzy set (SFS). The purpose of this hybrid structure is to express multipolar information in spherical fuzzy environment. An m-PSFS is a new approach towards computational intelligence and multi-criteria decision-making (MCDM) problems. We introduce the novel concepts of correlation measures and weighted correlation measures of m-PSFSs based on statistical notions of covariances and variances. Correlation measures estimate the linear relationship of the two quantitative objects. A correlation may be positive or negative depending on the direction of the relation between two objects and its value lies the interval $[-1, 1]$. The same concept is carried out towards m-polar spherical fuzzy (m-PSF) information. We investigate certain properties of covariances and the correlation measures to analyze that these concepts are extension of crisp correlation measures. The main advantage of proposed correlation measures is that these notions deal with uncertainty in the real-life problems efficiently with the help of m-PSF information. We discuss applications of m-polar spherical fuzzy sets and their correlation measures in pattern recognition and medical diagnosis. To discuss the superiority and efficiency of proposed correlation measures, we give a comparison analysis of proposed concepts with some existing concepts.

Keywords: m-polar spherical fuzzy set; correlation measures; weighted correlation measures; score function; pattern recognition; medical diagnosis
Mathematics Subject Classification: 03E72, 94D05, 90B50

1. Introduction

Crisp set theory and logic are key factors in the foundation of science and engineering. A crisp set describes the belongingness of an element in a digital manner, i.e., an element either belongs to a set or not. However, in many real-life problems, partial belongingness becomes necessary to describe uncertainty. In most of real-life problems, we have to deal with the situations where crisp set theory fails. For example, if a decision maker (DM) requires to seek a best teacher, sharp student, beautiful girl, young researcher, etc., the selection of such individual is vague. Since the criterion of best, sharp, beautiful, and young vary person to person. To deal with such problems, Zadeh [1] extended crisp set to fuzzy set and laid foundation of fuzzy set theory. A fuzzy set assigns partial grades which lies in unit closed interval $[0, 1]$ known as membership degree (MDs) to express the partial belongingness of an element to a set under a specific characteristics. A fuzzy set is strong set-theoretic model to express vague information. Fuzzy set got the attention of the researchers and this is broadly used in many fields like computational intelligence, information fusion, pattern recognition, engineering, medical diagnosis, artificial intelligence, machine learning, neural networks, and MCDM problems.

Atanassov [2] extended fuzzy set to the notion of intuitionistic fuzzy set (IFS). An IFS is very useful and efficient model which assigns MDs and non-membership degrees (NMDs) to each element in the universe of discourse with the constraint that sum of these grades lies between 0 and 1. Yager [3] introduced Pythagorean fuzzy set (PFS) with the constraint that sum of squares of MDs and NMDs is less or equal to 1. PFS is useful to deal with the uncertain situation where IFS fails to hold. For example, if the sum of MD and NMD is greater than 1 but the sum of their squares is less or equal to 1. This clearly shows that every PFS is an IFS but converse does not hold. Peng and Yang [4] presented some important results for PFSs. Peng *et al.* [5] proposed Pythagorean fuzzy information measures and their applications in pattern recognition, clustering analysis, and medical diagnosis. Peng and Selvachandran [6] introduced further interesting properties of PFSs. Yager [7] further introduced a strong model named as q-rung orthopair fuzzy set (q-ROFS) which is extension of both PFS and IFS.

Peng and Liu [8] introduced information measures for q-rung orthopair fuzzy sets. Molodtsov [9] introduced the concept of soft set (SS) which is a parameterized collection of subsets of a classical set. Maji *et al.* [10] extended IFS to intuitionistic fuzzy soft set (IFSS). Peng *et al.* [11], and Naeem *et al.* [12] introduced the idea of Pythagorean fuzzy soft sets (PFSSs) as a hybrid model of PFS and SS. Guleria and Bajaj [13] introduced matrix form of PFSSs and proposed a decision-making algorithm and its application.

Smarandache [14, 15] introduced the idea of neutrosophic set (NS) with three functions, truthness (truth membership) T, indeterminacy I, and falsity (false membership) F. Wang *et. al.* [16] introduced single-valued neutrosophic set (SVNS) with the constraints that three components, truth membership, indeterminacy, and falsity must lies in the unit closed interval $[0, 1]$. Cuong [17] introduced the concept of picture fuzzy set with three index, a membership grade, hesitancy, and a non-membership grade with the constraint that the sum of these grades is less or equal to 1. Gundogdu and Kahraman [18] further introduced the concept of spherical fuzzy set (SFS). These models have the ability to deal with uncertainties in the real-life problems with the interesting features that if one model fails to hold the other will manage the uncertain situation. A SFS is of great importance as it has the ability to express vague and uncertain information with the condition that the sum of squares of three index, a membership grade, hesitancy, and a non-membership grade, is less than or equal to 1. Ahmmad *et al.* [19] developed new average aggregation operators based on SFSs and their applications in MCDM.

Gundogdu and Kahraman [20–22] introduced several properties and arithmetic operations of spherical fuzzy sets, TOPSIS method for optimal site selection of electric vehicle charging station by using spherical fuzzy, and hospital performance assessment using interval-valued spherical fuzzy analytic hierarchy process. Gundogdu *et al.* [23] presented the analysis of usability test parameters affecting the mobile application designs by using spherical fuzzy sets. Shishavan *et al.* [24] proposed novel similarity measures in spherical fuzzy environment and their applications. Rafiq *et al.* [25] established new cosine similarity measures of spherical fuzzy sets and their applications in decision-making. Deli and Cagman [26] introduced spherical fuzzy numbers and spherical triangular fuzzy numbers and MCDM applications.

Garg and Arora [27] proposed extended TOPSIS method based on correlation coefficient for MCDM with IFSS information. Ashraf and Abdullah [28] also independently proposed the idea of spherical fuzzy set. Mahmood *et al.* [29] introduced some elementary operations on SFSs. Sitara *et al.* [30] decision-making analysis based on q-rung picture fuzzy graph structures. Akram *et al.* [31] proposed a hybrid decision-making framework under complex spherical fuzzy prioritized weighted aggregation operators. Recently, Akram *et al.* [32] introduced Extensions of Dombi aggregation operators with MCDM applications. Riaz and Hashmi [33] introduced a new extension of fuzzy sets named as linear Diophantine fuzzy set (LDFS). Kamaci [34] presented linear Diophantine fuzzy algebraic structures and their application to coding theory. Ayub *et al.* [35] introduced linear Diophantine fuzzy relations and their algebraic structures with decision making application. Shaheen *et al.* [36] analyzed risk analysis with generalized hesitant fuzzy rough sets (GHFRS).

Zhang [37] introduced bipolar fuzzy set (BFS) as an extension of FS. This set is used to deal with the data which is bipolar in nature that describes information and with its counter property. Lee [38] introduced bipolar-valued fuzzy sets and their elementary operations. Chen *et al*. [39] introduced m-polar fuzzy sets (m-PFS) as an extension of bipolar fuzzy sets. An m-PFS deals with multipolarity by assigning m degrees to each element of a crisp set. Recently, some hybrid structures of m-PFSs have been introduced for modeling uncertainties in multi-criteria decision-making (MCDM) problems. Naeem *et al.* [40] introduced the notion of Pythagorean m-polar fuzzy set (P-m-PFS) as a hybrid model of PFS and m-PFS. Riaz *et al.* [41] proposed Pythagorean m-polar fuzzy soft set (P-m-PFSS) as a hybrid model of PFS, m-PFS, and SS.

Correlation is a property that evaluates the mutual or reciprocal relation between two objects in connection to each other, while similarity measurement tests the proximity of two objects. In engineering and statistics, correlation have significant applications. Statistics have used probability approaches successfully to deal with many real space problems that rely on random data analysis. Similarly, the same probability theory has also tackled several mathematical issues in the scientific fields. These techniques, though, have certain inconveniences, as these procedures are dependent on a random choice of samples with a certain degree of confidence. If we gather data on a vast scale, it involves massive fluidity, which makes it impossible to refine idealism. Highlighting these challenges is how to achieve idealistic performance. A big question? It is very difficult to address this query. The first thing to note is that confusion still takes a position in every real situation, regardless of the technicality, medical or administrative problem. Correlation in statistics is one of our most essential topics of different subjects like business, finance, engineering, management, etc. The dilemma, though, is that this kind of correlation can only address the crisp data and we have resolved several real issues without considering complexity. Many ambiguous structures have been added to deal with uncertainty since the birth of the fuzzy set theory. Many extensions of fuzzy sets have been developed to express vague information until now. Consequently, correlation measures must be introduced in almost all extensions of fuzzy sets. Some correlation measures for IFSs was proposed by Ejegwa *et. al.* [42], Xu *et. al.* [43], Gerstenkorn and Manko [44], and Szmidt and Kacprzyk [45]. Later, Garg [46] suggested correlation measures in PFS environment. Liu *et. al.* [47] and Thao [48] observed the restrictions of the earlier defined correlations and proposed some new directional correlations for PFS environment. Correlation measures studied by Mahmood and Ali [49] to define an extended TOPSIS method based on PFS environment. Zulqarnain *et al.* [50] defined correlated TOPSIS for mask selection procedure. Joshi [51] defined some weighted correlations for MCDM.

1. In many real-life problems, we have to deal with the hybrid situations where multi-polarity of each of three index (membership, non-membership and neutral-membership degree) of the alternatives are required but with the help of existing models we can't handle this situations. To deal with this type of problems, we introduce the hybrid structure of m-polar fuzzy set and spherical fuzzy set named as m-polar spherical fuzzy set (m-PSFS). We introduce the idea of m-polar spherical fuzzy numbers (m-PSFNs). These concepts are more efficient to express vague and uncertain information in a realistic manner.
2. We introduce some basic operational laws of m-PSFSs and m-PSFNs. Based on these operational laws, we analyze certain properties of m-PSFSs and m-PSFNs.
3. We define the score function and certainty functions for comparison of mPSFNs and for finding the ranking of feasible alternatives.
4. We introduce new correlation measures and weighted correlation measures for m-PSFSs.
5. We present a MCDM method based on these correlation measures for medical diagnosis and pattern recognition.

This article is arranged as follows. In Section 2, we discuss basic concepts of m-PFS and SFS. In Section 3, the notion of m-PSFS is introduced which is a robust fusion of m-PFS and SFS. Some basic operations on m-PSFSs are introduced to analyze key properties of m-PSFSs. We also defined several novel features of m-PSFSs. In Section 4, the correlation measures and weighted correlation measures for m-PSFSs are proposed. The proposed correlation measures have their value in the interval $[-1, 1]$. In Section 5, we give applications of proposed correlation measures for m-PSFSs to pattern recognition and medical diagnosis. Lastly, the conclusion od current research work is summarized in Section 6.

2. Preliminaries

In this section, we review some preliminaries like spherical fuzzy set and m-polar fuzzy set that are necessary to understand many novel concepts in this manuscript.

Definition 2.1. Let X be the universe (set of objects or alternatives). A spherical fuzzy set (SFS) $\mathcal{T}$ on X having three index of membership degrees μ, indeterminacy π, and non-membership degree ν, which lies in the unit closed interval $[0, 1]$ for each object $\varsigma \in X$, can be expressed as

$$\mathcal{T} = \Big\{ < \varsigma, (\mu(\varsigma), \pi(\varsigma), \nu(\varsigma)) >: \varsigma \in X \Big\}$$

satisfying the restriction

$$0 \leq \mu^2(\varsigma) + \pi^2(\varsigma) + \nu^2(\varsigma) \leq 1$$

$\forall \varsigma \in X$. For the fixed ς, the triplet (μ, π, ν) is called spherical fuzzy number.

Definition 2.2. For any arbitrary cardinal number m, an m-polar fuzzy set (m-PFS) on X is characterized by the mapping $\Upsilon : X \to [0,1]^m$. An m-PFS can be expressed as

$$\mathcal{M} = \Big\{ < \varsigma, (\mu_1(\varsigma), \cdots, \mu_m(\varsigma)) >: \varsigma \in X \Big\}$$

where $0 \leq \mu_i(\varsigma) \leq 1$, for all $i = 1, \cdots, m$.

3. m-Polar spherical fuzzy set

In this section, first we introduce the notion of m-PSFS which is a robust fusion of m-PFS and SFS. Then we define some fundamental operations on m-PSFSs and their related key properties. Several novel features of m-PSFSs are also investigated.

Definition 3.1. An m-PSFSs $\mathcal{S}$ defined on X is characterized by the mappings $\mu^{(j)} : X \to [0,1]$ (known as membership function), $\pi^{(j)} : X \to [0,1]$ (known as hesitant function) and $\nu^{(j))} : X \to [0,1]$ (known as non-membership function), respectively with the restriction that sum of their squared values should not exceed unity, i.e.,

$$0 \leq (\mu^j(\varsigma))^2 + (\pi^j(\varsigma))^2 + (\nu^j(\varsigma))^2 \leq 1$$

for j=1,2,3,...,m.
Mathematically, An m-PSFS can be written as

$$\begin{aligned} \mathcal{S} &= \Big\{\Big\langle(\mu^{(1)}(\varsigma), \pi^{(1)}(\varsigma), \nu^{(1)}(\varsigma)), \cdots, (\mu^{(m)}(\varsigma), \pi^{(m)}(\varsigma), \nu^{(m)}(\varsigma))\Big\rangle : \varsigma \in X\Big\} \\ &= \Big\{\Big\langle\varsigma, (\mu^{(j)}(\varsigma), \pi^{(j)}(\varsigma), \nu^{(j)}(\varsigma))\Big\rangle_{j=1}^{m} : \varsigma \in X\Big\} \end{aligned}$$

and refusal degree is $\mathcal{R} = \sqrt{1 - \left(\mu^{(j)}(\varsigma)\right)^2 - \left(\pi^{(j)}(\varsigma)\right)^2 - \left(\nu^{(j)}(\varsigma)\right)^2}$, for j=1,2,3,...,m
For a fixed ς, an m-polar spherical fuzzy number (m-PSFN) can be written as

$$\begin{aligned} \mathcal{N} &= \Big\langle(\mu^{(1)}, \pi^{(1)}, \nu^{(1)}), \cdots, (\mu^{(m)}, \pi^{(m)}, \nu^{(m)})\Big\rangle \\ &= \Big\langle(\mu^{(j)}, \pi^{(j)}, \nu^{(j)})\Big\rangle_{j=1}^{m} \end{aligned}$$

If the cardinality of X is k, then the tabular representation of m-PSFS is given by Table 1:

Table 1. m-polar spherical fuzzy set.

$\mathcal{S}$	m$-PSFSs$
ς_1	$\Big(\mu^{(1)}(\varsigma_1), \pi^{(1)}(\varsigma_1), \nu^{(1)}(\varsigma_1)\Big)\Big(\mu^{(2)}(\varsigma_1), \pi^{(2)}(\varsigma_1), \nu^{(2)}(\varsigma_1)\Big)\cdots\Big(\mu^{(m)}(\varsigma_1), \pi^{(m)}(\varsigma_1), \nu^{(m)}(\varsigma_1)\Big)$
ς_2	$\Big(\mu^{(1)}(\varsigma_2), \pi^{(1)}(\varsigma_2), \nu^{(1)}(\varsigma_2)\Big)\Big(\mu^{(2)}(\varsigma_2), \pi^{(2)}(\varsigma_2), \nu^{(2)}(\varsigma_2)\Big)\cdots\Big(\mu^{(m)}(\varsigma_2), \pi^{(m)}(\varsigma_2), \nu^{(m)}(\varsigma_2)\Big)$
$\vdots$	$\vdots$
ς_k	$\Big(\mu^{(1)}(\varsigma_k), \pi^{(1)}(\varsigma_k), \nu^{(1)}(\varsigma_k)\Big)\Big(\mu^{(2)}(\varsigma_k), \pi^{(2)}(\varsigma_k), \nu^{(2)}(\varsigma_k)\Big)\cdots\Big(\mu^{(m)}(\varsigma_k), \pi^{(m)}(\varsigma_k), \nu^{(m)}(\varsigma_k)\Big)$

and m-polar spherical fuzzy set(m-PSFS) can be represented in matrix form as

$$\mathcal{S}=\begin{pmatrix} \left(\mu^{(1)}(\varsigma_1),\pi^{(1)}(\varsigma_1),\nu^{(1)}(\varsigma_1)\right) & \left(\mu^{(2)}(\varsigma_1),\pi^{(2)}(\varsigma_1),\nu^{(2)}(\varsigma_1)\right) & \cdots & \left(\mu^{(m)}(\varsigma_1),\pi^{(m)}(\varsigma_1),\nu^{(m)}(\varsigma_1)\right) \\ \left(\mu^{(1)}(\varsigma_2),\pi^{(1)}(\varsigma_2),\nu^{(1)}(\varsigma_2)\right) & \left(\mu^{(2)}(\varsigma_2),\pi^{(2)}(\varsigma_2),\nu^{(2)}(\varsigma_2)\right) & \cdots & \left(\mu^{(m)}(\varsigma_2),\pi^{(m)}(\varsigma_2),\nu^{(m)}(\varsigma_2)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\mu^{(1)}(\varsigma_k),\pi^{(1)}(\varsigma_k),\nu^{(1)}(\varsigma_k)\right) & \left(\mu^{(2)}(\varsigma_k),\pi^{(2)}(\varsigma_k),\nu^{(2)}(\varsigma_k)\right) & \cdots & \left(\mu^{(m)}(\varsigma_k),\pi^{(m)}(\varsigma_k),\nu^{(m)}(\varsigma_k)\right) \end{pmatrix}$$

The collection of all m-PSFSs on X is denoted by $m-PSFS(X)$.

