Mathematics

AIMS Mathematics, 6(10): 11187–11199.
DOI: 10.3934/math.2021649
Received: 16 March 2021
Accepted: 01 July 2021
Published: 03 August 2021

http://www.aimspress.com/journal/Math

Research article

An analysis for a special class of solution of a Duffing system with variable delays

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Abstract: In this study, we are concerned the existence of pseudo almost automorphic (PAA) solutions and globally exponential stability of a Duffing equation system with variable delays. Some differential inequalities and the well-known Banach fixed point theorem are used for the existence and uniqueness of PAA solutions. Also, with the help of Lyapunov functions, sufficient conditions are obtained for globally exponential stability of PAA solutions. Since the PAA is more general than the almost and pseudo almost periodicity, this work is new and complementary compared to previous studies. In addition, an example is given to show the correctness of our results.

Keywords: Lyapunov function; Pseudo almost automorphic; Banach fixed point theorem; exponential stability; Duffing equation system
Mathematics Subject Classification: 34K14, 34K40

1. Introduction

As it is known, Duffing equations explain the motion of a mechanical system in the field of twin well potential. There are many mathematical methods and techniques to solve Duffing equation and Duffing oscillator (see [18,19,22,23]). However, due to their potential applications in mechanics, physics, electric distribution lines, and engineering, topic of the dynamic behaviors of Duffing differential equations has also gained considerable importance. (see [4,5,7,11,14,17,20]). Considering literature, it can be seen that many studies have been applied on qualitative behavior of solutions such as oscillation, periodicity, almost periodicity (AP), and pseudo almost periodicity (PAP) [1–3,6–10,12,13,15,16,21,24,25]. Authors of [10] considered the following constant coefficients Duffing equation with variable delay:

$$\varsigma''(t) + c\varsigma'(t) - a\varsigma(t) + b(\varsigma^{n-1}(t - l(t)) = e(t),\ n > 1, \tag{1.1}$$

where $l(t)$ and $e(t)$ are almost periodic functions and n is an integer, a, b and c are constants. By defining $p(t) = \varsigma'(t) + \xi\varsigma(t) - \Psi_1(t)$ and $\Psi_2(t) = -\Psi_1'(t) + e(t) + (\xi - c)\Psi_1(t),\ \xi > 1$.

$$\begin{aligned} &\varsigma'(t) = \Psi_1(t) - \xi\vartheta'(t) + p(t), \\ &p'(t) = \Psi_2(t) - (c - \xi)p(t) + (a - \xi(\xi - c))\varsigma(t) - b\varsigma^n(t - l(t)). \end{aligned} \tag{1.2}$$

Researchers achieved some results for almost periodic solutions of (1.2). [9] extended (1.2) to the following system

$$\begin{aligned} &\varsigma'(t) = \Psi_1(t) - \varsigma(t)k_1(t) + p(t), \\ &p'(t) = \Psi_2(t) - p(t)k_2(t) + \varsigma(t)(\gamma(t) - k_2^2(t))) - \varsigma^n(t - l(t))b(t). \end{aligned} \tag{1.3}$$

where $b(t) \neq 0$ and $\gamma(t) > 0$. In [9], some suitable conditions were obtained on PAP solutions of (1.3). S. Bochner in [3] introduced the theory of AA to the literature. Additionally, Xiao et al. [12] introduced the concept of PAA, which is a generalization of the AP, AA, and PAP. If the considered output is not showing periodic, AP or AA can check whether its behavior is PAA or not [26].

In this study, we consider the following variable coefficient Duffing system:

$$\varsigma''(t) + k_1(t)\varsigma'(t) - k_2(t)\varsigma(t) + k_3(t)\varsigma^n(t - l(t)) = e(t),\ n > 1. \tag{1.4}$$

Let $\tilde{c} \in R^+$. Define $p(t) = \varsigma'(t) + \tilde{c}\varsigma(t)$, (1.4) can be converted into the following equivalent system

$$\begin{cases} \varsigma'(t) = p(t) - \tilde{c}\varsigma(t) \\ p'(t) = -k_1(t)p(t) + (\tilde{c}k_1(t) - \tilde{c}^2 + k_2(t))\varsigma(t) - \varsigma^n(t - l(t))k_3(t) + e(t). \end{cases} \tag{1.5}$$

As far as we know from the literature, there is no study related to the PAA solutions of the system (1.5). Our purpose in this study is to obtain some suitable criteria for the existence, uniqueness, and globally exponential stability of PAA solutions of system (1.5). The results obtained are new and complementary to the previous studies.

2. Preliminary results

Define the following notations: $\{w_i(t)\} = (w_1(t), w_2(t)) \in R^2$, $|w| = \{|w_i(t)|\}$ and $\|w(t)\| = \max_{1\le i\le 2}\{|w_i(t)|\}$. Let $BC(R,R)$ denote collection of bounded continuous functions. $BC(R,R)$ is Banach space with norm $\|\theta\|_\infty = \sup_{t\in R}|\theta(t)|$. Also, we use the notations $\theta^+ = \sup_{t\in R}|\theta(t)|$, $\theta^- = \inf_{t\in R}|\theta(t)|$, where $\theta(t) \in BC(\mathbb{R},\mathbb{R})$.

Definition 2.1. [8] Let $f \in C(\mathbb{R},\mathbb{R})$. For f to be AP it is necessary and sufficient that the family of functions $H = \{f^h\} = \{f(t+h)\},\ -\infty < h < \infty,$ is compact in $C(R)$.

