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Research article

Boundedness of multilinear Calderón-Zygmund singular operators on weighted Lebesgue spaces and Morrey-Herz spaces with variable exponents

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Abstract: In this paper, we introduce weighted Morrey-Herz spaces $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(w^{p(\cdot)})$ with variable exponent $p(\cdot)$. Then we prove the boundedness of multilinear Calderón-Zygmund singular operators on weighted Lebesgue spaces and weighted Morrey-Herz spaces with variable exponents.

Keywords: weighted Lebesgue spaces; Morrey-Herz spaces; variable exponents; multilinear Calderón-Zygmund singular operators; boundedness
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1. Introduction

In last decades there have been many works on the boundedness for the multilinear singular integral operators in the product of Lebesgue spaces, Morrey type spaces and Herz spaces etc. Let T be a multilinear singular integral operator which is initially defined on the m-fold product of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ with values in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ such that for $x \notin \cap_{i=1}^m \mathrm{supp} f_i$,

$$T(f_1, \cdots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \cdots, y_m) \prod_{i=1}^m f_i(y_i) dy_1 \cdots dy_m, \tag{1.1}$$

where $f_1, \cdots, f_m$ are in $L_c^\infty(\mathbb{R}^n)$, the space of compactly supported bounded functions. The kernel K is a function in $(\mathbb{R}^n)^{m+1}$ away from the diagonal $y_0 = y_1 = \cdots = y_m$ and satisfies the standard estimates

$$|K(x, y_1, \cdots, y_m)| \leq C(|x - y_1| + \cdots + |x - y_m|)^{-mn}, \tag{1.2}$$

and for some $\varepsilon > 0$,

$$|K(x, y_1, \cdots, y_m) - K(x', y_1, \cdots, y_m)| \leq \frac{C|x - x'|^\varepsilon}{(|x - y_1| + \cdots + |x - y_m|)^{mn+\varepsilon}}, \tag{1.3}$$

provided that $|x - x'| \leq \frac{1}{2}\max\{|x - y_1|, \cdots, |x - y_m|\}$ and

$$|K(x, y_1, \cdots, y_i, \cdots, y_m) - K(x, y_1, \cdots, y_i', \cdots, y_m)| \leq \frac{C|y_i - y_i'|^{\varepsilon}}{(|x - y_1| + \cdots + |x - y_m|)^{mn+\varepsilon}}, \tag{1.4}$$

provided that $|y_i - y_i'| \leq \frac{1}{2}\max\{|x - y_1|, \cdots, |x - y_m|\}$ for all $1 \leq i \leq m$.

Such kernels are called Calderón-Zygmund kernel and the collection of such functions is denoted by $m - CZK(C, \varepsilon)$ in [1]. Let T be as in (1.1) with an $m - CZK(C, \varepsilon)$ kernel. If T is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p for some $1 < p_1, \cdots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, then we say that T is an $m-$linear Calderón-Zygmund operator. If T is an $m-$linear Calderón-Zygmund operator, Grafakos and Torres in [1] proved its boundedness from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q for all $1 < q_1, \cdots, q_m < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ and from $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$. Curbera et al. in [2] obtained its weighted inequalities. P*é*rez and Torres in [3] presented a new proof of a weighted norm inequality for multilinear singular integral of Calderón-Zygmund type and studied an application of certain multilinear commutators. Recently, Tao and Zhang in [4] obtained the boundedness of the multilinear singular integral operators on weighted Morrey-Herz spaces and gave their weak estimates on endpoints.

Function spaces with variable exponents have been intensively studied in the recent years by a significant number of authors, e.g. [5–11]. The motivation for the increasing interest in such spaces comes not only from theoretical purpose, but also from applications to fluid dynamics, image restoration and PDE with non-standard growth conditions. The results above had been extended to the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and the variable exponent Morrey spaces $M^{p(\cdot)}_{q(\cdot)}(\Omega)$ over an open set $\Omega \subseteq \mathbb{R}^n$ in [12–14]. Lu and Zhu in [15] discussed the boundedness of multilinear Calderón-Zygmund singular operators on Morrey-Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ and Herz spaces $\dot{K}^{\alpha(\cdot)}_{q,p(\cdot)}(\mathbb{R}^n)$ with two variable exponents $\alpha(\cdot)$ and $p(\cdot)$.

Recently the generalized Muckenhoupt weights with variable exponents have been studied. Cruz-Uribe and Wang in [6] obtained the boundedness of fractional integrals on weighted Lebesgue spaces with variable exponents by applying the extrapolation. Izuki and Noi in [7] proved the the boundedness of fractional integrals on weighted Herz spaces with variable exponents by the theory of Banach functions spaces and Muckenhoupt theory with variable exponents. We refer readers [16–20] for more references about weights in variable exponent spaces.

Motivated by the results above, we give the definition of weighted Morrey-Herz spaces with variable exponents and prove the boundedness of multilinear Calderón-Zygmund singular operators on the product of weighted Lebesgue spaces and weighted Morrey-Herz spaces with variable exponents.

2. Preliminaries

2.1. Variable Lebesgue spaces

Definition 1. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \{f \text{ is measurable} : \rho_p(\frac{f}{\lambda}) < \infty \text{ for some constant } \lambda > 0\},$$

where $\rho_p(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx$.

$L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\{\lambda > 0 : \ \rho_p(\frac{f}{\lambda}) \leq 1\}.$$

We denote

$$p_- := \ \operatorname{ess\,inf}\{p(x) : \ x \in \mathbb{R}^n\}, \quad p_+ := \ \operatorname{ess\,sup}\{p(x) : \ x \in \mathbb{R}^n\}.$$

The set $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. $p'(\cdot)$ means the conjugate exponent of $p(\cdot)$, namely $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ holds. We also note that generalized Hölder's inequality

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$

is true for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, where $r_p := 1 + \frac{1}{p_-} - \frac{1}{p_+}$ ([5]).