Definition 3.2. An m-PSFS is said to be an empty or null m-PSFS if
$\mu^{(j)}(\varsigma)=0$,$\pi^{(j)}(\varsigma)=0$ and $\nu^{(j)}(\varsigma)=1$ for all j=1,2,3,..m. It is denoted as $\tilde{\Phi}$ and scripted as

$$\tilde{\Phi}=\left\{\left\langle\varsigma,\left((0,0,1),(0,0,1),...,(0,0,1)\right)\right\rangle:\varsigma\in X\right\}$$

Its matrix representation is

$$\tilde{\Phi}=\begin{bmatrix} (0,0,1) & (0,0,1) & \cdots & (0,0,1) \\ (0,0,1) & (0,0,1) & \cdots & (0,0,1) \\ \vdots & \vdots & \ddots & \vdots \\ (0,0,1) & (0,0,1) & \cdots & (0,0,1) \end{bmatrix}$$

Definition 3.3. An m-PSFS is said to be absolute m-PSFS, if
$\mu^{(j)}(\varsigma)=1$, $\pi^{(j)}(\varsigma)=0$ and $\nu^{(j)}(\varsigma)=0$ for all j=1,2,3,..m. It is denoted as $\tilde{X}$ and scripted as

$$\tilde{X}=\left\{\left\langle\varsigma,\left((1,0,0),(1,0,0),...,(1,0,0)\right)\right\rangle:\varsigma\in X\right\}$$

Its matrix representation is

$$\tilde{X}=\begin{bmatrix} (1,0,0) & (1,0,0) & \cdots & (1,0,0) \\ (1,0,0) & (1,0,0) & \cdots & (1,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (1,0,0) & (1,0,0) & \cdots & (1,0,0) \end{bmatrix}$$

Definition 3.4. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two m-PSFSs. Then $\mathcal{S}_1$ is subset of $\mathcal{S}_2$ written as $\mathcal{S}_1\subseteq\mathcal{S}_2$ if $\mu_1^{(j)}(\varsigma)\leq\mu_2^{(j)}(\varsigma),\pi_1^{(j)}(\varsigma)\leq\pi_2^{(j)}(\varsigma)$ and $\nu_1^{(j)}(\varsigma)\geq\nu_2^{(j)}(\varsigma)$ for all $\varsigma\in X$ and for $j=1,2,3,\cdots,m$.
Two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are said to be equal if $\mathcal{S}_1\subseteq\mathcal{S}_2$ and $\mathcal{S}_2\subseteq\mathcal{S}_1$.

Definition 3.5. The complement of m-PSFS

$$\mathcal{S}=\left\{\left\langle\varsigma,\left(\mu^{(j)}(\varsigma),\pi^{(j)}(\varsigma),\nu^{(j)}(\varsigma)\right)\right\rangle_{j=1}^{m}:\varsigma\in X\right\}$$

is defined as

$$\mathcal{S}^c=\left\{\left\langle\varsigma,\left(\nu^{(j)}(\varsigma),\pi^{(j)}(\varsigma),\mu^{(j)}(\varsigma)\right)\right\rangle_{j=1}^{m}:\varsigma\in X\right\}$$

It is interesting to note that information given by hesitancy part of m-PSFS is uncertain. It does not give exact information as given by membership and non-membership grades. Thus the h-complement of m-PSFS is denoted as C^* and defined as

$$\mathcal{S}^{c^*} = \left\{ \left\langle \varsigma, \left(\nu^{(j)}(\varsigma), 1 - (\pi^{(j)}(\varsigma))^2, \mu^{(j)}(\varsigma) \right) \right\rangle_{j=1}^{m} : \varsigma \in X \right\}$$

Notice that $(\mathcal{S}^c)^c = \mathcal{S}$, $\tilde{\Phi}^c = \tilde{X}$ and $\tilde{X}^c = \tilde{\Phi}$.

Definition 3.6. The union of two m-PSFSs $\mathcal{S}_1$ and $\mathcal{S}_2$ defined over the same universal set X is defined as

$$\mathcal{S}_1 \cup \mathcal{S}_2 = \left\{ \left\langle \varsigma, \left(\max\left(\mu_1^{(j)}(\varsigma), \mu_2^{(j)}(\varsigma)\right), \min\left(\pi_1^{(j)}(\varsigma), \pi_2^{(j)}(\varsigma)\right), \min\left(\nu_1^{(j)}(\varsigma), \nu_2^{(j)}(\varsigma)\right) \right) \right\rangle_{j=1}^{m} : \varsigma \in X \right\}.$$

Definition 3.7. The intersection of two m-PSFSs $\mathcal{S}_1$ and $\mathcal{S}_2$ defined over the same universal set X can be defined as

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \left\{ \left\langle \varsigma, \left(\min\left(\mu_1^{(j)}(\varsigma), \mu_2^{(j)}(\varsigma)\right), \min\left(\pi_1^{(j)}(\varsigma), \pi_2^{(j)}(\varsigma)\right), \max\left(\nu_1^{(j)}(\varsigma), \nu_2^{(j)}(\varsigma)\right) \right) \right\rangle_{j=1}^{m} : \varsigma \in X \right\}.$$

Definition 3.8. The difference of two m-PSFSs $\mathcal{S}_1$ and $\mathcal{S}_2$ defined over the same universal set X is defined as

$$\mathcal{S}_1 / \mathcal{S}_2 = \left\{ \left\langle \varsigma, \left(\min\left(\nu_1^{(j)}(\varsigma), \mu_2^{(j)}(\varsigma)\right), \min\left(\pi_1^{(j)}(\varsigma), \pi_2^{(j)}(\varsigma)\right), \max\left(\mu_1^{(j)}(\varsigma), \nu_2^{(j)}(\varsigma)\right) \right) \right\rangle_{j=1}^{m} : \varsigma \in X \right\}.$$

Definition 3.9. let $\mathcal{S}_1$ and $\mathcal{S}_2$ are m-PSFSs over X. Then some new operations are as follows.

1. $\mathcal{S}_1 \oplus \mathcal{S}_2 = \left\{ \left\langle \varsigma, \left(\left((\mu_1^{(j)}(\varsigma))^2 + (\mu_2^{(j)}(\varsigma))^2 - (\mu_1^{(j)}(\varsigma))^2 (\mu_2^{(j)}(\varsigma))^2 \right)^{1/2}, \left((1 - (\mu_2^{(j)}(\varsigma))^2)(\pi_1^{(j)}(\varsigma))^2 + (1 - (\mu_1^{(j)}(\varsigma))^2)(\pi_2^{(j)}(\varsigma))^2 - (\pi_1^{(j)}(\varsigma))^2 (\pi_2^{(j)}(\varsigma))^2 \right)^{1/2}, \nu_1^{(j)}(\varsigma)\nu_2^{(j)}(\varsigma) \right) \right\rangle_{j=1}^{m} : \varsigma \in X \right\}$
2. $\mathcal{S}_1 \otimes \mathcal{S}_2 = \left\{ \left\langle \varsigma, \left(\mu_1^{(j)}(\varsigma)\mu_2^{(j)}(\varsigma), \left((1 - (\nu_2^{(j)}(\varsigma))^2)(\pi_1^{(j)}(\varsigma))^2 + (1 - (\nu_1^{(j)}(\varsigma))^2)(\pi_2^{(j)}(\varsigma))^2 - (\pi_1^{(j)}(\varsigma))^2 (\pi_2^{(j)}(\varsigma))^2 \right)^{1/2}, \left((\nu_1^{(j)}(\varsigma))^2 + (\nu_2^{(j)}(\varsigma))^2 - (\nu_1^{(j)}(\varsigma))^2 (\nu_2^{(j)}(\varsigma))^2 \right)^{1/2} \right) \right\rangle_{j=1}^{m} : \varsigma \in X \right\}$
3. $\lambda \mathcal{S} = \left\{ \left\langle \varsigma, \left(\left(1 - \left(1 - (\mu^{(j)}(\varsigma))^2 \right)^{\lambda} \right)^{1/2}, \left(\left(1 - \mu^{(j)}(\varsigma))^2 \right)^{\lambda} - \left(1 - (\mu^{(j)}(\varsigma))^2 - (\pi^{(j)}(\varsigma))^2 \right)^{\lambda} \right)^{1/2}, (\nu^{(j)}(\varsigma))^{\lambda} \right) \right\rangle_{j=1}^{m} : \varsigma \in X \right\}, \ \lambda > 0$
4. $\mathcal{S}^{\lambda} = \left\{ \left\langle \varsigma, \left((\mu^{(j)}(\varsigma))^{\lambda}, \left((1 - (\nu^{(j)}(\varsigma))^2)^{\lambda} - \left(1 - (\nu^{(j)}(\varsigma))^2 - (\pi^{(j)}(\varsigma))^2 \right)^{\lambda} \right)^{1/2}, \left(1 - \left(1 - (\nu^{(j)}(\varsigma))^2 \right)^{\lambda} \right)^{1/2} \right) \right\rangle_{j=1}^{m} : \varsigma \in X \right\}, \ \lambda > 0$

Theorem 3.10. Let $\mathcal{S} = \left\langle(\mu_s^{(j)}, \pi_s^{(j)}, \nu_s^{(j)})\right\rangle_{j=1}^m$, $\mathcal{S}_1 = \left\langle(\mu_1^{(j)}, \pi_1^{(j)}, \nu_1^{(j)})\right\rangle_{j=1}^m$ $\mathcal{S}_2 = \left\langle(\mu_2^{(j)}, \pi_2^{(j)}, \nu_2^{(j)})\right\rangle_{j=1}^m$ and $\mathcal{T} = \left\langle(\mu_T^{(j)}, \pi_T^{(j)}, \nu_T^{(j)})\right\rangle_{j=1}^m$ be m-PSFNs. Then

1. If $\mathcal{S} \subseteq \mathcal{S}_1$ and $\mathcal{S} \subseteq \mathcal{S}_2$ then $\mathcal{S} \subseteq \mathcal{S}_1 \cap \mathcal{S}_2$.
2. If $\mathcal{S}_1 \subseteq \mathcal{S}$ and $\mathcal{S}_2 \subseteq \mathcal{S}$ then $\mathcal{S}_1 \cup \mathcal{S}_2 \subseteq \mathcal{S}$.
3. If $\mathcal{S}_1 \subseteq \mathcal{S}$ and $\mathcal{S}_2 \subseteq \mathcal{T}$ then $\mathcal{S}_1 \cup \mathcal{S}_2 \subseteq \mathcal{S} \cup \mathcal{T}$.
4. If $\mathcal{S}_1 \subseteq \mathcal{S}$ and $\mathcal{S}_2 \subseteq \mathcal{T}$ then $\mathcal{S}_1 \cap \mathcal{S}_2 \subseteq \mathcal{S} \cap \mathcal{T}$.

Proof. In the following, we shall prove (1) and (3) and (2) and (4) can be proved similar.
(1) If $\mathcal{S} \subseteq \mathcal{S}_1$ and $\mathcal{S} \subseteq \mathcal{S}_2$, then by definition

$$\mu_S^{(j)} \leq \mu_1^{(j)}, \pi_S^{(j)} \leq \pi_1^{(j)}, \nu_S^{(j)} \geq \nu_1^{(j)} \quad for\ all\ j. \tag{3.1}$$

$$\mu_S^{(j)} \leq \mu_2^{(j)}, \pi_S^{(j)} \leq \pi_2^{(j)}, \nu_S^{(j)} \geq \nu_2^{(j)} \quad for\ all\ j. \tag{3.2}$$

Suppose

$$\min\{\mu_1^{(j)}, \mu_2^{(j)}\} = \mu_1^{(j)}, \min\{\pi_1^{(j)}, \pi_2^{(j)}\} = \pi_1^{(j)} \quad and \quad \max\{\nu_1^{(j)}, \nu_2^{(j)}\} = \nu_2^{(j)}$$

then

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \left\langle\left(\min\{\mu_1^{(j)}, \mu_2^{(j)}\}, \min\{\pi_1^{(j)}, \pi_2^{(j)}\}, \max\{\nu_1^{(j)}, \nu_2^{(j)}\}\right)\right\rangle_{j=1}^m = \left\langle(\mu_1^{(j)}, \pi_1^{(j)}, \nu_2^{(j)})\right\rangle_{j=1}^m$$

Clearly from equations (3.1) and (3.2) $\mu_S^{(j)} \leq \mu_1^{(j)}, \pi_S^{(j)} \leq \pi_1^{(j)}$ and $\nu_S^{(j)} \geq \nu_2^{(j)}$. This Shows that $\mathcal{S} \subseteq \mathcal{S}_1 \cap \mathcal{S}_2$.

The rest of 7 cases may be discussed similarly.
(3). If $\mathcal{S}_1 \subseteq \mathcal{S}$ and $\mathcal{S}_2 \subseteq \mathcal{T}$, then by definition

$$\mu_1^{(j)} \leq \mu_S^{(j)}, \pi_1^{(j)} \leq \pi_S^{(j)}, \nu_1^{(j)} \geq \nu_S^{(j)} \quad for\ all\ j. \tag{3.3}$$

$$\mu_2^{(j)} \leq \mu_T^{(j)}, \pi_2^{(j)} \leq \pi_T^{(j)}, \nu_2^{(j)} \geq \nu_T^{(j)} \quad for\ all\ j. \tag{3.4}$$

Suppose

$$\max\{\mu_1^{(j)}, \mu_2^{(j)}\} = \mu_2^{(j)}, \min\{\pi_1^{(j)}, \pi_2^{(j)}\} = \pi_1^{(j)} \quad and \quad \min\{\nu_1^{(j)}, \nu_2^{(j)}\} = \nu_1^{(j)}$$

and

$$\min\{\mu_S^{(j)}, \mu_T^{(j)}\} = \mu_T^{(j)}, \min\{\pi_S^{(j)}, \pi_T^{(j)}\} = \pi_S^{(j)} \quad and \quad \max\{\nu_S^{(j)}, \nu_T^{(j)}\} = \nu_S^{(j)}$$

then

$$\mathcal{S}_1 \cup \mathcal{S}_2 = \left\langle\left(\max\{\mu_1^{(j)}, \mu_2^{(j)}\}, \min\{\pi_1^{(j)}, \pi_2^{(j)}\}, \min\{\nu_1^{(j)}, \nu_2^{(j)}\}\right)\right\rangle_{j=1}^m = \left\langle(\mu_2^{(j)}, \pi_1^{(j)}, \nu_1^{(j)})\right\rangle_{j=1}^m$$

$$\mathcal{S} \cup \mathcal{T} = \left\langle\left(\max\{\mu_S^{(j)}, \mu_T^{(j)}\}, \min\{\pi_S^{(j)}, \pi_T^{(j)}\}, \min\{\nu_S^{(j)}, \nu_T^{(j)}\}\right)\right\rangle_{j=1}^m = \left\langle(\mu_T^{(j)}, \pi_S^{(j)}, \nu_S^{(j)})\right\rangle_{j=1}^m$$

Clearly from equations (3.3) and (3.4)

$$\mu_2^{(j)} \leq \mu_T^{(j)}, \pi_1^{(j)} \leq \pi_S^{(j)}, \nu_1^{(j)} \geq \nu_S^{(j)} \quad for\ all\ j.$$

This shows that $\mathcal{S}_1 \cup \mathcal{S}_2 \subseteq \mathcal{S} \cup \mathcal{T}$.

The rest of 7 cases may be discussed similarly.

□

Theorem 3.11. Let $\mathcal{S}_1 = \left\langle(\mu_1^{(j)}, \pi_1^{(j)}, \nu_1^{(j)})\right\rangle_{j=1}^{m}$ and $\mathcal{S}_2 = \left\langle(\mu_2^{(j)}, \pi_2^{(j)}, \nu_2^{(j)})\right\rangle_{j=1}^{m}$ be two m-PSFNs and $\lambda > 0$, then

1. $\mathcal{S}_1 \subseteq \mathcal{S}_2 \Leftrightarrow (\mathcal{S}_2)^{c^*} \subseteq (\mathcal{S}_1)^{c^*}$
2. $\mathcal{S}_1 = \mathcal{S}_2 \Leftrightarrow (\mathcal{S}_2)^{c^*} = (\mathcal{S}_1)^{c^*}$
3. $\mathcal{S}_1 = \mathcal{S}_2 \Leftrightarrow (\mathcal{S}_2)^{c} = (\mathcal{S}_1)^{c}$
4. $\mathcal{S}_1 \subseteq \mathcal{S}_2 \Leftrightarrow \lambda(\mathcal{S}_1) \subseteq \lambda(\mathcal{S}_2)$
5. $\mathcal{S}_1 = \mathcal{S}_2 \Leftrightarrow \lambda(\mathcal{S}_1) = \lambda(\mathcal{S}_2)$

Proof. The proof is obvious.