Definition 2.2. [2] $Q \in C(R,R)$ is called AA if for every real sequence (v_n), there exists a subsequence (v_{n_k}) such that $\lim_{n\to\infty} Q(t + v_{n_k}) = P(t)$ and $\lim_{n\to\infty} P(t - v_{n_k}) = Q(t)$ for each $t \in R$.

Example 2.1. Let $\upsilon(t) = 2 + sin\sqrt{5}\,t + sin\,t$ and $g(t) \in \mathbb{R}$ such that $g(t) = cos\frac{1}{\upsilon(t)}$. Then $g(t) \in AA(R,R)$, but $g(t)$ is not $AP(R,R)$. The graph of $g(t)$ on the analytical plane can be seen in Figure 1.

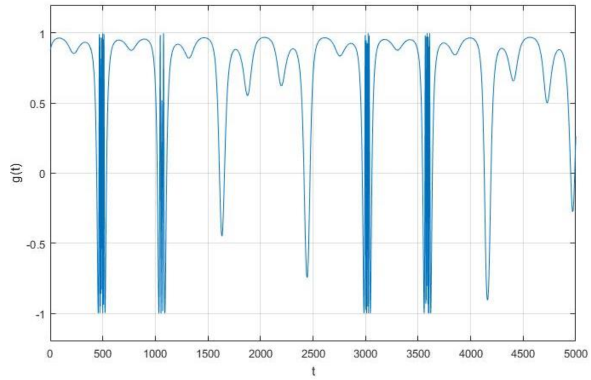

Figure 1. Graphic of $g(t)$.

Definition 2.3. [2] $f \in C(R,R)$ is called PAA if it can be noted as $f = f_1 + f_2$, with $f_1 \in AA(R,R)$ and $f_2 \in PAA_0(R,R)$ where space PAA_0 is defined by

$$PAA_0(R,R) := \left\{ f_2 \in BC(R,R) \middle| \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \| f_2(t) \| dt = 0 \right\}.$$

Lemma 2.1. [3] $PAA(R,R)$ is a Banach space with norm $\|\theta\|_\infty = \sup_{t\in R} |\theta(t)|$.

It is clear that $AA(R,R) \subset PAA(R,R) \subset BC(R,R)$.

Example 2.2. $f(t) = g(t) + e^{-2t}$ is PAA function but it is not $AA(R,R)$. The graph of $f(t)$ on the analytical plane can be seen in Figure 2.

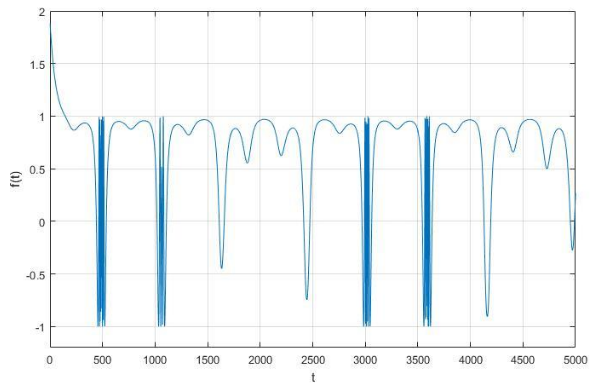

Figure 2. Graphic of $f(t)$.

Lemma 2.2. [1] If $l(t)\in AA(R,R)$ and $\Phi(t),\Upsilon(t)\in PAA(R,R)$, then $\Upsilon(t-k)$, $\Phi(t)\times\Upsilon(t),\Phi(t)+\Upsilon(t),\Upsilon(t-l(t))\in PAA(R,R)$.

Assume the following conditions for our main results.

(B_1) $n>1$ and $\bar{a}=\min\left(\tilde{c},\inf k_1(t)\right)$,

(B_2) $k_1(t),l(t)\in AA(R,R^+)$, and $k_2(t),k_3(t)\,,e(t)\in PAA(R,R)$ for all $t\in\mathbb{R}$,

(B_3) $\gamma=\dfrac{\sup_{t\in R}|e(t)|}{k_1^-}$, $\upsilon=\dfrac{1}{\bar{a}}\max\left\{1,\sup_{t\in R}\left|\tilde{c}k_1(t)-\tilde{c}^2+k_2(t)\right|+\sup_{t\in R}|k_3(t)|\right\}$, $\dfrac{\gamma}{1-\upsilon}<1$,

$\pi=\dfrac{1}{\bar{a}}\max\left\{1,\sup\limits_{t\in R}\left|\left[\left|\tilde{c}k_1(t)-\tilde{c}^2+k_2(t)\right|+n\left[2\gamma/(1-\upsilon)\right]^{n-1}\left|k_3(t)\right|\right],1\right|\right\}<1,$

(B_4) $\min\left\{\tilde{c},k_1^-+\sup\left(\left|\tilde{c}k_1(t)-\tilde{c}+k_2(t)\right|+n\left[2\gamma/(1-\upsilon)\right]^{n-1}|k_3(t)|\right)\right\}>0.$

3. PAA solutions

Theorem 3.1. Suppose that $(B_1)-(B_3)$ are satisfied. Define a nonlinear operator G for each $\varphi=\left(\varphi_1,\varphi_2\right)\in PAA(R,R^2)$, $\left(G\varphi\right):=\varphi(t)$ where

$$\varphi(t)=\left(\int_{-\infty}^{t}e^{\int_t^\eta \tilde{c}du}\gamma_1(\eta)d\eta,\int_{-\infty}^{t}e^{\int_t^\eta k_1(u)du}\gamma_2(\eta)d\eta\right)^T,$$

where

$$\gamma_1(\eta)=\varphi_2(\eta)$$
$$\gamma_2(\eta)=(\tilde{c}(k_1(\eta)-\tilde{c})+k_2(\eta))\varphi_1(\eta)-k_3(\eta)\varphi_1^m(\eta-\tau(\eta))+e(\eta).$$

Then $G\varphi\in PAA(R,R^2)$.

