Mathematics

Research article

A fractional Landweber iterative regularization method for stable analytic continuation

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Abstract: In this paper, we consider the problem of analytic continuation of the analytic function $g(z) = g(x + iy)$ on a strip domain $\Omega = \{z = x + iy \in \mathbb{C} \,|\, x \in \mathbb{R}, 0 < y < y_0\}$, where the data is given only on the line $y = 0$. This problem is a severely ill-posed problem. We propose the fraction Landweber iterative regularization method to deal with this problem. Under the a priori and a posteriori regularization parameter choice rule, we all obtain the error estimates between the regularization solution and the exact solution. Some numerical examples are given to verify the efficiency and accuracy of the proposed methods.

Keywords: stable analytic continuation; fractional Landweber regularization method; Ill-posed problem; inverse problem
Mathematics Subject Classification: 35R25, 35R30, 47A52

1. Introduction

The problem for analytic continuation of an analytic function is encountered in many practical applications (see,e.g., [1–4] and the references therein), and the numerical analytic continuation is a more interesting and hard problem. It is well known, in general, to be ill-posed in the sense that the solution does not depend continuously on the data. To obtain stable numerical algorithms for ill-posed problems, some effective regularization methods must be adopted, and several nonclassical methods have been developed rapidly in recent years. In [5], the authors used the fourier truncation regularization method to solve this problem. In [6], the authors used the modified kernel method to solve this problem. In [7], the authors used the generalized Tikhonov regularization method to solve this problem. In [8], the authors used the optimal filtering method to solve this problem and obtained the optimal error estimate. However in [5–8], the regularization parameter is chosen by the a priori choice which depends on the a priori bound E. But in practice, the a priori bound doesn't know for

sure, and working with a wrong constant E may lead to the bad regularization solution. In [9], the authors study a continuous fractional regularization method called FAR. In this work, it is strictly proved that FAR is an accelerated algorithm relative to the comparable order optimal regularization method if the fractional order is in the range of $(1, 2)$. In [10], the authors study the convergence of Landweber iteration for linear and nonlinear inverse problems in Hilbert scale. Different from the usual application of Hilbert scale in the framework of regularization method, the case of $s < 0$ (for Tikhonov regularization) corresponds to the weaker regularization standard in the regularization method. In [11], the authors propose a new iterative regularization method for solving ill-posed linear operator equations. The prototype of these iterative regularization methods is a second-order evolution equation with linear vanishing damping term. It can not only be regarded as an extension of asymptotic regularization, but also as a continuous simulation of Nesterov accelerated scheme. They also discussed the application of the newly developed accelerated iterative regularization method with a posteriori stopping rule in diffusion-based bioluminescence tomography, which is modeled as an inverse source problem of elliptic partial differential equations with both Dirichlet and Neumann boundary data. In [12,13], the authors used the modified Tikhonov regularization method to solve this problem, and gave the error estimates between the regularization solution and the exact solution under the a priori and a posteriori regularization parameter choice rules, respectively. However, under the a posteriori regularization selection rule, the form of error estimation is logarithmic. In [14–17], the authors used the iterative regularization method, wavelet regularization method, a modified Tikhonov regularization method and a modified Lavrentiev iterative regularization method to solve this problem. Under two regularization parameter choice rules, the Hölder type error estimates were all obtained. In this paper, we use fraction Landweber iterative regularization method and Landweber iterative regularization method to solve this problem. Landweber regularization method which is very useful to solve the inverse problem overcomes the saturation phenomenon of Tikhonov regularization method. The Landweber regularization method first comes from [18]. Now, the Landweber regularization method has been used to solve a lot of inverse problem, one can see [19–24]. In [25], Xiong firstly proposed Fractional Landweber regularization in 2017. Compared with the standard Landweber method, it reduces the step of the iteration greatly.

We can sum up the problem as follows. Suppose the domain Ω is

$$\Omega := \{z = x + iy \in \mathbb{C} | x \in \mathbb{R}, 0 < y < y_0, y_0 > 0\} \tag{1.1}$$

in complex plane $\mathbb{C}$, i is the imaginary unit and y_0 is a positive constant. The function $g(z) = g(x + iy)$ is an analytic function in Ω. $g(\cdot + iy) \in L^2(R^2)$ for all $y \in [0, y_0]$. The data at $y = 0$ is the given measurement data, i.e., $g(z)|_{y=0} = g(x) \in L^2(\mathbb{R})$. Let the noise data be $g^\delta(x)$, which belongs to $L^2(\mathbb{R})$. And $g(x)$, $g^\delta(x)$ satisfy

$$\|g(\cdot) - g^\delta(\cdot)\| \le \delta, \tag{1.2}$$

where δ is the noise level, $\| \cdot \|$ is $L^2(\mathbb{R})$ norm. Moreover, assume there holds the following a priori bound

$$\|g(\cdot + iy_0)\| \le E, \tag{1.3}$$

where E is a fixed positive constant.