□

Theorem 3.12. Let $\mathcal{S} = \left\langle(\mu^{(j)}, \pi^{(j)}, \nu^{(j)})\right\rangle_{j=1}^{m}$ and $\lambda > 0$, then following results hold

1. $(\mathcal{S}^{\lambda})^c = \lambda(\mathcal{S}^c)$
2. $(\lambda(\mathcal{S}^c))^c = (\mathcal{S})^{\lambda}$
3. $(\lambda\mathcal{S})^c = (\mathcal{S}^c)^{\lambda}$

Proof. (1) By definition,

$$\begin{aligned}
(\mathcal{S}^{\lambda})^c &= \left\langle\left((\mu^{(j)})^{\lambda},\ \sqrt{(1-(\nu^{(j)})^2)^{\lambda} - (1-(\nu^{(j)})^2-(\pi^{(j)})^2)^{\lambda}},\ \sqrt{1-(1-(\nu^{(j)})^2)^{\lambda}}\right)^c\right\rangle_{j=1}^{m} \\
&= \left\langle\left(\sqrt{1-(1-(\nu^{(j)})^2)^{\lambda}},,\ \sqrt{(1-(\nu^{(j)})^2)^{\lambda} - (1-(\nu^{(j)})^2-(\pi^{(j)})^2)^{\lambda}}, (\mu^{(j)})^{\lambda}\right)\right\rangle_{j=1}^{m} \\
&= \lambda\left\langle(\mu^{(j)}, \pi^{(j)}, \nu^{(j)})\right\rangle_{j=1}^{m} \\
&= \lambda(\mathcal{S}^c)
\end{aligned}$$

(2) By definition,

$$\begin{aligned}
(\lambda(\mathcal{S}^c))^c &= \left(\lambda\left\langle(\mu^{(j)}, \pi^{(j)}, \nu^{(j)})^c\right\rangle_{j=1}^{m}\right)^c \\
&= \left(\lambda\left\langle(\nu^{(j)}, \pi^{(j)}, \mu^{(j)})\right\rangle_{j=1}^{m}\right)^c \\
&= \left\langle\left(\sqrt{1-(1-(\nu^{(j)})^2)^{\lambda}},,\ \sqrt{(1-(\nu^{(j)})^2)^{\lambda} - (1-(\nu^{(j)})^2-(\pi^{(j)})^2)^{\lambda}}, (\mu^{(j)})^{\lambda}\right)^c\right\rangle_{j=1}^{m} \\
&= \left\langle\left((\mu^{(j)})^{\lambda},\ \sqrt{(1-(\nu^{(j)})^2)^{\lambda} - (1-(\nu^{(j)})^2-(\pi^{(j)})^2)^{\lambda}},\ \sqrt{1-(1-(\nu^{(j)})^2)^{\lambda}}\right)\right\rangle_{j=1}^{m} \\
&= \mathcal{S}^{\lambda}
\end{aligned}$$

(3) By definition,

$$\begin{aligned}
(\lambda\mathcal{S})^c &= \left(\lambda\left\langle(\mu^{(j)}, \pi^{(j)}, \nu^{(j)})\right\rangle_{j=1}^{m}\right)^c \\
&= \left(\left\langle\left(\sqrt{1-(1-(\mu^{(j)})^2)^{\lambda}},\ \sqrt{(1-(\mu^{(j)})^2)^{\lambda} - (1-(\mu^{(j)})^2-(\pi^{(j)})^2)^{\lambda}}, (\nu^{(j)})^{\lambda}\right)\right\rangle_{j=1}^{m}\right)^c
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle ((\nu^{(j)})^{\lambda},\ \sqrt{(1-(\mu^{(j)})^2)^{\lambda}-(1-(\mu^{(j)})^2-(\pi^{(j)})^2)^{\lambda}},\ \sqrt{1-(1-(\mu^{(j)})^2)^{\lambda}})\right\rangle_{j=1}^{m} \\
&= \left(\left\langle (\mu^{(j)},\pi^{(j)},\nu^{(j)})^{c}\right\rangle_{j=1}^{m}\right)^{\lambda} \\
&= (\mathcal{S}^{c})^{\lambda}
\end{aligned}
$$

□

Theorem 3.13. Let $\mathcal{S}_1 = \left\langle (\mu_1^{(j)},\pi_1^{(j)},\nu_1^{(j)})\right\rangle_{j=1}^{m}$, $\mathcal{S}_2 = \left\langle (\mu_2^{(j)},\pi_2^{(j)},\nu_2^{(j)})\right\rangle_{j=1}^{m}$ and $\lambda > 0$, then

1. $\lambda(\mathcal{S}_1 \cup \mathcal{S}_2) = \lambda\mathcal{S}_1 \cup \lambda\mathcal{S}_2$
2. $(\mathcal{S}_1 \cup \mathcal{S}_2)^{\lambda} = \mathcal{S}_1^{\lambda} \cup \mathcal{S}_2^{\lambda}$

Proof. Here, we shall prove (1) and (2) can be proved analogously.
Suppose the first possibility

$$\max\{\mu_1^{(j)},\mu_2^{(j)}\} = \mu_2^{(j)}, \min\{\pi_1^{(j)},\pi_2^{(j)}\} = \pi_1^{(j)} \quad \textit{and} \quad \min\{\nu_1^{(j)},\nu_2^{(j)}\} = \nu_1^{(j)}$$

(1) $\lambda\mathcal{S}_1 \cup \lambda\mathcal{S}_2 = \lambda\left\langle (\mu_1^{(j)},\pi_1^{(j)},\nu_1^{(j)})\right\rangle_{j=1}^{m} \cup \quad \lambda\left\langle (\mu_2^{(j)},\pi_2^{(j)},\nu_2^{(j)})\right\rangle_{j=1}^{m}$

$$
\begin{aligned}
&= \left\langle\left(\sqrt{1-(1-(\mu_1^{(j)})^2)^{\lambda}},\ \sqrt{(1-(\mu_1^{(j)})^2)^{\lambda}-(1-(\mu_1^{(j)})^2-(\pi_1^{(j)})^2)^{\lambda}}, (\nu_1^{(j)})^{\lambda}\right)\right\rangle_{j=1}^{m} \\
&\qquad \cup \left\langle\left(\sqrt{1-(1-(\mu_2^{(j)})^2)^{\lambda}},\ \sqrt{(1-(\mu_2^{(j)})^2)^{\lambda}-(1-(\mu_2^{(j)})^2-(\pi_2^{(j)})^2)^{\lambda}}, (\nu_2^{(j)})^{\lambda}\right)\right\rangle_{j=1}^{m} \\
&= \Big\langle \max\left\{ \sqrt{1-(1-(\mu_1^{(j)})^2)^{\lambda}},\ \sqrt{1-(1-(\mu_2^{(j)})^2)^{\lambda}}\right\}, \min\left\{ \sqrt{(1-(\mu_1^{(j)})^2)^{\lambda}-(1-(\mu_1^{(j)})^2-(\pi_1^{(j)})^2)^{\lambda}},\right. \\
&\left.\sqrt{(1-(\mu_2^{(j)})^2)^{\lambda}-(1-(\mu_2^{(j)})^2-(\pi_2^{(j)})^2)^{\lambda}}\right\}, \min\left\{(\nu_2^{(j)})^{\lambda},(\nu_2^{(j)})^{\lambda}\right\}\Big\rangle_{j=1}^{m} \\
&= \Big\langle \sqrt{1-(1-\max\{(\mu_1^{(j)})^2,(\mu_2^{(j)})^2\})^{\lambda}}, \\
&\sqrt{(1-\min\{(\mu_1^{(j)})^2),(\mu_2^{(j)})^2\}-\left(1-\min\{(\mu_1^{(j)})^2),(\mu_2^{(j)})^2)\}-\min\{(\pi_1^{(j)})^2),(\pi_2^{(j)})^2)\}\right)^{\lambda}}, \min\left\{(\nu_2^{(j)})^{\lambda},(\nu_2^{(j)})^{\lambda}\right\}\Big\rangle_{j=1}^{m} \\
&=\lambda\Big\langle \max\{\mu_1,\mu_2\},\min\{\pi_1,\pi_2\},\min\{\nu_1,\nu_2\}\Big\rangle \\
&=\lambda(\mathcal{S}_1 \cup \mathcal{S}_2)
\end{aligned}
$$

The rest of 7 possibilities may be discussed similarly. □

Theorem 3.14. *Let* $\mathcal{S}_1 = \left\langle (\mu_1^{(j)},\pi_1^{(j)},\nu_1^{(j)})\right\rangle_{j=1}^{m}$, $\mathcal{S}_2 = \left\langle (\mu_2^{(j)},\pi_2^{(j)},\nu_2^{(j)})\right\rangle_{j=1}^{m}$ *and* $\lambda > 0$*, then*

1. $\lambda(\mathcal{S}_1 \oplus \mathcal{S}_2) = \left(((\mathcal{S}_1 \oplus \mathcal{S}_2)^{c})^{\lambda}\right)^{c}$
2. $\lambda(\mathcal{S}_1 \otimes \mathcal{S}_2) = \left(((\mathcal{S}_1 \otimes \mathcal{S}_2)^{c})^{\lambda}\right)^{c}$
3. $\lambda(\mathcal{S}_1 \oplus \mathcal{S}_2) = \lambda(\mathcal{S}_1) \oplus \lambda(\mathcal{S}_2)$
4. $(\mathcal{S}_1 \otimes \mathcal{S}_2)^{\lambda} = \mathcal{S}_1^{\lambda} \otimes \mathcal{S}_2^{\lambda}$
5. $\lambda(\mathcal{S}_1) \otimes \lambda(\mathcal{S}_2) = \left(((\mathcal{S}_1)^{c})^{\lambda} \oplus ((\mathcal{S}_2)^{c})^{\lambda}\right)^{c}$

Proof. (1) $\Big(((\mathcal{S}_1\oplus\mathcal{S}_2)^c)^\lambda\Big)^c = \Bigg(\Bigg(\Big(\Big\langle\Big((\mu_1^{(j)})^2+(\mu_2^{(j)})^2-(\mu_1^{(j)})^2(\mu_2^{(j)})^2\Big)^{1/2},\Big(1-(\mu_2^{(j)})^2)(\pi_1^{(j)})^2+(1-(\mu_1^{(j)})^2)(\pi_2^{(j)})^2-$
$(\pi_1^{(j)})^2(\pi_2^{(j)})^2\Big)^{1/2},(\nu_1^{(j)})(\nu_2^{(j)})\Big\rangle_{j=1}^m\Big)^c\Bigg)^\lambda\Bigg)^c$

$$= \Big(\Big\langle\Big((\nu_1^{(j)})(\nu_2^{(j)}),\Big((1-(\mu_2^{(j)})^2)(\pi_1^{(j)})^2+(1-(\mu_1^{(j)})^2)(\pi_2^{(j)})^2-(\pi_1^{(j)})^2(\pi_2^{(j)})^2\Big)^{1/2},$$
$$\Big((\mu_1^{(j)})^2+(\mu_2^{(j)})^2-(\mu_1^{(j)})^2(\mu_2^{(j)})^2\Big)^{1/2}\Big\rangle_{j=1}^m\Big)^\lambda\Big)^c$$
$$= \Big(\Big\langle(\nu_1^{(j)})^\lambda(\nu_2^{(j)})^\lambda,\Big(\Big(1-((\mu_1^{(j)})^2+(\mu_2^{(j)})^2-(\mu_1^{(j)})^2(\mu_2^{(j)})^2)\Big)^\lambda-\Big(\Big(1-((\mu_1^{(j)})^2+(\mu_2^{(j)})^2-(\mu_1^{(j)})^2(\mu_2^{(j)})^2)\Big)-\Big(1-$$
$$(\mu_2^{(j)})^2)(\pi_1^{(j)})^2+(1-(\mu_1^{(j)})^2)(\pi_2^{(j)})^2-(\pi_1^{(j)})^2(\pi_2^{(j)})^2\Big)\Big)\Big)^{1/2},\Big(1-\Big(1-((\mu_1^{(j)})^2+(\mu_2^{(j)})^2-(\mu_1^{(j)})^2(\mu_2^{(j)})^2)\Big)^\lambda\Big)^{1/2}\Big\rangle_{j=1}^m\Big)^c$$
$$= \Big\langle\Big(1-\Big(1-((\mu_1^{(j)})^2+(\mu_2^{(j)})^2-(\mu_1^{(j)})^2(\mu_2^{(j)})^2)\Big)^\lambda\Big)^{1/2},\Big(\Big(1-((\mu_1^{(j)})^2+(\mu_2^{(j)})^2-(\mu_1^{(j)})^2(\mu_2^{(j)})^2)\Big)^\lambda-\Big(\Big(1-((\mu_1^{(j)})^2+$$
$$(\mu_2^{(j)})^2-(\mu_1^{(j)})^2(\mu_2^{(j)})^2)\Big)-\Big(1-(\mu_2^{(j)})^2)(\pi_1^{(j)})^2+(1-(\mu_1^{(j)})^2)(\pi_2^{(j)})^2-(\pi_1^{(j)})^2(\pi_2^{(j)})^2\Big)\Big)\Big)^{1/2},(\nu_1^{(j)})^\lambda(\nu_2^{(j)})^\lambda\Big\rangle_{j=1}^m$$
$$= \lambda\Big\langle\Big((\mu_1^{(j)})^2+(\mu_2^{(j)})^2-(\mu_1^{(j)})^2(\mu_2^{(j)})^2\Big)^{1/2},\Big(1-(\mu_2^{(j)})^2)(\pi_1^{(j)})^2+(1-(\mu_1^{(j)})^2)(\pi_2^{(j)})^2-$$
$$(\pi_1^{(j)})^2(\pi_2^{(j)})^2\Big)^{1/2},(\nu_1^{(j)})(\nu_2^{(j)})\Big\rangle_{j=1}^m$$
$$= \lambda(\mathcal{S}_1\oplus\mathcal{S}_2)$$

Thus (2) can be proved analogously.

(3) By definition

$$\lambda(S_1) = \Big\langle\Big(\sqrt{1-(1-(\mu_1^{(j)})^2)^\lambda},\ \sqrt{(1-(\mu_1^{(j)})^2)^\lambda-(1-(\mu_1^{(j)})^2-(\pi_1^{(j)})^2)^\lambda},(\nu_1^{(j)})^\lambda\Big)\Big\rangle_{j=1}^m$$

and

$$\lambda(S_2) = \Big\langle\Big(\sqrt{1-(1-(\mu_2^{(j)})^2)^\lambda},\ \sqrt{(1-(\mu_2^{(j)})^2)^\lambda-(1-(\mu_2^{(j)})^2-(\pi_2^{(j)})^2)^\lambda},(\nu_2^{(j)})^\lambda\Big)\Big\rangle_{j=1}^m.$$

$$\lambda(S_1)\oplus\lambda(S_2) = \Big(\Big(1-(1-(\mu_1^{(j)})^2)^\lambda\Big)+\Big(1-(1-(\mu_2^{(j)})^2)^\lambda\Big)-\Big(1-(1-(\mu_1^{(j)})^2)^\lambda\Big)\Big(1-(1-(\mu_2^{(j)})^2)^\lambda\Big)\Big)^{1/2},\Big(\Big(1-$$
$$(1-(\mu_2^{(j)})^2)^\lambda\Big)\Big((1-(\mu_1^{(j)})^2)^\lambda-(1-(\mu_1^{(j)})^2-(\pi_1^{(j)})^2)^\lambda\Big)+\Big(1-(1-(\mu_1^{(j)})^2)^\lambda\Big)\Big((1-(\mu_2^{(j)})^2)^\lambda-(1-(\mu_2^{(j)})^2-$$
$$(\pi_2^{(j)})^2)^\lambda\Big)-\Big(1-(1-(\mu_2^{(j)})^2)^\lambda\Big)\Big((1-(\mu_1^{(j)})^2)^\lambda-(1-(\mu_1^{(j)})^2-(\pi_1^{(j)})^2)^\lambda\Big)\Big(1-(1-(\mu_1^{(j)})^2)^\lambda\Big)\Big((1-(\mu_2^{(j)})^2)^\lambda-$$
$$(1-(\mu_2^{(j)})^2-(\pi_2^{(j)})^2)^\lambda\Big)\Big)^{1/2},(\nu_1^{(j)})^\lambda(\nu_2^{(j)})^\lambda\Big\rangle_{j=1}^m$$
$$= \Big(2-(1-(\mu_1^{(j)})^2)^\lambda-(1-(\mu_2^{(j)})^2)^\lambda-1+(1-(\mu_1^{(j)})^2)^\lambda+(1-(\mu_2^{(j)})^2)^\lambda-(1-(\mu_1^{(j)})^2)^\lambda(1-(\mu_2^{(j)})^2)^\lambda\Big)^{1/2},\Big((1-$$
$$(\mu_2^{(j)})^2)^\lambda(1-(\mu_1^{(j)})^2)^\lambda-(1-(\mu_2^{(j)})^2)^\lambda(1-(\mu_1^{(j)})^2-(\pi_1^{(j)})^2)^\lambda+(1-(\mu_1^{(j)})^2)^\lambda(1-(\mu_2^{(j)})^2)^\lambda-(1-(\mu_1^{(j)})^2)^\lambda(1-$$
$$(\mu_2^{(j)})^2-(\pi_2^{(j)})^2)^\lambda-(1-(\mu_1^{(j)})^2)^\lambda(1-(\mu_2^{(j)})^2)^\lambda+(1-(\mu_1^{(j)})^2)^\lambda(1-(\mu_2^{(j)})^2-(\pi_2^{(j)})^2)^\lambda+(1-(\mu_2^{(j)})^2)^\lambda(1-$$

$(\mu_1^{(j)})^2 - (\pi_1^{(j)})^2)^\lambda - (1 - (\mu_1^{(j)})^2 - (\pi_1^{(j)})^2)^\lambda(1 - (\mu_2^{(j)})^2 - (\pi_2^{(j)})^2)^\lambda\Big)^{1/2}, (\nu_1^{(j)})^\lambda(\nu_2^{(j)})^\lambda\Big\rangle_{j=1}^m$

$= \Big\langle\Big(1 - \Big(1 - ((\mu_1^{(j)})^2 + (\mu_2^{(j)})^2 - (\mu_1^{(j)})^2(\mu_2^{(j)})^2)\Big)^\lambda\Big)^{1/2}, \Big(\Big(1 - ((\mu_1^{(j)})^2 + (\mu_2^{(j)})^2 - (\mu_1^{(j)})^2(\mu_2^{(j)})^2)\Big)^\lambda - \Big(\Big(1 - ((\mu_1^{(j)})^2 +$

$(\mu_2^{(j)})^2 - (\mu_1^{(j)})^2(\mu_2^{(j)})^2)\Big) - \Big(1 - (\mu_2^{(j)})^2)(\pi_1^{(j)})^2 + (1 - (\mu_1^{(j)})^2)(\pi_2^{(j)})^2 - (\pi_1^{(j)})^2(\pi_2^{(j)})^2)\Big)\Big)^{1/2}, (\nu_1^{(j)})^\lambda(\nu_2^{(j)})^\lambda\Big\rangle_{j=1}^m$

$= \lambda\Big\langle\Big((\mu_1^{(j)})^2 + (\mu_2^{(j)})^2 - (\mu_1^{(j)})^2(\mu_2^{(j)})^2\Big)^{1/2}, \Big(1 - (\mu_2^{(j)})^2)(\pi_1^{(j)})^2 + (1 - (\mu_1^{(j)})^2)(\pi_2^{(j)})^2 -$

$(\pi_1^{(j)})^2(\pi_2^{(j)})^2\Big)^{1/2}, (\nu_1^{(j)})(\nu_2^{(j)})\Big\rangle_{j=1}^m$

$= \lambda(\mathcal{S}_1 \oplus \mathcal{S}_2)$

Thus (4) can be proved analogously.