Proof. Noting that $G\varphi\in PAA(R,R^2)$, then it follows from Definition 2.2 such that

$$\begin{pmatrix}\gamma_1(\eta)\\ \gamma_2(\eta)\end{pmatrix}=\begin{pmatrix}\gamma_{11}(\eta)+\gamma_{12}(\eta)\\ \gamma_{21}(\eta)+\gamma_{22}(\eta)\end{pmatrix}.$$

Thus,

$$\left(G\varphi\right)=\begin{pmatrix}\int_{-\infty}^{t}e^{\int_t^\eta \tilde{c}du}(\gamma_{11}(\eta)+\gamma_{12}(\eta))d\eta\\ \int_{-\infty}^{t}e^{\int_t^\eta k_1(u)du}(\gamma_{21}(\eta)+\gamma_{22}(\eta))d\eta\end{pmatrix}$$

$$=\begin{pmatrix}\int_{-\infty}^{t}e^{\int_t^\eta \tilde{c}du}\gamma_{11}(\eta)d\eta\\ \int_{-\infty}^{t}e^{\int_t^\eta k_1(u)du}\gamma_{21}(\eta)d\eta\end{pmatrix}+\begin{pmatrix}\int_{-\infty}^{t}e^{\int_t^\eta \tilde{c}du}\gamma_{12}(\eta)d\eta\\ \int_{-\infty}^{t}e^{\int_t^\eta k_1(u)du}\gamma_{22}(\eta)d\eta\end{pmatrix}=\left(G_1\varphi\right)+\left(G_2\varphi\right).$$

Firstly, we prove that $\left(G_1\varphi\right)\in AA(R,R^2)$.

Let $(a'_n) \subset R$ be a sequence. We can extract a subsequence (a_n) of (a'_n) such that

$$\begin{pmatrix} \lim\limits_{n\to+\infty} \gamma_{11}(\eta + a_n) \\ \lim\limits_{n\to+\infty} \gamma_{21}(\eta + a_n) \end{pmatrix} = \begin{pmatrix} \tilde{\gamma}_{11}(t) \\ \tilde{\gamma}_{21}(t) \end{pmatrix}, \quad \begin{pmatrix} \lim\limits_{n\to+\infty} \tilde{\gamma}_{11}(\eta - a_n) \\ \lim\limits_{n\to+\infty} \tilde{\gamma}_{21}(\eta - a_n) \end{pmatrix} = \begin{pmatrix} \gamma_{11}(\eta) \\ \gamma_{21}(\eta) \end{pmatrix}, \quad t \in R.$$

Define

$$\left(\tilde{G}_1\varphi\right)(t) = \begin{pmatrix} \int_{-\infty}^{t} e^{\int_t^\eta \tilde{c}du} \tilde{\gamma}_{11}(\eta)d\eta \\ \int_{-\infty}^{t} e^{\int_t^\eta a_1(u)du} \tilde{\gamma}_{21}(\eta)d\imath \end{pmatrix}.$$

Then, we have

$$\left(G_1\varphi\right)(t+a_n) - \left(\tilde{G}_1\varphi\right)(t) = \begin{pmatrix} \int_{-\infty}^{t+a_n} e^{\int_\eta^{t+a_n} \tilde{c}du} \gamma_{11}(\eta)d\eta \\ \int_{-\infty}^{t+b_n} e^{\int_\eta^{t+a_n} k_1(u)du} \gamma_{21}(\eta)d\eta \end{pmatrix} - \begin{pmatrix} \int_{-\infty}^{t} e^{\int_\eta^t \tilde{c}du} \tilde{\gamma}_{11}(\eta)d\eta \\ \int_{-\infty}^{t} e^{\int_\eta^t \tilde{a}_1(u)du} \tilde{\gamma}_{21}(\eta)d\eta \end{pmatrix}$$

$$= \begin{pmatrix} \int_{-\infty}^{t} e^{\int_\eta^{t+a_n} \tilde{c}du} \gamma_{11}(u+b_n)du - \int_{-\infty}^{t} e^{\int_\eta^{t+a_n} \tilde{c}du} \tilde{\gamma}_{11}(u)du \\ \int_{-\infty}^{t} e^{\int_u^t k_1(\tau+a_n)d\tau} \gamma_{21}(u+b_n)du + \int_{-\infty}^{t} e^{-\int_u^t k_1(\tau+a_n)d\tau} \tilde{\gamma}_{21}(u)du \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ -\int_t^{+\infty} e^{-\int_u^t k_1(\tau+a_n)d\tau} \tilde{\gamma}_{21}(u)ds + \int_t^{+\infty} e^{-\int_t^s k_1(\tau)d\tau} \tilde{\gamma}_{21}(u)du \end{pmatrix}$$

$$= \begin{pmatrix} \int_{-\infty}^{t} e^{\int_u^t a(\tau+a_n)d\tau} [\gamma_{11}(u+a_n) - \tilde{\gamma}_{11}(u)]du \\ \int_{-\infty}^{t} e^{-\int_u^t k_1(\tau+a_n)d\tau} [\gamma_{21}(u+a_n) - \tilde{\gamma}_{21}(u)]du \end{pmatrix} + \begin{pmatrix} 0 \\ \int_{-\infty}^{t} [e^{-\int_t^u k_1(\tau+a_n)d\tau} - e^{-\int_t^u k_1(\tau)d\tau}]\tilde{\gamma}_{21}(u)du \end{pmatrix}.$$

