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Research article

On the characterization of Pythagorean fuzzy subgroups

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Abstract: Pythagorean fuzzy environment is the modern tool for handling uncertainty in many decisions making problems. In this paper, we represent the notion of Pythagorean fuzzy subgroup (PFSG) as a generalization of intuitionistic fuzzy subgroup. We investigate various properties of our proposed fuzzy subgroup. Also, we introduce Pythagorean fuzzy coset and Pythagorean fuzzy normal subgroup (PFNSG) with their properties. Further, we define the notion of Pythagorean fuzzy level subgroup and establish related properties of it. Finally, we discuss the effect of group homomorphism on Pythagorean fuzzy subgroup.

Keywords: Pythagorean fuzzy set; Pythagorean fuzzy subgroup; Pythagorean fuzzy coset; Pythagorean fuzzy normal subgroup; Pythagorean fuzzy level subgroup
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1. Introduction

Group theory has various applications in many fields of Mathematics such as Algebraic geometry, Cryptography, Harmonic analysis, Algebraic number theory, etc. Uncertainty is a part of our daily life. There is an uncertainty in almost every problems we face day to day. In 1965, fuzzy set theory was first invented by Zadeh [1] to handle uncertainty in real life problems. In 1971 using the concept of fuzzy set, Rosenfeld [2] first defined the notion of fuzzy subgroup. In 1979, using t-norm the notion of fuzzy subgroup was redefined by Anthony and Sherwood [3, 4]. Das [5] introduced the concept of fuzzy level subgroup. In 1988, Choudhury et al. [6] proved various properties of fuzzy subgroups and fuzzy homomorphism. In 1990, Dixit et al. [7] discussed fuzzy level subgroups and union of fuzzy subgroups. The notion of anti-fuzzy subgroups was first proposed by Biswas [8]. Ajmal and Prajapati [9] gave the idea of fuzzy normal subgroup, fuzzy coset and fuzzy quotient subgroup. Chakraborty and

Khare [10] studied various properties of fuzzy homomorphism. Many more results on fuzzy subgroup ware introduced by Mukherjee [11, 12] and Bhattacharya [13].

In recent years many researchers studied various properties of fuzzy subgroups. In 2015, Tarnauceanu [14] classified fuzzy normal subgroup of finite groups. In 2016, Onasanya [15] reviewed some anti fuzzy properties of fuzzy subgroups. Shuaib [16] and Shaheryar [17] studied the properties of omicron fuzzy subgroup and omicron anti fuzzy subgroup. In 2018, Addis [18] developed fuzzy homomorphism theorems on groups. In 1986, Atanassov [19] invented Intuitionistic fuzzy set. In 1996, Intuitionistic fuzzy subgroup was first studied by Biswas [20]. Zhan and Tan [21] introduced intuitionistic fuzzy M-group. Furthermore, researchers developed intuitionistic fuzzy subgroup in many ways [22, 23, 24].

In 2013, Yager [25] introduced Pythagorean fuzzy set, where the sum of square of the membership degree and non membership degree lies between 0 and 1. Pythagorean fuzzy set is more fruitful in many decision making problems. This concept is perfectly designed to represent vagueness and uncertainty in mathematical way and to produce a formalized tool to handle imprecision to real problems. In 2018, Naz et al. [26] proposed a novel approach to decision making problem using Pythagorean fuzzy set. In 2019, Akram and Naz [27] applied complex Pythagorean fuzzy set in decision making problems. Ejegwa [28] gave an application of Pythagorean fuzzy set in career placements based on academic performance using max-min-max composition. Some results related to it were given by Peng [29] and Yang [30].

Pythagorean fuzzy set gives a modern way to model vagueness and uncertainty with high precision and accuracy compared to intuitionistic fuzzy set. Group symmetry plays a vital role to analyse molecule structures. Isotope molecules decays with a certain rate, so the fuzzy sense comes into it. If decay rate follows the criteria of Pythagorean fuzzy environment then we can not opted for intuitionistic fuzzy subgroup to analyse the structure of that isotope at certain time. For this type of situations where we can not opted for intuitionistic fuzzy subgroup, it is very important to introduce Pythagorean fuzzy subgroup as a bigger class of intuitionistic fuzzy subgroup. But till now no algebraic structure is defined on Pythagorean fuzzy environment. In this article, we proved that intuitionistic fuzzy subgroup is a subclass of Pythagorean fuzzy subgroup.

This paper is organized as follows: Pythagorean fuzzy subgroup is described in Section 3. Pythagorean fuzzy coset and Pythagorean fuzzy normal subgroup are discussed in Section 4. Pythagorean fuzzy level subgroup and its properties are given in Section 5. Finally, effect of group homomorphism on Pythagorean fuzzy subgroup is discussed in Section 6 and a conclusion is given in Section 7.

2. Preliminaries

In this section, we recap some definitions and concepts which are very much important to develop later sections.

Definition 2.1. *[1] Let C be a crisp set. Then $\mu : C \to [0, 1]$ is called a fuzzy subset of C. Here $\mu(m)$ is called degree of membership of $m \in C$.*

Definition 2.2. *[2] Let $\mu : C \to [0, 1]$ be a fuzzy subset of a group $(C, \circ)$. Then μ is said to be a fuzzy subgroup of $(C, \circ)$ if the following conditions hold:*

(i) $\mu(m \circ n) \geq \mu(m) \wedge \mu(n)$ $\forall m, n \in C$,
(ii) $\mu(m^{-1}) \geq \mu(m)$ $\forall m \in C$.

Definition 2.3. *[19] Let C be a crisp set. An intuitionistic fuzzy set (IFS) I on C is defined by* $I = \{(m, \mu(m), \nu(m)) | m \in C\}$ *where* $\mu(m) \in [0, 1]$ *and* $\nu(m) \in [0, 1]$ *are the degree of membership and non membership of* $m \in C$ *respectively, which satisfy the condition* $0 \leq \mu(m) + \nu(m) \leq 1$ $\forall m \in C$.

Definition 2.4. *[20] Let* $I = \{(m, \mu(m), \nu(m)) | m \in C\}$ *be a IFS of a group* $(C, \circ)$. *Then I is said to be an intuitionistic fuzzy subgroup (IFSG) of C if the following conditions hold:*
(i) $\mu(m \circ n) \geq \mu(m) \wedge \mu(n)$ *and* $\nu(m \circ n) \leq \nu(m) \vee \nu(n)$ $\forall m, n \in C$,
(ii) $\mu(m^{-1}) \geq \mu(m)$ *and* $\nu(m^{-1}) \leq \nu(m)$ $\forall m \in C$.

In 2013, Yager [25] defined Pythagorean fuzzy subset (PFS) as a generalization of IFS.

Definition 2.5. *[25] Let C be a crisp set. A Pythagorean fuzzy set (PFS)* ψ *on C is defined by* $\psi = \{(m, \mu(m), \nu(m)) | m \in C\}$ *where* $\mu(m) \in [0, 1]$ *and* $\nu(m) \in [0, 1]$ *are the degree of membership and non membership of* $m \in C$ *respectively, which satisfy the condition* $0 \leq \mu^2(m) + \nu^2(m) \leq 1$ $\forall m \in C$.

Some operations on PFSs [31] are stated below.
Let $\psi_1 = \{(m, \mu_1(m), \nu_1(m)) | m \in C\}$ and $\psi_2 = \{(m, \mu_2(m), \nu_2(m)) | m \in C\}$ be two PFSs of C. Then the following holds:

- $\psi_1 \cup \psi_2 = \{(m, \mu_1(m) \vee \mu_2(m), \nu_1(m) \wedge \nu_2(m)) \, | m \in C\}$.
- $\psi_1 \cap \psi_2 = \{(m, \mu_1(m) \wedge \mu_2(m), \nu_1(m) \vee \nu_2(m)) \, | m \in C\}$.
- $\psi_1^c = \{(m, \nu_1(m), \mu_1(m)) | m \in C\}$.
- $\psi_1 \subseteq \psi_2$ if $\mu_1(m) \leq \mu_2(m)$ and $\nu_1(m) \geq \nu_2(m)$ for all $m \in C$.
- $\psi_1 = \psi_2$ if $\mu_1(m) = \mu_2(m)$ and $\nu_1(m) = \nu_2(m)$ for all $m \in C$.

Throughout this paper, we will write Pythagorean fuzzy subset as PFS and we will write $\psi = (\mu, \nu)$ instead of $\psi = \{(m, \mu(m), \nu(m)) | m \in C\}$.

3. Pythagorean fuzzy subgroup

In this section, we define Pythagorean fuzzy subgroup (PFSG) as an extension of fuzzy subgroup and intuitionistic fuzzy subgroup (IFSG).