(5) $\Big(((\mathcal{S}_1)^c)^\lambda \oplus ((\mathcal{S}_2)^c)^\lambda\Big)^c = \Big(\Big(\Big\langle(\nu_1^{(j)}, \pi_1^{(j)}, \mu_1^{(j)})\Big\rangle_{j=1}^m\Big)^\lambda \oplus \Big(\Big\langle(\nu_1^{(j)}, \pi_1^{(j)}, \mu_1^{(j)})\Big\rangle_{j=1}^m\Big)^\lambda\Big)^c$

$= \Big(\Big\langle\Big((\nu_1^{(j)})^\lambda, \Big((1 - (\mu_1^{(j)})^2)^\lambda - (1 - (\mu_1^{(j)})^2 - \pi_1^{(j)})^2)^\lambda\Big)^{1/2}, \Big(1 - (1 - (\mu_1^{(j)})^2)^\lambda\Big)^{1/2}\Big) \oplus \Big((\nu_2^{(j)})^\lambda, \Big((1 - (\mu_2^{(j)})^2)^\lambda -$

$(1 - (\mu_2^{(j)})^2 - \pi_2^{(j)})^2)^\lambda\Big)^{1/2}, \Big(1 - (1 - (\mu_2^{(j)})^2)^\lambda\Big)^{1/2}\Big)\Big\rangle_{j=1}^m\Big)^c$

$= \Big(\Big\langle\Big(((\nu_1^{(j)})^2 + (\nu_2^{(j)})^2 - (\nu_1^{(j)})^2(\nu_2^{(j)})^2)^\lambda\Big)^{1/2}, \Big((1 - (\nu_2^{(j)})^2)^\lambda)((1 - (\mu_1^{(j)})^2)^\lambda - (1 - (\mu_1^{(j)})^2 - (\pi_1^{(j)})^2)^\lambda) + (1 -$

$(\nu_1^{(j)})^2)^\lambda)((1 - (\mu_2^{(j)})^2)^\lambda - (1 - (\mu_2^{(j)})^2 - (\pi_2^{(j)})^2)^\lambda - \Big((1 - (\mu_1^{(j)})^2)^\lambda - (1 - (\mu_1^{(j)})^2 - (\pi_1^{(j)})^2\Big)^\lambda\Big((1 - (\mu_2^{(j)})^2)^\lambda -$

$(1 - (\mu_2^{(j)})^2 - (\pi_2^{(j)})^2)^\lambda\Big)\Big)^{1/2}, \Big(1 - (1 - (\mu_1^{(j)})^2)^\lambda\Big)^{1/2}\Big(1 - (1 - (\mu_2^{(j)})^2)^\lambda\Big)^{1/2}\Big\rangle_{j=1}^m\Big)^c$

$= \Big(\Big\langle\Big(((\nu_1^{(j)})^2 + (\nu_2^{(j)})^2 - (\nu_1^{(j)})^2(\nu_2^{(j)})^2)^\lambda\Big)^{1/2}, (1 - (\nu_2^{(j)})^2)^\lambda)(1 - (\mu_1^{(j)})^2)^\lambda - (1 - (\nu_2^{(j)})^2)^\lambda)(1 - (\mu_1^{(j)})^2 - (\pi_1^{(j)})^2)^\lambda) +$
$(1 - (\nu_1^{(j)})^2)^\lambda)(1 - (\mu_2^{(j)})^2)^\lambda - (1 - (\nu_1^{(j)})^2)^\lambda)(1 - (\mu_2^{(j)})^2 - (\pi_2^{(j)})^2)^\lambda) - (1 - (\nu_1^{(j)})^2)^\lambda)(1 - (\mu_2^{(j)})^2)^\lambda) + (1 -$
$(\mu_1^{(j)})^2)^\lambda)(1 - (\mu_2^{(j)})^2 - (\pi_2^{(j)})^2)^\lambda + (1 - (\mu_2^{(j)})^2)^\lambda)(1 - (\mu_1^{(j)})^2 - (\pi_1^{(j)})^2)^\lambda + (1 - (\mu_1^{(j)})^2 - (\pi_1^{(j)})^2)^\lambda(1 - (\mu_2^{(j)})^2 -$
$(\pi_2^{(j)})^2)^\lambda, \Big(1 - (1 - (\mu_2^{(j)})^2)^\lambda) - (1 - (\mu_1^{(j)})^2)^\lambda) + (1 - (\mu_1^{(j)})^2)^\lambda)(1 - (\mu_2^{(j)})^2)^\lambda)\Big)\Big\rangle_{j=1}^m\Big)^c$

$= \Big\langle\Big(1 - (1 - (\mu_2^{(j)})^2)^\lambda) - (1 - (\mu_1^{(j)})^2)^\lambda) + (1 - (\mu_1^{(j)})^2)^\lambda)(1 - (\mu_2^{(j)})^2)^\lambda)\Big), (1 - (\nu_2^{(j)})^2)^\lambda)(1 - (\mu_1^{(j)})^2)^\lambda - (1 -$
$(\nu_2^{(j)})^2)^\lambda)(1 - (\mu_1^{(j)})^2 - (\pi_1^{(j)})^2)^\lambda) + (1 - (\nu_1^{(j)})^2)^\lambda)(1 - (\mu_2^{(j)})^2)^\lambda - (1 - (\nu_1^{(j)})^2)^\lambda)(1 - (\mu_2^{(j)})^2 - (\pi_2^{(j)})^2)^\lambda) - (1 -$
$(\nu_1^{(j)})^2)^\lambda)(1 - (\mu_2^{(j)})^2)^\lambda) + (1 - (\mu_1^{(j)})^2)^\lambda)(1 - (\mu_2^{(j)})^2 - (\pi_2^{(j)})^2)^\lambda + (1 - (\mu_2^{(j)})^2)^\lambda)(1 - (\mu_1^{(j)})^2 - (\pi_1^{(j)})^2)^\lambda + (1 -$
$(\mu_1^{(j)})^2 - (\pi_1^{(j)})^2)^\lambda(1 - (\mu_2^{(j)})^2 - (\pi_2^{(j)})^2)^\lambda, \Big(((\nu_1^{(j)})^2 + (\nu_2^{(j)})^2 - (\nu_1^{(j)})^2(\nu_2^{(j)})^2)^\lambda\Big)^{1/2}\Big\rangle_{j=1}^m$

$= \Big\langle\Big(1 - (1 - (\mu_1^{(j)})^2)^\lambda\Big)^{1/2}, \Big((1 - (\mu_1^{(j)})^2)^\lambda - (1 - (\mu_1^{(j)})^2 - \pi_1^{(j)})^2)^\lambda\Big)^{1/2}, (\nu_1^{(j)})^\lambda\Big\rangle_{j=1}^m \otimes \Big\langle\Big(1 - (1 - (\mu_2^{(j)})^2)^\lambda\Big)^{1/2}, \Big((1 -$

$(\mu_2^{(j)})^2)^\lambda - (1 - (\mu_2^{(j)})^2 - \pi_2^{(j)})^2)^\lambda\Big)^{1/2}, (\nu_2^{(j)})^\lambda\Big\rangle_{j=1}^m$

$= \lambda(\mathcal{S}_1) \otimes \lambda(\mathcal{S}_1)$

□

Theorem 3.15. *let* $\mathcal{S}_1 = \left\langle(\mu_1^{(j)}, \pi_1^{(j)}, \nu_1^{(j)})\right\rangle_{j=1}^m$, $\mathcal{S}_2 = \left\langle(\mu_2^{(j)}, \pi_2^{(j)}, \nu_2^{(j)})\right\rangle_{j=1}^m$ *and* $\mathcal{S}_3 = \left\langle(\mu_3^{(j)}, \pi_3^{(j)}, \nu_3^{(j)})\right\rangle_{j=1}^m$, *then*

1. $(\mathcal{S}_1 \cup \mathcal{S}_2) \cap \mathcal{S}_2 = \mathcal{S}_2$
2. $(\mathcal{S}_1 \cap \mathcal{S}_2) \cup \mathcal{S}_2 = \mathcal{S}_2$
3. $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 = \mathcal{S}_1 \cup \mathcal{S}_3 \cup \mathcal{S}_2$
4. $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3 = \mathcal{S}_1 \cap \mathcal{S}_3 \cap \mathcal{S}_2$

Proof. The proof is obvious.

□

Theorem 3.16. *Let* $\mathcal{S}_1 = \left\langle(\mu_1^{(j)}, \pi_1^{(j)}, \nu_1^{(j)})\right\rangle_{j=1}^m$, $\mathcal{S}_2 = \left\langle(\mu_2^{(j)}, \pi_2^{(j)}, \nu_2^{(j)})\right\rangle_{j=1}^m$ *and* $\mathcal{S}_3 = \left\langle(\mu_3^{(j)}, \pi_3^{(j)}, \nu_3^{(j)})\right\rangle_{j=1}^m$, *then*

1. $(\mathcal{S}_1 \cup \mathcal{S}_2) \oplus \mathcal{S}_3 = (\mathcal{S}_1 \oplus \mathcal{S}_3) \cup (\mathcal{S}_2 \oplus \mathcal{S}_3)$
2. $(\mathcal{S}_1 \cap \mathcal{S}_2) \oplus \mathcal{S}_3 = (\mathcal{S}_1 \oplus \mathcal{S}_3) \cap (\mathcal{S}_2 \oplus \mathcal{S}_3)$
3. $(\mathcal{S}_1 \cup \mathcal{S}_2) \otimes \mathcal{S}_3 = (\mathcal{S}_1 \otimes \mathcal{S}_3) \cup (\mathcal{S}_2 \otimes \mathcal{S}_3)$
4. $(\mathcal{S}_1 \cap \mathcal{S}_2) \otimes \mathcal{S}_3 = (\mathcal{S}_1 \otimes \mathcal{S}_3) \cap (\mathcal{S}_2 \otimes \mathcal{S}_3)$

Proof. Suppose the first possibility

$$\max\{\mu_1^{(j)}, \mu_2^{(j)}\} = \mu_2^{(j)}, \max\{\mu_1^{(j)}, \mu_3^{(j)}\} = \mu_3^{(j)}, \max\{\mu_2^{(j)}, \mu_3^{(j)}\} = \mu_3^{(j)}$$

$$\min\{\pi_1^{(j)}, \pi_2^{(j)}\} = \pi_1^{(j)}, \min\{\pi_1^{(j)}, \pi_3^{(j)}\} = \pi_1^{(j)}, \min\{\pi_2^{(j)}, \pi_3^{(j)}\} = \pi_2^{(j)}$$

$$\min\{\nu_1^{(j)}, \nu_2^{(j)}\} = \nu_1^{(j)}, \min\{\nu_1^{(j)}, \nu_3^{(j)}\} = \nu_1^{(j)}, \min\{\nu_2^{(j)}, \nu_3^{(j)}\} = \nu_2^{(j)}$$

(1)

$$(\mathcal{S}_1 \oplus \mathcal{S}_3) \cup (\mathcal{S}_2 \oplus \mathcal{S}_3) = \left(\left\langle\left((\mu_1^{(j)})^2 + (\mu_3^{(j)})^2 - (\mu_1^{(j)})^2(\mu_3^{(j)})^2\right)^{1/2}, \left(1 - (\mu_3^{(j)})^2)(\pi_1^{(j)})^2 + (1 - (\mu_1^{(j)})^2)(\pi_3^{(j)})^2 - (\pi_1^{(j)})^2(\pi_3^{(j)})^2\right)^{1/2}, (\nu_1^{(j)})(\nu_3^{(j)})\right\rangle_{j=1}^m\right) \cup \left(\left\langle\left((\mu_2^{(j)})^2 + (\mu_3^{(j)})^2 - (\mu_2^{(j)})^2(\mu_3^{(j)})^2\right)^{1/2}, \left(1 - (\mu_3^{(j)})^2)(\pi_2^{(j)})^2 + (1 - (\mu_2^{(j)})^2)(\pi_3^{(j)})^2 - (\pi_2^{(j)})^2(\pi_3^{(j)})^2\right)^{1/2}, (\nu_2^{(j)})(\nu_3^{(j)})\right\rangle_{j=1}^m\right)$$

$$= \left\langle \max\left\{\left((\mu_1^{(j)})^2 + (\mu_3^{(j)})^2 - (\mu_1^{(j)})^2(\mu_3^{(j)})^2\right)^{1/2}, \left((\mu_2^{(j)})^2 + (\mu_3^{(j)})^2 - (\mu_2^{(j)})^2(\mu_3^{(j)})^2\right)^{1/2}\right\}, \min\left\{\left(1 - (\mu_3^{(j)})^2)(\pi_1^{(j)})^2 + (1 - (\mu_1^{(j)})^2)(\pi_3^{(j)})^2 - (\pi_1^{(j)})^2(\pi_3^{(j)})^2\right)^{1/2}, \left(1 - (\mu_3^{(j)})^2)(\pi_2^{(j)})^2 + (1 - (\mu_2^{(j)})^2)(\pi_3^{(j)})^2 - (\pi_2^{(j)})^2(\pi_3^{(j)})^2\right)^{1/2}\right\}, \min\left\{(\nu_1^{(j)})(\nu_3^{(j)}), (\nu_2^{(j)})(\nu_3^{(j)})\right\}\right\rangle_{j=1}^m$$

$$= \left\langle\left((\mu_2^{(j)})^2 + (\mu_3^{(j)})^2 - (\mu_2^{(j)})^2(\mu_3^{(j)})^2\right)^{1/2}, \left(1 - (\mu_3^{(j)})^2)(\pi_1^{(j)})^2 + (1 - (\mu_2^{(j)})^2)(\pi_3^{(j)})^2 - (\pi_1^{(j)})^2(\pi_3^{(j)})^2\right)^{1/2}, (\nu_1^{(j)})(\nu_3^{(j)})\right\rangle_{j=1}^m$$

$$= \left\langle(\mu_2^{(j)}, \pi_1^{(j)}, \nu_1^{(j)})\right\rangle_{j=1}^m \oplus \left\langle(\mu_3^{(j)}, \pi_3^{(j)}, \nu_3^{(j)})\right\rangle_{j=1}^m$$

$= \left\langle \max\{\mu_1^{(j)}, \mu_2^{(j)}\}, \min\{\pi_1^{(j)}, \pi_2^{(j)}\}, \min\{\nu_1^{(j)}, \nu_2^{(j)}\} \right\rangle_{j=1}^{m} \oplus \left\langle (\mu_3^{(j)}, \pi_3^{(j)}, \nu_3^{(j)}) \right\rangle_{j=1}^{m}$
$= (\mathcal{S}_1 \cup \mathcal{S}_2) \oplus \mathcal{S}_3$
The rest of possibilities may be considered accordingly.
(3)
$(\mathcal{S}_1 \otimes \mathcal{S}_3) \cup (\mathcal{S}_2 \otimes \mathcal{S}_3) = \left(\left\langle (\mu_1^{(j)})(\mu_3^{(j)}), \left(1 - (\nu_3^{(j)})^2)(\pi_1^{(j)})^2 + (1 - (\nu_1^{(j)})^2)(\pi_3^{(j)})^2 - (\pi_1^{(j)})^2(\pi_3^{(j)})^2\right)^{1/2}, \left((\nu_1^{(j)})^2 + (\nu_3^{(j)})^2 - (\nu_1^{(j)})^2(\nu_3^{(j)})^2\right)^{1/2} \right\rangle_{j=1}^{m} \right) \cup \left(\left\langle (\mu_2^{(j)})(\mu_3^{(j)}), \left(1 - (\nu_3^{(j)})^2)(\pi_2^{(j)})^2 + (1 - (\nu_2^{(j)})^2)(\pi_3^{(j)})^2 - (\pi_2^{(j)})^2(\pi_3^{(j)})^2\right)^{1/2}, \left((\nu_2^{(j)})^2 + (\nu_3^{(j)})^2 - (\nu_2^{(j)})^2(\nu_3^{(j)})^2\right)^{1/2} \right\rangle_{j=1}^{m} \right)$
$= \left\langle \max\left\{(\mu_1^{(j)})(\mu_3^{(j)}), (\mu_2^{(j)})(\mu_3^{(j)})\right\}, \min\left\{\left(1 - (\nu_3^{(j)})^2)(\pi_1^{(j)})^2 + (1 - (\nu_1^{(j)})^2)(\pi_3^{(j)})^2 - (\pi_1^{(j)})^2(\pi_3^{(j)})^2\right)^{1/2}, \left(1 - (\nu_3^{(j)})^2)(\pi_2^{(j)})^2 + (1 - (\nu_2^{(j)})^2)(\pi_3^{(j)})^2 - (\pi_2^{(j)})^2(\pi_3^{(j)})^2\right)^{1/2}, \right\},$
$\min\left\{\left((\nu_1^{(j)})^2 + (\nu_3^{(j)})^2 - (\nu_1^{(j)})^2(\nu_3^{(j)})^2\right)^{1/2}, \left((\nu_2^{(j)})^2 + (\nu_3^{(j)})^2 - (\nu_2^{(j)})^2(\nu_3^{(j)})^2\right)^{1/2}\right\} \right\rangle_{j=1}^{m}$
$= \left\langle (\mu_2^{(j)})(\mu_3^{(j)}), \left(1 - (\nu_3^{(j)})^2)(\pi_1^{(j)})^2 + (1 - (\nu_1^{(j)})^2)(\pi_3^{(j)})^2 - (\pi_1^{(j)})^2(\pi_3^{(j)})^2\right)^{1/2}, \left((\nu_1^{(j)})^2 + (\nu_3^{(j)})^2 - (\nu_1^{(j)})^2(\nu_3^{(j)})^2\right)^{1/2} \right\rangle_{j=1}^{m}$
$= \left\langle \max\{\mu_1^{(j)}, \mu_2^{(j)}\}, \min\{\pi_1^{(j)}, \pi_2^{(j)}\}, \min\{\nu_1^{(j)}, \nu_2^{(j)}\} \right\rangle_{j=1}^{m} \otimes \left\langle (\mu_3^{(j)}, \pi_3^{(j)}, \nu_3^{(j)}) \right\rangle_{j=1}^{m}$
$= (\mathcal{S}_1 \cup \mathcal{S}_2) \otimes \mathcal{S}_3$
The rest of possibilities may be considered accordingly.

□

4. Correlation coefficient for *m*-PSFSs

The correlation coefficient is a statistical analysis of how strong an association exists between two variables' variations. The scope of values is -1.0 to 1.0. There was an error in the correlation calculation if the computed number was greater than 1.0 or less than -1.0. A negative correlation of -1.0 indicates a complete negative correlation, while a positive correlation of 1.0 indicates a complete positive correlation. A correlation of 0.0 means that the changes of the two variables have no linear relationship. In this section, we define correlation coefficient in *m*-PSFSs. We analyze their usefulness and address some of their characteristics.