We say that a function $p : \mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous, if it satisfies

$$|p(x) - p(y)| \leq \frac{-C}{\log(|x-y|)} \quad \text{for } x, y \in \mathbf{R}^n, \ |x-y| \leq \frac{1}{2}. \tag{2.1}$$

If

$$|p(x) - p_\infty| \leq \frac{C}{\log(e+|x|)} \quad \text{for some } \ p_\infty \geq 1 \text{ and all } x \in \mathbf{R}^n \tag{2.2}$$

we say p satisfies the log-Hölder decay condition at infinity. The set of $p(\cdot)$ satisfying (2.1) and (2.2) is denoted by $LH(\mathbb{R}^n)$. It is also well known that the Hardy-Littlewood maximal operator M defined by

$$Mf(x) = \sup_{B:\text{ball}, x \in B} \frac{1}{|B|} \int_B |f(y)|dy,$$

is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ whenever $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$.

2.2. The Muckenhoupt weights with variable exponents

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w be a weight. The weighted variable exponent Lebesgue space $L^{p(\cdot)}(w)$ is the set of all complex-valued measurable functions f such that $fw^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(\mathbb{R}^n)$. The space $L^{p(\cdot)}(w)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(w)} := \|fw^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}}.$$

Now we define the Muckenhoupt classes. We begin with the classical Muckenhoupt A_1 weights.

Definition 2. A weight is said to be a Muckenhoupt A_1 weight if $Mw(x) \leq Cw(x)$ holds for almost every $x \in \mathbb{R}^n$. The set A_1 consists of all Muckenhoupt A_1 weights.

The original Muckenhoupt A_p class with constant exponent $p \in (1, \infty)$ established by Muckenhoupt [21] can be generalized in terms of a variable exponent as follows.

Definition 3. ([7, 20]) Suppose $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A weight w is said to be an $A_{p(\cdot)}$ weight if

$$\sup_{B:\text{ball}} \frac{1}{|B|} \|w^{\frac{1}{p(\cdot)}} \chi_B\|_{L^{p(\cdot)}} \|w^{-\frac{1}{p(\cdot)}} \chi_B\|_{L^{p'(\cdot)}} < \infty.$$

Next we state the monotone property of the class $A_{p(\cdot)}$.

Lemma 1. *([7] Corollary 1) If* $p(\cdot),\ q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$ *and* $p(\cdot) \le q(\cdot)$*, then we have*

$$A_1 \subset A_{p(\cdot)} \subset A_{q(\cdot)}.$$

Definition 4. ([7]) Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and a weight w, we say $(p(\cdot), w)$ is an M-pair if the maximal operator M is bounded on both $L^{p(\cdot)}(w^{p(\cdot)})$ and $L^{p'(\cdot)}(w^{-p'(\cdot)})$.

Lemma 2. *If* $(p(\cdot), w)$ *is an M-pair, then we have that for all balls B in* $\mathbb{R}^n$,

$$C^{-1} \le \frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|\chi_B\|_{L^{p'(\cdot)}(w^{-p'(\cdot)})} \le C.$$

We introduce the sharp maximal function

$$M^\sharp f(x) := \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y) - f_Q|dy,$$

where the supremum is taken over all cubes Q containing x, as usual, f_Q denotes the average of over Q. For $\delta > 0$, we denote by $M^\sharp_\delta$ the operator

$$M^\sharp_\delta f = M^\sharp(|f|^\delta)^{1/\delta}.$$

Similarly as above, for $\delta > 0$, we denote by M_δ the operator $M_\delta(f) = M(|f|^\delta)^{1/\delta}$.

The next lemma gives an estimate for the norm $\|\cdot\|_{L^{p(\cdot)}(w^{p(\cdot)})}$ in terms of $M^\sharp$.

Lemma 3. *If* $(p(\cdot), w)$ *is an M-pair, then we have that for all* $f \in L^{p(\cdot)}(w^{p(\cdot)})$,

$$\|f\|_{L^{p(\cdot)}(w^{p(\cdot)})} \le C\|M^\sharp f\|_{L^{p(\cdot)}(w^{p(\cdot)})}.$$

The two Lemmas above are the generalization of Lemmas 2 and 4 in [22] to the weighted case, which can be found in [16] and [6] respectively.

Lemma 4. *Suppose* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$ *and* $w \in A_{p(\cdot)}$*, then we can take constants* $\delta_1, \delta_2 \in (0, 1)$ *such that*

$$\frac{\|\chi_E\|_{(L^{p(\cdot)}(w^{p(\cdot)}))'}}{\|\chi_B\|_{(L^{p(\cdot)}(w^{p(\cdot)}))'}} = \frac{\|\chi_E\|_{L^{p'(\cdot)}(w^{-p'(\cdot)})}}{\|\chi_B\|_{L^{p'(\cdot)}(w^{-p'(\cdot)})}} \le C(\frac{|E|}{|B|})^{\delta_1}, \tag{2.3}$$

$$\frac{\|\chi_E\|_{(L^{p(\cdot)}(w^{p(\cdot)}))'}}{\|\chi_B\|_{(L^{p(\cdot)}(w^{p(\cdot)}))'}} \le C(\frac{|E|}{|B|})^{\delta_2}, \tag{2.4}$$

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})}}{\|\chi_E\|_{L^{p(\cdot)}(w^{p(\cdot)})}} \le C\frac{|B|}{|E|}, \tag{2.5}$$

for all balls B and all measurable sets $E \subset B$.

Equations (2.3) and (2.4) are from [7]. (2.5) is the generalization of inequality (6) of Lemma 1 in [23] to the weighted case. Their proofs are similar, we omit it here.

