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Research article

Rotational periodic solutions for fractional iterative systems

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Abstract: In this paper, we devoted to deal with the rotational periodic problem of some fractional iterative systems in the sense of Caputo fractional derivative. Under one sided-Lipschtiz condition on nonlinear term, the existence and uniqueness of solution for a fractional iterative equation is proved by applying the Leray-Schauder fixed point theorem and topological degree theory. Furthermore, the well posedness for a nonlinear control system with iteration term and a multivalued disturbance is established by using set-valued theory. The existence of solutions for a iterative neural network system is demonstrated at the end.

Keywords: existence; rotational periodic; fractional iterative systems; neural network
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1. Introduction

The research on iterative differential equations can be traced back to 1965, Petuhov studied a periodic boundary value problem of second-order differential iterative equation

$$x'' = \lambda x^{[2]}(t),\ x(0) = x(T) = \alpha \in R,$$

with $x^{[2]}(t) = x(x(t))$ in [1], where the author obtained the existence and uniqueness of solution of the equation when the parameters λ and α were in different ranges. Compared with ordinary differential equations, the appearance of iteration terms brings certain difficulties to study this type of iterative differential equations, and the fixed point theory is a common method to deal with such problems. Recently, Kaufmann considered the boundary value problem of a class of second-order differential iterative equations in [2], and used the Schauder fixed point theorem to show the existence of solution. In [3–6], researchers use different fixed point theories, including Krasnoselskii fixed point theorem, Schauder fixed point theorem, etc., to investigate the existence and uniqueness of periodic solutions or quasi-periodic solutions of several types of iterative differential equations.

In recent years, fractional differential equations has aroused the interest of many mathematicians and researchers in the fields of engineering, chemistry, and physics, etc. However, there are few results on boundary value and periodic problems of fractional iterative differential equations. It is worth noting that when the nonlinear function satisfies the Lipschitz condition, Ibrahim [7] extended some results of integer order to fractional iterative differential equations. However, to the best of the authors' knowledge, there is no report about vector iterative equations in R^N including integer order, since one-dimensional iterative differential systems were handled in previous works.

In this paper, we first consider the following rotational periodic boundary value problem (RPBVP) of fractional differential system:

$$\begin{cases} {}^{C}D^{\alpha}x(t) + Ax(t) = f(t, x(t), x_{+}^{[2]}(t)), \quad t \in I := [0, b], \\ x(b) = Qx(0), \end{cases} \tag{1.1}$$

where ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative with $\alpha \in (0, 1)$, $A : R^N \to R^N$ is a linear operator, $f : I \times R^N \times R^N \to R^N$ is a measurable function, and $Q \in O(n)$, where $O(n)$ denotes the group of orthogonal matrix. When Q-rotating periodic are in different ranges, different solutions are derived, such as periodic solutions if we let $Q = E_N$, where E_N represents the identity matrix in R^N, anti-periodic solutions if we let $Q = -E_N$, subharmonic solutions if we let $Q^k = E_N$ for some $k \in Z$, and quasi-periodic solutions if we let $Q = \mathrm{diag}(W(\theta_1), \cdots, W(\theta_k))$ for $N = 2k$ with $k \in Z$, or $Q = \mathrm{diag}(W(\theta_1), \cdots, W(\theta_k), \pm 1)$ for $N = 2k + 1$, where $W(\theta_i) = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$ and $\theta_i \in [0, 2\pi]$ $(i = 1, 2, \cdots, k)$.

The first objective of this work is to prove that the RPBVP (1.1) has a unique rotational periodic solution by using topology-degree theory and the Leray-Schauder fixed point theorem. It is worth noting that the solution x is called rotational periodic solution if x satisfies $x(t+T) = Qx(t)$ for all $t \in R$. Recently, more attention has been paid to the rotational periodic solutions of differential equations. The existence of rotating periodic solutions for second-order differential equations are demonstrated in [8,9] by using the coincidence degree method. The existence of rotating periodic solutions for second-order Hamiltonian system is investigated in [10] via the technique of penalized functionals and Morse theory, where the resonance condition at infinity is satisfied. Applying the homotopy continuation method, the rotating periodic problems of second order differential equation is considered in [11], where the nonlinearity satisfies the Hartman-type condition. The interested readers are referred to see [12, 13] for physical investigation on the rotational periodic.

Furthermore, we consider a fractional nonlinear control system of the following form:

$$\begin{cases} {}^{C}D^{\alpha}x(t) + Ax(t) \in f(t, x(t), x_{+}^{[2]}(t)) + U(t) + d(t, x), \quad t \in I \\ U(t) \in F(t, x) \\ x(b) = Qx(0) \end{cases} \tag{1.2}$$

where A, Q, f are present as in RPBVP (1.1), $U : I \to R^N$ is a control input, $F : I \times R^N \to 2^{R^N} \setminus \{\emptyset\}$ is a multifunction of observation data, and $d : I \times R^N \to 2^{R^N} \setminus \{\emptyset\}$ is a multivalued disturbance function. Motivated by [14, 15], we use the techniques of functional analysis and set-value theory to get the existence of solutions for RPBVP (1.2).