Definition 3.1. *Let* $(C, \circ)$ *be a group and* $\psi = (\mu, \nu)$ *be a PFS of C. Then* ψ *is said to be a PFSG of C if the following conditions hold:*
(i) $\mu^2(m \circ n) \geq \mu^2(m) \wedge \mu^2(n)$ *and* $\nu^2(m \circ n) \leq \nu^2(m) \vee \nu^2(n)$ $\forall m, n \in C$,
(ii) $\mu^2(m^{-1}) \geq \mu^2(m)$ *and* $\nu^2(m^{-1}) \leq \nu^2(m)$ $\forall m \in C$.
Here, $\mu^2(m) = \{\mu(m)\}^2$ *and* $\nu^2(m) = \{\nu(m)\}^2$ *for all* $m \in C$.

Example 3.1. *Let us take the set* $C = \{1, -1, i, -i\}$. *Then* $(C, .)$ *is a group, where '.' is the usual multiplication.*
Define a PFS $\psi = (\mu, \nu)$ *on C by*

$$\mu(1) = 0.8, \; \mu(-1) = 0.6, \; \mu(i) = 0.5, \; \mu(-i) = 0.5,$$
$$\nu(1) = 0.1, \; \nu(-1) = 0.2, \; \nu(i) = 0.4, \; \nu(-i) = 0.4.$$

Here, $\mu^2(i.-i) = \mu^2(1) = (0.8)^2 = 0.64$ *and* $\nu^2(i.-i) = \nu^2(1) = (0.1)^2 = 0.01$.
Now, $\mu^2(i) \wedge \mu^2(-i) = \min\{0.25, 0.25\} = 0.25$ *and* $\nu^2(i) \vee \nu^2(-i) = \max\{0.16, 0.16\} = 0.16$.
So, $\mu^2(i.-i) > \mu^2(i) \wedge \mu^2(-i)$ *and* $\nu^2(i.-i) < \nu^2(i) \vee \nu^2(-i)$.
Also, $\mu^2(i) = \mu^2(-i)$ *and* $\nu^2(i) = \nu^2(-i)$.
In the same manner it can be shown that $\mu^2(m.n) \geq \mu^2(m) \wedge \mu^2(n)$, $\nu^2(m.n) \leq \nu^2(m) \vee \nu^2(n)$ *for all* $m, n \in C$ *and* $\mu^2(m^{-1}) \geq \mu^2(m)$, $\nu^2(m^{-1}) \leq \nu^2(m)$ *for all* $m \in C$. *Hence* ψ *is a PFSG of the group* $(C, .)$.

Proposition 3.1. *Let* $\psi = (\mu, \nu)$ *be a PFSG of a group* $(C, \circ)$. *Then the following holds:*
(i) $\mu^2(e) \geq \mu^2(m)$ *and* $\nu^2(e) \leq \nu^2(m)$ $\forall m \in C$ *and*
(ii) $\mu^2(m^{-1}) = \mu^2(m)$ *and* $\nu^2(m^{-1}) = \nu^2(m)$ $\forall m \in C$ *where, e is the identity element in* C.

Proof. Since $\psi = (\mu, \nu)$ is a PFSG of a group $(C, \circ)$, then
$\mu^2(m \circ n) \geq \mu^2(m) \wedge \mu^2(n)$, $\nu^2(m \circ n) \leq \nu^2(m) \vee \nu^2(n)$ and
$\mu^2(m^{-1}) \geq \mu^2(m)$, $\nu^2(m^{-1}) \leq \nu^2(m)$ for all $m, n \in C$.
(i) Now, $\mu^2(e) = \mu^2(m \circ m^{-1}) \geq \mu^2(m) \wedge \mu^2(m^{-1}) = \mu^2(m)$.
Also, $\nu^2(e) = \nu^2(m \circ m^{-1}) \leq \nu^2(m) \vee \nu^2(m^{-1}) = \nu^2(m)$ for all $m \in C$.
(ii) We have $\mu^2(m^{-1}) \geq \mu^2(m)$ and $\nu^2(m^{-1}) \leq \nu^2(m)$ for all $m \in C$.
Putting m^{-1} in place of m, we obtain
$\mu^2\left((m^{-1})^{-1}\right) \geq \mu^2(m^{-1}) \Rightarrow \mu^2(m) \geq \mu^2(m^{-1})$ for all $m \in C$.
Again, $\nu^2\left((m^{-1})^{-1}\right) \leq \nu^2(m^{-1}) \Rightarrow \nu^2(m) \leq \nu^2(m^{-1})$ for all $m \in C$.
Hence combining all the results we get $\mu^2(m^{-1}) = \mu^2(m)$ and $\nu^2(m^{-1}) = \nu^2(m)$ $\forall m \in C$. □

Now, we will show that every intuitionistic fuzzy subgroup (IFSG) of a group $(C, \circ)$ is also a Pythagorean fuzzy subgroup (PFSG) of the group $(C, \circ)$. But the converse is not true.

Theorem 3.1. *If* $\psi = (\mu, \nu)$ *is a IFSG of a group* $(C, \circ)$, *then* ψ *is a PFSG of the group* $(C, \circ)$.

Proof. Since $\psi = (\mu, \nu)$ is a IFSG of a group $(C, \circ)$, then
$\mu(m \circ n) \geq \mu(m) \wedge \mu(n)$, $\nu(m \circ n) \leq \nu(m) \vee \nu(n)$ and
$\mu(m^{-1}) \geq \mu(m)$, $\nu(m^{-1}) \leq \nu(m), \forall m, n \in C$.
Here $\mu(m) \in [0, 1]$ and $\nu(m) \in [0, 1]$ for all $m \in C$. Many case arises.
Case 1: Let $\mu(m) > \mu(n)$ and $\nu(m) > \nu(n)$ for all $m, n \in C$.
So, $\mu^2(m) > \mu^2(n)$ and $\nu^2(m) > \nu^2(n)$ for all $m, n \in C$.
Now, $\mu(m \circ n) \geq \mu(m) \wedge \mu(n) = \mu(n) \Rightarrow \mu^2(m \circ n) \geq \mu^2(n) = \mu^2(m) \wedge \mu^2(n)$
i.e., $\mu^2(m \circ n) \geq \mu^2(m) \wedge \mu^2(n)$.
Again, $\nu(m \circ n) \leq \nu(m) \vee \nu(n) = \nu(m) \Rightarrow \nu^2(m \circ n) \leq \nu^2(m) = \nu^2(m) \vee \nu^2(n)$
i.e., $\nu^2(m \circ n) \leq \nu^2(m) \vee \nu^2(n)$.
Case 2: Let $\mu(m) < \mu(n)$ and $\nu(m) < \nu(n)$ for all $m, n \in C$.
So, $\mu^2(m) < \mu^2(n)$ and $\nu^2(m) < \nu^2(n)$ for all $m, n \in C$.
Now, $\mu(m \circ n) \geq \mu(m) \wedge \mu(n) = \mu(m) \Rightarrow \mu^2(m \circ n) \geq \mu^2(m) = \mu^2(m) \wedge \mu^2(n)$
i.e., $\mu^2(m \circ n) \geq \mu^2(m) \wedge \mu^2(n)$.
Again, $\nu(m \circ n) \leq \nu(m) \vee \nu(n) = \nu(n) \Rightarrow \nu^2(m \circ n) \leq \nu^2(n) = \nu^2(m) \vee \nu^2(n)$
i.e., $\nu^2(m \circ n) \leq \nu^2(m) \vee \nu^2(n)$.
Case 3: Let $\mu(m) = \mu(n)$ and $\nu(m) = \nu(n)$ for all $m, n \in C$.
So, $\mu^2(m) = \mu^2(n)$ and $\nu^2(m) = \nu^2(n)$ for all $m, n \in C$.