Using Lebesgue Dominated Convergence Theorem in [6], we get

$$\lim_{n\to\infty} \left(G_1\varphi\right)(t+a_n) = \left(\tilde{G}_1\varphi\right)(t).$$

Similarly, we can write

$$\lim_{n\to\infty} \left(\tilde{G}_1\varphi\right)(t-a_n) = \left(G_1\varphi\right)(t),$$

which implies that $(G_1\varphi) \in AA(R, R^2)$.

Secondly, we have

$$\lim_{x\to\infty} \frac{1}{2x}\int_{-x}^{x} \left|\left(G_2\varphi\right)(t)\right| dt = \left(\lim_{x\to\infty} \frac{1}{2x}\int_{-x}^{x} \left| \int_{-\infty}^{t} e^{-\int_t^\eta \tilde{c}du} \gamma_{12}(\eta)d\eta \right| dt, \lim_{x\to\infty} \frac{1}{2x}\int_{-x}^{x} \left| \int_{-\infty}^{t} e^{-\int_t^\eta k_1(u)du} \gamma_{22}(\eta)d\eta \right| dt \right).$$

It can be clearly seen that

$$\lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x}\left|\int_{-\infty}^{t}e^{-\int_t^{\eta}\tilde{c}du}\gamma_{12}(\eta)d\eta\right|dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-x}^{x}\left|\int_{-\infty}^{t}e^{-\tilde{c}(\eta-t)}\gamma_{12}(\eta)d\eta\right|dt$$
$$\leq\lim_{x\to\infty}\frac{1}{2x}\int_{-T}^{T}dt\left|\int_{-\infty}^{t}e^{-\tilde{c}(\eta-t)}d\eta\right|\sup_{t\in R}\left|\gamma_{12}(t)\right|$$
$$=0.$$

Similarly, we get

$$\lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x}\left|\int_{-\infty}^{t}e^{-\int_\eta^{t}k_1(u)du}\gamma_{22}(\eta)d\eta\right|dt\leq\lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x}\left|\int_{-T}^{t}e^{-\int_s^{t}k_1(u)du}\gamma_{22}(\eta)d\eta\right|dt$$
$$+\lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x}\left|\int_{-\infty}^{-x}e^{-\int_s^{t}k_1(u)du}\gamma_{22}(\eta)d\eta\right|dt$$
$$\leq\lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x}\left\|\gamma_{22}(t)\right\|dt\int_{-T}^{t}e^{-k_1^{-}(\eta-t)}d\eta$$
$$+\lim_{x\to\infty}\frac{\sup\left|\gamma_{22}(t)\right|}{2x}\int_{-x}^{x}dt\int_{-\infty}^{-x}\left|e^{-k_1^{-}(\eta-t)}\right|d\eta$$
$$=\lim_{x\to\infty}\frac{\sup\left|\gamma_{22}(t)\right|}{2x(k_1^{-})^2}(1-e^{-2x})=0.$$

Then,

$$\lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x}\left|\left(G_2\varphi\right)(t)\right|dt=0,$$

which implies that $\left(G_2\varphi\right)\in AA_0(R,R^2)$. Thus, $G\varphi\in PAA(R,R^2)$.

Theorem 3.2. Let conditions of $(B_1)-(B_3)$ hold. Then, system (1.5) has a unique PAA solution in the following region:

$$\tilde{C}=\left\{\varphi\left|\ \left\|\varphi-\varphi_0\right\|\leq\frac{\gamma\nu}{1-\nu},\ \varphi=\left(\varphi_1,\varphi_2\right)\in PAA(R,R^2)\subseteq BC(R,R^2)\right.\right\},$$

where

$$\varphi_0=\left(0,\int_{-\infty}^{t}e^{-\int_s^{t}k_1(u)du}e(s)ds\right).$$

Proof. Since PAA functions are uniformly continuous on $\mathbb{R}$, one can get $\tilde{C}\subseteq PAA(R,R^2)$.

Let $\varphi_1=\left(\varphi_{11},\varphi_{12}\right)^T\in\tilde{C}$. According to Lemma 2.2 and condition (B_2), we have

$$\begin{pmatrix} \varphi_{12}(t) \\ (\tilde{c}k_1(t)+k_2(t)-(\tilde{c})^2)\varphi_{11}(t)-\varphi_{11}^n(t-\tau(t))k_3(t)+e(t) \end{pmatrix} \in PAA(R,R^2).$$

Thus, from Lemma 1 and Lemma 2 in [2], we know that the auxiliary the system (1.5) has a unique PAA solution:

$$\varphi_1(t)=\begin{pmatrix} \int_{-\infty}^{t} e^{-\int_{\eta}^{t}\tilde{c}du}\varphi_{12}(\eta)d\eta \\ \int_{-\infty}^{t} e^{-\int_{\eta}^{t}k_1(u)du}\left[(\tilde{c}(k_1(\eta)-\tilde{c})+k_2(\eta))\varphi_{11}(\eta)-k_3(s)\varphi_{11}^n(\eta-\tau(\eta))+e(\eta)\right]d\eta \end{pmatrix}^T.$$

Define a mapping $\Sigma:\tilde{C}\to PAA(R,R^2)$, by setting $\left(\Theta\varphi_1\right)=\left(\varphi_{11},\varphi_{12}\right)^T$.