It is well known that dynamic neural networks systems are widely used in combinatorial optimization, associative memory, image and signal processing, pattern recognition and other fields. It should be mentioned that the existence and stability of solution for a fractional neural networks is studied in [16], where neuron activation functions are required to be continuous. Inspired by this, we also consider the fractional neural networks where neuron activations are discontinuous, and establish the existence of equilibrium point of this system at the end.

The remainder of this paper is organized as follows. some basic definitions and auxiliary results corresponding to the fractional calculus are introduced in section 2. In section 3, the existence and uniqueness of solution for RPBVP (1.1) is provided by applying the Leray-Schauder fixed point theorem and topological degree theory. In section 4, the well posedness for a nonlinear control system (1.2) is established by using set-valued theory, followed up the existence of solution for a iterative neural network system in section 5.

2. Preliminaries

Let $I := [0, b]$, where b is greater than a constant given later. Let R^N be an N-dimensional Euclid space, where the inner product and norm of R^N are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let $C([0, b]; R^N)$ denote the space composed of all continuous functions from $[0, b]$ to R^N with $\|x\|_C = \max_{t\in I} \|x(t)\|$, $\forall x \in C([0, b]; R^N)$. Let $\|\cdot\|_\infty$ denote the norm of $L^\infty[0, b]$. For the basic properties of fractional calculus, we refer the readers to see [17–19].

Definition 2.1. *The fractional integral of order $\alpha > 0$ of a function y is defined as*

$$^{C}I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau, \qquad t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *The fractional derivative of order α in the sense of Caputo for a function $y \in C^n[0, b]$ is defined as*

$$^{C}D^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}y^{(n)}(\tau)d\tau, \qquad t > 0,$$

where $\alpha \in (n-1, n)$, $\forall n \in N$.

A property regarding the Caputo fractional derivative is given as follows:

Proposition 2.1. *If $y \in C^n[0, T]$, then*

$$^{C}I^{\alpha C}D^{\alpha}y(t) = y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}t^k,$$

where $\alpha \in (n-1, n)$, $\forall n \in N$. In particular, if $y \in C^1[0, T]$ and $\alpha \in (0, 1)$, then

$$^{C}I^{\alpha C}D^{\alpha}y(t) = y(t) - y(0).$$

Now, we state the fractional Gronwall's inequality which will be used later.

Lemma 2.1. [20] *Suppose that the nonnegative integrable function $m(t)$ is defined on I, the nonnegative function $p(t)$ is non-decreasing on I, and the nonnegative integrable function $Y(t)$ satisfies*

$$Y(t) \leq m(t) + p(t)\int_0^t (t-s)^{\alpha-1} Y(s)ds, \quad \forall t \in I,\ \alpha > 0.$$

Then

$$Y(t) \leq m(t)E_\alpha[\Gamma(\alpha)p(t)t^\alpha],$$

where $E_\alpha(w) := \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\alpha k+1)}$, $\forall w \in R$, is called the single parameter Mittag-Leffler function.

Several inequalities on fractional derivative are given in the following.

Lemma 2.2. [21] *Assume that $Y : I \to R^N$ is a continuous differentiable function, and the matrix $P \in R^{N\times N}$ is a positive definite. Then we have*

$$\frac{1}{2}{}^C D^\alpha[Y^T(t)PY(t)] \leq Y^T(t)P{}^C D^\alpha Y(t),$$

for any $\alpha \in (0,1)$.

Lemma 2.3. [22] *Suppose $W : R \to R^+$ is a continuous function satisfying*

$${}^C D^\alpha W(t) \leq -\omega W(t), \quad \forall \alpha \in (0,1),$$

where ω is a positive constant. Then the following inequality holds:

$$W(t) \leq W(0)E_\alpha(-\omega t^\alpha), \quad \forall t \geq 0.$$

We present Leray-Schauder alternative theorem which plays an crucial role in our proofs.

Lemma 2.4. *[23] Assume that $\mathcal{X}$ is a Banach space, the set $\mathcal{Q} \subseteq \mathcal{X}$ is nonempty and convex with $0 \in \mathcal{Q}$, and $G : \mathcal{Q} \to \mathcal{Q}$ is an upper semicontinuous multifunction which has compact convex value and maps bounded sets to relatively compact sets, then one of the following arguments is valid:*
(i) $\Gamma = \{x \in \mathcal{Q} : z \in \beta G(z), \beta \in (0,1)\}$ is an unbounded set;
(ii) the multifunction $G(\cdot)$ has a fixed point, i.e. there exists $x \in \mathcal{Q}$ such that $z \in G(z)$.