Now, $\mu(m \circ n) = \mu(m) = \mu(n) \Rightarrow \mu^2(m \circ n) = \mu^2(m) = \mu^2(n)$
i.e., $\mu^2(m \circ n) = \mu^2(m) \wedge \mu^2(n)$.
Again, $\nu(m \circ n) = \nu(m) = \nu(n) \Rightarrow \nu^2(m \circ n) = \nu^2(m) = \nu^2(n)$
i.e., $\nu^2(m \circ n) = \nu^2(m) \vee \nu^2(n)$.
In this way, considering all the cases we can easily show that
$\mu^2(m \circ n) \geq \mu^2(m) \wedge \mu^2(n)$ and $\nu^2(m \circ n) \leq \nu^2(m) \vee \nu^2(n) \forall m, n \in C$.
Again, $\mu(m^{-1}) \geq \mu(m)$ and $\nu(m^{-1}) \leq \nu(m)$ for all $m \in C$.
Since $\mu(m) \in [0, 1]$ and $\nu(m) \in [0, 1]$,
$\mu^2(m^{-1}) \geq \mu^2(m)$ and $\nu^2(m^{-1}) \leq \nu^2(m)$ for all $m \in C$.
Hence $\psi = (\mu, \nu)$ is a PFSG of the group $(C, \circ)$. □

Example 3.2. *Let us consider the Klein's 4-group $C = \{e, a, b, c\}$, where $a^2 = b^2 = c^2 = e$ and $ab = c$, $bc = a$, $ca = b$.*
Define a PFS $\psi = (\mu, \nu)$ on C by

$$\mu(e) = 0.9,\ \mu(c) = 0.8,\ \mu(a) = 0.6,\ \mu(b) = 0.6,$$
$$\nu(e) = 0.2,\ \nu(c) = 0.3,\ \nu(a) = 0.4,\ \nu(b) = 0.4.$$

We can easily verify that $\psi = (\mu, \nu)$ is a PFSG of C.

But here $\mu(e) + \nu(e) = 1.1$, which is greater than one. So ψ is not a IFS of C. Therefore ψ is not a IFSG of C.

This example shows that PFSG may not be an IFSG.

Remark 3.1. *Every IFSG of a group $(C, \circ)$ is a PFSG of $(C, \circ)$, but the converse need not be true.*

Proposition 3.2. *Let $\psi = (\mu, \nu)$ be a PFS of a group $(C, \circ)$. Then ψ is a PFSG of $(C, \circ)$ iff $\mu^2(m \circ n^{-1}) \geq \mu^2(m) \wedge \mu^2(n)$ and $\nu^2(m \circ n^{-1}) \leq \nu^2(m) \vee \nu^2(n)$ $\forall m, n \in C$.*

Proof. Let $\psi = (\mu, \nu)$ be a PFSG of a group $(C, \circ)$.
So, $\mu^2(m \circ n^{-1}) \geq \mu^2(m) \wedge \mu^2(n^{-1}) = \mu^2(m) \wedge \mu^2(n)$ and
$\nu^2(m \circ n^{-1}) \leq \nu^2(m) \vee \nu^2(n^{-1}) = \nu^2(m) \vee \nu^2(n)$ $\forall m, n \in C$.
Conversely, let us assume that $\mu^2(m \circ n^{-1}) \geq \mu^2(m) \wedge \mu^2(n)$ and
$\nu^2(m \circ n^{-1}) \leq \nu^2(m) \vee \nu^2(n)$ $\forall m, n \in C$.
Now $\mu^2(e) = \mu^2(m \circ m^{-1}) \geq \mu^2(m) \wedge \mu^2(m^{-1}) = \mu^2(m)$ and
$\nu^2(e) = \nu^2(m \circ m^{-1}) \leq \nu^2(m) \vee \nu^2(m^{-1}) = \nu^2(m)$ $\forall m \in C$, where e is the identity element of C.
Again, $\mu^2(m^{-1}) = \mu^2(e \circ m^{-1}) \geq \mu^2(e) \wedge \mu^2(m^{-1}) = \mu^2(m)$ and
$\nu^2(m^{-1}) = \nu^2(e \circ m^{-1}) \leq \nu^2(e) \vee \nu^2(m^{-1}) = \nu^2(m)$ $\forall m \in C$.
Therefore, $\mu^2(m \circ n) = \mu^2\left(m \circ (n^{-1})^{-1}\right) \geq \mu^2(m) \wedge \mu^2(n^{-1}) \geq \mu^2(m) \wedge \mu^2(n)$ and
$\nu^2(m \circ n) = \nu^2\left(m \circ (n^{-1})^{-1}\right) \leq \nu^2(m) \vee \mu^2(n^{-1}) \leq \nu^2(m) \vee \nu^2(n)$ $\forall m, n \in C$.
Hence $\psi = (\mu, \nu)$ is PFSG of the group $(C, \circ)$. □

Now we will check whether the union and intersection of two PFSGs of a group $(C, \circ)$ is a PFSG of C.

Theorem 3.2. *Intersection of two PFSGs of a group $(C, \circ)$ is a PFSG of the group $(C, \circ)$.*

Proof. Let $\psi_1 = (\mu_1, \nu_1)$ and $\psi_2 = (\mu_2, \nu_2)$ be two PFSGs of a group $(C, \circ)$.
Then $\psi = \psi_1 \cap \psi_2 = (\mu, \nu)$, where $\mu(m) = \mu_1(m) \wedge \mu_2(m)$ and $\nu(m) = \nu_1(m) \vee \nu_2(m)$ for all $m \in C$.

Now for all $m, n \in C$

$$\begin{aligned}
\mu^2(m \circ n^{-1}) &= \mu_1^2(m \circ n^{-1}) \wedge \mu_2^2(m \circ n^{-1}) \\
&\geq \left(\mu_1^2(m) \wedge \mu_1^2(n)\right) \wedge \left(\mu_2^2(m) \wedge \mu_2^2(n)\right) \\
&= \left(\mu_1^2(m) \wedge \mu_2^2(m)\right) \wedge \left(\mu_1^2(n) \wedge \mu_2^2(n)\right) \\
&= \mu^2(m) \wedge \mu^2(n)
\end{aligned}$$

and

$$\begin{aligned}
\nu^2(m \circ n^{-1}) &= \nu_1^2(m \circ n^{-1}) \vee \nu_2^2(m \circ n^{-1}) \\
&\leq \left(\nu_1^2(m) \vee \nu_1^2(n)\right) \vee \left(\nu_2^2(m) \vee \nu_2^2(n)\right) \\
&= \left(\nu_1^2(m) \vee \nu_2^2(m)\right) \vee \left(\nu_1^2(n) \vee \nu_2^2(n)\right) \\
&= \nu^2(m) \vee \nu^2(n).
\end{aligned}$$

Therefore $\psi = \psi_1 \cap \psi_2$ is a PFSG of $(C, \circ)$.

Hence intersection of two PFSGs of a group is also a PFSG of the group. □

Corollary 3.1. *Intersection of a family of PFSGs of a group* $(C, \circ)$ *is also a PFSG of the group* $(C, \circ)$.

Proof. Let $W = \{\psi_1, \psi_2, \ldots, \psi_p\}$ be a family of PFSGs of $(C, \circ)$.

We have to show that $\psi = \cap_{i=1}^{p} \psi_i$ is a PFSG of $(C, \circ)$.

Then $\psi = (\mu, \nu)$ is given by $\mu(m) = \mu_1(m) \wedge \mu_2(m) \wedge \ldots \wedge \mu_p(m)$ and
$\nu(m) = \nu_1(m) \vee \nu_2(m) \vee \ldots \vee \nu_p(m)$ for all $m \in C$.

Now for all $m, n \in C$

$$\begin{aligned}
\mu^2(m \circ n^{-1}) &= \mu_1^2(m \circ n^{-1}) \wedge \mu_2^2(m \circ n^{-1}) \wedge \ldots \wedge \mu_p^2(m \circ n^{-1}) \\
&\geq \left(\mu_1^2(m) \wedge \mu_1^2(n)\right) \wedge \left(\mu_2^2(m) \wedge \mu_2^2(n)\right) \wedge \ldots \wedge \left(\mu_p^2(m) \wedge \mu_p^2(n)\right) \\
&= \left(\mu_1^2(m) \wedge \mu_2^2(m) \wedge \ldots \wedge \mu_p^2(m)\right) \wedge \left(\mu_1^2(n) \wedge \mu_2^2(n) \wedge \ldots \wedge \mu_p^2(n)\right) \\
&= \mu^2(m) \wedge \mu^2(n)
\end{aligned}$$

and

$$\begin{aligned}
\nu^2(m \circ n^{-1}) &= \nu_1^2(m \circ n^{-1}) \vee \nu_2^2(m \circ n^{-1}) \vee \ldots \vee \nu_p^2(m \circ n^{-1}) \\
&\leq \left(\nu_1^2(m) \vee \nu_1^2(n)\right) \vee \left(\nu_2^2(m) \vee \nu_2^2(n)\right) \vee \ldots \vee \left(\nu_p^2(m) \vee \nu_p^2(n)\right) \\
&= \left(\nu_1^2(m) \vee \nu_2^2(m) \vee \ldots \vee (\nu_p^2(m)\right) \vee \left(\nu_1^2(n) \vee \nu_2^2(n) \vee \ldots \vee \nu_p^2(n)\right) \\
&= \nu^2(m) \vee \nu^2(n).
\end{aligned}$$

Therefore $\psi = (\mu, \nu)$ is a PFSG of the group $(C, \circ)$.