Lemma 5. *([3]) Let T be an m-linear Calderón-Zygmund operator and let* $0 < \delta < 1/m$*. Then, there exists a constant* $C > 0$ *such that for any vector function* $\vec{f} = (f_1, \dots, f_m)$*, where each* f_j *is a smooth function and with compact support, the following inequality holds*

$$M^\sharp_\delta(T(f_1, \dots, f_m))(x) \le C\prod_{j=1}^m M(f_j)(x).$$

2.3. *Weighted Morrey-Herz spaces with variable exponents*

Let $E \subset \mathbb{R}^n$ be a measurable set and w a positive and locally integrable function on E. The set $L^{p(\cdot)}_{\text{loc}}(E, w)$ consists of all functions f satisfying the following condition: For all compact sets $K \subset E$, there exists a constant $\lambda > 0$ such that

$$\int_K |\frac{f(x)}{\lambda}|^{p(x)} w(x) dx < \infty.$$

Next we define weighted Morrey-Herz spaces with variable exponents motivated by [7,24]. We use the following notations. For each $k \in \mathbb{Z}$, we denote

$$B_k =: \ \{x \in \mathbb{R}^n : \ |x| \le 2^k\}, \quad C_k =: \ B_k \setminus B_{k-1}, \quad \text{and} \ \chi_k =: \ \chi_{C_k}.$$

Definition 5. Let $0 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 \le \lambda < \infty$ and $\alpha \in \mathbb{R}$. The weighted Morrey-Herz space with variable exponents $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(w^{p(\cdot)})$ is defined by

$$M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(w^{p(\cdot)}) = \{f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \backslash \{0\}, w) : \|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(w^{p(\cdot)})} < \infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(w^{p(\cdot)})} =: \sup_{L\in\mathbb{Z}} 2^{-L\lambda} (\sum_{k=-\infty}^{L} 2^{k\alpha q} \|f\chi_k\|^q_{L^{p(\cdot)}(w)})^{\frac{1}{q}}.$$

It obviously follows that $M\dot{K}^{\alpha,0}_{q,p(\cdot)}(w^{p(\cdot)})$ coincides with the weighted Herz spaces with variable exponents $\dot{K}^{\alpha,q}_{p(\cdot)}(w)$ defined in [7].

3. The main results

Theorem 1. Let $(p_j(\cdot), w)$ be M-pairs, $j = 1, 2, \ldots, m$, and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$ satisfy $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \cdots + \frac{1}{p_m(x)}$. Let w be a weight in $A_{p_0(\cdot)}$ where $p_0(\cdot) = \min\{p_1(\cdot), \ldots, p_m(\cdot)\}$. Then, there exists a constant $C > 0$ so that for all $\vec{f} = (f_1, \ldots, f_m)$, where each f_j is a smooth function and with compact support, the m-linear Calderón-Zygmund operator T is bounded on the product of weighted variable exponent Lebesgue spaces. Moreover,

$$\|T(f_1, \ldots, f_m)\|_{L^{p(\cdot)}(w^{p(\cdot)})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j(\cdot)}(w^{p_j(\cdot)})}. \tag{3.1}$$

Proof. By using Lemma 3, Lemma 5, Hölder's inequality and the boundedness of M on $L^{p_j(\cdot)}(w^{p_j(\cdot)})$, we get

$$\begin{aligned}
&\|T(f_1, \cdots, f_m)\|_{L^{p(\cdot)}(w^{p(\cdot)})} = \|T(f_1, \cdots, f_m)(x)w(x)\|_{L^{p(\cdot)}} \\
&\le C\|M^{\sharp}_{\delta}(T(f_1, \cdots, f_m))(x)w(x)\|_{L^{p(\cdot)}} \\
&\le C\|\prod_{j=1}^{m} \|M(f_j)(x)w(x)\|_{L^{p(\cdot)}} \le C \prod_{j=1}^{m} \|M(f_j)w(x)\|_{L^{p_j(\cdot)}} \\
&= C \prod_{j=1}^{m} \|M(f_j)\|_{L^{p_j(\cdot)}(w^{p_j(\cdot)})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j(\cdot)}(w^{p_j(\cdot)})}.
\end{aligned}$$

This yields (3.1).

Theorem 2. Let $(p_j(\cdot), w)$ be M-pairs, $j = 1, 2, \ldots, m$, and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$ satisfy $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \cdots + \frac{1}{p_m(x)}$. Let w be a weight in $A_{p_0(\cdot)}$ where $p_0(\cdot) = \min\{p_1(\cdot), \ldots, p_m(\cdot)\}$, $\delta_1 \in (0, 1)$ be the constant appearing in (2.3). Suppose that $\lambda = \sum_{i=1}^m \lambda_i$, $\alpha = \sum_{i=1}^m \alpha_i$, $0 < \lambda_i < \alpha_i < n\delta_1$, $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i}$. Then, there exists a constant $C > 0$ so that for all $\vec{f} = (f_1, \ldots, f_m)$, where each f_j is a smooth function and with compact support, the $m-$linear Calderón-Zygmund operator T is bounded on the product of weighted variable exponent Morrey-Herz spaces. Moreover, we have

$$\|T(\vec{f})\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(w^{p(\cdot)})} \leq C \prod_{j=1}^{m} \|f_j\|_{M\dot{K}^{\alpha_j,\lambda_j}_{q_j,p_j(\cdot)}(w^{p_j(\cdot)})}. \tag{3.2}$$

Let $\lambda_i = 0$, we immediately get the boundedness of the multilinear Calderón-Zygmund integral operator on the product of weighted Herz spaces with variable exponents.

Proof. Without loss of generality, we only consider the case $m = 2$. Actually, the similar procedure works for all $m \in N$. When $m = 2$, we have

$$T(f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2.$$

Write

$$f_i(x) = \sum_{l_i=-\infty}^{\infty} f_i(x)\chi_{l_i}(x) =: \sum_{l_i=-\infty}^{\infty} f_{l_i}(x), \quad i = 1,\ 2.$$