Let $E_{\alpha,\beta}(w) := \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\alpha k+\beta)}$, $\forall w \in R$ be the two parameter Mittag-Leffler function. For notational convenience, set $M_\alpha = \max_{t\in I} \|E_\alpha(At^\alpha)\|$, $\hat{M}_\alpha = \max_{t\in I} \|E_{\alpha,\alpha}(At^\alpha)\|$, $M_E = \|(Q - E_\alpha(Ab^\alpha)^{-1}\|$. Throughout the article, we assume $b > \mathcal{M}^{\frac{1}{1-\alpha}}$ with $\mathcal{M} := \frac{(M_E M_\alpha+1)\hat{M}_\alpha M_\lambda}{\alpha}$, where M_λ is a positive constant.

3. Fractional iterative differential equations

Consider the following fractional iterative vector differential equations

$$\begin{aligned} &{}^C D^\alpha x(t) + Ax(t) = f(t, x(t), x_+^{[2]}(t)), \quad t \in I, \\ &x(b) = Qx(0), \end{aligned} \tag{3.1}$$

where $x_+^{[2]}(t) = (x_1(\|x\|), x_2(\|x\|), \cdots, x_n(\|x\|))$, $A : R^N \to R^N$ is a linear operator, $f : I \times R^N \times R^N \to R^N$ is a Carathéodory function. The required assumptions are given below.

$H(A)$: $A : R^N \to R^N$ is a bounded, linear positive definite operator, that is, for any $y \in R^N$, there exists a constant $c \in R^+$ such that $\langle Ay, y\rangle \geq c\|y\|^2$.

$H(f)$: $f : I \times R^N \times R^N \to R^N$ is a Carathéodory function such that

(i) for any $v, u \in R^N$, there exists a nonnegative function $\lambda \in L^\infty[0, b]$ with $\|\lambda\|_\infty \leq \frac{M_\lambda}{2}$, such that $\|f(t, v, u)\| \leq \lambda(t),\ \forall t \in [0, b]$;

(ii) for any $t \in I$, and $v_1, v_2, u_1, u_2 \in R^N$, there exists a nonnegative function $\mu \in L^\infty[0, b]$, such that

$$\langle f(t, v_1, u_1) - f(t, v_2, u_2), v_1 - v_2\rangle \leq \mu(t)\|v_1 - v_2\|^2,$$

where $\|\mu\|_\infty < c$, c is the positive constant provided in assumption $H(A)$.

Theorem 3.1. *Assume that $H(A)$ and $H(f)$ hold, then the fractional iterative differential system* (3.1) *has a unique solution.*

Proof. It is easy to verify that the problem (3.1) is equivalent to the following integral iterative equation [24]

$$x(t) = E_\alpha(At^\alpha)x(0) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^\alpha)f(\tau, x(\tau), x_+^{[2]}(\tau))d\tau,\ \forall t \in I. \tag{3.2}$$

Define an operator $T_1 : C([0, b]; R^N) \to C([0, b]; R^N)$ by

$$T_1x(t) = E_\alpha(At^\alpha)x(0) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^\alpha)f(\tau, x(\tau), x_+^{[2]}(\tau))d\tau. \tag{3.3}$$

We divide the proof process into three steps.

Step 1. The priori boundedness of the solutions for problem (3.1).

Invoke the definition of operator T_1 and the hypothesis $H(f)(i)$, to deduce

$$\begin{aligned} \|T_1x\|_C &\leq \|x(0)\|\|E_\alpha(At^\alpha)\|_C + \max_{t\in I}\|E_{\alpha,\alpha}(At^\alpha)\| \max_{t\in I}\int_0^t (t-s)^{\alpha-1}|f(s, x(s), x^{[2]}(s))|ds \\ &\leq \|x(0)\|M_\alpha + \|\lambda\|_\infty \hat{M}_\alpha \max_{t\in I}\int_0^t (t-s)^{\alpha-1}ds \\ &\leq \|x(0)\|M_\alpha + \frac{\|\lambda\|_\infty \hat{M}_\alpha}{\alpha}b^\alpha, \end{aligned} \tag{3.4}$$

where $M_\alpha = \max_{t\in I}\|E_\alpha(At^\alpha)\|$, $\hat{M}_\alpha = \max_{t\in I}\|E_{\alpha,\alpha}(At^\alpha)\|$, for any $t \in I$. Now, we estimate the initial value $\|x(0)\|$. In Eq (3.2), taking $t = b$, we have

$$x(b) = E_\alpha(Ab^\alpha)x(0) + \int_0^b (b-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(b-\tau)^\alpha)f(\tau, x(\tau), x_+^{[2]}(\tau))d\tau.$$

Since $x(b) = Qx(0)$ and hypothesis $H(A)$, it is easy to check that the determinant $|Q - E_\alpha(Ab^\alpha)| \neq 0$, so

$$x(0) = (Q - E_\alpha(Ab^\alpha))^{-1}\int_0^b (b-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(b-\tau)^\alpha)f(\tau, x(\tau), x_+^{[2]}(\tau))d\tau.$$