Hence intersection of a family of PFSGs of a group is also a PFSG of that group. □

Remark 3.2. *Union of two PFSGs of a group may not be a PFSG of that group.*

Example 3.3. *Let us take the group C=($\mathbb{Z}$,+), the group of integers under usual addition and let $\psi_1 = (\mu_1, \nu_1)$, $\psi_2 = (\mu_2, \nu_2)$ be two PFSGs of C defined by*

$$\mu_1(a) = \begin{cases} 0.3, & \text{when } a \in 5\mathbb{Z} \\ 0, & \text{elsewhere} \end{cases}$$

$$\nu_1(a) = \begin{cases} 0, & \text{when } a \in 5\mathbb{Z} \\ 0.5, & \text{elsewhere} \end{cases}$$

$$\mu_2(a) = \begin{cases} 0.15, & \text{when } a \in 3\mathbb{Z} \\ 0, & \text{elsewhere} \end{cases}$$

$$\nu_2(a) = \begin{cases} 0.2, & \text{when } a \in 3\mathbb{Z} \\ 0.3 & \text{elsewhere.} \end{cases}$$

Let $\psi = \psi_1 \cup \psi_2 = (\mu, \nu)$, where

$$\mu(a) = \begin{cases} 0.3, & \text{when } a \in 5\mathbb{Z} \\ 0.15, & \text{when } a \in 3\mathbb{Z} - 5\mathbb{Z} \\ 0, & \text{elsewhere} \end{cases}$$

$$\nu(a) = \begin{cases} 0, & \text{when } a \in 5\mathbb{Z} \\ 0.2, & \text{when } a \in 3\mathbb{Z} - 5\mathbb{Z} \\ 0.3, & \text{elsewhere.} \end{cases}$$

Here, $\mu^2(5 + (-3)) = \mu^2(2) = 0$, but $\mu^2(5) \wedge \mu^2(-3) = \min\{0.3^2, 0.15^2\} = 0.15^2$.

So, $\mu^2(5 + (-3)) \ngeq \mu^2(5) \wedge \mu^2(-3)$.

Again, $\nu^2(5 + (-3)) = \nu^2(2) = 0.09$, but $\nu^2(5) \vee \nu^2(-3) = \max\{0, 0.04\} = 0.04$.

So, $\nu^2(5 + (-3)) \nleq \nu^2(5) \vee \nu^2(-3)$.

Hence $\psi = \psi_1 \cup \psi_2 = (\mu, \nu)$ is not a PFSG of $C = (\mathbb{Z}, +)$.

Proposition 3.3. *If $\psi = (\mu, \nu)$ is a PFSG of a group $(C, \circ)$. Then $\mu^2(m^k) \geq \mu^2(m)$ and $\nu^2(m^k) \leq \nu^2(m)$ for all $m \in C$ and $k \in \mathbb{N}$. Here $m^k = m \circ m \circ \cdots \circ m$ (k times).*

Proof. Since $\psi = (\mu, \nu)$ is a PFSG of a group $(C, \circ)$, then

$\mu^2(m \circ n) \geq \mu^2(m) \wedge \mu^2(n)$ and $\nu^2(m \circ n) \leq \nu^2(m) \vee \nu^2(n)$ for all $m, n \in C$.

So, $\mu^2(m^2) = \mu^2(m \circ m) \geq \mu^2(m) \wedge \mu^2(m) = \mu^2(m)$ and

$\nu^2(m^2) = \nu^2(m \circ m) \leq \nu^2(m) \vee \nu^2(m) = \nu^2(m)\ \forall m \in C$.

Thus by induction, we can show that

$\mu^2(m^k) \geq \mu^2(m)$ and $\nu^2(m^k) \leq \nu^2(m)$ for all $m \in C$ and $k \in \mathbb{N}$. □

The next result produce the condition when equality occurs in the definition of PFSG.

Proposition 3.4. *Let $\psi = (\mu, \nu)$ be a PFSG of a group $(C, \circ)$. If $\mu(m) \neq \mu(n)$ and $\nu(m) \neq \nu(n)$, then $\mu^2(m \circ n) = \mu^2(m) \wedge \mu^2(n)$ and $\nu^2(m \circ n) = \nu^2(m) \vee \nu^2(n)$ respectively $\forall m, n \in C$.*

Proof. Let us assume that $\mu(m) > \mu(n)$ and $\nu(m) < \nu(n)$.

So, $\mu^2(m) > \mu^2(n)$ and $\nu^2(m) < \nu^2(n)$ for all $m, n \in C$.

Now

$$\begin{aligned} \mu^2(n) &= \mu^2(m^{-1} \circ m \circ n) \\ &\geq \mu^2(m^{-1}) \wedge \mu^2(m \circ n) \\ &= \mu^2(m) \wedge \mu^2(m \circ n) \\ &\geq \mu^2(m \circ n), \text{otherwise } \mu^2(n) \geq \mu^2(m), \text{ a contradiction.} \end{aligned}$$

This shows that $\mu^2(n) \geq \mu^2(m \circ n)$.

Again $\mu^2(m \circ n) \geq \mu^2(m) \wedge \mu^2(n) = \mu^2(n)$.

Therefore $\mu^2(m \circ n) \geq \mu^2(n)$.

So, $\mu^2(m \circ n) = \mu^2(n) = \mu^2(m) \wedge \mu^2(n)$ for all $m, n \in C$.

Similarly, when $\mu(m) < \mu(n)$ this result also holds.

Also,

$$\begin{aligned} \nu^2(n) &= \nu^2(m^{-1} \circ m \circ n) \\ &\leq \nu^2(m^{-1}) \vee \nu^2(m \circ n) \\ &= \nu^2(m) \vee \nu^2(m \circ n) \\ &\leq \nu^2(m \circ n), \text{otherwise } \nu^2(n) \leq \nu^2(m), \text{ a contradiction.} \end{aligned}$$

Thus $\nu^2(n) \leq \nu^2(m \circ n)$.

Again $\nu^2(m \circ n) \leq \nu^2(m) \vee \nu^2(n) = \nu^2(n)$.

Therefore $\nu^2(m \circ n) \leq \nu^2(n)$.

Therefore $\nu^2(m \circ n) = \nu^2(n) = \nu^2(m) \vee \nu^2(n)$ for all $m, n \in C$.

Similarly, for $\nu(m) > \nu(n)$ this result also holds.

Hence $\mu^2(m \circ n) = \mu^2(m) \wedge \mu^2(n)$ and $\nu^2(m \circ n) = \nu^2(m) \vee \nu^2(n)$, when $\mu(m) \neq \mu(n)$ and $\nu(m) \neq \nu(n)$ respectively for all $m, n \in C$. □

Proposition 3.5. *Let $\psi = (\mu, \nu)$ be a PFSG of a group $(C, \circ)$ with e as the identity element and $m \in C$. If $\mu^2(m) = \mu^2(e)$ then $\mu^2(m \circ n) = \mu^2(n)$ $\forall n \in C$ and if $\nu^2(m) = \nu^2(e)$ then $\nu^2(m \circ n) = \nu^2(n)$ $\forall n \in C$.*

Proof. Let us assume that $\mu^2(m) = \mu^2(e)$ and $\nu^2(m) = \nu^2(e)$.

So, $\mu^2(m \circ n) \geq \mu^2(m) \wedge \mu^2(n) = \mu^2(e) \wedge \mu^2(n) = \mu^2(n)$, since $\mu^2(e) \geq \mu^2(n)$ $\forall n \in C$.

Also,

$$\begin{aligned} \mu^2(n) &= \mu^2(m^{-1} \circ m \circ n) \\ &\geq \mu^2(m^{-1}) \wedge \mu^2(m \circ n) \\ &= \mu^2(m) \wedge \mu^2(m \circ n) \\ &= \mu^2(e) \wedge \mu^2(m \circ n) \\ &= \mu^2(m \circ n), \text{ since } \mu^2(e) \geq \mu^2(n)\ \forall n \in G. \end{aligned}$$

Therefore $\mu^2(m \circ n) = \mu^2(n)$ $\forall n \in C$, when $\mu^2(m) = \mu^2(e)$.

Again, $\nu^2(m \circ n) \leq \nu^2(m) \vee \nu^2(n) = \nu^2(e) \vee \nu^2(n) = \nu^2(n)$, since $\nu^2(e) \leq \nu^2(n)$ $\forall n \in C$.

In the same way we can show that $\nu^2(n) \leq \nu^2(m \circ n)$.