$L^{p(\cdot)}(w^{p(\cdot)})$ is a Banach space, hence we have

$$\begin{aligned}
\|T(f_1, f_2)\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(w^{p(\cdot)})} &= \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha q} \|T(f_1, f_2)\chi_k\|^q_{L^{p(\cdot)}(w^{p(\cdot)})} \Big)^{1/q} \\
&\leq \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha q} \Big\| \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} T(f_{l_1}, f_{l_2}))\chi_k \Big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})} \Big)^{1/q} \\
&\leq C \sum_{i=1}^{9} I_i.
\end{aligned}$$

where

$$I_1 = \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha q} \Big\| \sum_{l_1=-\infty}^{k-2} \sum_{l_2=-\infty}^{k-2} T(f_{l_1}, f_{l_2}))\chi_k \Big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})} \Big)^{1/q},$$

$$I_2 = \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha q} \Big\| \sum_{l_1=-\infty}^{k-2} \sum_{l_2=k-1}^{k+1} T(f_{l_1}, f_{l_2}))\chi_k \Big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})} \Big)^{1/q},$$

$$I_3 = \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha q} \Big\| \sum_{l_1=-\infty}^{k-2} \sum_{l_2=k+2}^{\infty} T(f_{l_1}, f_{l_2}))\chi_k \Big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})} \Big)^{1/q},$$

$$I_4 = \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha q} \Big\| \sum_{l_1=k-1}^{k+1} \sum_{l_2=-\infty}^{k-2} T(f_{l_1}, f_{l_2}))\chi_k \Big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})} \Big)^{1/q},$$

$$I_5 = \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha q}\Big\|\sum_{l_1=k-1}^{k+1}\sum_{l_2=k-1}^{k+1} T(f_{l_1}, f_{l_2}))\chi_k\Big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})}\Big)^{1/q},$$

$$I_6 = \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha q}\Big\|\sum_{l_1=k-1}^{k+1}\sum_{l_2=k+2}^{\infty} T(f_{l_1}, f_{l_2}))\chi_k\Big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})}\Big)^{1/q},$$

$$I_7 = \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha q}\Big\|\sum_{l_1=k+2}^{\infty}\sum_{l_2=-\infty}^{k-2} T(f_{l_1}, f_{l_2}))\chi_k\Big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})}\Big)^{1/q},$$

$$I_8 = \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha q}\Big\|\sum_{l_1=k+2}^{\infty}\sum_{l_2=k-1}^{k+1} T(f_{l_1}, f_{l_2}))\chi_k\Big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})}\Big)^{1/q},$$

$$I_9 = \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha q}\Big\|\sum_{l_1=k+2}^{\infty}\sum_{l_2=k+2}^{\infty} T(f_{l_1}, f_{l_2}))\chi_k\Big\|^q_{L^{p(\cdot)}(w^{p(\cdot)})}\Big)^{1/q}.$$

Because of the symmetry of f_1 and f_2, we see that the estimate of I_2 is analogous to that of I_4, the estimate of I_3 is similar to that of I_7 and the estimate of I_6 is similar to that of I_8. We will estimate $I_1 - I_3$, I_5, I_6 and I_9 respectively.

(i) To estimate I_1, we note $l_i \le k-2$ for $i = 1,\ 2$, and

$$|x - y_i| \ge ||x| - |y_i|| > 2^{k-1} - 2^{l_i} > 2^{k-2},\ \text{for } x \in C_k,\ y_i \in C_{l_i}.$$

Thus, for $x \in C_k$, we get

$$\begin{aligned}|T(f_{l_1}, f_{l_2})(x)| &\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_{l_1}(y_1)||f_{l_2}(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}dy_1dy_2\\ &\le C2^{-2kn}\int_{\mathbb{R}^n}|f_{l_1}(y_1)|dy_1\int_{\mathbb{R}^n}|f_{l_2}(y_2)|dy_2.\end{aligned}$$

Applying the generalized Hölder's inequality to the last two integrals, we obtain

$$\begin{aligned}&\Big\|\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2} T(f_{l_1}, f_{l_2})\chi_k\Big\|_{L^{p(\cdot)}(w^{p(\cdot)})}\\ &\le C\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2} 2^{-2kn}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{l_1}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\\ &\cdot\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_{l_2}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\|\chi_k\|_{L^{p(\cdot)}(w^{p(\cdot)})}.\end{aligned}$$

By $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$, using the generalized Hölder's inequality and Lemmas 2 and 4 , we get

$$\begin{aligned}&\|\chi_{l_1}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{l_2}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\|\chi_k\|_{L^{p(\cdot)}(w^{p(\cdot)})}\\ &\le \|\chi_{B_{l_1}}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_{l_2}}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\|\chi_{B_k}\|_{L^{p(\cdot)}(w^{p(\cdot)})}\\ &\le C\|\chi_{B_{l_1}}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_{l_2}}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{B_k}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\ &\le C\|\chi_{B_{l_1}}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_{l_2}}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}|B_k|\|\chi_{B_k}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}|B_k|\|\chi_{B_k}\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\end{aligned}$$

$$
\begin{aligned}
&\le C2^{2kn}\Big(\frac{|B_{l_1}|}{|B_k|}\Big)^{\delta_1}\Big(\frac{|B_{l_2}|}{|B_k|}\Big)^{\delta_1}\\
&= C2^{2kn}2^{(l_1-k)n\delta_1}2^{(l_2-k)n\delta_1}.
\end{aligned}
$$

Therefore, we arrive at the inequality

$$
\Big\|\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}T(f_{l_1},f_{l_2})\chi_k\Big\|_{L^{p(\cdot)}(w^{p(\cdot)})}\le C\prod_{i=1}^{2}\sum_{l_i=-\infty}^{k-2}2^{(l_i-k)n\delta_1}\|f_{l_i}\|_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}.
$$

Since $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, we have

$$
\begin{aligned}
I_1&\le C\sup_{L\in\mathbb{Z}}2^{-L\lambda}\Big\{\sum_{k=-\infty}^{L}2^{k\alpha q}\Big(\prod_{i=1}^{2}\sum_{l_i=-\infty}^{k-2}2^{(l_i-k)n\delta_1}\|f_{l_i}\|_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\Big)^{q}\Big\}^{\frac{1}{q}}\\
&\le C\prod_{i=1}^{2}\sup_{L\in\mathbb{Z}}2^{-L\lambda_i}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_i=-\infty}^{k-2}2^{(l_i-k)(n\delta_1-\alpha_i)}2^{l_i\alpha_i}\|f_{l_i}\|_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\Big)^{q_i}\Big\}^{\frac{1}{q_i}}\\
&:=I_{11}I_{12}.
\end{aligned}
$$