Definition 4.1. Let $\mathcal{S} = \left\{ \left\langle \varsigma, (\mu^{(j)}(\varsigma), \pi^{(j)}(\varsigma), \nu^{(j)}(\varsigma)) \right\rangle_{j=1}^{m} : \varsigma \in X \right\}$ be a *m*-PSFS over a universal set X. Then the average or mean of $\mathcal{S}$ is defined as

$$\overline{\mathcal{S}} = \left\{ \left\langle (\overline{\mu^{(j)}}, \overline{\pi^{(j)}}, \overline{\nu^{(j)}}) \right\rangle_{j=1}^{m} \right\}$$

$$= \left\{ \left\langle \left(\frac{1}{|X|} \sum_{\varsigma} \mu^{(j)}(\varsigma), \frac{1}{|X|} \sum_{\varsigma} \pi^{(j)}(\varsigma), \frac{1}{|X|} \sum_{\varsigma} \nu^{(j)}(\varsigma) \right) \right\rangle_{j=1}^{m} \right\}$$

If we take X=$\{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5\}$ and 3PSFSs as

$$\mathcal{S} = \begin{pmatrix} \left(\mu^{(1)}(\varsigma_1), \pi^{(1)}(\varsigma_1), \nu^{(1)}(\varsigma_1)\right) & \left(\mu^{(2)}(\varsigma_1), \pi^{(2)}(\varsigma_1), \nu^{(2)}(\varsigma_1)\right) & \left(\mu^{(3)}(\varsigma_1), \pi^{(3)}(\varsigma_1), \nu^{(3)}(\varsigma_1)\right) \\ \left(\mu^{(1)}(\varsigma_2), \pi^{(1)}(\varsigma_2), \nu^{(1)}(\varsigma_2)\right) & \left(\mu^{(2)}(\varsigma_2), \pi^{(2)}(\varsigma_2), \nu^{(2)}(\varsigma_2)\right) & \left(\mu^{(3)}(\varsigma_2), \pi^{(3)}(\varsigma_2), \nu^{(3)}(\varsigma_2)\right) \\ \left(\mu^{(1)}(\varsigma_3), \pi^{(1)}(\varsigma_3), \nu^{(1)}(\varsigma_3)\right) & \left(\mu^{(2)}(\varsigma_3), \pi^{(2)}(\varsigma_3), \nu^{(2)}(\varsigma_3)\right) & \left(\mu^{(3)}(\varsigma_3), \pi^{(3)}(\varsigma_3), \nu^{(3)}(\varsigma_3)\right) \\ \left(\mu^{(1)}(\varsigma_4), \pi^{(1)}(\varsigma_4), \nu^{(1)}(\varsigma_4)\right) & \left(\mu^{(2)}(\varsigma_4), \pi^{(2)}(\varsigma_4), \nu^{(2)}(\varsigma_4)\right) & \left(\mu^{(3)}(\varsigma_4), \pi^{(3)}(\varsigma_4), \nu^{(3)}(\varsigma_4)\right) \\ \left(\mu^{(1)}(\varsigma_5), \pi^{(1)}(\varsigma_5), \nu^{(1)}(\varsigma_5)\right) & \left(\mu^{(2)}(\varsigma_5), \pi^{(2)}(\varsigma_5), \nu^{(2)}(\varsigma_5)\right) & \left(\mu^{(3)}(\varsigma_5), \pi^{(3)}(\varsigma_5), \nu^{(3)}(\varsigma_5)\right) \end{pmatrix}$$

then the average or mean of $\mathcal{S}$ is given by

$$\overline{\mathcal{S}} = \left(\left(\overline{\mu^{(1)}}, \overline{\pi^{(1)}}, \overline{\nu^{(1)}}\right), \left(\overline{\mu^{(2)}}, \overline{\pi^{(2)}}, \overline{\nu^{(2)}}\right), \left(\overline{\mu^{(3)}}, \overline{\pi^{(3)}}, \overline{\nu^{(3)}}\right)\right)$$

, where

$$\begin{array}{lll} \overline{\mu^{(1)}} = \frac{1}{5}\sum_{i=1}^{5}\mu^{(1)}(\varsigma_i), & \overline{\pi^{(1)}} = \frac{1}{5}\sum_{i=1}^{5}\pi^{(1)}(\varsigma_i), & \overline{\nu^{(1)}} = \frac{1}{5}\sum_{i=1}^{5}\nu^{(1)}(\varsigma_i) \\ \overline{\mu^{(2)}} = \frac{1}{5}\sum_{i=1}^{5}\mu^{(2)}(\varsigma_i), & \overline{\pi^{(2)}} = \frac{1}{5}\sum_{i=1}^{5}\pi^{(2)}(\varsigma_i), & \overline{\nu^{(2)}} = \frac{1}{5}\sum_{i=1}^{5}\nu^{(2)}(\varsigma_i) \\ \overline{\mu^{(3)}} = \frac{1}{5}\sum_{i=1}^{5}\mu^{(3)}(\varsigma_i), & \overline{\pi^{(3)}} = \frac{1}{5}\sum_{i=1}^{5}\pi^{(3)}(\varsigma_i), & \overline{\nu^{(3)}} = \frac{1}{5}\sum_{i=1}^{5}\nu^{(3)}(\varsigma_i) \end{array}$$

Definition 4.2. Let

$$\mathcal{S}_A = \begin{pmatrix} \left(\mu_A^{(1)}(\varsigma_1), \pi_A^{(1)}(\varsigma_1), \nu_A^{(1)}(\varsigma_1)\right) & \left(\mu_A^{(2)}(\varsigma_1), \pi_A^{(2)}(\varsigma_1), \nu_A^{(2)}(\varsigma_1)\right) & \cdots & \left(\mu_A^{(m)}(\varsigma_1), \pi_A^{(m)}(\varsigma_1), \nu_A^{(m)}(\varsigma_1)\right) \\ \left(\mu_A^{(1)}(\varsigma_2), \pi_A^{(1)}(\varsigma_2), \nu_A^{(1)}(\varsigma_2)\right) & \left(\mu_A^{(2)}(\varsigma_2), \pi_A^{(2)}(\varsigma_2), \nu_A^{(2)}(\varsigma_2)\right) & \cdots & \left(\mu_A^{(m)}(\varsigma_2), \pi_A^{(m)}(\varsigma_2), \nu_A^{(m)}(\varsigma_2)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\mu_A^{(1)}(\varsigma_k), \pi_A^{(1)}(\varsigma_k), \nu_A^{(1)}(\varsigma_k)\right) & \left(\mu_A^{(2)}(\varsigma_k), \pi_A^{(2)}(\varsigma_k), \nu_A^{(2)}(\varsigma_k)\right) & \cdots & \left(\mu_A^{(m)}(\varsigma_k), \pi_A^{(m)}(\varsigma_k), \nu_A^{(m)}(\varsigma_k)\right) \end{pmatrix}$$

and

$$\mathcal{S}_B = \begin{pmatrix} \left(\mu_B^{(1)}(\varsigma_1), \pi_B^{(1)}(\varsigma_1), \nu_B^{(1)}(\varsigma_1)\right) & \left(\mu_B^{(2)}(\varsigma_1), \pi_B^{(2)}(\varsigma_1), \nu_B^{(2)}(\varsigma_1)\right) & \cdots & \left(\mu_B^{(m)}(\varsigma_1), \pi_B^{(m)}(\varsigma_1), \nu_B^{(m)}(\varsigma_1)\right) \\ \left(\mu_B^{(1)}(\varsigma_2), \pi_B^{(1)}(\varsigma_2), \nu_B^{(1)}(\varsigma_2)\right) & \left(\mu_B^{(2)}(\varsigma_2), \pi_B^{(2)}(\varsigma_2), \nu_B^{(2)}(\varsigma_2)\right) & \cdots & \left(\mu_B^{(m)}(\varsigma_2), \pi_B^{(m)}(\varsigma_2), \nu_B^{(m)}(\varsigma_2)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\mu_B^{(1)}(\varsigma_k), \pi_B^{(1)}(\varsigma_k), \nu_B^{(1)}(\varsigma_k)\right) & \left(\mu_B^{(2)}(\varsigma_k), \pi_B^{(2)}(\varsigma_k), \nu_B^{(2)}(\varsigma_k)\right) & \cdots & \left(\mu_B^{(m)}(\varsigma_k), \pi_B^{(m)}(\varsigma_k), \nu_B^{(m)}(\varsigma_k)\right) \end{pmatrix}$$

be m-PSFSs over X. Then covariance of $\mathcal{S}_A$ and $\mathcal{S}_B$ can be defined as

$$I_{(\mathcal{S}_A \to \mathcal{S}_B)} = \frac{1}{3m}\sum_{j=1}^{m}\left(I^{(j)}_{\mu(\mathcal{S}_A \to \mathcal{S}_B)} + I^{(j)}_{\pi(\mathcal{S}_A \to \mathcal{S}_B)} + I^{(j)}_{\nu(\mathcal{S}_A \to \mathcal{S}_B)}\right)$$

Where,

$$I^{(j)}_{\mu(\mathcal{S}_A \to \mathcal{S}_B)} = \sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i) - \overline{\mu_A^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i) - \overline{\mu_B^{(j)}}\right),$$

$$I^{(j)}_{\pi(\mathcal{S}_A\to\mathcal{S}_B)} = \sum_{i=1}^{k}\left(\pi^{(j)}_A(\varsigma_i) - \overline{\pi^{(j)}_A}\right)\left(\pi^{(j)}_B(\varsigma_i) - \overline{\pi^{(j)}_B}\right),$$

$$I^{(j)}_{\nu(\mathcal{S}_A\to\mathcal{S}_B)} = \sum_{i=1}^{k}\left(\nu^{(j)}_A(\varsigma_i) - \overline{\nu^{(j)}_A}\right)\left(\nu^{(j)}_B(\varsigma_i) - \overline{\nu^{(j)}_B}\right)$$

for j=1,2,3,...,m.

Definition 4.3. Let $\mathcal{S}_A$ be a m-PSFS on X as mentioned in definition 4.2. Then Variance of $\mathcal{S}_A$ can be defined as

$$\mathcal{V}(\mathcal{S}_A) = \frac{1}{3m}\sum_{j=1}^{m}\left(\mathcal{V}^{(j)}_\mu(\mathcal{S}_A) + \mathcal{V}^{(j)}_\pi(\mathcal{S}_A) + \mathcal{V}^{(j)}_\nu(\mathcal{S}_A)\right)$$

Where,

$$\mathcal{V}^{(j)}_\mu(\mathcal{S}_A) = \sum_{i=1}^{k}\left(\mu^{(j)}_A(\varsigma_i) - \overline{\mu^{(j)}_A}\right)^2,$$

$$\mathcal{V}^{(j)}_\pi(\mathcal{S}_A) = \sum_{i=1}^{k}\left(\pi^{(j)}_A(\varsigma_i) - \overline{\pi^{(j)}_A}\right)^2,$$

$$\mathcal{V}^{(j)}_\nu(\mathcal{S}_A) = \sum_{i=1}^{k}\left(\nu^{(j)}_A(\varsigma_i) - \overline{\nu^{(j)}_A}\right)^2$$

for j=1,2,3,...,m.

Proposition 4.4. The covariance and variance for m-PSFSs of $\mathcal{S}_A$ and $\mathcal{S}_B$ follow the following properties.

(a) $I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_A)} = \mathcal{V}^{(j)}_\mu(\mathcal{S}_A)$, $I^{(j)}_{\pi(\mathcal{S}_A\to\mathcal{S}_A)} = \mathcal{V}^{(j)}_\pi(\mathcal{S}_A)$, $I^{(j)}_{\nu(\mathcal{S}_A\to\mathcal{S}_A)} = \mathcal{V}^{(j)}_\nu(\mathcal{S}_A)$

(b) $I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)} = I^{(j)}_{\mu(\mathcal{S}_B\to\mathcal{S}_A)}$, $I^{(j)}_{\pi(\mathcal{S}_A\to\mathcal{S}_B)} = I^{(j)}_{\pi(\mathcal{S}_B\to\mathcal{S}_A)}$, $I^{(j)}_{\nu(\mathcal{S}_A\to\mathcal{S}_B)} = I^{(j)}_{\nu(\mathcal{S}_B\to\mathcal{S}_A)}$

(c) $\mid I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)} \mid\leq \sqrt{\mathcal{V}^{(j)}_\mu(\mathcal{S}_A)\mathcal{V}^{(j)}_\mu(\mathcal{S}_B)}$, $\mid I^{(j)}_{\pi(\mathcal{S}_A\to\mathcal{S}_B)} \mid\leq \sqrt{\mathcal{V}^{(j)}_\pi(\mathcal{S}_A)\mathcal{V}^{(j)}_\pi(\mathcal{S}_B)}$,
$\mid I^{(j)}_{\nu(\mathcal{S}_A\to\mathcal{S}_B)} \mid\leq \sqrt{\mathcal{V}^{(j)}_\nu(\mathcal{S}_A)\mathcal{V}^{(j)}_\nu(\mathcal{S}_B)}$.

Proof. (a) and (b) are simple to prove.
(c) We will use Cauchy-Schwarz inequality to prove our result. Cauchy-Schwarz inequality states that

$$(\sum_{l=1}^{n} u_l v_l)^2 \leq (\sum_{l=1}^{n} u_l^2)(\sum_{l=1}^{n} v_l^2)$$

for $u_l, v_l \in R$ Consider

$$\begin{aligned}\left(I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)}\right)^2 &= \left(\sum_{i=1}^{k}(\mu^{(j)}_A(\varsigma_i) - \overline{\mu^{(j)}_A})(\mu^{(j)}_B(\varsigma_i) - \overline{\mu^{(j)}_B})\right)^2\\ &\leq \sum_{i=1}^{k}(\mu^{(j)}_A(\varsigma_i) - \overline{\mu^{(j)}_A})^2\sum_{i=1}^{k}(\mu^{(j)}_B(\varsigma_i) - \overline{\mu^{(j)}_B})^2\end{aligned}$$

$$= \mathcal{V}_{\mu}^{(j)}(\mathcal{S}_A)\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_B)$$

$$\Rightarrow \mid I_{\mu(\mathcal{S}_A\to\mathcal{S}_B)}^{(j)} \mid \leq \sqrt{\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_A)\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_B)}$$

The other two are simple to prove. □

Definition 4.5. Let $\mathcal{S}_A$ and $\mathcal{S}_B$ be m-PSFSs over X as mentioned in Definition 4.2. Then correlation coefficient of $\mathcal{S}_A$ and $\mathcal{S}_B$ can be defined by

$$\rho(\mathcal{S}_A,\mathcal{S}_B) = \frac{1}{3m}\sum_{j=1}^{m}\left(\rho_{\mu}^{(j)}(\mathcal{S}_A,\mathcal{S}_B) + \rho_{\pi}^{(j)}(\mathcal{S}_A,\mathcal{S}_B) + \rho_{\nu}^{(j)}(\mathcal{S}_A,\mathcal{S}_B)\right) \tag{4.1}$$

Where,

$$\rho_{\mu}^{(j)}(\mathcal{S}_A,\mathcal{S}_B) = \frac{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i) - \overline{\mu_A^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i) - \overline{\mu_B^{(j)}}\right)}{\sqrt{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i) - \overline{\mu_A^{(j)}}\right)^2 \sum_{i=1}^{k}\left(\mu_B^{(j)}(\varsigma_i) - \overline{\mu_B^{(j)}}\right)^2}},$$

$$\rho_{\pi}^{(j)}(\mathcal{S}_A,\mathcal{S}_B) = \frac{\sum_{i=1}^{k}\left(\pi_A^{(j)}(\varsigma_i) - \overline{\pi_A^{(j)}}\right)\left(\pi_B^{(j)}(\varsigma_i) - \overline{\pi_B^{(j)}}\right)}{\sqrt{\sum_{i=1}^{k}\left(\pi_A^{(j)}(\varsigma_i) - \overline{\pi_A^{(j)}}\right)^2 \sum_{i=1}^{k}\left(\pi_B^{(j)}(\varsigma_i) - \overline{\pi_B^{(j)}}\right)^2}},$$

$$\rho_{\nu}^{(j)}(\mathcal{S}_A,\mathcal{S}_B) = \frac{\sum_{i=1}^{k}\left(\nu_A^{(j)}(\varsigma_i) - \overline{\nu_A^{(j)}}\right)\left(\nu_B^{(j)}(\varsigma_i) - \overline{\nu_B^{(j)}}\right)}{\sqrt{\sum_{i=1}^{k}\left(\nu_A^{(j)}(\varsigma_i) - \overline{\nu_A^{(j)}}\right)^2 \sum_{i=1}^{k}\left(\nu_B^{(j)}(\varsigma_i) - \overline{\nu_B^{(j)}}\right)^2}}$$

for j=1,2,3,...,m.
It is very critical to note that if, for some j(= $1,2,3,...,m$), $\mu_A^{(j)}(\varsigma_i)$ = constant for all i=1,2,3,...,n, then $\overline{\mu_A^{(j)}}$ = constant and hence $I_{\mu(\mathcal{S}_A=0}^{(j)}$ and $\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_A) = 0$ and hence $I_{\mu(\mathcal{S}_A\to\mathcal{S}_B)}^{(j)} = 0$ and $\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_A) = 0$. Therefore, the correlation $\rho_{\mu}^{(j)}(\mathcal{S}_A,\mathcal{S}_B)$ can not be defined. Similarly if $\pi_A^{(j)}(\varsigma_i)$ = constant for all i=1,2,3,...,n, then $\overline{\pi_A^{(j)}}$ = constant and hence $I_{\pi(\mathcal{S}_A=0}^{(j)}$ and $\mathcal{V}_{\pi}^{(j)}(\mathcal{S}_A) = 0$ and hence $I_{\pi(\mathcal{S}_A\to\mathcal{S}_B)}^{(j)} = 0$ and $\mathcal{V}_{\pi}^{(j)}(\mathcal{S}_A) = 0$. Therefore, the correlation $\rho_{\pi}^{(j)}(\mathcal{S}_A,\mathcal{S}_B)$ can not be defined. And similarly if $\nu_A^{(j)}(\varsigma_i)$ = constant for all i=1,2,3,...,n, then $\overline{\nu_A^{(j)}}$ = constant and hence $I_{\nu(\mathcal{S}_A=0}^{(j)}$ and $\mathcal{V}_{\nu}^{(j)}(\mathcal{S}_A) = 0$ and hence $I_{\nu(\mathcal{S}_A\to\mathcal{S}_B)}^{(j)} = 0$ and $\mathcal{V}_{\nu}^{(j)}(\mathcal{S}_A) = 0$. Therefore, the correlation $\rho_{\nu}^{(j)}(\mathcal{S}_A,\mathcal{S}_B)$ can not be defined. In either case, $\rho(\mathcal{S}_A,\mathcal{S}_B)$ is meaningless. Therefore, our m-PSF information should be diverse in nature.

Example 4.6. Let Table 2

Table 2. 3-polar spherical fuzzy set.