Hence $\nu^2(m \circ n) = \nu^2(n)$ $\forall n \in C$, when $\nu^2(m) = \mu^2(e)$. □

Theorem 3.3. *Let $\psi = (\mu, \nu)$ be a PFSG of a group $(C, \circ)$. Then the set $N = \{m \in C | \mu^2(e) = \mu^2(m)$ and $\nu^2(e) = \nu^2(m)\}$ forms a subgroup of the group $(C, \circ)$, where e is the identity element in C.*

Proof. Here $N = \{m \in C | \mu^2(e) = \mu^2(m)$ and $\nu^2(e) = \nu^2(m)\}$. Clearly N is non empty, as $e \in N$.

To show $(N, \circ)$ is a subgroup of $(C, \circ)$, we have to show that $m \circ n^{-1} \in N$ for all $m, n \in C$.

Let $m, n \in N$. Then $\mu^2(m) = \mu^2(e) = \mu^2(n)$, $\nu^2(m) = \nu^2(e) = \nu^2(n)$.

Since $\psi = (\mu, \nu)$ is a PFSG of $(C, \circ)$, then

$$\begin{aligned} \mu^2(m \circ n^{-1}) &\geq \mu^2(m) \wedge \mu^2(n^{-1}) \\ &= \mu^2(m) \wedge \mu^2(n) \\ &= \mu^2(e) \wedge \mu^2(e) \\ &= \mu^2(e). \end{aligned}$$

Similarly, we can show that, $\nu^2(m \circ n^{-1}) \leq \nu^2(e)$.

Again from Proposition 3.1, we have $\mu^2(e) \geq \mu^2(m \circ n^{-1})$ and $\nu^2(e) \leq \nu^2(m \circ n^{-1})$.

Therefore $\mu^2(m \circ n^{-1}) = \mu^2(e)$ and $\nu^2(m \circ n^{-1}) = \nu^2(e)$. So, $m \circ n^{-1} \in N$.

Hence $(N, \circ)$ is a subgroup of $(C, \circ)$. □

4. Pythagorean fuzzy coset and Pythagorean fuzzy normal subgroup

In this section, we will define Pythagorean fuzzy coset and Pythagorean fuzzy normal subgroup. Further, we will describe properties related to Pythagorean fuzzy normal subgroup.

Definition 4.1. *Let $\psi = (\mu, \nu)$ be a PFSG of a group $(C, \circ)$. Then for $m \in C$, the Pythagorean fuzzy left coset of ψ is the PFS $m\psi = (m\mu, m\nu)$, defined by $(m\mu)^2(u) = \mu^2(m^{-1} \circ u)$, $(m\nu)^2(u) = \nu^2(m^{-1} \circ u)$ and the Pythagorean fuzzy right coset of ψ is the PFS $\psi m = (\mu m, \nu m)$, defined by $(\mu m)^2(u) = \mu^2(u \circ m^{-1})$, $(\nu m)^2(u) = \nu^2(u \circ m^{-1})$ for all $u \in C$.*

Definition 4.2. *Let $\psi = (\mu, \nu)$ be a PFSG of a group $(C, \circ)$. Then ψ is a Pythagorean fuzzy normal subgroup (PFNSG) of the group $(C, \circ)$ if every Pythagorean fuzzy left coset of ψ is also a Pythagorean fuzzy right coset of ψ in C.*

Equivalently, $m\psi = \psi m$ for all $m \in C$.

Example 4.1. *Let us take the group $C = (\mathbb{Z}_3, +_3)$, where '$+_3$' is addition of integers modulo* 3.

Define a PFS $\psi = (\mu, \nu)$ on C by

$$\mu(0) = 0.9,\ \mu(1) = 0.7,\ \mu(2) = 0.7,$$
$$\nu(0) = 0.1,\ \nu(1) = 0.2,\ \nu(2) = 0.2.$$

We can easily verify that $\psi = (\mu, \nu)$ is a PFSG of C.

For $m = 1 \in C$, the Pythagorean fuzzy left coset of ψ is the PFS $1\psi = (1\mu, 1\nu)$, defined by $(1\mu)^2(u) = \mu^2(1^{-1} +_3 u)$, $(1\nu)^2(u) = \nu^2(1^{-1} +_3 u)$ and the Pythagorean fuzzy right coset of ψ is the PFS $\psi 1 = (\mu 1, \nu 1)$, defined by $(\mu 1)^2(u) = \mu^2(u +_3 1^{-1})$, $(\nu 1)^2(u) = \nu^2(u +_3 1^{-1})$ for all $u \in C$.

When $u = 0$, $(1\mu)^2(0) = \mu^2(1^{-1} +_3 0) = \mu^2(2 +_3 0) = \mu^2(2) = 0.49$ and

$(\mu 1)^2(0) = \mu^2(0 +_3 1^{-1}) = \mu^2(0 +_3 2) = \mu^2(2) = 0.49$.

Also, $(1\nu)^2(0) = \nu^2(1^{-1} +_3 0) = \nu^2(2 +_3 0) = \nu^2(2) = 0.04$ *and*
$(\nu 1)^2(0) = \nu^2(0 +_3 1^{-1}) = \nu^2(0 +_3 2) = \nu^2(2) = 0.04$.
So, $(1\mu)^2(0) = (\mu 1)^2(0)$ *and* $(1\nu)^2(0) = (\nu 1)^2(0)$.
Similarly, we can check that the result holds when $u = 1$ *and* 2.
Therefore $(1\mu)^2(u) = (\mu 1)^2(u)$ *and* $(1\nu)^2(u) = (\nu 1)^2(u)$ *for all* $u \in C$.
That is $1\psi = \psi 1$.
In the same manner it can be shown that $m\psi = \psi m$ *for all* $m \in C$.
Hence $\psi = (\mu, \nu)$ *is a PFNSG of the group* $(\mathbb{Z}_3, +_3)$.

Proposition 4.1. *Let* $\psi = (\mu, \nu)$ *be a PFSG of a group* $(C, \circ)$. *Then* ψ *is a PFNSG of* C *iff* $\mu^2(m \circ n) = \mu^2(n \circ m)$ *and* $\nu^2(m \circ n) = \nu^2(n \circ m)$ *for all* $m, n \in C$.

Proof. Let $\psi = (\mu, \nu)$ be a PFNSG of a group $(C, \circ)$. Then $m\psi = \psi m$ for all $m \in C$.
That is $(m\mu)^2(u) = (\mu m)^2(u)$ and $(m\nu)^2(u) = (\nu m)^2(u)$ for all $m, u \in C$.
Therefore $\mu^2(m^{-1} \circ u) = \mu^2(u \circ m^{-1})$ and $\nu^2(m^{-1} \circ u) = \nu^2(u \circ m^{-1})$ for all $m, u \in C$.
So, $\mu^2(m \circ n) = \mu^2\left(m \circ (n^{-1})^{-1}\right) = \mu^2\left((n^{-1})^{-1} \circ m\right) = \mu^2(n \circ m)$ and
$\nu^2(m \circ n) = \nu^2\left(m \circ (n^{-1})^{-1}\right) = \nu^2\left((n^{-1})^{-1} \circ m\right) = \nu^2(n \circ m)\ \forall m, n \in C$.
Conversely, let $\mu^2(m \circ n) = \mu^2(n \circ m)$ and $\nu^2(m \circ n) = \nu^2(n \circ m)$ for all $m, n \in C$.
This gives $\mu^2\left(m \circ (n^{-1})^{-1}\right) = \mu^2\left((n^{-1})^{-1} \circ m\right)$ and $\nu^2\left(m \circ (n^{-1})^{-1}\right) = \nu^2\left((n^{-1})^{-1} \circ m\right)\ \forall m, n \in C$.
Put $n^{-1} = d$. Then $\mu^2(m \circ d^{-1}) = \mu^2(d^{-1} \circ m)$ and $\nu^2(m \circ d^{-1}) = \nu^2(d^{-1} \circ m)\ \forall m, d \in C$.
So, $(\mu d)^2(m) = (d\mu)^2(m)$ and $(\nu d)^2(m) = (d\nu)^2(m)\ \forall m, d \in C$.
This implies that $\mu d = d\mu$ and $\nu d = d\nu\ \forall d \in C$.
Therefore $\psi d = d\psi$ for all $d \in C$. Hence ψ is a PFNSG of the group $(C, \circ)$. □

Proposition 4.2. *Let* $\psi = (\mu, \nu)$ *be a PFSG of a group* $(C, \circ)$. *Then* ψ *is a PFNSG of* C *iff* $\mu^2(k \circ u \circ k^{-1}) = \mu^2(u)$ *and* $\nu^2(k \circ u \circ k^{-1}) = \nu^2(u)$ *for all* $u, k \in C$.