From the hypothesis $H(f)(i)$, in a similar fashion as (3.4), we derive directly that

$$\|x(0)\| \leq \frac{M_E\hat{M}_\alpha\|\lambda\|_\infty b^\alpha}{\alpha}, \tag{3.5}$$

where $M_E = \|(Q - E_\alpha(Ab^\alpha))^{-1}\|$. Substitute (3.5) into (3.4) to obtain

$$\|T_1 x\|_C \leq \frac{(M_E M_\alpha + 1)\hat{M}_\alpha \|\lambda\|_\infty}{\alpha} b^\alpha \leq \mathcal{M} b^\alpha, \tag{3.6}$$

where $\mathcal{M} = \frac{(M_E M_\alpha+1)\hat{M}_\alpha M_\lambda}{\alpha}$. Due to $b > \mathcal{M}^{\frac{1}{1-\alpha}}$, one has $\|T_1 x\|_C \leq \mathcal{M} b^\alpha < b$.

Step 2. The existence of the solution to problem (3.1).

To begin with, we show that $T_1 x \in C([0,b]; R^N)$ for any $x \in C([0,b]; R^N)$. For any $t, t+\delta \in [0,b]$, and $\delta > 0$, it follows from (3.3) that

$$\begin{aligned}
&|T_1 x(t+\delta) - T_1 x(t)| \\
&\leq \| \int_0^{t+\delta} (t+\delta-s)^{\alpha-1} E_{\alpha,\alpha}(A(t+\delta-s)^\alpha) f(s, x(s), x^{[2]}(s)) ds \\
&\quad - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) f(s, x(s), x^{[2]}(s)) ds \| \\
&\quad + \|[E_\alpha(A(t+\delta)^\alpha) - E_\alpha(At^\alpha)]x(0)\| \\
&\leq \|\lambda\|_\infty \hat{M}_\alpha | \int_0^{t+\delta} (t+\delta-s)^{\alpha-1} ds + \int_0^t (t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1} ds | \\
&\quad + \|[E_\alpha(A(t+\delta)^\alpha) - E_\alpha(At^\alpha)]x(0)\| \\
&\leq \frac{2\|\lambda\|_\infty \hat{M}_\alpha}{\alpha} \delta^\alpha + \frac{2\|\lambda\|_\infty \hat{M}_\alpha}{\alpha} |(t+\delta-a)^\alpha - (t-a)^\alpha| \\
&\quad + \|[E_\alpha(A(t+\delta)^\alpha) - E_\alpha(At)^\alpha]x(0)\|.
\end{aligned}$$

When $\delta \to 0$, we have $|T_1 x(t+\delta) - T_1 x(t)| \to 0$, therefore $T_1 x \in C([0,b]; R^N)$. Taking $x_n \to x \in C([0,b]; R^N)$ where $x_n(t) := (x_{1n}(t), x_{2n}(t), \cdots, x_{nn}(t)) \in R^N$ and $x(t) := (x_1(t), x_2(t), \cdots, x_n(t)) \in R^N$ for $t \in [0,b]$, we arrive at $x_{in}(\|x\|) \to x_i(\|x\|)$ for each $i = 1, 2, \cdots, n$, which together with the continuity of $(s, v) \to f(t, s, v)$, yields $|T_1 x_n - T_1 x| \to 0$. Hence, $T_1 : C([0,b]; R^N) \to C([0,b]; R^N)$ is continuous. According to the prior estimation (Step.1) and applying Arzela-Ascoli theorem, we obtain that the operator $T_1 : \Omega \to \Omega$ is completely continuous, where

$$\Omega = \{u \in C([0,b]; R^N) : \|u\|_C \leq b + 1\}.$$

Thus, the existence of solutions for the differential iterative system (3.1) can be transformed into a fixed point problem of T_1. Define the mapping $h_\varepsilon(x) = x - \varepsilon T_1(x)$ for $x \in C([0,b]; R^N)$, where $\varepsilon \in [0,1]$. Let $p \notin h(\partial\Omega)$, for any $\varepsilon \in [0,1]$, this allows us to get

$$\deg(h_\varepsilon, \Omega, p) = \deg(h_1, \Omega, p) = \deg(I_E - T_1, \Omega, p) = \deg(h_0, \Omega, p) = \deg(I_E, \Omega, p) = 1 \neq 0,$$

where I_E is the identity map. Therefore, T_1 has a fixed point on Ω, namely $x = T_1 x$, so the existence of the solution x for differential iterative system (3.1) follows.

Step 3. The uniqueness of the solution for problem (3.1).