It is clear that

$$\|\varphi_0\|\le\sup_{t\in R}\max\left\{0,\int_{-\infty}^{t} e^{-\int_{\eta}^{t}k_1(u)du}e(\eta)d\eta\right\}$$
$$\le\frac{1}{k_1^-}\max\left\{\sup_{t\in R}|e(\mathrm{t})|,0\right\}\le(\tilde{a})^-\max\left\{\sup_{t\in R}|e(\mathrm{t})|,0\right\}=\gamma.$$

Also, $\|\varphi_1\|_\infty\le\|\varphi_0\|_\infty+\|\varphi_1-\varphi_0\|\le\gamma+\frac{\gamma\nu}{1-\nu}=\frac{\gamma}{1-\nu}<1.$

Therefore, we get

$$\|\Theta\varphi_1-\varphi_0\|_\infty=\left(\left|\int_{-\infty}^{t} e^{-\int_{\eta}^{t}\tilde{c}du}\varphi_{12}(\eta)d\eta\right|,\left|\int_{-\infty}^{t} e^{-\int_{s}^{t}k_1(u)du}\gamma^\eta(\eta)d\eta\right|\right), \tag{1.6}$$

where $\gamma^\eta(\eta)=(\tilde{c}(k_1(\eta)-\tilde{c})+k_2(\eta))\varphi_{11}(\eta)-k_3(\eta)\varphi_{11}^n(\eta-\tau(\eta)).$ Hence,

$$\|\Theta\varphi_1\|_\infty\le\|\Theta\varphi_1-\varphi_0\|_\infty+\|\varphi_0\|_\infty=\left(\left|\int_{-\infty}^{t} e^{-\int_{\eta}^{t}\tilde{c}du}\varphi_{12}(\eta)d\eta\right|,\left|\int_{-\infty}^{t} e^{-\int_{t}^{\eta}k_1(u)du}\gamma^\eta(\eta)d\eta\right|\right)+\|\varphi_0\|_\infty$$
$$\le\sup_{t\in R}\max\left\{\int_{-\infty}^{t} e^{-\int_{t}^{\eta}\tilde{c}du}d\eta,\sup_{t\in R}+|k_3(t)|)\int_{-\infty}^{t} e^{-\int_{\mu}^{t}k_1(u)du}d\eta\right\}\|\varphi_1\|_\infty+\gamma$$
$$\le\max\left\{\frac{1}{\tilde{c}},\frac{\sup_{t\in R}|\tilde{c}k_1(t)-\tilde{c}^2+k_2(t))|+\sup_{t\in R}|k_3(t)|}{k_1^-}\right\}\|\varphi_1\|_\infty$$
$$\le\frac{1}{\bar{a}}\max\left\{1,\sup_{t\in R}|\tilde{c}k_1(t)-\tilde{c}^2+k_2(t))|+\sup_{t\in R}|k_3(t)|\right\}\|\varphi_1\|_\infty$$
$$=\upsilon\|\varphi_1\|_\infty\le\frac{\upsilon\gamma}{1-\upsilon}+\gamma=\frac{\gamma}{1-\upsilon}<1.$$

Therefore, $\left(\Sigma\varphi_1\right)\in\tilde{C}$. So $\Sigma:\tilde{C}\to\tilde{C}$ is a self-mapping. For all $\varphi_1,\varphi_2\in\tilde{C}$

$$\left|(\Sigma\varphi_1)(t)-(\Sigma\varphi_2)(t)\right| = \left(\left|\left((\Sigma\varphi_1)(t)-(\Sigma\varphi_2)(t)\right)_1\right|, \left|\left((\Sigma\varphi_1)(t)-(\Sigma\varphi_2)(t)\right)_2\right|\right)^T$$

$$= \left(\left|\int_{-\infty}^{t} e^{-\int_s^t \tilde{c}du}\left(\varphi_{12}(s)-\psi_{22}(s)\right)ds\right|,\right.$$

$$\left.\left|\int_{-\infty}^{t} e^{-\int_\eta^t k_1(u)du}\left[(\tilde{c}(k_1(\eta)-\tilde{c})+k_2(\eta))\varphi_{11}(\eta)-k_3(\eta)\varphi_1^n(\eta-\tau(\eta))+e(\eta)\right]d\eta\right|\right)^T$$

$$= \left(\left|\int_{-\infty}^{t} e^{-\int_s^t \tilde{c}du}\left(\varphi_{12}(\eta)-\varphi_{22}(\eta)\right)d\eta\right|, \left|\int_{-\infty}^{t} e^{-\int_s^t k_1(u)du}\left[(\tilde{c}(k_1(\eta)-\tilde{c})+k_2(\eta))\left(\varphi_{11}(\eta)-\varphi_{22}(\eta)\right)\right.\right.\right.$$

$$\left.-k_3(\eta)n\left(\varphi_{21}(\eta-\tau(\eta))\right)+m(\eta)\left(\varphi_{11}(\eta-\tau(\eta))-\varphi_{21}(\eta-\tau(\eta))\right)^{n-1}\right]d\eta$$