Proof. Let $\psi = (\mu, \nu)$ be a PFNSG of a group $(C, \circ)$.
Then $\mu^2(m \circ n) = \mu^2(n \circ m)$ and $\nu^2(m \circ n) = \nu^2(n \circ m)$ for all $m, n \in C$.
Now for all $u, k \in C$,

$$\begin{aligned}
\mu^2(k \circ u \circ k^{-1}) &= \mu^2\left((k \circ u) \circ k^{-1}\right) \\
&= \mu^2\left(k^{-1} \circ (k \circ u)\right) \text{(using the above condition)} \\
&= \mu^2(k^{-1} \circ k \circ u) \\
&= \mu^2(e \circ u) \\
&= \mu^2(u).
\end{aligned}$$

Similarly, we get $\nu^2(k \circ u \circ k^{-1}) = \nu^2(u)$.
Therefore $\mu^2(k \circ u \circ k^{-1}) = \mu^2(u)$ and $\nu^2(k \circ u \circ k^{-1}) = \nu^2(u)$ for all $u, k \in C$.
Conversely, let $\mu^2(k \circ u \circ k^{-1}) = \mu^2(u)$ and $\nu^2(k \circ u \circ k^{-1}) = \nu^2(u)$ for all $u, k \in C$.
Now for all $m, n \in C$,

$$\begin{aligned}
\mu^2(m \circ n) &= \mu^2(n^{-1} \circ n \circ m \circ n) \\
&= \mu^2\left((n^{-1}) \circ (n \circ m) \circ (n^{-1})^{-1}\right)
\end{aligned}$$

$$= \mu^2(n \circ m) \text{ (using the above condition)}.$$

Similarly, we get $\nu^2(m \circ n) = \nu^2(n \circ m)$. Therefore by Proposition 4.1, $\psi = (\mu, \nu)$ is a PFNSG of the group $(C, \circ)$. □

Theorem 4.1. *Let $\psi = (\mu, \nu)$ be a PFNSG of a group $(C, \circ)$. Then the set $N = \{m \in C | \mu^2(e) = \mu^2(m)$ and $\nu^2(e) = \nu^2(m)\}$ forms a normal subgroup of the group $(C, \circ)$, where e is the identity element in C.*

Proof. Clearly N is non empty, as $e \in N$. By Proposition 3.3, we have N is a subgroup of the group $(C, \circ)$.

Let $k \in C$ and $u \in N$.
As $u \in N$, $\mu^2(e) = \mu^2(u)$ and $\nu^2(e) = \nu^2(u)$.
Since $\psi = (\mu, \nu)$ is a PFNSG of the group $(G, \circ)$, by Proposition 4.2 we have
$\mu^2(k \circ u \circ k^{-1}) = \mu^2(u)$ and $\nu^2(k \circ u \circ k^{-1}) = \nu^2(u)$ for all $u, k \in C$.
Consequently, $\mu^2(k \circ u \circ k^{-1}) = \mu^2(e)$ and $\nu^2(k \circ u \circ k^{-1}) = \nu^2(e)$ for all $u, k \in C$.
Therefore $k \circ u \circ k^{-1} \in N$. Hence $(N, \circ)$ is a normal subgroup of the group $(C, \circ)$. □

5. Pythagorean fuzzy level subgroup

Definition 5.1. *Let C be a crisp set. Let $\psi = (\mu, \nu)$ be a PFS of the set C. For θ and $\tau \in [0, 1]$, the set $\psi_{(\theta,\tau)} = \{m \in C | \mu^2(m) \geq \theta$ and $\nu^2(m) \leq \tau\}$ is called a Pythagorean fuzzy level subset (PFLS) of the PFS ψ of C, where $0 \leq \theta^2 + \tau^2 \leq 1$.*

Proposition 5.1. *Let $\psi' = (\mu', \nu')$ and $\psi'' = (\mu'', \nu'')$ be two PFSs of the universal set C. Then*
(i) $\psi'_{(\theta,\tau)} \subseteq \psi'_{(\epsilon,\delta)}$ if $\epsilon \leq \theta$ and $\tau \leq \delta$ for ϵ, τ, θ and $\delta \in [0, 1]$,
(ii) $\psi' \subseteq \psi'' \Rightarrow \psi'_{(\theta,\tau)} \subseteq \psi''_{(\theta,\tau)}$ for θ and $\tau \in [0, 1]$.

Proof. (*i*) Let $m \in \psi'_{(\theta,\tau)} \Rightarrow (\mu')^2(m) \geq \theta, (\nu')^2(m) \leq \tau$.
We have $\epsilon \leq \theta$ and $\tau \leq \delta$. So, $\epsilon \leq \theta \leq (\mu')^2(m)$ and $(\nu')^2(m) \leq \tau \leq \delta$.
Therefore, $m \in \psi'_{(\epsilon,\delta)}$.
Hence $\epsilon \leq \theta$ and $\tau \leq \delta \Rightarrow \psi'_{(\theta,\tau)} \subseteq \psi'_{(\epsilon,\delta)}$.
(*ii*) Since $\psi' \subseteq \psi''$, so $\mu'(m) \leq \mu''(m)$ and $\nu'(m) \geq \nu''(m)$ for all $m \in C$.
$\Rightarrow (\mu')^2(m) \leq (\mu'')^2(m)$ and $(\nu')^2(m) \geq (\nu'')^2(m)$ for all $m \in C$.
Let $m \in \psi'_{(\theta,\tau)}$. This implies that $(\mu')^2(m) \geq \theta$ and $(\nu')^2(m) \leq \tau$.
So, $\theta \leq (\mu')^2(m) \leq (\mu'')^2(m)$ and $(\nu'')^2(m) \leq (\nu')^2(m) \leq \tau$.
This shows that $\theta \leq (\mu'')^2(m)$ and $(\nu'')^2(m) \leq \tau$.
Therefore $m \in \psi''_{(\theta,\tau)}$. Hence $\psi'_{(\theta,\tau)} \subseteq \psi''_{(\theta,\tau)}$. □

Proposition 5.2. *Let $\psi = (\mu, \nu)$ be a PFSG of a group $(C, \circ)$. Then the Pythagorean fuzzy level subset $\psi_{(\theta,\tau)}$ forms a subgroup of the group $(C, \circ)$, where $\theta \leq \mu^2(e)$ and $\tau \geq \nu^2(e)$, e is the identity element in C.*

Proof. Here, $\psi_{(\theta,\tau)} = \{m \in C | \mu^2(m) \geq \theta$ and $\nu^2(m) \leq \tau\}$.
Clearly $\psi_{(\theta,\tau)}$ is non empty, as $e \in \psi_{(\theta,\tau)}$.
To show that $\psi_{(\theta,\tau)}$ is a subgroup of $(C, \circ)$, we have to show that for $m, n \in \psi_{(\theta,\tau)}$, $m \circ n^{-1} \in \psi_{(\theta,\tau)}$. Let $m, n \in \psi_{(\theta,\tau)}$.

Then $\mu^2(m) \geq \theta$ & $\nu^2(m) \leq \tau$ and $\mu^2(n) \geq \theta$ & $\nu^2(b) \leq \tau$.
Since $\psi = (\mu, \nu)$ is a PFSG of the group $(C, \circ)$, then

$$\begin{array}{rcl}\mu^2(m \circ n^{-1}) & \geq & \mu^2(m) \wedge \mu^2(n^{-1}) \\ & = & \mu^2(m) \wedge \mu^2(n)) \\ & \geq & \theta \wedge \theta \\ & = & \theta\end{array}$$

and

$$\begin{array}{rcl}\nu^2(m \circ n^{-1}) & \leq & \nu^2(m) \vee \nu^2(n^{-1}) \\ & = & \nu^2(m) \vee \nu^2(n)) \\ & \leq & \tau \vee \tau \\ & = & \tau.\end{array}$$

Therefore $m \circ n^{-1} \in \psi_{(\theta,\tau)}$. Hence $\psi_{(\theta,\tau)}$ is a subgroup of the group $(C, \circ)$. □

Definition 5.2. *The subgroup $\psi_{(\theta,\tau)}$ of the group $(C, \circ)$ is called Pythagorean fuzzy level subgroup (PFLSG) of the PFSG $\psi = (\mu, \nu)$.*

Proposition 5.3. *Let $\psi = (\mu, \nu)$ be a PFS of a group $(C, \circ)$. If the PFLS $\psi_{(\theta,\tau)}$ is a subgroup of the group $(C, \circ)$, where $\theta \leq \mu^2(e)$ and $\tau \geq \nu^2(e)$ then $\psi = (\mu, \nu)$ is a PFSG of the group $(C, \circ)$.*