Suppose that $x_1, x_2 \in C([0,b]; R^N)$ are two solutions of the problem (3.1). Substitute x_1 and x_2 into (3.1) respectively, then take a difference and the inner product with $x_1 - x_2$, to obtain

$$\begin{aligned}
&\langle x_1(t)-x_2(t), {}^CD^\alpha(x_1(t)-x_2(t))\rangle + \langle x_1(t)-x_2(t), A(x_1(t)-x_2(t))\rangle \\
&\qquad = \langle x_1(t)-x_2(t), f(t,x_1(t),x_{1+}^{[2]}(t)) - f(t,x_2(t),x_{2+}^{[2]}(t))\rangle.
\end{aligned}$$

By virtue of the hypotheses $H(A)$ and $H(f)$(ii), invoking Lemma 2.2, one gets

$$\begin{aligned}
D^\alpha\|x_1(t)-x_2(t)\|^2 &\leq 2\langle x_1(t)-x_2(t), D^\alpha(x_1(t)-x_2(t))\rangle \\
&\leq 2\mu(t)\|x_1-x_2\|^2 - 2c\|x_1-x_2\|^2.
\end{aligned}$$

Set $\mathcal{U}(t) = \|x_1(t)-x_2(t)\|^2$ for brevity, the above estimate can be simplified as

$${}^CD^\alpha\mathcal{U}(t) \leq 2(\mu(t)-c)\mathcal{U}(t).$$

Apply Lemma 2.3 to present

$$\mathcal{U}(t) \leq \mathcal{U}(0)E_\alpha(2(\|\mu\|_\infty - c)t^\alpha), \quad \forall t \in I. \tag{3.7}$$

Taking $t = b$ in (3.7), one obtains

$$\mathcal{U}(b) \leq \mathcal{U}(0)E_\alpha((2\|\mu\|_\infty - c)b^\alpha). \tag{3.8}$$

Since boundary condition $x_i(b) = Qx_i(0)$ $(i = 1,2)$, one can find

$$\begin{aligned}
\mathcal{U}(b) &= \|x_1(b)-x_2(b)\|^2 \\
&= \|Qx_1(0)-Qx_2(0)\|^2 \\
&= \langle Q(x_1(0)-x_2(0)), Q(x_1(0)-x_2(0))\rangle \\
&= (x_1(0)-x_2(0))^TQ^TQ(x_1(0)-x_2(0)) \\
&= \|x_1(0)-x_2(0)\|^2 \\
&= \mathcal{U}(0).
\end{aligned} \tag{3.9}$$

Hence, it follows from (3.8) that

$$\mathcal{U}(0)\{1 - E_\alpha[2(\|\mu\|_\infty - c)b^\alpha]\} \leq 0.$$

Due to the monotonicity of Mittag-Leffler function $E_\alpha(t)(t \geq 0)$ and $\|\mu\|_\infty < c$, we can conclude that $E_\alpha[2(\|\mu\|_\infty - c)b^\alpha] < 1$. Because $\mathcal{U}(0) = \|x_1(0)-x_2(0)\|^2 \geq 0$, we can derive $\mathcal{U}(0) = 0$. From (3.7), we have $\mathcal{U}(t) \leq 0$, and then $\mathcal{U}(t) \equiv 0$, that is, $x_1 \equiv x_2$, so the iterative differential equations (3.1) admits a unique solution, which our desired result follows.

4. Nonlinear control problem with a multivalued disturbance

In this section, consider the following nonlinear iterative control with a multivalued disturbance:

$$\begin{cases}
{}^CD^\alpha x(t) + Ax(t) \in f(t,x(t),x_+^{[2]}(t)) + U(t) + d(t,x), \quad t \in I, \\
U(t) \in F(t,x), \\
x(b) = Qx(0),
\end{cases} \tag{4.1}$$

where $A, Q\ f$ are present as in RPBVP (3.1), $U : I \to R^N$ is a control input, $F : I \times R^N \to 2^{R^N} \setminus \{\emptyset\}$ is a multifunction of observation data, and $d : I \times R^N \to 2^{R^N} \setminus \{\emptyset\}$ is a multivalued disturbance function. The hypotheses on F and d are given as follows:

$H(F)$: $F : I \times R^N \to 2^{R^N} \setminus \{\emptyset\}$ is a multivalued mapping with closed convex value such that

(i) $(t, w) \to F(t, w)$ is graph measurable for every $(t, w) \in I \times R^N$;

(ii) for almost all $t \in I$, $w \to F(t, w)$ has a closed graph;

(iii) for every $w \in R^N$ and all $t \in I$, there exists a nonnegative function $\lambda_1 \in L^\infty[0, b]$ such that

$$|F| = \sup\{\|f\|; f \in F\} \leq \lambda_1(t),$$

with $\|\lambda_1\|_\infty < \frac{1}{4}M_\lambda$.

$H(d)$ $d : I \times R^N \to 2^{R^N} \setminus \{\emptyset\}$ is a multivalued mapping with closed convex value such that

(i) $\forall\ x \in R^N,\ t \to d(t, x)$ is measurable;

(ii) $\forall\ t \in I,\ x \to d(t, x)$ is upper semicontinuity;

(iii) for all $x \in R^N$ and $t \in I$, $|d(t, x)| \leq \frac{1}{4}M_\lambda$.