We consider the two cases $0<q_i\le 1$ and $1<q_i<\infty$. When $0<q_i\le 1,$ we apply inequality

$$
\Big(\sum_{h=1}^{\infty}a_h\Big)^{q_i}\le\sum_{h=1}^{\infty}a_h^{q_i},\qquad (a_1,a_2\ldots\ge 0), \tag{3.3}
$$

and obtain

$$
\begin{aligned}
I_{1i}&\le C\sup_{L\in\mathbb{Z}}2^{-L\lambda_i}\Big\{\sum_{k=-\infty}^{L}\sum_{l_i=-\infty}^{k-2}2^{(l_i-k)(n\delta_1-\alpha_i)q_i}2^{l_i\alpha_iq_i}\|f_{l_i}\|^{q_i}_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\Big\}^{\frac{1}{q_i}}\\
&= C\sup_{L\in\mathbb{Z}}2^{-L\lambda_i}\Big\{\sum_{l_i=-\infty}^{L-2}2^{l_i\alpha_iq_i}\|f_{l_i}\|^{q_i}_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\sum_{k=l_i+2}^{L}2^{(l_i-k)(n\delta_1-\alpha_i)q_i}\Big\}^{\frac{1}{q_i}}\\
&\le C\sup_{L\in\mathbb{Z}}2^{-L\lambda_i}\Big\{\sum_{l_i=-\infty}^{L-2}2^{l_i\alpha_iq_i}\|f_{l_i}\|^{q_i}_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\Big\}^{\frac{1}{q_i}}\\
&\le C\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(w^{p_i(\cdot)})}.
\end{aligned}
$$

When $1<q_i<\infty$, by Hölder's inequality we obtain

$$
\begin{aligned}
I_{1i}&= C\sup_{L\in\mathbb{Z}}2^{-L\lambda_i}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_i=-\infty}^{k-2}2^{(l_i-k)(n\delta_1-\alpha_i)}2^{l_i\alpha_i}\|f_{l_i}\|_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\Big)^{q_i}\Big\}^{\frac{1}{q_i}}\\
&\le C\sup_{L\in\mathbb{Z}}2^{-L\lambda_i}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_i=-\infty}^{k-2}2^{(l_i-k)(n\delta_1-\alpha_i)\frac{q_i}{2}}2^{l_i\alpha_iq_i}\|f_{l_i}\|^{q_i}_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\Big)\\
&\times\Big(\sum_{l_i=-\infty}^{k-2}2^{(l_i-k)(n\delta_1-\alpha_i)\frac{q_i'}{2}}\Big)^{\frac{q_i}{q_i'}}\Big\}^{\frac{1}{q_i}}
\end{aligned}
$$

$$\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda_i}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_i=-\infty}^{k-2}2^{(l_i-k)(n\delta_1-\alpha_i)\frac{q_i}{2}}2^{l_i\alpha_iq_i}\|f_{l_i}\|^{q_i}_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\Big)\Big\}^{\frac{1}{q_i}}$$

$$=C\sup_{L\in\mathbb{Z}}2^{-L\lambda_i}\Big\{\sum_{l_i=-\infty}^{L-2}2^{l_i\alpha_iq_i}\|f_{l_i}\|^{q_i}_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\sum_{k=l_i+2}^{L}2^{(l_i-k)(n\delta_1-\alpha_i)\frac{q_i}{2}}\Big\}^{\frac{1}{q_i}}$$

$$\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda_i}\Big\{\sum_{l_i=-\infty}^{L-2}2^{l_i\alpha_iq_i}\|f_{l_i}\|^{q_i}_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\Big\}^{\frac{1}{q_i}}$$

$$\leq C\|f_i\|_{M\dot{K}^{\alpha_i,\lambda_i}_{q_i,p_i(\cdot)}(w^{p_i(\cdot)})}.$$

Therefore, for any $0<q_i<\infty$, we could obtain

$$I_1\leq C\|f_1\|_{M\dot{K}^{\alpha_1,\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.$$

(ii) To estimate I_2, we have $l_1\leq k-2$, and $k-1\leq l_2\leq k+1$, then

$$|x-y_1|+|x-y_2|\geq|x-y_1|\geq||x|-|y_1||>2^{k-2},\ \text{for } x\in C_k,\ y_i\in C_{l_i},\ i=1,\ 2.$$

From the inequality above, we obtain

$$\Big\|\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{k+1}T(f_{l_1},f_{l_2})\chi_k\Big\|_{L^{p(\cdot)}(w^{p(\cdot)})}$$
$$\leq C\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{k+1}2^{(l_1-k)n\delta_1}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\frac{\|\chi_{B_{l_2}}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

By Lemma 4, we have

if $l_2=k-1$, $\frac{\|\chi_{B_{l_2}}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}\leq C\Big(\frac{|B_{l_2}|}{|B_k|}\Big)^{\delta_1}=C2^{(l_2-k)n\delta_1}=C2^{-n\delta_1}$,

if $l_2=k$, $\frac{\|\chi_{B_{l_2}}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}=1$,

if $l_2=k+1$, $\frac{\|\chi_{B_{l_2}}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}\leq C\frac{|B_{l_2}|}{|B_k|}=C2^{(l_2-k)n}=C2^n$.

Combing the estimates above, we arrive the inequality

$$\frac{\|\chi_{B_{l_2}}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}\leq C,\qquad \text{for}\quad k-1\leq l_2\leq k+1,$$

where C is independent of l_2 and k.