Proof. Let $m, n \in C$. Given that $\psi = (\mu, \nu)$ is a PFS of C and $\psi_{(\theta,\tau)}$ is a subgroup of the group $(C, \circ)$.
Let us assume that $\mu^2(m) = \theta_1$, $\mu^2(n) = \theta_2$ with $\theta_1 < \theta_2$ and $\nu^2(m) = \tau_1$, $\nu^2(n) = \tau_2$ with $\tau_1 > \tau_2$.
This implies that $m \in \psi_{(\theta_1,\tau_1)}$ and $n \in \psi_{(\theta_2,\tau_2)}$.
Since, $\theta_1 < \theta_2$ and $\tau_1 > \tau_2$, then $\psi_{(\theta_2,\tau_2)} \subseteq \psi_{(\theta_1,\tau_1)}$. So, $n \in \psi_{(\theta_1,\tau_1)}$.
Now, $m \in \psi_{(\theta_1,\tau_1)}$ and $n \in \psi_{(\theta_1,\tau_1)}$.
So, $m \circ n \in \psi_{(\theta_1,\tau_1)}$, since $\psi_{(\theta_1,\tau_1)}$ is a subgroup of $(C, \circ)$.
Therefore $\mu^2(m \circ n) \geq \theta_1$ and $\nu^2(m \circ n) \leq \tau_1$
$\Rightarrow \mu^2(m \circ n) \geq \theta_1 \wedge \theta_2$ and $\nu^2(m \circ n) \leq \tau_1 \vee \tau_2$, since $\theta_1 < \theta_2$ and $\tau_1 > \tau_2$
$\Rightarrow \mu^2(m \circ n) \geq \mu^2(m) \wedge \mu^2(n)$ and $\nu^2(m \circ n) \leq \nu^2(m) \vee \nu^2(n)$.
Again, $m \in \psi_{(\theta_1,\tau_1)} \Rightarrow m^{-1} \in \psi_{(\theta_1,\tau_1)}$, since $\psi_{(\theta_1,\tau_1)}$ is a subgroup of $(C, \circ)$
$\Rightarrow \mu^2(m^{-1}) \geq \theta_1$ and $\nu^2(m^{-1}) \leq \tau_1$
$\Rightarrow \mu^2(m^{-1}) \geq \mu^2(m)$ and $\nu^2(m^{-1}) \leq \nu^2(m)$.
Since $m, n \in C$ are arbitrary,
$\mu^2(m \circ n) \geq \mu^2(m) \wedge \mu^2(n)$, $\nu^2(m \circ n) \leq \nu^2(m) \vee \nu^2(n)$ for all $m, n \in C$ and
$\mu^2(m^{-1}) \geq \mu^2(a)$, $\nu^2(m^{-1}) \leq \nu^2(m)$ for all $m \in C$.
Hence $\psi = (\mu, \nu)$ is a PFSG of the group $(C, \circ)$. □

Proposition 5.4. *Let $\psi = (\mu, \nu)$ be a PFNSG of a group $(C, \circ)$. Then the Pythagorean fuzzy level subset $\psi_{(\theta,\tau)}$ forms a normal subgroup of the group $(C, \circ)$, where $\theta \leq \mu^2(e)$ and $\tau \geq \nu^2(e)$, e is the identity element in C.*

Proof. Since $\psi = (\mu, \nu)$ is a PFNSG of the group $(C, \circ)$, then for all $u, k \in C$,
$\mu^2(k \circ u \circ k^{-1}) = \mu^2(u)$ and $\nu^2(k \circ u \circ k^{-1}) = \nu^2(u)$.
By Proposition 5.2, $\psi_{(\theta,\tau)} = \{m \in G | \mu^2(m) \geq \theta$ and $\nu^2(m) \leq \tau\}$ is a subgroup of $(C, \circ)$.
Let $k \in G$ and $u \in \psi_{(\theta,\tau)}$. Then $\mu^2(k \circ u \circ k^{-1}) = \mu^2(u) \geq \theta$ and $\nu^2(k \circ u \circ k^{-1}) = \nu^2(u) \leq \tau$.
Therefore $k \circ u \circ k^{-1} \in \psi_{(\theta,\tau)}$. Hence $\psi_{(\theta,\tau)}$ is a normal subgroup of the group $(C, \circ)$. □

6. Homomorphism on Pythagorean fuzzy subgroup

In this section, we will discuss the effect of group homomorphism on Pythagorean fuzzy subgroup.

Theorem 6.1. *Let $(C_1, \circ_1)$ and $(C_2, \circ_2)$ be two groups. Let g be a surjective homomorphism from $(C_1, \circ_1)$ to $(C_2, \circ_2)$ and $\psi = (\mu, \nu)$ be a PFSG of $(C_1, \circ_1)$. Then $g(\psi) = (g(\mu), g(\nu))$ is a PFSG of $(C_2, \circ_2)$.*

Proof. Since $g : C_1 \to C_2$ is a surjective homomorphism, then $g(C_1) = C_2$.
Let m_2 and n_2 be two elements of C_2.
Suppose $m_2 = g(m_1)$ and $n_2 = g(n_1)$ for some $m_1, n_1 \in C_1$.
We have $g(\psi) = \{(c, g(\mu)(c), g(\nu)(c)) | c \in C_2\}$. Now

$$\begin{aligned}
(g(\mu))^2(m_2 \circ_2 n_2) &= \{(g(\mu))(m_2 \circ_2 n_2)\}^2 \\
&= [\vee \{\mu(t) | t \in C_1, g(t) = m_2 \circ_2 n_2\}]^2 \\
&= \vee \left\{\mu^2(t) | t \in C_1, g(t) = m_2 \circ_2 n_2\right\} \\
&\geq \vee \left\{\mu^2(m_1 \circ_1 n_1) | m_1, n_1 \in C_1 \text{ and } g(m_1) = m_2, g(n_1) = n_2\right\} \\
&\geq \vee \left\{\mu^2(m_1) \wedge \mu^2(n_1) | m_1, n_1 \in C_1 \text{ and } g(m_1) = m_2, g(n_1) = n_2\right\} \\
&= \left(\vee \left\{\mu^2(m_1) | m_1 \in C_1 \text{ and } g(m_1) = m_2\right\}\right) \wedge \left(\vee \left\{\mu^2(n_1) | n_1 \in C_1 \text{ and } g(n_1) = n_2\right\}\right) \\
&= \{(g(\mu))(m_2)\}^2 \wedge \{(g(\mu))(n_2)\}^2 \\
&= (g(\mu))^2(m_2) \wedge (g(\mu))^2(n_2).
\end{aligned}$$

Therefore $(g(\mu))^2(m_2 \circ_2 n_2) \geq (g(\mu))^2(m_2) \wedge (g(\mu))^2(n_2)$ for all m_2 and $n_2 \in C_2$.
Similarly, we can prove that $(g(\nu))^2(m_2 \circ_2 n_2) \leq (g(\nu))^2(m_2) \vee (g(\nu))^2(n_2)$ for all m_2 and $n_2 \in C_2$. Again

$$\begin{aligned}
(g(\mu))^2(m_2^{-1}) &= \{g(\mu)(m_2^{-1})\}^2 \\
&= \left[\vee \left\{\mu(m) | m \in C_1 \text{ and } g(m) = (m_2^{-1})\right\}\right]^2 \\
&= \left[\vee \left\{\mu(m^{-1}) | m^{-1} \in C_1 \text{ and } g(m^{-1}) = m_2\right\}\right]^2 \\
&= \{g(\mu)(m_2)\}^2 \\
&= (g(\mu))^2(m_2).
\end{aligned}$$

Therefore $(g(\mu))^2(m_2^{-1}) = (g(\mu))^2(m_2)$ for all $m_2 \in C_2$.
Similarly, we can show that $(g(\nu))^2(m_2^{-1}) = (g^{-1}(\nu))^2(m_2)$ for all $m_2 \in C_2$. Hence $g(\psi) = (g(\mu), g(\nu))$ is a PFSG of $(C_2, \circ_2)$. □

Theorem 6.2. *Let $(C_1, \circ_1)$ and $(C_2, \circ_2)$ be two groups. Let g be a bijective homomorphism from $(C_1, \circ_1)$ to $(C_2, \circ_2)$ and $\psi = (\mu, \nu)$ be a PFSG of $(C_2, \circ_2)$. Then $g^{-1}(\psi) = (g^{-1}(\mu), g^{-1}(\nu))$ is a PFSG of $(C_1, \circ_1)$.*

Proof. Let m_1 and n_1 be two arbitrary elements of C_1, then $m_1 \circ_1 n_1 \in C_1$. We have $g^{-1}(\psi) = \{(c, g^{-1}(\mu)(c), g^{-1}(\nu)(c)) | c \in C_1\}$. Now

$$\begin{aligned}
(g^{-1}(\mu))^2(m_1 \circ_1 n_1) &= \{g^{-1}(\mu)(m_1 \circ_1 n_1)\}^2 \\
&= \{\mu\,(g(m_1 \circ_1 n_1))\}^2 \\
&= \{\mu\,(g(m_1) \circ_2 g(n_1))\}^2 \text{ (since } g \text{ is a homomorphism)} \\
&= \mu^2\,(g(m_1) \circ_2 g(n_1)) \\
&\geq \mu^2(g(m_1)) \wedge \mu^2(g(n_1)) \text{ (since } \psi = (\mu, \nu) \text{ is a PFSG of } (C_2, \circ_2)) \\
&= (g^{-1}(\mu))^2(m_1) \wedge (g^{-1}(\mu))^2(n_1).
\end{aligned}$$

Therefore $(g^{-1}(\mu))^2(m_1 \circ_1 n_1) \geq (g^{-1}(\mu))^2(m_1) \wedge (g^{-1}(\mu))^2(n_1)$ for all m_1 and $n_1 \in C_1$.