$\mathcal{S}_A$	3-PSFS
ς_1	$\Big(0.28, 0.49, 0.55\Big), \Big(0.32, 0.42, 0.20\Big), \Big(0.45, 0.37, 0.06\Big)$
ς_2	$\Big(0.55, 0.32, 0.42\Big), \Big(0.52, 0.33, 0.22\Big), \Big(0.49, 0.55, 0.09\Big)$
ς_3	$\Big(0.42, 0.38, 0.49\Big), \Big(0.34, 0.56, 0.48\Big), \Big(0.36, 0.49, 0.23\Big)$

and Table 3

Table 3. 3-polar spherical fuzzy set.

$\mathcal{S}_B$	3-PSFS
ς_1	$\Big(0.20, 0.60, 0.25\Big), \Big(0.34, 0.41, 0.17\Big), \Big(0.52, 0.41, 0.32\Big)$
ς_2	$\Big(0.55, 0.72, 0.08\Big), \Big(0.39, 0.45, 0.16\Big), \Big(0.32, 0.15, 0.48\Big)$
ς_3	$\Big(0.37, 0.25, 0.42\Big), \Big(0.54, 0.32, 0.19\Big), \Big(0.49, 0.18, 0.31\Big)$

be 3-PSFSs defined over X. Then $\overline{\mathcal{S}_A} = \Big((0.42, 0.38, 0.42), (0.39, 0.44, 0.30), (0.43, 0.47, 0.13)\Big)$,

$$\overline{\mathcal{S}_B} = \Big((0.37, 0.52, 0.22), (0.42, 0.39, 0.17), (0.44, 0.25, 0.37)\Big),$$

$$\mathcal{V}(\mathcal{S}_A) = 0.02, \mathcal{V}(\mathcal{S}_B) = 0.04, I_{(\mathcal{S}_A \to \mathcal{S}_B)} = 0.003$$

Hence,

$$\rho(\mathcal{S}_A, \mathcal{S}_B) = 0.11$$

Theorem 4.7. For the m-PSFSs $\mathcal{S}_A$ and $\mathcal{S}_B$, we have
(a) The correlation ρ is symmetrical, that is, $\rho(\mathcal{S}_A, \mathcal{S}_B) = \rho(\mathcal{S}_B, \mathcal{S}_A)$
(b) The correlation ρ lies between -1 and 1, that is, $-1 \leq \rho(\mathcal{S}_A, \mathcal{S}_B) \leq 1$
(c) $\rho(\mathcal{S}_A, \mathcal{S}_B) = 1$ if $A = B$
(d) $\rho(\mathcal{S}_A, \mathcal{S}_B) = 1$ if $A = (\alpha)B, \alpha > 0$.

Proof. (a) It is easy to prove.
(b) From the proposition 4.4, we have $| I^{(j)}_{\mu(\mathcal{S}_A \to \mathcal{S}_B)} | \leq \sqrt{\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A)\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_B)}$

$$\Rightarrow -\sqrt{\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A)\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_B)} \leq I^{(j)}_{\mu(\mathcal{S}_A \to \mathcal{S}_B)} \leq \sqrt{\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A)\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_B)}$$

$$\Rightarrow -1 \leq \frac{I^{(j)}_{\mu(\mathcal{S}_A \to \mathcal{S}_B)}}{\sqrt{\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A)\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_B)}} \leq 1$$

$$\Rightarrow -1 \leq \rho_{\mu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) \leq 1$$

Similarly, $-1 \leq \rho_{\pi}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) \leq 1$ and $-1 \leq \rho_{\nu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) \leq 1$ and hence $-1 \leq \rho(\mathcal{S}_A, \mathcal{S}_B) \leq 1$.
(c) If A=B then $\mu_A^{(j)}(\varsigma_i) = \mu_B^{(j)}(\varsigma_i)$, $\pi_A^{(j)}(\varsigma_i) = \pi_B^{(j)}(\varsigma_i)$ and $\nu_A^{(j)}(\varsigma_i) = \nu_B^{(j)}(\varsigma_i)$, then

$$\rho_{\mu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = \frac{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)}{\sqrt{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)^2 \sum_{i=1}^{k}\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)^2}} = \frac{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)}{\sqrt{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)^2 \sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)^2}} = 1$$

Similarly $\rho_{\pi}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = 1$ and $\rho_{\nu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = 1$ and hence $\rho(\mathcal{S}_A, \mathcal{S}_B) = 1$.

(d) If A=(α)B then $\mu_A^{(j)}(\varsigma_i) = (\alpha)\mu_B^{(j)}(\varsigma_i)$, $\pi_A^{(j)}(\varsigma_i) = (\alpha)\pi_B^{(j)}(\varsigma_i)$ and $\nu_A^{(j)}(\varsigma_i) = (\alpha)\nu_B^{(j)}(\varsigma_i)$, then

$$\rho_{\mu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = \frac{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)}{\sqrt{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)^2 \sum_{i=1}^{k}\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)^2}}$$

$$= \frac{\sum_{i=1}^{k}\left((\alpha)\mu_B^{(j)}(\varsigma_i)-(\alpha)\overline{\mu_B^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)}{\sqrt{\sum_{i=1}^{k}\left((\alpha)\mu_B^{(j)}(\varsigma_i)-(\alpha)\overline{\mu_B^{(j)}}\right)^2 \sum_{i=1}^{k}\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)^2}} = 1$$

Similarly $\rho_{\pi}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = 1$ and $\rho_{\nu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = 1$ and hence $\rho(\mathcal{S}_A, \mathcal{S}_B) = 1$. □

Definition 4.8. Let $\mathcal{S}_A$ and $\mathcal{S}_B$ be m-PSFSs over X as mentioned in definition 4.2. Then correlation coefficient of $\mathcal{S}_A$ and $\mathcal{S}_B$ can also be defined as

$$\widetilde{\varrho}(\mathcal{S}_A, \mathcal{S}_B) = \frac{1}{3m}\sum_{j=1}^{m}\left(\widetilde{\varrho}_{\mu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) + \widetilde{\varrho}_{\pi}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) + \widetilde{\varrho}_{\nu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B)\right) \tag{4.2}$$

Where,

$$\widetilde{\varrho}_{\mu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = \frac{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)}{\max\left\{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)^2, \sum_{i=1}^{k}\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)^2\right\}},$$

$$\widetilde{\varrho}_{\pi}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = \frac{\sum_{i=1}^{k}\left(\pi_A^{(j)}(\varsigma_i)-\overline{\pi_A^{(j)}}\right)\left(\pi_B^{(j)}(\varsigma_i)-\overline{\pi_B^{(j)}}\right)}{\max\left\{\sum_{i=1}^{k}\left(\pi_A^{(j)}(\varsigma_i)-\overline{\pi_A^{(j)}}\right)^2, \sum_{i=1}^{k}\left(\pi_B^{(j)}(\varsigma_i)-\overline{\pi_B^{(j)}}\right)^2\right\}},$$

and

$$\widetilde{\varrho}_{\nu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = \frac{\sum_{i=1}^{k}\left(\nu_A^{(j)}(\varsigma_i)-\overline{\nu_A^{(j)}}\right)\left(\nu_B^{(j)}(\varsigma_i)-\overline{\nu_B^{(j)}}\right)}{\max\left\{\sum_{i=1}^{k}\left(\nu_A^{(j)}(\varsigma_i)-\overline{\nu_A^{(j)}}\right)^2, \sum_{i=1}^{k}\left(\nu_B^{(j)}(\varsigma_i)-\overline{\nu_B^{(j)}}\right)^2\right\}}.$$

Theorem 4.9. *For the m-PSFSs $\mathcal{S}_A$ and $\mathcal{S}_B$, we have*
(a) The correlation $\widetilde{\varrho}$ is symmetrical, that is, $\widetilde{\varrho}(\mathcal{S}_A, \mathcal{S}_B) = \widetilde{\varrho}(\mathcal{S}_B, \mathcal{S}_A)$
(b) The correlation $\widetilde{\varrho}$ lies between -1 and 1, that is, $-1 \leq \widetilde{\varrho}(\mathcal{S}_A, \mathcal{S}_B) \leq 1$
(c) $\widetilde{\varrho}(\mathcal{S}_A, \mathcal{S}_B) = 1$ if $A = B$
(d) $\widetilde{\varrho}(\mathcal{S}_A, \mathcal{S}_B) = 1$ if $A = (\alpha)B$, $\alpha > 0$.

Proof. (a) It is easy to prove.
(b) From the proposition 4.4, we have $| I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)} | \leq \sqrt{\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A)\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_B)}$

$$\Rightarrow -\sqrt{\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A)\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_B)} \leq I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)} \leq \sqrt{\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A)\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_B)}$$

$$\Rightarrow -1 \leq \frac{I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)}}{\sqrt{\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A)\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_B)}} \leq 1$$

$$\Rightarrow -1 \leq \widetilde{\varrho}^{(j)}_{\mu}(\mathcal{S}_A,\mathcal{S}_B) \leq 1$$

Similarly, $-1 \leq \widetilde{\varrho}^{(j)}_{\pi}(\mathcal{S}_A,\mathcal{S}_B) \leq 1$ and $-1 \leq \widetilde{\varrho}^{(j)}_{\nu}(\mathcal{S}_A,\mathcal{S}_B) \leq 1$ and hence $-1 \leq \widetilde{\varrho}(\mathcal{S}_A,\mathcal{S}_B) \leq 1$.
(c) If A=B then $\mu^{(j)}_A(\varsigma_i) = \mu^{(j)}_B(\varsigma_i)$, $\pi^{(j)}_A(\varsigma_i) = \pi^{(j)}_B(\varsigma_i)$ and $\nu^{(j)}_A(\varsigma_i) = \nu^{(j)}_B(\varsigma_i)$, then

$$\widetilde{\varrho}^{(j)}_{\mu}(\mathcal{S}_A,\mathcal{S}_B) = \frac{\sum_{i=1}^{k}\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)}{\max\left\{\sum_{i=1}^{k}\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)^2,\sum_{i=1}^{k}\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)^2\right\}} = \frac{\sum_{i=1}^{k}\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)}{\max\left\{\sum_{i=1}^{k}\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)^2,\sum_{i=1}^{k}\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)^2\right\}} = 1$$ Similarly

$\widetilde{\varrho}^{(j)}_{\pi}(\mathcal{S}_A,\mathcal{S}_B) = 1$ and $\widetilde{\varrho}^{(j)}_{\nu}(\mathcal{S}_A,\mathcal{S}_B) = 1$ and hence $\widetilde{\varrho}(\mathcal{S}_A,\mathcal{S}_B) = 1$.
(d) If A=(α)B then $\mu^{(j)}_A(\varsigma_i) = (\alpha)\mu^{(j)}_B(\varsigma_i)$, $\pi^{(j)}_A(\varsigma_i) = (\alpha)\pi^{(j)}_B(\varsigma_i)$ and $\nu^{(j)}_A(\varsigma_i) = (\alpha)\nu^{(j)}_B(\varsigma_i)$, then

$$\widetilde{\varrho}^{(j)}_{\mu}(\mathcal{S}_A,\mathcal{S}_B) = \frac{\sum_{i=1}^{k}\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)}{\max\left\{\sum_{i=1}^{k}\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)^2,\sum_{i=1}^{k}\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)^2\right\}}$$

$$= \frac{\sum_{i=1}^{k}\left((\alpha)\mu^{(j)}_B(\varsigma_i)-(\alpha)\overline{\mu^{(j)}_B}\right)\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)}{\max\left\{\sum_{i=1}^{k}\left((\alpha)\mu^{(j)}_B(\varsigma_i)-(\alpha)\overline{\mu^{(j)}_B}\right)^2,\sum_{i=1}^{k}\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)^2\right\}} = 1$$

Similarly $\widetilde{\varrho}^{(j)}_{\pi}(\mathcal{S}_A,\mathcal{S}_B) = 1$ and $\widetilde{\varrho}^{(j)}_{\nu}(\mathcal{S}_A,\mathcal{S}_B) = 1$ and hence $\widetilde{\varrho}(\mathcal{S}_A,\mathcal{S}_B) = 1$. □

Two main aspects of most real-world issues should be addressed: complexity and weights. Uncertainty is a major factor that can influence our decisions and analyses when dealing with such issues. Furthermore, weights have an effect on the majority of unknown mathematical models. We describe weighted Correlation coefficients (WCCs) to deal with such issues.

Definition 4.10. Let $\mathcal{S}_A$ and $\mathcal{S}_B$ be m-PSFSs defined over X as mentioned in definition 4.2. The alternatives are effected by $\varsigma_1,\varsigma_2,\cdots,\varsigma_n$ are affected by the weights $\omega_1,\omega_2,\cdots,\omega_n$ with the condition $\sum_{i=1}^{n}\omega_i = 1$ and $0 \leq \omega_i \leq 1$. Then weighted covariance of $\mathcal{S}_A$ and $\mathcal{S}_B$ can be defined as

$$^{\omega}I_{(\mathcal{S}_A\to\mathcal{S}_B)} = \frac{1}{3m}\sum_{j=1}^{m}\left({}^{\omega}I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)} + {}^{\omega}I^{(j)}_{\pi(\mathcal{S}_A\to\mathcal{S}_B)} + {}^{\omega}I^{(j)}_{\nu(\mathcal{S}_A\to\mathcal{S}_B)}\right) \tag{4.3}$$

Where,

$$^{\omega}I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)} = \sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right),$$

$$^{\omega}I^{(j)}_{\pi(\mathcal{S}_A\to\mathcal{S}_B)} = \sum_{i=1}^{k}\omega_i\left(\pi^{(j)}_A(\varsigma_i)-\overline{\pi^{(j)}_A}\right)\left(\pi^{(j)}_B(\varsigma_i)-\overline{\pi^{(j)}_B}\right),$$

$$^{\omega}I^{(j)}_{\nu(\mathcal{S}_A\to\mathcal{S}_B)} = \sum_{i=1}^{k}\omega_i\Big(\nu^{(j)}_A(\varsigma_i)-\overline{\nu^{(j)}_A}\Big)\Big(\nu^{(j)}_B(\varsigma_i)-\overline{\nu^{(j)}_B}\Big)$$

for j=1,2,3,...,m.

Definition 4.11. Let $\mathcal{S}_A$ be a m-PSFS on X as mentioned in definition 4.2. Then weighted variance of $\mathcal{S}_A$ can be defined as

$$^{\omega}\mathcal{V}(\mathcal{S}_A) = \frac{1}{3m}\sum_{j=1}^{m}\Big(\ ^{\omega}\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A) +\ ^{\omega}\mathcal{V}^{(j)}_{\pi}(\mathcal{S}_A) +\ ^{\omega}\mathcal{V}^{(j)}_{\nu}(\mathcal{S}_A)\Big)$$

Where,

$$^{\omega}\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A) = \sum_{i=1}^{k}\omega_i\Big(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\Big)^2,$$

$$^{\omega}\mathcal{V}^{(j)}_{\pi}(\mathcal{S}_A) = \sum_{i=1}^{k}\omega_i\Big(\pi^{(j)}_A(\varsigma_i)-\overline{\pi^{(j)}_A}\Big)^2,$$

$$^{\omega}\mathcal{V}^{(j)}_{\nu}(\mathcal{S}_A) = \sum_{i=1}^{k}\omega_i\Big(\nu^{(j)}_A(\varsigma_i)-\overline{\nu^{(j)}_A}\Big)^2$$

for j=1,2,3,...,m.

Proposition 4.12. *The covariance and variance for m-PSFSs of $\mathcal{S}_A$ and $\mathcal{S}_B$ follow the following properties.*

(a) $\ ^{\omega}I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_A)} =\ ^{\omega}\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A),\ \ ^{\omega}I^{(j)}_{\pi(\mathcal{S}_A\to\mathcal{S}_A)} =\ ^{\omega}\mathcal{V}^{(j)}_{\pi}(\mathcal{S}_A),\ \ ^{\omega}I^{(j)}_{\nu(\mathcal{S}_A\to\mathcal{S}_A)} =\ ^{\omega}\mathcal{V}^{(j)}_{\nu}(\mathcal{S}_A)$

(b) $\ ^{\omega}I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)} =\ ^{\omega}I^{(j)}_{\mu(\mathcal{S}_B\to\mathcal{S}_A)},\ \ ^{\omega}I^{(j)}_{\pi(\mathcal{S}_A\to\mathcal{S}_B)} =\ ^{\omega}I^{(j)}_{\pi(\mathcal{S}_B\to\mathcal{S}_A)},\ \ ^{\omega}I^{(j)}_{\nu(\mathcal{S}_A\to\mathcal{S}_B)} =\ ^{\omega}I^{(j)}_{\nu(\mathcal{S}_B\to\mathcal{S}_A)}$

(c) $|\ ^{\omega}I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)}| \leq \sqrt{^{\omega}\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A)\ ^{\omega}\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_B)}$, $|\ ^{\omega}I^{(j)}_{\pi(\mathcal{S}_A\to\mathcal{S}_B)}| \leq \sqrt{^{\omega}\mathcal{V}^{(j)}_{\pi}(\mathcal{S}_A)\ ^{\omega}\mathcal{V}^{(j)}_{\pi}(\mathcal{S}_B)}$,
$|\ ^{\omega}I^{(j)}_{\nu(\ ^{\omega}\mathcal{S}_A\to\mathcal{S}_B)}| \leq \sqrt{^{\omega}\mathcal{V}^{(j)}_{\nu}(\mathcal{S}_A)\ ^{\omega}\mathcal{V}^{(j)}_{\nu}(\mathcal{S}_B)}$.