For I_2, we get

$$I_2\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda}\Big\{\sum_{k=-\infty}^{L}2^{k\alpha q}\Big(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{k+1}2^{(l_1-k)n\delta_1}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big)^q\Big\}^{\frac{1}{q}}$$

$$\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda_1}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_1=-\infty}^{k-2}2^{(l_1-k)(n\delta_1-\alpha_1)}2^{l_1\alpha_1}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big)^{q_1}\Big\}^{\frac{1}{q_1}}$$

$$\times \sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_2=k-1}^{k+1} 2^{k\alpha_2}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big)^{q_2}\Big\}^{\frac{1}{q_2}}$$
$$=: CI_{21}I_{22}.$$

Note that

$$I_{21} = I_{11} \le C\|f_1\|_{M\dot{K}^{\alpha_1,\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}.$$

$$\begin{aligned}
I_{22} &= \sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L} 2^{k\alpha_2 q_2}\Big(\sum_{l_2=k-1}^{k+1}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big)^{q_2}\Big\}^{\frac{1}{q_2}}\\
&\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L} 2^{k\alpha_2 q_2}\|f_2\chi_{2^{k-1}\le|\cdot|\le 2^{k+1}}(\cdot)\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big\}^{\frac{1}{q_2}}\\
&\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L+1} 2^{k\alpha_2 q_2}\|f_2\chi_k\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big\}^{\frac{1}{q_2}}\\
&\le C\|f_2\|_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}$$

(iii) To estimate I_3, for $x\in C_k$, $y_i\in C_{l_i}$ and $l_1 \le k-2$, $l_2 \ge k+2$, we have

$$|x-y_1| \ge |x|-|y_1| > 2^{k-2},\quad |x-y_2| \ge |y_2|-|x| > 2^{l_2-2}.$$

Thus, for $x\in C_k$, we get

$$|T(f_{l_1}, f_{l_2}(x)| \le C2^{-kn}\int_{\mathbb{R}^n}|f_{l_1}(y_1)|dy_1 2^{-l_2 n}\int_{\mathbb{R}^n}|f_{l_2}(y_2)|dy_2.$$

By Hölder's inequality, Lemmas 2 and 4, we obtain

$$\begin{aligned}
&\Big\|\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty} T(f_{l_1}, f_{l_2})\chi_k\Big\|_{L^{p(\cdot)}(w^{p(\cdot)})}\\
&\le C\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty} 2^{(k-l_2)n}\frac{\|\chi_{B_{l_1}}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}\frac{\|\chi_{B_{l_2}}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\le C\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty} 2^{(k-l_2)n}\Big(\frac{|B_{l_1}|}{|B_k|}\Big)^{\delta_1}\frac{|B_{l_2}|}{|B_k|}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\le C\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty} 2^{(l_1-k)n\delta_1}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}$$

For I_3, we get

$$\begin{aligned}
I_3 &\leq C \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\Big\{ \sum_{k=-\infty}^{L} 2^{k\alpha q}\Big(\sum_{l_1=-\infty}^{k-2} \sum_{l_2=k+2}^{\infty} 2^{(l_1-k)n\delta_1} \|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big)^q \Big\}^{\frac{1}{q}} \\
&\leq C \sup_{L\in\mathbb{Z}} 2^{-L\lambda_1}\Big\{ \sum_{k=-\infty}^{L} \Big(\sum_{l_1=-\infty}^{k-2} 2^{(l_1-k)(n\delta_1-\alpha_1)} 2^{l_1\alpha_1} \|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big)^{q_1} \Big\}^{\frac{1}{q_1}} \\
&\times \sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{ \sum_{k=-\infty}^{L} \Big(\sum_{l_2=k+2}^{\infty} 2^{k\alpha_2} \|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big)^{q_2} \Big\}^{\frac{1}{q_2}} \\
&=: C I_{31} I_{32}.
\end{aligned}$$

Note that

$$I_{31} = I_{21} = I_{11} \leq C\|f_1\|_{M\dot{K}^{\alpha_1,\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}.$$

As for I_{32}, we write

$$\begin{aligned}
I_{32} &= \sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{ \sum_{k=-\infty}^{L} 2^{k\alpha_2 q_2}\Big(\sum_{l_2=k+2}^{\infty} \|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big)^{q_2} \Big\}^{\frac{1}{q_2}} \\
&\leq \sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{ \sum_{k=-\infty}^{L} \Big(\sum_{l_2=k+2}^{L-1} 2^{(k-l_2)\alpha_2} 2^{l_2\alpha_2} \|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big)^{q_2} \Big\}^{\frac{1}{q_2}} \\
&+ \sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{ \sum_{k=-\infty}^{L} \Big(\sum_{l_2=L}^{\infty} 2^{(k-l_2)\alpha_2} 2^{l_2\alpha_2} \|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big)^{q_2} \Big\}^{\frac{1}{q_2}} \\
&=: I_{32}^1 + I_{32}^2.
\end{aligned}$$

Now, we estimate I_{32}^1 and I_{32}^2 respectively. For I_{32}^1, using similar methods as that for I_{1i}. Observing that $\alpha_2 > 0$, we consider the following two cases:

When $0 < q_2 \leq 1$, by (3.3), it follows that

$$\begin{aligned}
I_{32}^1 &\leq C \sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{ \sum_{k=-\infty}^{L} \sum_{l_2=k+2}^{L-1} 2^{(k-l_2)\alpha_2 q_2} 2^{l_2\alpha_2 q_2} \|f_{l_2}\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \Big\}^{\frac{1}{q_2}} \\
&= C \sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{ \sum_{l_2=-\infty}^{L-1} 2^{l_2\alpha_2 q_2} \|f_{l_2}\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \sum_{k=-\infty}^{l_2-2} 2^{(k-l_2)\alpha_2 q_2} \Big\}^{\frac{1}{q_2}} \\
&\leq C \sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{ \sum_{l_2=-\infty}^{L-1} 2^{l_2\alpha_2 q_2} \|f_{l_2}\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \Big\}^{\frac{1}{q_2}} \\
&\leq C\|f_2\|_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}$$