$\hat{g}$ is the Fourier transform of the function $g(x)$, which is expressed as follows

$$\hat{g}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} g(x) dx, \tag{1.4}$$

and the inverse Fourier transform of the function $\hat{g}(\xi)$ is

$$g(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi}\hat{g}(\xi)d\xi. \tag{1.5}$$

The $L^2(\mathbb{R})$ norm for $g(x)$ is defined by

$$\|g\|_{L^2(\mathbb{R})} := \left(\int_{-\infty}^{\infty} |g(x)|^2 dx\right)^{\frac{1}{2}}, \tag{1.6}$$

and by utilizing the Parseval formula,

$$\|g\|_{L^2(\mathbb{R})} := \left(\int_{-\infty}^{\infty} |g(x)|^2 dx\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} |\hat{g}(\xi)|^2 d\xi\right)^{\frac{1}{2}}. \tag{1.7}$$

The problem that needs to be solved is wielding the measurement data $g(x + iy)|_{y=0}$ to recover the data $g(x + iy)$ for $0 < y < y_0$. By using the inverse Fourier transform with the respect to the variable x,

$$g(z) = g(x + iy) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(x+iy)\xi}\hat{g}(\xi)d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y\xi}e^{ix\xi}\hat{g}(\xi)d\xi. \tag{1.8}$$

Therefore, we get the following equation

$$\widehat{g(\cdot + iy)}(\xi, y) = e^{-y\xi}\hat{g}(\xi), \tag{1.9}$$

i.e.,

$$e^{y\xi}\widehat{g(\cdot + iy)}(\xi, y) = \hat{g}(\xi). \tag{1.10}$$

Problem (1.10) also can be denoted by the linear operator equation

$$\hat{K}(\xi)\widehat{g(\cdot + iy)}(\xi, y) = \hat{g}(\xi). \tag{1.11}$$

where $\hat{K}(\xi) := e^{y\xi}$ is a multiplication operator. By [27], we know that problem (1.10) is a linear ill-posed operator equation.

The clue of this paper is as follows. In section 2, fractional Landweber iterative regularization method and the a priori error estimation are proposed. In section 3, the a posteriori error estimations between the exact solution and the approximate solution are given. In section 4, several examples are selected to show the effectiveness of this method for solving this problem.

2. The convergence error estimate with an a priori parameter choice rule

Now, we take on a regularization method which is called Landweber regularization method. And the Eq (1.11) could be described as the following operator equation

$$\widehat{g(\cdot + iy)}(\xi, y) = (I - a\hat{K} * \hat{K})\widehat{g(\cdot + iy)}(\xi, y) + a\hat{K}\hat{g}(\xi) \tag{2.1}$$

to replace $\hat{K}\widehat{g(\cdot + iy)}(\xi, y) = \hat{g}(\xi)$, where $0 < a < \frac{1}{\|K\|^2}$. Therefore, the form of iterative scheme is as follows:

$$\begin{cases} g^{0,\widehat{\delta}(\cdot + iy)}(\xi, y) = 0, \\ g^{m,\widehat{\delta}(\cdot + iy)}(\xi, y) = (I - a\hat{K} * \hat{K})g^{m-1,\widehat{\delta}(\cdot + iy)}(\xi, y) + a\hat{K}\hat{g}^{\delta}(\xi), \end{cases} \tag{2.2}$$

$m = 1, 2, 3, \cdots$ is the iterative steps. Combined with noise disturbance term $\hat{g}^\delta(\xi)$, the approximate solution obtained by Landweber iterative regularization method can be expressed as

$$g^{m_1,\delta}\widehat{(\cdot + iy)}(\xi, y) = R_{m_1}(\xi, y)\hat{g}^\delta(\xi), \tag{2.3}$$

where

$$R_{m_1}(\xi, y) = \begin{cases} e^{-y\xi}, & \xi \geq 0, \\ \frac{1-(1-ae^{2y\xi})^{m_1}}{e^{y\xi}}, & \xi < 0. \end{cases} \tag{2.4}$$