$$\left.\left.\times\left(\varphi_{11}(\eta-\tau(\eta))-\varphi_{21}(\eta-\tau(\eta))\right)\right]d\eta\right|\right)^T$$

$$= \left(\left|\int_{-\infty}^{t} e^{-\int_s^t \tilde{c}du}\left(\varphi_{12}(\eta)-\varphi_{22}(\eta)\right)d\eta\right|, \left|\int_{t}^{+\infty} e^{-\int_\eta^t k_1(u)du}\right.\right.$$

$$\times\left[(\tilde{c}(k_1(\eta)-\tilde{c})+k_2(\eta))\left(\varphi_{11}(\eta)-\varphi_{22}(\eta)\right)\right.$$

$$\left.-k_3(\eta)n(1-m(\eta))\left(\varphi_{21}(\eta-\tau(\eta))\right)+m(\eta)\varphi_{11}(\eta-\tau(\eta))^{n-1}\right]$$

$$\left.\left(\varphi_{11}(\eta-\tau(\eta))-\varphi_{21}(\eta-\tau(\eta))\right)\right]d\eta\Big)^T,$$

where $0 < m(\eta) < 1$. Then,

$$\left|(\Theta\varphi_1)(t)-(\Theta\varphi_2)(t)\right| \le \left(\int_{-\infty}^{t} e^{-\int_\eta^t \tilde{c}du}ds\sup_{t\in R}\left|\varphi_{12}(t)-\varphi_{22}(t)\right|,\right.$$

$$\int_{-\infty}^{t} e^{-\int_\eta^t k_1(u)du}\left[\left|\tilde{c}(k_1(\eta)-\tilde{c})+k_2(\eta)\right|\sup_{t\in R}\left|\varphi_{11}(t)-\varphi_{22}(t)\right|\right.$$

$$\left.\left.+\left|k_3(\eta)\right|n\left[\sup_{t\in R}\left|\varphi_{11}(t-l(t))\right|+\sup_{t\in R}\varphi_{21}\left|(\eta-l(\eta))\right|\right]^{n-1}\times\left|\varphi_{11}(\eta-\tau(\eta))-\varphi_{21}(\eta-\tau(\eta))\right|\right]d\eta\right)^T$$

$$\le \left(\int_{-\infty}^{t} e^{-\int_\eta^t \tilde{c}du}d\eta dt, \int_{-\infty}^{t} e^{-\int_\eta^t k_1(u)du}d\eta\right.$$

$$\left.\sup_{t\in R}\left[\left(\left|\tilde{c}(k_1(t)-\tilde{c})+k_2(t)\right|+n\left[2\gamma/(1-\upsilon)\right]^{n-1}\left|k_3(t)\right|\right)\right]\right)^T\left\|\varphi_1-\varphi_2\right\|_\infty$$

$$= \left(\int_{-\infty}^{t} e^{\tilde{c}(t-\eta)}d\eta, \sup_{t\in R}\left[\left(\left|\tilde{c}k_1(t)-\tilde{c}^2+k_2(t)\right|+n\left[2l/(1-\theta)\right]^{n-1}\left|k_3(t)\right|\right)\right]\right)^T\left\|\varphi_1-\varphi_2\right\|_\infty$$

$$\le \max\left\{\frac{1}{\tilde{c}}, \frac{\sup_{t\in R}\left[\left|\tilde{c}k_1(t)-\tilde{c}^2+k_2(t)\right|+n\left[2\gamma/(1-\upsilon)\right]^{n-1}\left|k_3(t)\right|\right]}{k_1^-}\right\}\left\|\varphi_1-\varphi_2\right\|_\infty$$

$$\le \frac{1}{\bar{a}}\max\left\{1,\sup_{t\in R}\left\|\left[\left|\tilde{c}k_1(t)-\tilde{c}^2+k_2(t)\right|+\left|k_3(t)\right|n\left[2\gamma/(1-\upsilon)\right]^{n-1}\right],1\right\|\right\}\left\|\varphi_1-\varphi_2\right\|_\infty = \pi\left\|\varphi_1-\varphi_2\right\|_\infty.$$

Since $\pi < 1$, Σ is a contraction mapping. Therefore, Σ has a fixed point $\tilde{s}(t) = (\tilde{\varsigma}(t), \tilde{\rho}(t))$ in the set $\tilde{C}$ such that $\Sigma \tilde{s} = \tilde{s}$ which is the PAA solution.

4. Globally exponential stability of PAA solutions

Theorem 4.1. Let $\tilde{s}(t) = (\tilde{\varsigma}(t), \tilde{\rho}(t))$ be a unique PAA solution of (1.5) with initial function $(\tilde{\varphi}_1(t), \tilde{\varphi}_2(t))$ and suppose conditions of Theorem 3.2. Then, $\tilde{s}(t)$ solution of (1.5) is globally exponential stability.