Proof. (a) and (b) are simple to prove.
(c) We will use Cauchy-Schwarz inequality to prove our result. Cauchy-Schwarz inequality states that

$$(\sum_{l=1}^{n}u_l v_l)^2 \leq (\sum_{l=1}^{n}u_l^2)(\sum_{l=1}^{n}v_l^2)$$

for $u_l, v_l \in R$ Consider

$$\begin{aligned}\Big(\ ^{\omega}I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)}\Big)^2 &= \Big(\sum_{i=1}^{k}\omega_i(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A})(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B})\Big)^2\\ &= \Big(\sum_{i=1}^{k}\sqrt{\omega_i}(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A})\sqrt{\omega_i}(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B})\Big)^2\\ &\leq \sum_{i=1}^{k}\omega_i(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A})^2\sum_{i=1}^{k}\omega_i(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B})^2\end{aligned}$$

$$= \ {}^{\omega}\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_A)\ {}^{\omega}\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_B)$$

$$\Rightarrow \left| \ {}^{\omega}I_{\mu(\mathcal{S}_A \to \mathcal{S}_B)}^{(j)} \right| \leq \sqrt{{}^{\omega}\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_A)\ {}^{\omega}\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_B)}$$

The other two are simple to prove. □

Definition 4.13. Let $\mathcal{S}_A$ and $\mathcal{S}_B$ be m-PSFSs over X as mentioned in definition 4.2. Then weighted correlation coefficient of $\mathcal{S}_A$ and $\mathcal{S}_B$ can be defined by

$$ {}^{\omega}\rho(\mathcal{S}_A, \mathcal{S}_B) = \frac{1}{3m} \sum_{j=1}^{m} \left({}^{\omega}\rho_{\mu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) + {}^{\omega}\rho_{\pi}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) + {}^{\omega}\rho_{\nu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) \right) \tag{4.4}$$

Where,

$$ {}^{\omega}\rho_{\mu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = \frac{\sum_{i=1}^{k} \omega_i \left(\mu_A^{(j)}(\varsigma_i) - \overline{\mu_A^{(j)}} \right) \left(\mu_B^{(j)}(\varsigma_i) - \overline{\mu_B^{(j)}} \right)}{\sqrt{\sum_{i=1}^{k} \omega_i \left(\mu_A^{(j)}(\varsigma_i) - \overline{\mu_A^{(j)}} \right)^2 \sum_{i=1}^{k} \omega_i \left(\mu_B^{(j)}(\varsigma_i) - \overline{\mu_B^{(j)}} \right)^2}},$$

$$ {}^{\omega}\rho_{\pi}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = \frac{\sum_{i=1}^{k} \omega_i \left(\pi_A^{(j)}(\varsigma_i) - \overline{\pi_A^{(j)}} \right) \left(\pi_B^{(j)}(\varsigma_i) - \overline{\pi_B^{(j)}} \right)}{\sqrt{\sum_{i=1}^{k} \omega_i \left(\pi_A^{(j)}(\varsigma_i) - \overline{\pi_A^{(j)}} \right)^2 \sum_{i=1}^{k} \omega_i \left(\pi_B^{(j)}(\varsigma_i) - \overline{\pi_B^{(j)}} \right)^2}},$$

$$ {}^{\omega}\rho_{\nu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = \frac{\sum_{i=1}^{k} \omega_i \left(\nu_A^{(j)}(\varsigma_i) - \overline{\nu_A^{(j)}} \right) \left(\nu_B^{(j)}(\varsigma_i) - \overline{\nu_B^{(j)}} \right)}{\sqrt{\sum_{i=1}^{k} \omega_i \left(\nu_A^{(j)}(\varsigma_i) - \overline{\nu_A^{(j)}} \right)^2 \sum_{i=1}^{k} \omega_i \left(\nu_B^{(j)}(\varsigma_i) - \overline{\nu_B^{(j)}} \right)^2}}$$

for j=1,2,3,...,m.

Theorem 4.14. For the m-PSFSs $\mathcal{S}_A$ and $\mathcal{S}_B$, we have
(a) The weighted correlation ${}^{\omega}\rho$ is symmetrical, that is, ${}^{\omega}\rho(\mathcal{S}_A, \mathcal{S}_B) = {}^{\omega}\rho(\mathcal{S}_B, \mathcal{S}_A)$
(b) The weighted correlation ${}^{\omega}\rho$ lies between -1 and 1, that is, $-1 \leq {}^{\omega}\rho(\mathcal{S}_A, \mathcal{S}_B) \leq 1$
(c) ${}^{\omega}\rho(\mathcal{S}_A, \mathcal{S}_B) = 1$ if $A = B$
(d) ${}^{\omega}\rho(\mathcal{S}_A, \mathcal{S}_B) = 1$ if $A = (\alpha)B, \alpha > 0$.

Proof. (a) It is easy to prove.
(b) From the proposition 4.12, we have $\left| \ {}^{\omega}I_{\mu(\mathcal{S}_A \to \mathcal{S}_B)}^{(j)} \right| \leq \sqrt{{}^{\omega}\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_A)\ {}^{\omega}\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_B)}$

$$\Rightarrow -\sqrt{{}^{\omega}\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_A)\ {}^{\omega}\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_B)} \leq {}^{\omega}I_{\mu(\mathcal{S}_A \to \mathcal{S}_B)}^{(j)} \leq \sqrt{{}^{\omega}\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_A)\ {}^{\omega}\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_B)}$$

$$\Rightarrow -1 \leq \frac{{}^{\omega}I_{\mu(\mathcal{S}_A \to \mathcal{S}_B)}^{(j)}}{\sqrt{{}^{\omega}\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_A)\ {}^{\omega}\mathcal{V}_{\mu}^{(j)}(\mathcal{S}_B)}} \leq 1$$

$$\Rightarrow -1 \leq {}^{\omega}\rho_{\mu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) \leq 1$$

Similarly, $-1 \leq {}^{\omega}\rho_{\pi}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) \leq 1$ and $-1 \leq {}^{\omega}\rho_{\nu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) \leq 1$ and hence $-1 \leq {}^{\omega}\rho(\mathcal{S}_A, \mathcal{S}_B) \leq 1$.
(c) If A=B then $\mu_A^{(j)}(\varsigma_i) = \mu_B^{(j)}(\varsigma_i)$, $\pi_A^{(j)}(\varsigma_i) = \pi_B^{(j)}(\varsigma_i)$ and $\nu_A^{(j)}(\varsigma_i) = \nu_B^{(j)}(\varsigma_i)$, then

$$ {}^{\omega}\rho_{\mu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = \frac{\sum_{i=1}^k \omega_i\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)}{\sqrt{\sum_{i=1}^k \omega_i\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)^2 \sum_{i=1}^k \omega_i\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)^2}} = \frac{\sum_{i=1}^k \omega_i\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)}{\sqrt{\sum_{i=1}^k \omega_i\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)^2 \sum_{i=1}^k \omega_i\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)^2}} = 1 $$

Similarly ${}^{\omega}\rho_{\pi}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = 1$ and ${}^{\omega}\rho_{\nu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = 1$ and hence ${}^{\omega}\rho(\mathcal{S}_A, \mathcal{S}_B) = 1$.

(d) If A=(α)B then $\mu_A^{(j)}(\varsigma_i) = (\alpha)\mu_B^{(j)}(\varsigma_i)$, $\pi_A^{(j)}(\varsigma_i) = (\alpha)\pi_B^{(j)}(\varsigma_i)$ and $\nu_A^{(j)}(\varsigma_i) = (\alpha)\nu_B^{(j)}(\varsigma_i)$, then

$$ \begin{aligned} {}^{\omega}\rho_{\mu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) &= \frac{\sum_{i=1}^k \omega_i\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)}{\sqrt{\sum_{i=1}^k \omega_i\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)^2 \sum_{i=1}^k \omega_i\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)^2}} \\ &= \frac{\sum_{i=1}^k \omega_i\left((\alpha)\mu_B^{(j)}(\varsigma_i)-(\alpha)\overline{\mu_B^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)}{\sqrt{\sum_{i=1}^k \omega_i\left((\alpha)\mu_B^{(j)}(\varsigma_i)-(\alpha)\overline{\mu_B^{(j)}}\right)^2 \sum_{i=1}^k \omega_i\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)^2}} = 1 \end{aligned} $$

Similarly ${}^{\omega}\rho_{\pi}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = 1$ and ${}^{\omega}\rho_{\nu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = 1$ and hence ${}^{\omega}\rho(\mathcal{S}_A, \mathcal{S}_B) = 1$. □

Definition 4.15. Let $\mathcal{S}_A$ and $\mathcal{S}_B$ be m-PSFSs over X as mentioned in definition 4.2. Then weighted correlation coefficient of $\mathcal{S}_A$ and $\mathcal{S}_B$ can also be defined as

$$ {}^{\omega}\widetilde{\varrho}(\mathcal{S}_A, \mathcal{S}_B) = \frac{1}{3m}\sum_{j=1}^{m}\left({}^{\omega}\widetilde{\varrho}_{\mu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) + {}^{\omega}\widetilde{\varrho}_{\pi}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) + {}^{\omega}\widetilde{\varrho}_{\nu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B)\right) $$

Where,

$$ {}^{\omega}\widetilde{\varrho}_{\mu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = \frac{\sum_{i=1}^k \omega_i\left(\mu_A^{(j)}(\varsigma_i) - \overline{\mu_A^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i) - \overline{\mu_B^{(j)}}\right)}{\max\left\{ \sum_{i=1}^k \omega_i\left(\mu_A^{(j)}(\varsigma_i) - \overline{\mu_A^{(j)}}\right)^2, \sum_{i=1}^k \omega_i\left(\mu_B^{(j)}(\varsigma_i) - \overline{\mu_B^{(j)}}\right)^2\right\}}, $$

$$ {}^{\omega}\widetilde{\varrho}_{\pi}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = \frac{\sum_{i=1}^k \omega_i\left(\pi_A^{(j)}(\varsigma_i) - \overline{\pi_A^{(j)}}\right)\left(\pi_B^{(j)}(\varsigma_i) - \overline{\pi_B^{(j)}}\right)}{\max\left\{ \sum_{i=1}^k \omega_i\left(\pi_A^{(j)}(\varsigma_i) - \overline{\pi_A^{(j)}}\right)^2, \sum_{i=1}^k \omega_i\left(\pi_B^{(j)}(\varsigma_i) - \overline{\pi_B^{(j)}}\right)^2\right\}}, $$

and

$$ {}^{\omega}\widetilde{\varrho}_{\nu}^{(j)}(\mathcal{S}_A, \mathcal{S}_B) = \frac{\sum_{i=1}^k \omega_i\left(\nu_A^{(j)}(\varsigma_i) - \overline{\nu_A^{(j)}}\right)\left(\nu_B^{(j)}(\varsigma_i) - \overline{\nu_B^{(j)}}\right)}{\max\left\{ \sum_{i=1}^k \omega_i\left(\nu_A^{(j)}(\varsigma_i) - \overline{\nu_A^{(j)}}\right)^2, \sum_{i=1}^k \omega_i\left(\nu_B^{(j)}(\varsigma_i) - \overline{\nu_B^{(j)}}\right)^2\right\}}. $$

Theorem 4.16. For the m-PSFSs $\mathcal{S}_A$ and $\mathcal{S}_B$, we have
(a) The correlation ${}^{\omega}\widetilde{\varrho}$ is symmetrical, that is, ${}^{\omega}\widetilde{\varrho}(\mathcal{S}_A, \mathcal{S}_B) = {}^{\omega}\widetilde{\varrho}(\mathcal{S}_B, \mathcal{S}_A)$
(b) The correlation $\widetilde{\varrho}$ lies between -1 and 1, that is, $-1 \leq {}^{\omega}\widetilde{\varrho}(\mathcal{S}_A, \mathcal{S}_B) \leq 1$
(c) ${}^{\omega}\widetilde{\varrho}(\mathcal{S}_A, \mathcal{S}_B) = 1$ if $A = B$
(d) ${}^{\omega}\widetilde{\varrho}(\mathcal{S}_A, \mathcal{S}_B) = 1$ if $A = (\alpha)B, \alpha > 0$.

Proof. (a) It is easy to prove.
(b) From the proposition 4.4, we have $\mid {}^{\omega}I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)} \mid\leq \sqrt{{}^{\omega}\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A)\,{}^{\omega}\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_B)}$

$$\Rightarrow -\sqrt{{}^{\omega}\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A)\,{}^{\omega}\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_B)} \leq {}^{\omega}I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)} \leq \sqrt{{}^{\omega}\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A)\,{}^{\omega}\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_B)}$$

$$\Rightarrow -1 \leq \frac{{}^{\omega}I^{(j)}_{\mu(\mathcal{S}_A\to\mathcal{S}_B)}}{\sqrt{{}^{\omega}\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_A)\,{}^{\omega}\mathcal{V}^{(j)}_{\mu}(\mathcal{S}_B)}} \leq 1$$

$$\Rightarrow -1 \leq {}^{\omega}\widetilde{\varrho}^{(j)}_{\mu}(\mathcal{S}_A,\mathcal{S}_B) \leq 1$$

Similarly, $-1 \leq {}^{\omega}\widetilde{\varrho}^{(j)}_{\pi}(\mathcal{S}_A,\mathcal{S}_B) \leq 1$ and $-1 \leq {}^{\omega}\widetilde{\varrho}^{(j)}_{\nu}(\mathcal{S}_A,\mathcal{S}_B) \leq 1$ and hence $-1 \leq {}^{\omega}\widetilde{\varrho}(\mathcal{S}_A,\mathcal{S}_B) \leq 1$.
(c) If A=B then $\mu^{(j)}_A(\varsigma_i) = \mu^{(j)}_B(\varsigma_i)$, $\pi^{(j)}_A(\varsigma_i) = \pi^{(j)}_B(\varsigma_i)$ and $\nu^{(j)}_A(\varsigma_i) = \nu^{(j)}_B(\varsigma_i)$, then

$${}^{\omega}\widetilde{\varrho}^{(j)}_{\mu}(\mathcal{S}_A,\mathcal{S}_B) = \frac{\sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)}{\max\left\{\sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)^2, \sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)^2\right\}}$$
$$= \frac{\sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)}{\max\left\{\omega_i\sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)^2, \sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)^2\right\}} = 1$$

Similarly ${}^{\omega}\widetilde{\varrho}^{(j)}_{\pi}(\mathcal{S}_A,\mathcal{S}_B) = 1$ and ${}^{\omega}\widetilde{\varrho}^{(j)}_{\nu}(\mathcal{S}_A,\mathcal{S}_B) = 1$ and hence ${}^{\omega}\widetilde{\varrho}(\mathcal{S}_A,\mathcal{S}_B) = 1$.
(d) If A=(α)B then $\mu^{(j)}_A(\varsigma_i) = (\alpha)\mu^{(j)}_B(\varsigma_i)$, $\pi^{(j)}_A(\varsigma_i) = (\alpha)\pi^{(j)}_B(\varsigma_i)$ and $\nu^{(j)}_A(\varsigma_i) = (\alpha)\nu^{(j)}_B(\varsigma_i)$, then

$${}^{\omega}\widetilde{\varrho}^{(j)}_{\mu}(\mathcal{S}_A,\mathcal{S}_B) = \frac{\sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)}{\max\left\{\sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)^2, \sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)^2\right\}}$$
$$= \frac{\sum_{i=1}^{k}\omega_i\left((\alpha)\mu^{(j)}_B(\varsigma_i)-(\alpha)\overline{\mu^{(j)}_B}\right)\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)}{\max\left\{\sum_{i=1}^{k}\omega_i\left((\alpha)\mu^{(j)}_B(\varsigma_i)-(\alpha)\overline{\mu^{(j)}_B}\right)^2, \sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)^2\right\}} = 1$$

Similarly ${}^{\omega}\widetilde{\varrho}^{(j)}_{\pi}(\mathcal{S}_A,\mathcal{S}_B) = 1$ and ${}^{\omega}\widetilde{\varrho}^{(j)}_{\nu}(\mathcal{S}_A,\mathcal{S}_B) = 1$ and hence ${}^{\omega}\widetilde{\varrho}(\mathcal{S}_A,\mathcal{S}_B) = 1$. □

The weights $(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$ is changeless in correlation. So, once such weights occur, we can just ignore them and proceed unweightedly. In other words, the weighted alternatives can be considered uneven. We may observe that

$${}^{\omega}\widetilde{\varrho}^{(j)}_{\mu}(\mathcal{S}_A,\mathcal{S}_B) = \frac{\sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)}{\max\left\{\sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)^2, \sum_{i=1}^{k}\omega_i\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)^2\right\}},$$

$$= \frac{\sum_{i=1}^{k}\frac{1}{n}\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)}{\max\left\{\sum_{i=1}^{k}\frac{1}{n}\left(\mu^{(j)}_A(\varsigma_i)-\overline{\mu^{(j)}_A}\right)^2, \sum_{i=1}^{k}\frac{1}{n}\left(\mu^{(j)}_B(\varsigma_i)-\overline{\mu^{(j)}_B}\right)^2\right\}},$$

$$= \frac{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)}{\max\left\{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)^2,\sum_{i=1}^{k}\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)^2\right\}} = \widetilde{\varrho}_{\mu}^{(j)}(\mathcal{S}_A,\mathcal{S}_B)$$

and

$$^{\omega}\rho_{\mu}^{(j)}(\mathcal{S}_A,\mathcal{S}_B) = \frac{\sum_{i=1}^{k}\omega_i\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)}{\sqrt{\sum_{i=1}^{k}\omega_i\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)^2\sum_{i=1}^{k}\omega_i\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)^2}},$$

$$= \frac{\sum_{i=1}^{k}\frac{1}{n}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)}{\sqrt{\sum_{i=1}^{k}\frac{1}{n}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)^2\sum_{i=1}^{k}\frac{1}{n}\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)^2}},$$

$$= \frac{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)}{\sqrt{\sum_{i=1}^{k}\left(\mu_A^{(j)}(\varsigma_i)-\overline{\mu_A^{(j)}}\right)^2\sum_{i=1}^{k}\left(\mu_B^{(j)}(\varsigma_i)-\overline{\mu_B^{(j)}}\right)^2}} = \rho_{\mu}^{(j)}(\mathcal{S}_A,\mathcal{S}_B)$$

5. Applications of proposed correlation measures

In statistics and engineering the correlation plays a major role. The cumulative relation between two variables can be analysed using the level of stability of the two variables by means of correlation analysis. Correlation actions are prevalent indicators and appropriate methodologies to MCDM problems. Several intellectuals have used correlations in different uncertain conditions to solve these problems quickly and productively. The correlations defined in this article will be used in this section to precisely recognize patterns and medical diagnosis.

Pattern recognition is automatic identification of collection and aggregation in statistics. It is usually classified based on the type of learning experience for producing the value of output. Pattern recognition methods usually help to identify all the independent variables moderately and to make "most likely" the insights to match, taking their statistical variance into consideration. Pattern recognition is usually divided into the classifications of learning process of creating new value of the output.

5.1. Application to pattern recognition

Now we develop Algorithm 1 for pattern recognition with the help of proposed correlation based on *m*-PSF information.