Theorem 4.1. *If the assumptions $H(A), H(f), H(F)$ and $H(d)$ are satisfied, then the problem (4.1) admits at least one solution $x \in C([0, b]; R^N)$.*

Proof. First, introduce a closed convex subset $\mathcal{K}$ in $L^\infty(I; R^N)$ defined by

$$\mathcal{K} := \left\{u \in L^\infty(I; R^N);\ \|u\|_\infty \leq \frac{M_\lambda}{2}\right\}.$$

According to Theorem 3.1, it is straightforward to deduce that the following equation

$$\begin{cases} {}^C D^\alpha x(t) - f(t, x(t), x_+^{[2]}(t)) + Ax(t) = g(t), & t \in I, \\ x(b) = Qx(0), \end{cases} \tag{4.2}$$

has a unique solution $x_g \in C([0, b]; R^N)$ for each $g \in \mathcal{K}$. Define an operator

$$\mathcal{L} : D(\mathcal{L}) \subset C([0, b]; R^N) \to L^\infty([0, b]; R^N),$$

by

$$\mathcal{L}x = D^\alpha x - f(t, x(t), x_+^{[2]}(t)) + Ax, \quad x \in D(\mathcal{L}), \tag{4.3}$$

where $D(\mathcal{L}) := \{x \in C([0, b]; R^N),\ x(b) = Qx(0)\}$. Since $\mathcal{L} : D(\mathcal{L}) \to \mathcal{K}(\subset L^\infty([0, T]; R^N))$ is a one-to-one mapping, then it holds that $\mathcal{L}^{-1} : \mathcal{K} \to D(\mathcal{L})$ exists. Now, we show that the operator

$$\mathcal{L}^{-1} : \mathcal{K} \to D(\mathcal{L})$$

is completely continuous. For this, we will claim that $\mathcal{L}^{-1} : \mathcal{K} \to D(\mathcal{L})$ is continuous. Assume $g_m \to g$ in $\mathcal{K}$ as $m \to \infty$, it remains to show that $x_m = \mathcal{L}^{-1}(g_m) \to x = \mathcal{L}^{-1}(g)$ in $D(\mathcal{L})(\subset C([0, b]; R^N))$. Replace x with x_m in (4.2), then subtract (4.2) to get

$$\begin{aligned} {}^C D^\alpha(x_m(t) - x(t)) + A(x_m(t) - x(t)) &= g_m(t) - g(t) + f(t, x_m(t), x_{m+}^{[2]}(t)) \\ &\quad - f(t, x(t), x_+^{[2]}(t)). \end{aligned}$$

Taking the inner product with $x_m - x$ on the both side of above equation and using Lemma 2.2, implies

$$\begin{aligned}\frac{1}{2}{}^{C}D^{\alpha}\|x_m - x\|^2 &\leq \langle x_m(t) - x(t), {}^{C}D^{\alpha}(x_m(t) - x(t))\rangle \\ &\leq \langle x_1(t) - x_2(t), f(t, x_1(t), x_{1+}^{[2]}(t)) - f(t, x_2(t), x_{2+}^{[2]}(t)) \\ &\quad -\langle x_m(t) - x(t), A(x_m(t) - x(t))\rangle \\ &\quad -\langle x_m(t) - x(t), g_m(t) - g(t)\rangle. \end{aligned} \tag{4.4}$$

Now integrate in time, invoking Proposition 2.1, to derive

$$\begin{aligned}&\frac{1}{2}\|x_m - x\|^2 \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\langle x_m(\tau) - x(\tau), f(t, x_m(\tau), x_{m+}^{[2]}(\tau)) - f(t, x(\tau), x_{+}^{[2]}(\tau)\rangle d\tau \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\langle x_m(\tau) - x(\tau), A(x_m(\tau) - x(\tau))\rangle d\tau \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\langle x_m(\tau) - x(\tau), g_m(\tau) - g(\tau)\rangle d\tau \\ &\quad + \frac{1}{2}\|x_m(0) - x(0)\|^2. \end{aligned} \tag{4.5}$$

Analogous the priori estimate of the solution to those of Theorem 3.1, one can find that $\|x_m\|_C \leq b$, which with $x_m \in C([0, b]; R^N)$ and Arzela-Ascoli theorem together, implies that there exists a subsequence x_m (still denoted by itself) such that $x_m \to \hat{x}$ in $D(\mathcal{L})$ as $m \to \infty$. Taking the limit in (4.5), this allow us to get

$$\begin{aligned}\frac{1}{2}\|\hat{x} - x\|^2 &\leq \|\hat{x}(0) - x(0)\|^2 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\langle \hat{x}(\tau) - x(\tau), A(\hat{x}(\tau) - x(\tau))\rangle d\tau \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\langle \hat{x}(\tau) - x(\tau), f(\tau, \hat{x}(\tau), \hat{x}_{+}^{[2]}(\tau)) - f(\tau, x(\tau), x_{+}^{[2]}(\tau))\rangle d\tau. \end{aligned} \tag{4.6}$$