The fractional Landweber regularization solution is

$$g^{m,\delta}\widehat{(\cdot + iy)}(\xi, y) = R_m(\xi, y)\hat{g}^\delta(\xi), \tag{2.5}$$

where

$$R_m(\xi, y) = \begin{cases} e^{-y\xi}, & \xi \geq 0, \\ \frac{[1-(1-ae^{2y\xi})^m]^\gamma}{e^{y\xi}}, & \xi < 0, \end{cases} \tag{2.6}$$

where $0 < \gamma < 1$ is a constant.

Now, we first give some useful Lemmas.

Lemma 2.1. Take $p > 0, q > 0$, then

$$\sup_{s \leq 0}(1 - ae^{ps})^m e^{qs} \leq \left(\frac{q}{aq + apm}\right)^{\frac{q}{p}}, \tag{2.7}$$

where m, a are both positive constants and a satisfies $0 < a < \frac{1}{\|K\|^2}$.

Proof. Let $F(s) := (1 - ae^{ps})^m e^{qs}$. Based on $\lim_{s\to 0} F(s) = \lim_{s\to-\infty} = 0$, thus we can presume

$$F(s) \leq \sup_{s \leq 0} F(s) \leq F(s_0),$$

where $s_0 < 0$, satisfies $F'(s_0) = 0$. We can easily calculate $s_0 = \frac{1}{p} \ln \frac{q}{aq+amp}$, so we have

$$F(s) \leq \left(1 - \frac{q}{q + mp}\right)^m \left(\frac{q}{aq + amp}\right)^{\frac{q}{p}} \leq \left(\frac{q}{aq + amp}\right)^{\frac{q}{p}}.$$

Lemma 2.2 [5]. If $\|g(\cdot + iy_0)\| \leq E$, we obtain

$$\|g(\cdot + iy)\| \leq E^{\frac{y}{y_0}} \delta^{1-\frac{y}{y_0}}(1 + o(1)). \tag{2.8}$$

Lemma 2.3. For $\frac{1}{2} < \gamma < 1, 0 < a < \frac{1}{\|K\|^2}$ and $m \geq 1$, we can get

$$\sup_{\xi < 0} [1 - (1 - ae^{2y\xi})^m]^\gamma \frac{1}{e^{y\xi}} \leq a^{\frac{1}{2}} m^{\frac{1}{2}}.$$

Proof. Because $0 < a < \frac{1}{\|K\|^2}$, so we obtain $0 < ae^{2y\xi} < 1$. We define two functions with $\beta^2 = ae^{2y\xi}$:

$$\varphi(\beta) = a[1 - (1 - \beta^2)^m]^{2\gamma}\beta^{-2}$$

and

$$\psi(\beta) = [1 - (1 - \beta^2)^m]^{2\gamma}\beta^{-2}.$$

Consequently, $\varphi(\beta) = a\psi(\beta)$. These two functions are continuous when $\beta \in (0, 1)$.
For $\frac{1}{2} < \gamma < 1$ and $\beta \in (0, 1)$, by Lemma 3.3 in [26], we obtain

$$\psi(\beta) \leq m.$$

Therefore,

$$\sup_{\xi<0} [1 - (1 - ae^{2y\xi})^m]^\gamma \frac{1}{e^{y\xi}} \leq a^{\frac{1}{2}} m^{\frac{1}{2}}.$$

Theorem 2.3. Assume (1.2) and (1.3) hold, if we select

$$m = [\frac{E}{\delta}]^{\frac{2y}{y_0}}, \tag{2.9}$$

then we obtain,

$$\|g^{m,\delta}(\cdot + iy) - g(\cdot + iy)\| \leq C_1 E^{\frac{y}{y_0}} \delta^{1 - \frac{y}{y_0}} + \delta, \tag{2.10}$$

where $[m]$ denotes the largest integer less than or equal to m, $C_1 = (\frac{y_0-y}{ay})^{\frac{y_0-y}{2y}} + \sqrt{a}$.
Proof. By the triangle inequality and Parseval equality, we have

$$\|g^{m,\delta}(\cdot + iy)(\xi, y) - g(\cdot + iy)(\xi, y)\|$$
$$= \|g^{m,\widehat{\delta}(\cdot + iy)}(\xi, y) - g\widehat{(\cdot + iy)}(\xi, y)\|$$
$$\leq \|g^{m,\widehat{\delta}(\cdot + iy)}(\xi, y) - g^m\widehat{(\cdot + iy)}(\xi, y)\| + \|g^m\widehat{(\cdot + iy)}(\xi, y) - g\widehat{(\cdot + iy)}(\xi, y)\|.$$