Algorithm 1 (Pattern Recognition)

Step 1: Read the pattern to be recognized as $\mathcal{P}$ and express in the form of *m*-PSFS.
Step 2: Arrange the known patterns in the form of a sequence $\mathcal{P}^{(1)}, \mathcal{P}^{(2)}, \cdots, \mathcal{P}^{(n)}$.
Step 3 (i): By using the formula (4.1) or (4.2), find the correlation coefficients of the pattern $\mathcal{P}$ from the known patterns $\mathcal{P}^{(1)}, \mathcal{P}^{(2)}, \cdots, \mathcal{P}^{(n)}$.
Step 3 (ii): If the universe X for the *m*-PSF model contains weighted alternatives, find the respective

weighted correlation coefficients using the formula (4.3) or (4.4).
Step 4: Pattern $\mathcal{P}$ belongs to $\mathcal{P}^{(i)}$, if any pattern $\mathcal{P}^{(i)}$ finds the greatest correlation with $\mathcal{P}$.

Example 5.1. Pattern recognition is one of the most important and fundamental concern in robotics, machine learning and security management. We consider a perticular problem of pattern recognition, namely, facial recognition. Facial recognition is a mechanism through which an individual's identification can be confirmed by his face. In photos, videos and even in real time, facial recognition systems can be used to identify people. One of the main category of biometric safety is facial identification. Suppose a machine, which is already feeded with three patterns $\mathcal{P}^{(1)}, \mathcal{P}^{(2)}$ and $\mathcal{P}^{(3)}$ of three different persons in the form of m-PSFSs, has to detect a face of an unknown person. (We are considering only three persons in this example because our aim is to give a mathematical model only. However, a certain machine can be feeded with hundreds and thousands of the patterns of different persons at a time). The three patterns are expressed in in terms of 3-PSFS in Table 4, Table 5, Table 6, respectively.

Table 4. Assessment of pattern $\mathcal{P}^{(1)}$ in terms of 3-PSFS.

$\mathcal{P}^{(1)}$	3-PSFS
ς_1	$\Big(0.19, 0.24, 0.35\Big), \Big(0.24, 0.36, 0.49\Big), \Big(0.22, 0.33, 0.45\Big)$
ς_2	$\Big(0.24, 0.33, 0.47\Big), \Big(0.14, 0.32, 0.48\Big), \Big(0.30, 0.47, 0.58\Big)$
ς_3	$\Big(0.25, 0.37, 0.54\Big), \Big(0.20, 0.31, 0.50\Big), \Big(0.25, 0.39, 0.59\Big)$

Table 5. Assessment of pattern $\mathcal{P}^{(2)}$ in terms of 3-PSFS.

$\mathcal{P}^{(2)}$	3-PSFS
ς_1	$\Big(0.25, 0.34, 0.47\Big), \Big(0.17, 0.34, 0.40\Big), \Big(0.21, 0.34, 0.38\Big)$
ς_2	$\Big(0.08, 0.14, 0.25\Big), \Big(0.26, 0.38, 0.47\Big), \Big(0.24, 0.32, 0.49\Big)$
ς_3	$\Big(0.27, 0.36, 0.49\Big), \Big(0.15, 0.33, 0.63\Big), \Big(0.26, 0.43, 0.52\Big)$

and

Table 6. Assessment of pattern $\mathcal{P}^{(3)}$ in terms of 3-PSFS.

$\mathcal{P}^{(3)}$	3-PSFS
ς_1	$\Big(0.18, 0.22, 0.33\Big), \Big(0.24, 0.55, 0.67\Big), \Big(0.32, 0.49, 0.56\Big)$
ς_2	$\Big(0.25, 0.34, 0.50\Big), \Big(0.37, 0.48, 0.62\Big), \Big(0.25, 0.34, 0.57\Big)$
ς_3	$\Big(0.14, 0.27, 0.37\Big), \Big(0.32, 0.46, 0.53\Big), \Big(0.14, 0.28, 0.47\Big)$

The machine first detects the face of an unknown person, analyzes it and converts it into a m-PSF model, say $\mathcal{P}$, which is given in Table 7 as follows.

Table 7. m-PSF model.

$\mathcal{P}$	3-PSFS
ς_1	$\Big(0.20, 0.44, 0.66\Big), \Big(0.06, 0.19, 0.50\Big), \Big(0.32, 0.46, 0.55\Big)$
ς_2	$\Big(0.42, 0.53, 0.65\Big), \Big(0.32, 0.44, 0.59\Big), \Big(0.37, 0.43, 0.50\Big)$
ς_3	$\Big(0.29, 0.38, 0.52\Big), \Big(0.27, 0.36, 0.64\Big), \Big(0.23, 0.32, 0.40\Big)$

It, then, finds the correlations of the unknown pattern from the known ones and decides that which pattern is closest to the unknown pattern. This whole facial recognition process can be performed through the following Algorithm 1, that is, the machine can be feeded with the proposed algorithm in order to detect the person.

The correlation coefficients of $\mathcal{P}$ from $\mathcal{P}^{(1)}, \mathcal{P}^{(2)}, \mathcal{P}^{(3)}$ (using the formula (4.1) and (4.2) respectively) are given by

$\rho(\mathcal{P}, \mathcal{P}^{(1)}) = 0.09$	$\rho(\mathcal{P}, \mathcal{P}^{(2)}) = -0.29$	$\rho(\mathcal{P}, \mathcal{P}^{(3)}) = 0.36$
$\widetilde{\varrho}(\mathcal{P}, \mathcal{P}^{(1)}) = 0.07$	$\widetilde{\varrho}(\mathcal{P}, \mathcal{P}^{(2)}) = -0.28$	$\widetilde{\varrho}(\mathcal{P}, \mathcal{P}^{(3)}) = 0.23$

However, if the alternatives $\varsigma_1, \varsigma_2, \varsigma_3$ pursue some weights $\omega = (0.24, 0.36, 0.40)$, then we find the respective weighted correlation coefficients (using the formula (4.3) and (4.4) respectively) as follows.

${}^{\omega}\rho(\mathcal{P}, \mathcal{P}^{(1)}) = 0.10$	${}^{\omega}\rho(\mathcal{P}, \mathcal{P}^{(2)}) = -0.31$	${}^{\omega}\rho(\mathcal{P}, \mathcal{P}^{(3)}) = 0.39$
${}^{\omega}\widetilde{\varrho}(\mathcal{P}, \mathcal{P}^{(1)}) = 0.06$	${}^{\omega}\widetilde{\varrho}(\mathcal{P}, \mathcal{P}^{(2)}) = -0.29$	${}^{\omega}\widetilde{\varrho}(\mathcal{P}, \mathcal{P}^{(3)}) = 0.27$

The correlation flow of $\mathcal{P}$ from the known patterns $\mathcal{P}^{(1)}$, $\mathcal{P}^{(2)}$ and $\mathcal{P}^{(3)}$ under the proposed correlations are given in the following table.

Table 8. Comparison analysis.

Correlation measures	Ranking	The optimal pattern
ρ	$\mathcal{P}^{(3)} \succ \mathcal{P}^{(1)} \succ \mathcal{P}^{(2)}$	$\mathcal{P}^{(3)}$
$\widetilde{\varrho}$	$\mathcal{P}^{(3)} \succ \mathcal{P}^{(1)} \succ \mathcal{P}^{(2)}$	$\mathcal{P}^{(3)}$
${}^{\omega}\rho$	$\mathcal{P}^{(3)} \succ \mathcal{P}^{(1)} \succ \mathcal{P}^{(2)}$	$\mathcal{P}^{(3)}$
${}^{\omega}\widetilde{\varrho}$	$\mathcal{P}^{(3)} \succ \mathcal{P}^{(1)} \succ \mathcal{P}^{(2)}$	$\mathcal{P}^{(3)}$

All of the above results show that the pattern $\mathcal{P}$ finds the greatest correlation from $\mathcal{P}^{(3)}$. Therefore, the facial recognition machine detects the person $\mathcal{P}$ as $\mathcal{P}^{(3)}$, and then the examiners has to decide the physical existence of the unknown person.

5.2. Medical diagnosis

Now we develop Algorithm 2 for medical diagnosis with the help of proposed correlation for m-PSF information.

Algorithm 2 (Medical Diagnosis)

Step 1: Consider the set of m symptoms $\{\varsigma_1, \varsigma_2, \cdots \varsigma_m\}$ and the set of n diseases $\{\beta^{(1)}, \beta^{(2)} \cdots \beta^{(n)}\}$.
Step 2: Compute assessment matrices for diseases $\beta^{(1)}, \beta^{(2)} \cdots \beta^{(n)}$ and assessment matrix for the patient $\mathcal{P}$ in terms of m-PSFS.
Step 3 (i): By using the formula (4.1) or (4.2), find the correlation coefficients of the matrix $\mathcal{P}$ with each matrix $\beta^{(1)}, \beta^{(2)} \cdots \beta^{(n)}$.
Step 3 (ii): Compute weighted correlation coefficients using the formula (4.3) or (4.4).
Step 4: The disease $\beta^{(i)}$ showing highest correlation with patient $\mathcal{P}$ is the actual disease.

Example 5.2. In this example, we use the Algorithm 2 to analyze the actual disease of a patient. Suppose a patient has to be diagnosed in a healthcare center and healthcare experts diagnosis the patient with the following symptoms as follows:

$$\varsigma_1 = \text{headache}, \varsigma_2 = \text{fatigue}, \varsigma_3 = \text{chest pain}, \varsigma_4 = \text{breathing difficulty}$$

and the possible diseases as follows:

$$\beta^{(1)} = \text{fever}, \beta^{(2)} = \text{typhoid}, \beta^{(3)} = \text{corona}, \beta^{(4)} = \text{heart disease}$$

The assessment matrices for $\beta^{(1)}$, $\beta^{(2)}$,$\beta^{(3)}$ and $\beta^{(4)}$ are expressed in Table 9, Table 10, Table 11, Table 12, respectively.

Table 9. Assessment matrix for $\beta^{(1)}$.

$\beta^{(1)}$	3-PSFS
ς_1	$\left(0.18, 0.26, 0.39\right), \left(0.15, 0.24, 0.46\right), \left(0.19, 0.36, 0.42\right)$
ς_2	$\left(0.24, 0.38, 0.44\right), \left(0.36, 0.48, 0.54\right), \left(0.33, 0.41, 0.57\right)$
ς_3	$\left(0.11, 0.22, 0.44\right), \left(0.12, 0.35, 0.47\right), \left(0.24, 0.51, 0.62\right)$
ς_4	$\left(0.22, 0.34, 0.48\right), \left(0.10, 0.15, 0.20\right), \left(0.13, 0.27, 0.46\right)$

Table 10. Assessment matrix for $\beta^{(2)}$.

$\beta^{(2)}$	3-PSFS
ς_1	$\left(0.20, 0.35, 0.45\right), \left(0.18, 0.29, 0.43\right), \left(0.15, 0.29, 0.37\right)$
ς_2	$\left(0.10, 0.16, 0.27\right), \left(0.26, 0.37, 0.48\right), \left(0.20, 0.35, 0.50\right)$
ς_3	$\left(0.23, 0.33, 0.46\right), \left(0.32, 0.39, 0.45\right), \left(0.25, 0.37, 0.46\right)$
ς_4	$\left(0.29, 0.38, 0.44\right), \left(0.27, 0.40, 0.56\right), \left(0.17, 0.34, 0.62\right)$

Table 11. Assessment matrix for $\beta^{(3)}$.

$\beta^{(3)}$	3-PSFS
ς_1	$\Big(0.43, 0.50, 0.62\Big), \Big(0.18, 0.26, 0.34\Big), \Big(0.30, 0.48, 0.64\Big)$
ς_2	$\Big(0.14, 0.28, 0.32\Big), \Big(0.17, 0.32, 0.54\Big), \Big(0.15, 0.40, 0.57\Big)$
ς_3	$\Big(0.16, 0.30, 0.45\Big), \Big(0.24, 0.37, 0.58\Big), \Big(0.18, 0.26, 0.44\Big)$
ς_4	$\Big(0.13, 0.29, 0.48\Big), \Big(0.32, 0.46, 0.62\Big), \Big(0.23, 0.38, 0.50\Big)$

and

Table 12. Assessment matrix for $\beta^{(4)}$.

$\beta^{(4)}$	3-PSFS
ς_1	$\Big(0.16, 0.24, 0.36\Big), \Big(0.19, 0.33, 0.45\Big), \Big(0.28, 0.35, 0.43\Big)$
ς_2	$\Big(0.28, 0.37, 0.54\Big), \Big(0.26, 0.53, 0.65\Big), \Big(0.39, 0.49, 0.54\Big)$
ς_3	$\Big(0.17, 0.26, 0.39\Big), \Big(0.34, 0.47, 0.55\Big), \Big(0.20, 0.49, 0.62\Big)$
ς_4	$\Big(0.18, 0.46, 0.59\Big), \Big(0.39, 0.45, 0.60\Big), \Big(0.13, 0.35, 0.49\Big)$

Assume the expert opinion for the patient and his symptoms is evaluated in terms 3PSFS as given in Table 13.

Table 13. Expert opinion in terms of 3-PSFS.

$\mathcal{P}$	3-PSFS
ς_1	$\Big(0.20, 0.30, 0.40\Big), \Big(0.17, 0.25, 0.36\Big), \Big(0.09, 0.32, 0.41\Big)$
ς_2	$\Big(0.15, 0.16, 0.18\Big), \Big(0.22, 0.35, 0.43\Big), \Big(0.15, 0.26, 0.44\Big)$
ς_3	$\Big(0.21, 0.32, 0.52\Big), \Big(0.25, 0.34, 0.47\Big), \Big(0.33, 0.45, 0.68\Big)$
ς_4	$\Big(0.23, 0.38, 0.42\Big), \Big(0.28, 0.49, 0.65\Big), \Big(0.26, 0.40, 0.60\Big)$

In order to check the actual disease of patient, we find the correlation coefficients of β from $\beta^{(1)}, \beta^{(2)}$ using formula (4.1) and (4.2) respectively as follows.

$\rho(\mathcal{P}, \beta^{(1)}) = -0.21$	$\rho(\mathcal{P}, \beta^{(2)}) = 0.81$	$\rho(\mathcal{P}, \beta^{(3)}) = 0.15$	$\rho(\mathcal{P}, \beta^{(4)}) = 0.29$
$\widetilde{\varrho}(\mathcal{P}, \beta^{(1)}) = -0.15$	$\widetilde{\varrho}(\mathcal{P}, \beta^{(2)}) = 0.60$	$\widetilde{\varrho}(\mathcal{P}, \beta^{(3)}) = 0.18$	$\widetilde{\varrho}(\mathcal{P}, \beta^{(4)}) = 0.20$

However, if the patients $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4$ are assigned some weights $\omega = (0.19, 0.22, 0.26, 0.33)$ from the healthcare team, then we find the respective weighted correlation coefficients using the formula (4.3) and (4.4) respectively as follows.

${}^{\omega}\rho(\mathcal{P},\beta^{(1)}) = -0.23$	${}^{\omega}\rho(\mathcal{P},\beta^{(2)}) = 0.80$	${}^{\omega}\rho(\mathcal{P},\beta^{(3)}) = 0.13$	${}^{\omega}\rho(\mathcal{P},\beta^{(4)}) = 0.29$
${}^{\omega}\widetilde{\varrho}(\mathcal{P},\beta^{(1)}) = -0.16$	${}^{\omega}\widetilde{\varrho}(\mathcal{P},\beta^{(2)}) = 0.60$	${}^{\omega}\widetilde{\varrho}(\mathcal{P},\beta^{(3)}) = 0.16$	${}^{\omega}\widetilde{\varrho}(\mathcal{P},\beta^{(4)}) = 0.19$

The correlation flow of $\mathcal{P}$ from $\beta^{(1)}, \beta^{(2)}$ and $\beta^{(3)}$ under the suggested correlation measures are given in the Table 14. From these results, we find that the patient $\mathcal{P}$ shows the greatest correlation with $\beta^{(2)}$. So, the patient is suffering from typhoid and the necessary measures are to be taken in this regard.

Table 14. Comparison analysis.

Correlation measures	Ranking	The optimal diagnosis
ρ	$\beta^{(2)} \succ \beta^{(4)} \succ \beta^{(3)} \succ \beta^{(1)}$	$\beta^{(2)}$
$\widetilde{\varrho}$	$\beta^{(2)} \succ \beta^{(4)} \succ \beta^{(3)} \succ \beta^{(1)}$	$\beta^{(2)}$
${}^{\omega}\rho$	$\beta^{(2)} \succ \beta^{(4)} \succ \beta^{(3)} \succ \beta^{(1)}$	$\beta^{(2)}$
${}^{\omega}\widetilde{\varrho}$	$\beta^{(2)} \succ \beta^{(4)} \succ \beta^{(3)} \succ \beta^{(1)}$	$\beta^{(2)}$

6. Conclusions

MCDM has been studied by a large number of scholars and researchers. The methods designed for this purpose are generally influenced by the judgment operational framework used. The most of its key issues are attached to vague, imprecise, and multi-polar data that can't be adequately explained by fuzzy set alone. An m-polar fuzzy set (m-PFS) has the ability to deal with vagueness by multi-polarity and the spherical fuzzy set (SFS) deal with uncertainty by using three independent grades (membership degree, neutral-membership degree, and non-membership degree).

In order to deal with real-life vague circumstances when decision makers require a new mathematical model to deal with multi-polarity as well as three independent spherical index, we introduced a new hybrid model named as m-polar spherical fuzzy set (m-PSFS) as a robust fusion of SFS and m-PFS. The existing models namely m-PFS and SFS are the special cases of suggested hybrid m-PSFS. To ensure the algebraic structures of this robust extension, we developed fundamental operations on m-PSFSs and investigated their related results. A suitable numbers of illustrations are presented to explain suggested notions and results. We introduced the correlation measures and weighted correlation measures for m-PSFSs. Certain properties of covariances and the correlation measures are proposed to analyze that suggested concepts are novel extension of crisp correlation measures. The main advantage of proposed correlation measures is that these notions deal with vagueness and uncertainty in the real-life problems efficiently with the help of m-PSF information. We discussed applications of m-PSFSs and their correlation measures in pattern recognition and medical diagnosis. To discuss the superiority and efficiency of proposed correlation measures, we give a comparison analysis of proposed concepts with some existing concepts.

Conflict of interest

The authors declare no conflict of interest.

Acknowledgement

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University, Abha 61413, Saudi Arabia for funding this work through research groups program under grant number R.G. P-2/29/42.

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