Analogous analysis to (3.7), set $\mathcal{Y} = \|\hat{x} - x\|^2$, then it follows from (4.6) that

$$\mathcal{Y} \leq \mathcal{Y}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} 2(\mu(\tau) - c)\mathcal{Y}(\tau)d\tau, \tag{4.7}$$

which together with Lemma 2.1 leads to

$$\mathcal{Y}(t) \leq \mathcal{Y}(0)E_{\alpha}(2(\|\mu\|_{\infty} - c)t^{\alpha}), \quad \forall t \in I. \tag{4.8}$$

Arguing as in (3.8), we can conclude that $\mathcal{Y}(t) \equiv 0$, i.e., $\hat{x} \equiv x$ in $D(\mathcal{L})$. This gives the continuity of operator $\mathcal{L}^{-1}$. In light of a priori estimate of the solution, it is easy to verify that $L^{-1}(\mathcal{K})$ is a bounded set in $C([0, b]; R^N)$. Thanks to Arzela-Ascoli theorem, $\mathcal{L}^{-1}(\mathcal{K}) \subset L^{\infty}([0, b]; R^N)$ is relatively compact. Therefore, $\mathcal{L}^{-1} : \mathcal{K} \to L^{\infty}([0, b]; R^N)$ is completely continuous.

Define a multivalued Nemitsky operator $\mathcal{N} : L^{\infty}([0, b]; R^N) \to 2^{L^{\infty}([0,b];R^N)}$ in terms of $F(t, x)$ and $d(t, x)$ given by

$$\mathcal{N}(x) = \{v \in L^{\infty}([0, b]; R^N);\ v(t) \in F(t, x) + d(t, x),\ a.e.\ t \in [0, b]\}.$$

Thanks to hypotheses $H(F)$ and $H(d)$, it results that the multivalued Nemitsky operator $\mathcal{N}(\cdot)$ is nonempty, closed, convex value, and upper hemicontinuous (Theorem 3.2, [25]). So we can see that $\mathcal{L}^{-1} \circ \mathcal{N} : \mathcal{K} \to L^{\infty}([0,T]; R^N)$ is a upper hemicontinuous multifunction with closed, convex value, and map a bounded set into a relatively compact set. Now consider the following fixed points problem

$$x \in \mathcal{L}^{-1} \circ \mathcal{N}(x). \tag{4.9}$$

For this, by using Lemma 2.4, it remains to show that the set $\Omega := \{x \in L^{\infty}([0,b]; R^N) : x \in \xi\mathcal{L}^{-1} \circ \mathcal{N}(x), \xi \in (0,1)\}$ is bounded. Let $x \in \Omega$, then $\mathcal{L}(\frac{x}{\xi}) \in \mathcal{N}(x)$, which gives

$$^{C}D^{\alpha}\frac{x}{\xi} - f(t, \frac{x(t)}{\xi}, \frac{x_+^{[2]}(t)}{\xi}) + \frac{Ax}{\xi} = g_1(t) + g_2(t), \tag{4.10}$$

where $g_1(t) \in F(x,t)$ and $g_2(t) \in d(x,t)$ for all $t \in I$. Likewise as in (3.2), the Eq (4.10) can be rewritten as

$$\begin{aligned} x(t) &= E_{\alpha}(At^{\alpha})x(0) + \xi \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^{\alpha}) f(\tau, \frac{x(\tau)}{\xi}, \frac{x_+^{[2]}(\tau)}{\xi}) d\tau \\ &\quad + \xi \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^{\alpha})(g_1(\tau) + g_2(\tau)) d\tau. \end{aligned} \tag{4.11}$$

The same arguments as in (3.4) and by taking into account $H(F)$(iii) and $H(d)$(iii), it follows from (4.11) that

$$\begin{aligned} \|x\|_C &\leq \|x(0)\| M_{\alpha} + (\|\lambda\|_{\infty} + \|\lambda_1\|_{\infty} + \frac{M_{\lambda}}{4}) \hat{M}_{\alpha} \max_{t \in I} \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \|x(0)\| M_{\alpha} + \frac{M_{\lambda} \hat{M}_{\alpha}}{\alpha} b^{\alpha}. \end{aligned} \tag{4.12}$$

Similar to the estimate $\|x(0)\|$ of (3.4), it holds that

$$\|x(0)\| \leq \frac{M_E \hat{M}_{\alpha} M_{\lambda} b^{\alpha}}{\alpha}, \tag{4.13}$$

which with (4.12) together gives that $\|x(t)\|$ is uniformly bounded in I. Invoking Lemma 2.4, there exists $x \in D(\mathcal{L})$, such that $x \in \mathcal{L}^{-1} \circ \mathcal{N}(x)$. Obviously, x is the solution of problem (3.1). This proof is thus complete.