Proof. Due to the condition (B_4), a constant $\zeta > 0$ can be found such that

$$\min\left\{\tilde{c} - \zeta, k_1^- + \sup\left(\left|\tilde{c}k_1(t) - \tilde{c}^2 + k_2(t)\right| + n\left[2\gamma/(1-\upsilon)\right]^{n-1}\left|k_3(t)\right|\right) - \zeta\right\} > 0. \tag{1.7}$$

Let $s(t) = (\varsigma(t), \rho(t))$ be an arbitrary solution of (1.5) with initial function $(\varphi_1(t), \varphi_2(t))$. Set $A_1(t) = \varsigma(t) - \tilde{\varsigma}(t)$, $A_2(t) = p(t) - \tilde{p}(t)$. Then,

$$\begin{cases} A_1'(t) = -\tilde{c}A_1(t) + A_2(t) \\ A_2'(t) = -k_1(t)A_2(t) + (\tilde{c}k_1(t) - \tilde{c}^2 + k_2(t))A_1(t) - (\varsigma^n\left(t - l(t)\right) - \tilde{\varsigma}^n\left(t - l(t)\right))k_3(t) + e(t). \end{cases} \tag{1.8}$$

Consider the following Lyapunov functions

$$U_1(t) = \left|A_1(t)\right|e^{\zeta t}, \quad U_2(t) = \left|A_2(t)\right|e^{\zeta t}. \tag{1.9}$$

Computing $\left(U_i'(t)\right)^+$ along the solution $(A_1(t), A_2(t))$ of system (1.5) with the initial function $(\varphi_1(t) - \varphi_2(t), \tilde{\varphi}_1(t) - \tilde{\varphi}_2(t))$, we get

$$\begin{aligned} D^+(U_1(t)) &= \left|A_1(t)\right|\zeta e^{\zeta t} + A_1'(t)sign(A_1(t))e^{\zeta t} \\ &= \left|A_1(t)\right|\zeta e^{\zeta t} + sign(A_1(t))[-\tilde{c}A_1(t) + A_2(t)]e^{\zeta t} \\ &\le e^{\zeta t}\left((\zeta - \tilde{c})\left|A_1(t)\right| + \left|A_2(t)\right|\right), \end{aligned} \tag{1.10}$$

$$\begin{aligned} &D^+(U_2(t)) = \left|A_2(t)\right|\zeta e^{\zeta t} + A_2'(t)sign\left(A_2(t)\right)e^{\zeta t} \\ &\le \left|A_2(t)\right|\zeta e^{\zeta t} + sign\left(A_2(t)\right)\left[-k_1(t)A_2(t) + (\tilde{c}(k_1(t) - \tilde{c}) + k_2(t))A_1(t)\right. \\ &\left. -(\varsigma^n\left(t - l(t)\right) - \tilde{\varsigma}^n\left(t - l(t)\right))k_3(t) + e(t)\right]e^{\zeta t} \\ &\le e^{\zeta t}\left\{\left(\zeta - k_1^-\right)\left|A_2(t)\right| + \sup\left(\left|(\tilde{c}k_1(t) - \tilde{c}^2) + k_2(t))\right| + \left(2\gamma/(1-\upsilon)\right)^{n-1}\right)\left|k_3(t)\right|\left|A_1(t)\right|\right\}. \end{aligned} \tag{1.11}$$

Let $N > 1$ is constant and set

$$\Xi \equiv \max\left\{\left\|\varphi_1 - \varphi_2\right\|, \left|\tilde{\varphi}_1 - \tilde{\varphi}_2\right|\right\} > 0$$

It follows from (1.8) that

$$U_1(t) = \left|A_1(t)\right|e^{\zeta t} < N\Xi, \qquad U_2(t) = \left|A_2(t)\right|e^{\zeta t} < N\Xi, \quad \text{for all } t \in [-l, 0].$$

We claim that

$$U_1(t)=|A_1(t)|e^{\zeta t}<N\Xi,\quad U_2(t)=|A_2(t)|e^{\zeta t}<N\Xi,\quad \text{for all } t>0. \tag{1.12}$$

Otherwise, the following two situations arise.
Case a. $t_1>0$ can be found as

$$U_1(t_1)=N\Xi,\quad U_i(t)=<N\Xi, \text{ for all } t\in[-l,t_1), i=1,2. \tag{1.13}$$

Case b. $t_2>0$ can be found as

$$U_2(t_2)=N\Xi,\quad U_i(t)=<N\Xi,\quad \text{for all } t\in[-l,t_2), i=1,2. \tag{1.14}$$

If Case a holds, (1.7), (1.9), and (1.12) shows that

$$\begin{aligned} D^+(U_1(t_1)) &\le \mathrm{e}^{\zeta t_1}\left((\zeta-\tilde{c})|A_1(t)|+|A_2(t)|\right) \\ &\le (\zeta-\tilde{c})N\Xi. \end{aligned} \tag{1.15}$$

Thus, $\lambda\ge-\tilde{c}$ and this contrasts with (1.7).
If Case b holds, (1.7), (1.10), and (1.13) refers that

$$\begin{aligned} D^+(U_2(t_2)) &\le e^{\zeta t_2}\left\{\left(\lambda-\inf\left(k_1(t_2)\right)\right)|A_2(t_2)|\right\} \\ &+\sup\left((\tilde{c}k_1(t_2)-\tilde{c}^2+k_2(t_2))+|k_3(t_2)|n\left(2\gamma/(1-\upsilon)\right)^{n-1}\right)|A_1(t_2)|e^{\zeta t_2} \\ &\le\left(\zeta-k_1^-+\sup\left((\tilde{c}k_1(t_2)+k_2(t_2)-\tilde{c}^2)+\left(2\gamma/(1-\upsilon)\right)^{n-1}|k_3(t_2)|\right)\right)N\Xi. \end{aligned} \tag{1.16}$$