When $1 < q_2 < \infty$, we use Hölder's inequality and obtain

$$
\begin{aligned}
I_{32}^1 &\leq C\sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_2=k+2}^{L-1} 2^{(k-l_2)\alpha_2\frac{q_2}{2}} 2^{l_2\alpha_2 q_2}\|f_{l_2}\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big)\\
&\times\Big(\sum_{l_2=k+2}^{L-1} 2^{(k-l_2)\alpha_2)\frac{q_2'}{2}}\Big)^{\frac{q_2}{q_2'}}\Big\}^{\frac{1}{q_2}}\\
&\leq C\sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_2=k+2}^{L-1} 2^{(k-l_2)\alpha_2\frac{q_2}{2}} 2^{l_2\alpha_2 q_2}\|f_{l_2}\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big)\Big\}^{\frac{1}{q_2}}\\
&= C\sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{l_2=-\infty}^{L-1} 2^{l_2\alpha_2 q_2}\|f_{l_2}\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\sum_{k=-\infty}^{l_2-2} 2^{(k-l_2)\alpha_2\frac{q_2}{2}}\Big\}^{\frac{1}{q_2}}\\
&\leq C\|f_2\|_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}
$$

For I_{32}^2, when $0 < q_2 \leq 1$, by the fact that $0 < \lambda_2 < \alpha_2$, we have

$$
\begin{aligned}
I_{32}^2 &\leq C\sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}\sum_{l_2=L}^{\infty} 2^{(k-l_2)\alpha_2 q_2} 2^{l_2\alpha_2 q_2}\|f_{l_2}\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big\}^{\frac{1}{q_2}}\\
&\leq C\sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}\sum_{l_2=L}^{\infty} 2^{(k-l_2)\alpha_2 q_2} 2^{l_2\lambda_2 q_2} 2^{-l_2\lambda_2 q_2}\sum_{j=-\infty}^{l_2} 2^{j\alpha_2 q_2}\|f_{2j}\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big\}^{\frac{1}{q_2}}\\
&\leq C\sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}\sum_{l_2=L}^{\infty} 2^{(k-l_2)\alpha_2 q_2} 2^{l_2\lambda_2 q_2}\|f_2\|^{q_2}_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}\Big\}^{\frac{1}{q_2}}\\
&= C\sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L} 2^{k\alpha_2 q_2}\sum_{l_2=L}^{\infty} 2^{l_2(\lambda_2-\alpha_2)q_2}\|f_2\|^{q_2}_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}\Big\}^{\frac{1}{q_2}}\\
&= C\sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{2^{L\alpha_2 q_2} 2^{L(\lambda_2-\alpha_2)q_2}\|f_2\|^{q_2}_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}\Big\}^{\frac{1}{q_2}}\\
&= \|f_2\|_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}
$$

When $1 < q_2 < \infty$, because $\alpha_2 - \lambda_2 > 0$, we have

$$
\begin{aligned}
I_{32}^2 &= \sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_2=L}^{\infty} 2^{l_2\alpha_2}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} 2^{\frac{(k-l_2)(\alpha_2+\lambda_2)}{2}} 2^{\frac{(k-l_2)(\alpha_2-\lambda_2)}{2}}\Big)^{q_2}\Big\}^{\frac{1}{q_2}}\\
&\leq C\sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_2=L}^{\infty} 2^{l_2\alpha_2 q_2}\|f_{l_2}\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} 2^{\frac{(k-l_2)(\alpha_2+\lambda_2)q_2}{2}}\Big)\\
&\times\Big(\sum_{l_2=L}^{\infty} 2^{\frac{(k-l_2)(\alpha_2-\lambda_2)q_2'}{2}}\Big)^{\frac{q_2}{q_2'}}\Big\}^{\frac{1}{q_2}}\\
&\leq C\sup_{L\in\mathbb{Z}} 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}\sum_{l_2=L}^{\infty} 2^{l_2\alpha_2 q_2}\|f_{l_2}\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} 2^{\frac{(k-l_2)(\alpha_2+\lambda_2)q_2}{2}}\Big\}^{\frac{1}{q_2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}\sum_{l_2=L}^{\infty}2^{\frac{(k-l_2)(\alpha_2+\lambda_2)q_2}{2}}2^{l_2\lambda_2q_2}2^{-l_2\lambda_2q_2}\sum_{j=-\infty}^{l_2}2^{j\alpha_2q_2}\|f_{2j}\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big\}^{\frac{1}{q_2}}\\
&\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}\sum_{l_2=L}^{\infty}2^{\frac{(k-l_2)(\alpha_2+\lambda_2)q_2}{2}}2^{l_2\lambda_2q_2}\|f_2\|^{q_2}_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}\Big\}^{\frac{1}{q_2}}\\
&= C\sup_{L\in\mathbb{Z}}2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}2^{k\lambda_2q_2}\sum_{l_2=L}^{\infty}2^{\frac{(k-l_2)(\alpha_2-\lambda_2)q_2}{2}}\|f_2\|^{q_2}_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}\Big\}^{\frac{1}{q_2}}\\
&\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}2^{k\lambda_2q_2}\|f_2\|^{q_2}_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}\Big\}^{\frac{1}{q_2}}\\
&\leq C\|f_2\|_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}
$$

Therefore we get

$$I_3\leq C\|f_1\|_{M\dot{K}^{\alpha_1,\lambda_1}_{q_1,p_1(\cdot)}(w^{p_2(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.$$

(iv) To estimate the term I_5, by Theorem 1, the $L^{p(\cdot)}(w^{p(\cdot)})$- boundedness of T, we note that

$$\|T(f_{l_1},f_{l_2})\|_{L^{p(\cdot)}(w^{p(\cdot)})}\leq C\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

Since $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, we have

$$
\begin{aligned}
I_5&\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda}\Big\{\sum_{k=-\infty}^{L}2^{k\alpha q}\Big(\prod_{i=1}^{2}\sum_{l_i=k-1}^{k+1}\|f_{l_i}\|_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\Big)^{q}\Big\}^{\frac{1}{q}}\\
&\leq C\sup_{L\in\mathbb{Z}}\prod_{i=1}^{2}2^{-L\lambda_i}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_i=k-1}^{k+1}2^{k\alpha_i}\|f_{l_i}\|_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\Big)^{q_i}\Big\}^{\frac{1}{q_i}}\\
&\leq C\sup_{L\in\mathbb{Z}}\prod_{i=1}^{2}2^{-L\lambda_i}\Big\{\sum_{k=-\infty}^{L+1}2^{k\alpha_iq_i}\|f_i\chi_k\|^{q_i}_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\Big\}^{\frac{1}{q_i}}\\
&\leq C\|f_1\|_{M\dot{K}^{\alpha_1,\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}
$$