5. Rotating boundary value problem of a fractional iterative neural network system

Consider the fractional iterative neural network model (FINN) described as follows:

$$D^{\alpha}x(t) + \hat{A}x(t) = g(x(t), x_+^{[2]}(t)) + \mathcal{I}(t), \ t \in I, \tag{5.1}$$

where $x : [0,b] \to R^N$ is the vector of neuron system; $\hat{A} = \mathrm{diag}(k_1, k_2, \cdots, k_N)$ is a constant diagonal matrix with $k_i > 0 (i = 1, 2, \cdots, N)$; $g : R^N \times R^N \to R^N$ represents the neuron input-output continuous activation function satisfying the following assumptions:

(i) For any $u, v \in R^N$, there exists a nonnegative function $\widehat{\lambda} \in L^\infty[0, b]$ with $\|\widehat{\lambda}\|_\infty \le \frac{M_\lambda}{2}$, such that $\|g(u, v)\| \le \widehat{\lambda}(t),\ \forall t \in [0, b]$;

(ii) For any $u_1, u_2, v_1, v_2 \in R^N$, there exists a nonnegative function $\widehat{\mu} \in L^\infty[0, b]$, such that

$$\langle g(u_1, v_1) - g(u_2, v_2), u_1 - u_2\rangle \le \widehat{\mu}(t)\|u_1 - u_2\|^2,$$

where $\|\widehat{\mu}\|_\infty < \min\{k_i : i = 1, 2, \cdots, N\}$; the mapping of neuron inputs $\mathcal{I} : [0, b] \to R^N$ is continuous with $\|\mathcal{I}\|_C \le \frac{1}{2}M_\lambda$.

The initial value problem of this system (5.1) without iteration was studied in [16], where the existence and uniqueness of equilibrium point was established. Here however considering the iterative term and the rotating periodic boundary value condition, we guarantee the existence of a unique rotational periodic boundary value solution to system (5.1) by using our results. It is easy to check that all assumptions of Theorem 3.1 holds, so the following result for system (5.1) is present.

Theorem 5.1. *Under the above assumptions, the problem (5.1) has a unique rotational periodic boundary value solution.*

It should be pointed out that the system (5.1) without iteration was examined in Song et al. [16] where input function $\mathcal{I}(t)$ is continuous. Naturally, a question is whether the system (5.1) has a solution if $\mathcal{I}(t) = (\mathcal{I}_1(t), \cdots, \mathcal{I}_N(t))$ is discontinuous. The following work is to answer this question. For this, we further hypothesize that $\mathcal{I}_j \in \Phi, (j = 1, 2, \cdots, N)$ are nondecreasing monotone bounded, where $\Phi : R \to R$ represents the class of functions which have at most finite jumping discontinuities in every closed interval. If there are only isolated jump discontinuities for any $\mathcal{I}_j (j = 1, 2, \cdots, N)$, then we deduce

$$\mathcal{R}(\mathcal{I}(t)) := ([\underline{\mathcal{I}_1}, \overline{\mathcal{I}_1}], [\underline{\mathcal{I}_2}, \overline{\mathcal{I}_2}], \cdots, [\underline{\mathcal{I}_N}, \overline{\mathcal{I}_N}])$$

where $\underline{\mathcal{I}_j} \le \mathcal{I}_j \le \overline{\mathcal{I}_j}, \underline{\mathcal{I}_j} = \varliminf_{\varepsilon \to t_j} \mathcal{I}_j(\varepsilon), \overline{\mathcal{I}_j} = \varlimsup_{\varepsilon \to t_j} \mathcal{I}_j(\varepsilon) (j = 1, 2, \cdots, N)$. Hence, in this way, the problem (5.1) can be rewritten as the following differential inclusion:

$$D^\alpha x(t) + \hat{A}x(t) \in g(x(t), x_+^{[2]}(t)) + \mathcal{R}(\mathcal{I}(t)). \tag{5.2}$$

Here $\mathcal{R}(\mathcal{I}(t))$ can be treated as a multivalued disturbance item of problem (4.1). Then similar to the argument of theorem 4.1, we can conclude the following theorem.

Theorem 5.2. *Under the given assumptions, then the solution set of problem (5.2) is nonempty.*

6. Conclusions

The rotational periodic problems of some fractional iterative systems in the sense of Caputo fractional derivative are investigated in this paper. First we use the Leray-Schauder fixed point theorem and topological degree theory to establish the existence and uniqueness of solution for a fractional iterative equation with one sided-Lipschtiz condition on nonlinear term. Furthermore, applying set-valued theory, the well posedness of a nonlinear control system with iteration term and a multivalued disturbance is completed. Finally, to reflect the application of fractional iterative systems, the existence of solutions for a iterative neural network system is demonstrated. Our future concerns are to examine the stability for these fractional iterative systems which is still an open problem.

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Conflict of interest

The authors declare that there are no conflict of interest.

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