Similarly, we can show that $(g^{-1}(\nu))^2(m_1 \circ_1 n_1) \leq (g^{-1}(\nu))^2(m_1) \vee (g^{-1}(\nu))^2(n_1)$ for all m_1 and $n_1 \in C_1$. Again

$$\begin{aligned}
(g^{-1}(\mu))^2(m_1^{-1}) &= \{(g^{-1}(\mu))(m_1^{-1})\}^2 \\
&= \{\mu\left(g(m_1^{-1})\right)\}^2 \\
&= \mu^2\left(g(m_1^{-1})\right) \\
&= \mu^2\left(g(m_1)^{-1}\right) \\
&= \mu^2\,(g(m_1)) \\
&= (g^{-1}(\mu))^2(m_1).
\end{aligned}$$

Therefore $(g^{-1}(\mu))^2(m_1^{-1}) = (g^{-1}(\mu))^2(m_1)$ for all $m_1 \in C_1$.

Similarly, we can show that $(g^{-1}(\nu))^2(m_1^{-1}) = (g^{-1}(\nu))^2(m_1)$ for all $m_1 \in C_1$. Hence $g^{-1}(\psi) = (g^{-1}(\mu), g^{-1}(\nu))$ is a PFSG of $(C_1, \circ_1)$. □

Theorem 6.3. *Let $(C_1, \circ_1)$ and $(C_2, \circ_2)$ be two groups. Let g be a surjective homomorphism from $(C_1, \circ_1)$ to $(C_2, \circ_2)$ and $\psi = (\mu, \nu)$ be a PFNSG of $(C_1, \circ_1)$. Then $g(\psi) = (g(\mu), g(\nu))$ is a PFNSG of $(C_2, \circ_2)$.*

Proof. In view of Theorem 6.1, we can state that $g(\psi) = (g(\mu), g(\nu))$ is a PFSG of $(C_2, \circ_2)$.

Since $\psi = (\mu, \nu)$ is a PFNSG of $(C_1, \circ_1)$, then by Proposition 4.1 $\mu^2(m_1 \circ_1 n_1) = \mu^2(n_1 \circ_1 m_1)$ and $\nu^2(m_1 \circ_1 n_1) = \nu^2(n_1 \circ_1 m_1)$ for all $m_1, n_1 \in C_1$.

Let m_2 and n_2 be two elements of C_2.

Suppose that there exist unique m_1 and $n_1 \in C_1$, such that $m_2 = g(m_1)$ and $n_2 = g(n_1)$. Now

$$\begin{aligned}
(g(\mu))^2(m_2 \circ_2 n_2) &= \{g(\mu)(m_2 \circ_2 n_2)\}^2 \\
&= [\vee \{\mu(l) | l \in C_1, g(l) = m_2 \circ_2 n_2\}]^2 \\
&= [\vee \{\mu(m_1 \circ_1 n_1) | m_1, n_1 \in C_1 \text{ and } g(m_1) = m_2, g(n_1) = n_2\}]^2 \\
&= \vee \left\{\mu^2(m_1 \circ_1 n_1) | m_1, n_1 \in C_1 \text{ and } g(m_1) = m_2, g(n_1) = n_2\right\} \\
&= \vee \left\{\mu^2(n_1 \circ_1 m_1) | m_1, n_1 \in C_1 \text{ and } g(m_1) = m_2, g(n_1) = n_2\right\} \\
&= [\vee \{\mu(l) | l \in C_1, g(l) = n_2 \circ_2 m_2\}]^2
\end{aligned}$$

$$= (g(\mu))^2(n_2 \circ_2 m_2).$$

Therefore $(g(\mu))^2(m_2 \circ_2 n_2) = (g(\mu))^2(n_2 \circ_2 m_2)$ for all m_2 and $n_2 \in C_2$.

Similarly, we can prove that $(g(\nu))^2(m_2 \circ_2 n_2) = (g(\nu))^2(n_2 \circ_2 m_2)$ for all m_2 and $n_2 \in C_2$.

Hence by Proposition 4.1, $g(\psi) = (g(\mu), g(\nu))$ is a PFNSG of $(C_2, \circ_2)$. □

Theorem 6.4. *Let $(C_1, \circ_1)$ and $(C_2, \circ_2)$ be two groups. Let g be a bijective homomorphism from $(C_1, \circ_1)$ to $(C_2, \circ_2)$ and $\psi = (\mu, \nu)$ be a PFNSG of $(C_2, \circ_2)$. Then $g^{-1}(\psi) = (g^{-1}(\mu), g^{-1}(\nu))$ is a PFNSG of $(C_1, \circ_1)$.*

Proof. In view of Theorem 6.2, we can state that $g^{-1}(\psi) = (g^{-1}(\mu), g^{-1}(\nu))$ is a PFSG of $(C_1, \circ_1)$.

Since $\psi = (\mu, \nu)$ is a PFNSG of $(C_2, \circ_2)$, then by Proposition 4.1
$\mu^2(m_2 \circ_2 n_2) = \mu^2(n_2 \circ_2 m_2)$ and $\nu^2(m_2 \circ_2 n_2) = \nu^2(n_2 \circ_2 m_2)$ for all $m_2, n_2 \in C_2$.

Let m_1 and n_1 be two elements of C_1. Then

$$\begin{aligned}
(g^{-1}(\mu))^2(m_1 \circ_1 n_1) &= \{g^{-1}(\mu)(m_1 \circ_1 n_1)\}^2 \\
&= \{\mu\,(g(m_1 \circ_1 n_1))\}^2 \\
&= \mu^2\,(g(m_1 \circ_1 n_1)) \\
&= \mu^2\,(g(m_1) \circ_2 g(n_1)) \ \text{(Since } g \text{ is a homomorphism)} \\
&= \mu^2\,(g(n_1) \circ_2 g(m_1)) \\
&= \mu^2\,(g(n_1 \circ_1 m_1)) \\
&= \{g^{-1}(\mu)(n_1 \circ_1 m_1)\}^2 \\
&= (g^{-1}(\mu))^2(n_1 \circ_1 m_1).
\end{aligned}$$

Therefore $(g^{-1}(\mu))^2(m_1 \circ_1 n_1) = (g^{-1}(\mu))^2(n_1 \circ_1 m_1)$ for all m_1 and $n_1 \in C_1$.

Similarly, we can prove that $(g^{-1}(\nu))^2(m_1 \circ_1 n_1) = (g^{-1}(\nu))^2(n_1 \circ_1 m_1)$ for all m_1 and $n_1 \in C_1$.

Hence by Proposition 4.1, $g^{-1}(\psi) = (g^{-1}(\mu), g^{-1}(\nu))$ is a PFNSG of $(C_1, \circ_1)$. □

7. Conclusion

The purpose of this paper is to initiate the study of Pythagorean fuzzy subgroup. We have discussed various algebraic attributes of Pythagorean fuzzy subgroup. We have proved that intuitionistic fuzzy subgroup of any group is a Pythagorean fuzzy subgroup of that group. We have introduced the notion of Pythagorean fuzzy coset and Pythagorean fuzzy normal subgroup. We have presented the necessary and sufficient condition for Pythagorean fuzzy subgroup to be a Pythagorean fuzzy normal subgroup. Further, we have proved that Pythagorean fuzzy level subset is a normal subgroup of the given group. Moreover, we have studied the effect of group homomorphism on Pythagorean fuzzy subgroup. In our future work, we will work on Pythagorean fuzzy quotient group and order of Pythagorean fuzzy subgroup. We will also work on the Lagrange theorem in Pythagorean fuzzy subgroup.

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Conflict of interest

All authors declare that there is no conflict of interest.

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