(v) To estimate the term I_6, for C_k, $y_i\in C_{l_i}$, $k-1\leq l_1\leq k+1$, $l_2\geq k+2$, we have

$$|T(f_{l_1},f_{l_2}(x)|\leq C2^{-kn}\int_{\mathbb{R}^n}|f_{l_1}(y_1)|dy_1 2^{-l_2n}\int_{\mathbb{R}^n}|f_{l_2}(y_2)|dy_2.$$

By the generalized Hölder's inequality, Lemmas 2 and 4, we obtain

$$
\begin{aligned}
&\Big\|\sum_{l_1=k-1}^{k+1}\sum_{l_2=k+2}^{\infty}T(f_{l_1},f_{l_2})\chi_k\Big\|_{L^{p(\cdot)}(w^{p(\cdot)})}\\
&\leq C\sum_{l_1=k-1}^{k+1}\sum_{l_2=k+2}^{\infty}2^{(k-l_2)n}\frac{\|\chi_{B_{l_1}}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}\frac{\|\chi_{B_{l_2}}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}
\end{aligned}
$$

$$\leq C\sum_{l_1=k-1}^{k+1}\sum_{l_2=k+2}^{\infty}2^{(k-l_2)n}\frac{|B_{l_2}|}{|B_k|}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}$$
$$\leq C\sum_{l_1=k-1}^{k+1}\sum_{l_2=k+2}^{\infty}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})},$$

where

$$\frac{\|\chi_{B_{l_1}}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}\leq C,\qquad \text{for}\quad k-1\leq l_1\leq k+1,$$

with the constant C independent of l_1 and k.

Hence

$$I_6\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda}\Big\{\sum_{k=-\infty}^{L}2^{k\alpha q}\Big(\sum_{l_1=k-1}^{k+1}\sum_{l_2=k+2}^{\infty}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big)^q\Big\}^{\frac{1}{q}}$$
$$\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda_1}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_1=k-1}^{k+1}2^{k\alpha_1}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big)^{q_1}\Big\}^{\frac{1}{q_1}}$$
$$\times\sup_{L\in\mathbb{Z}}2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_2=k+2}^{\infty}2^{k\alpha_2}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big)^{q_2}\Big\}^{\frac{1}{q_2}}$$
$$:=CI_{61}I_{62}.$$

Here the estimate of I_{61} is equal to that of I_{22} and $I_{62}=I_{32}$.

(vi) Finally to estimate the term I_9, we note $l_2\geq k+2$ and $|x-y_i|>2^{l_i-2}$, for $x\in C_k,\ y_i\in C_{l_i}$,

$$|T(f_{l_1},f_{l_2}(x)|\leq C2^{-l_1n}\int_{\mathbb{R}^n}|f_{l_1}(y_1)|dy_1 2^{-l_2n}\int_{\mathbb{R}^n}|f_{l_2}(y_2)|dy_2.$$

Applying Hölder's inequality to the last integral, we obtain

$$\Big\|\sum_{l_1=k+2}^{\infty}\sum_{l_2=k+2}^{\infty}T(f_{l_1},f_{l_2})\chi_k\Big\|_{L^{p(\cdot)}(w^{p(\cdot)}))}$$
$$\leq C\sum_{l_1=k+2}^{\infty}\sum_{l_2=k+2}^{\infty}2^{(k-l_1)n}2^{(k-l_2)n}\frac{\|\chi_{B_{l_1}}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}\frac{\|\chi_{B_{l_2}}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}$$
$$\times\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}$$
$$\leq C\sum_{l_1=k+2}^{\infty}\sum_{l_2=k+2}^{\infty}2^{(k-l_1)n}\frac{|B_{l_1}|}{|B_k|}2^{(k-l_2)n}\frac{|B_{l_2}|}{|B_k|}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}$$
$$\leq C\sum_{l_1=k+2}^{\infty}\sum_{l_2=k+2}^{\infty}\|f_{l_1}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|f_{l_2}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

Thus

$$\begin{aligned} I_9 &\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda}\Big\{\sum_{k=-\infty}^{L} 2^{k\alpha q}\Big(\prod_{i=1}^{2}\sum_{l_i=k+2}^{\infty}\|f_{l_i}\|_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\Big)^q\Big\}^{\frac{1}{q}} \\ &\le C\sup_{L\in\mathbb{Z}}\prod_{i=1}^{2} 2^{-L\lambda_i}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l_i=k+2}^{\infty} 2^{k\alpha_i}\|f_{l_i}\|_{L^{p_i(\cdot)}(w^{p_i(\cdot)})}\Big)^{q_i}\Big\}^{\frac{1}{q_i}} \\ &=: CI_{91}I_{92} \le C\|f_1\|_{M\dot{K}^{\alpha_1,\lambda_1}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\|f_2\|_{M\dot{K}^{\alpha_2,\lambda_2}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}, \end{aligned}$$

where the estimate of I_{9i} $(i = 1,\ 2)$ is equal to that of I_{32}.

Combining all the estimates for I_i together $(i = 1, 2, \cdots, 9)$, we get

$$\|T(f_1, f_1)\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(w^{p(\cdot)})} \le C\prod_{j=1}^{m}\|f_j\|_{M\dot{K}^{\alpha_j,\lambda_j}_{q_j,p_j(\cdot)}(w^{p_j(\cdot)})}.$$

Consequently, we have proved the Theorem 2.

4. Conclusions

In this paper we define the Muckenhoupt weights with variable exponent and introduce weighted Morrey-Herz spaces $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(w^{p(\cdot)})$ with variable exponent $p(\cdot)$. We investigate the boundedness of multi-linear Calderon-Zygmund operator on weighted Lebesgue spaces and weighted Morrey-Herz spaces with variable exponents. The results we obtain are the generalization of these in the references, and they could be applied more widely.

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Conflict of interest

The authors declare no conflict of interest.

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