Hence, $\varsigma-k_1^-+\sup\left(\left|\tilde{c}k_1(t_2)-\tilde{c}^2+k_2(t_2)\right|+\left(2\gamma/(1-\upsilon)\right)^{n-1}|k_3(t_2)|\right)\ge0$ and this contrasts with (1.7).
Thus, (1.11) holds. As a result, we get

$$\max\left\{|\varsigma(t)-\tilde{\varsigma}(t)|,|p(t)-\tilde{p}(t)|\right\}\le e^{-\zeta t}\max\left\{\|\varphi_1-\tilde{\varphi}_1\|,|\varphi_1-\tilde{\varphi}_2|\right\},\quad \forall t>0.$$

5. An application

Let $n=2$, $l(t)=|\sin t|$, $\tilde{c}=25$, $k_1(t)=25+\cos\frac{1}{2+\omega(t)}$, $k_2(t)=\cos\frac{1}{2+\omega(t)}+e^{-2t}$, $k_3(t)=\sin\frac{1}{2+\omega(t)}+e^{-2t}$, $e(t)=\cos\frac{1}{2+\omega(t)}+e^{-2t}$, $\omega(t)=\sin t+\sin\sqrt{2}t$. If these notions are written in (1.4),

$$\begin{aligned} &\varsigma''(t)+(25+\cos\frac{1}{2+\omega(t)})\varsigma'(t)+(\cos\frac{1}{2+\omega(t)}+e^{-2t})\varsigma(t) \\ &+(\sin\frac{1}{2+\omega(t)}+e^{-2t})\varsigma^2\left(t-|\sin t|\right)=\cos\frac{1}{2+\omega(t)}+e^{-2t}, \end{aligned} \tag{1.17}$$

is obtained. Then, (1.17) has exactly one PAA solution.
Set $p(t)=\varsigma'(t)+25\varsigma(t)$, then, (1.17) can be converted to the following system such that

$$
\begin{aligned}
&\varsigma'(t) = -25\theta(t) + p(t) \\
&p'(t) = -(25 + \cos\frac{1}{2+\omega(t)})p(t) + (\cos\frac{1}{2+\omega(t)} + e^{-2t} - 625)\varsigma(t) \\
&\qquad -(\sin\frac{1}{2+\omega(t)} + e^{-2t})\varsigma^2(t - |\sin t|) + \cos\frac{1}{2+\omega(t)} + e^{-2t}.
\end{aligned}
\tag{1.18}
$$

Then, $\gamma = \frac{2}{25} < 1$, $\upsilon = \frac{4}{25} < 1$ and $\pi = \frac{22}{175} < 1$.

Consequently, whole conditions of Theorem 3.3 are satisfied; therefore, (1.18) has a unique globally exponential stable PAA solution in the following region:

$$
\tilde{C} = \left\{ \varphi \middle| \, \|\varphi - \varphi_0\| \le \frac{8}{525},\ \varphi = (\varphi_1, \varphi_2) \in PAA(R, R^2) \subseteq BC(R, R^2) \right\}.
$$

The numerical simulations for the state vectors of (1.18) are as follows in Figures 3 and 4:

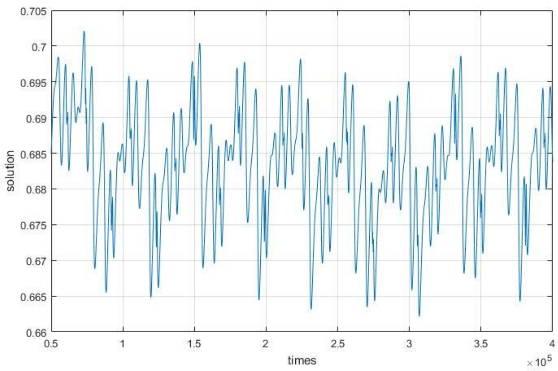

Figure 3. Trajectory of component for $\varsigma(t)$.

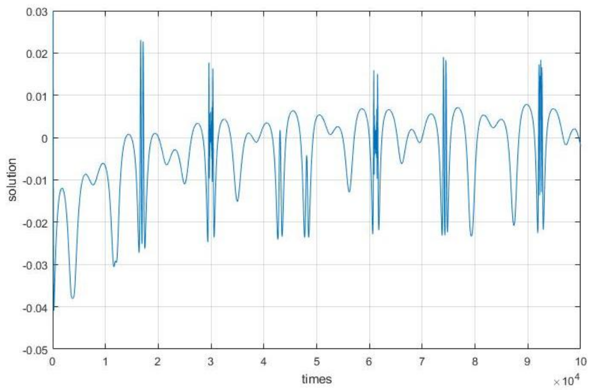

Figure 4. Trajectory of component for $p(t)$.

6. Conclusions

We remark that there is no phenomenon that is purely periodic in nature, and this gives the idea of the AP oscillation, the PAP oscillation, the WPAP oscillation, the automorphic oscillation, and the

PAA oscillation. As seen in Example 2.2, the set of PAA functions is more extensive than the known sets of AP, PAP, and AA. Therefore, in this study, some important results are obtained regarding the PAA solutions of the discussed Duffing differential equation model. Some differential inequalities and the well-known Banach fixed point theorem are used for the existence and uniqueness of PAA solutions. Also, with the help of Lyapunov functions, sufficient conditions are obtained for globally exponential stability of PAA solutions. As a result, the main results of the study are new and complementary to the previous studies.

Conflict of interest

The author declares no conflicts of interest in this paper.

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