Using (2.6), we obtain

$$\|g^{m,\widehat{\delta}(\cdot + iy)}(\xi, y) - g^m\widehat{(\cdot + iy)}(\xi, y)\|$$
$$= (\int_{-\infty}^{\infty} |R_m(\xi, y)(\hat{g}^\delta(\xi) - \hat{g}(\xi))|^2 d\xi)^{\frac{1}{2}}$$
$$= (\int_0^{+\infty} |e^{-y\xi}(\hat{g}^\delta(\xi) - \hat{g}(\xi))|^2 d\xi)^{\frac{1}{2}}$$
$$+ (\int_{-\infty}^0 |\frac{[1 - (1 - ae^{2y\xi})^m]^\gamma}{e^{y\xi}}(\hat{g}^\delta(\xi) - \hat{g}(\xi))|^2 d\xi)^{\frac{1}{2}}$$
$$\leq (\sup_{\xi>0} e^{-y\xi})(\int_0^{+\infty} |\hat{g}^\delta(\xi) - \hat{g}(\xi)|^2 d\xi)^{\frac{1}{2}}$$
$$+ (\sup_{\xi<0} (\frac{[1 - (1 - ae^{2y\xi})^m]^\gamma}{e^{y\xi}})^2)^{\frac{1}{2}} (\int_{-\infty}^0 |\hat{g}^\delta(\xi) - \hat{g}(\xi)|^2 d\xi)^{\frac{1}{2}}$$
$$\leq [(\sup_{\xi\geq0} e^{-y\xi}) + (\sup_{\xi<0} \frac{[1 - (1 - ae^{2y\xi})^m]^\gamma}{e^{y\xi}})](\int_{-\infty}^{+\infty} |(\hat{g}^\delta(\xi) - \hat{g}(\xi))|^2 d\xi)^{\frac{1}{2}}$$
$$\leq (1 + \sup_{\xi<0} B_1(\xi))\delta,$$

where $B_1(\xi) := \frac{[1-(1-ae^{2y\xi})^m]^\gamma}{e^{y\xi}}$. According to Lemma 2.3, we obtain

$$\sup_{\xi<0} B_1(\xi) \leq \sqrt{am},$$

then,

$$\|g^{m,\delta}\widehat{(\cdot + iy)}(\xi, y) - g^m\widehat{(\cdot + iy)}(\xi, y)\| \le \sqrt{am}\delta + \delta. \tag{2.11}$$

Because $0 < 1 - (1 - ae^{2y\xi})^m < 1$ and $\frac{1}{2} < \gamma < 1$, so $[1 - (1 - ae^{2y\xi})^m]^\gamma > [1 - (1 - ae^{2y\xi})^m]$. Futher, we can get $1 - [1 - (1 - ae^{2y\xi})^m]^\gamma < 1 - [1 - (1 - ae^{2y\xi})^m] = (1 - ae^{2y\xi})^m$. By (1.10), lemma 2.1, and (1.3), (2.6), we get

$$\|g^m\widehat{(\cdot + iy)}(\xi, y) - g\widehat{(\cdot + iy)}(\xi, y)\|$$

$$= (\int_{-\infty}^{\infty} |R_m(\xi, y)\hat{g}(\xi) - \frac{1}{e^{y\xi}}\hat{g}(\xi)|^2 d\xi)^{\frac{1}{2}}$$

$$= (\int_{-\infty}^{0} |(\frac{[1 - [1 - (1 - ae^{2y\xi})^m]^\gamma]}{e^{y\xi}})\hat{g}(\xi)|^2 d\xi)^{\frac{1}{2}}$$

$$\le (\int_{-\infty}^{0} |\frac{(1 - ae^{2y\xi})^m}{e^{y\xi - y_0\xi}} g\widehat{(\cdot + iy_0)}(\xi, y_0)|^2 d\xi)^{\frac{1}{2}}$$

$$= (\int_{-\infty}^{0} |\frac{(1 - ae^{2y\xi})^m}{e^{y\xi - y_0\xi}} g\widehat{(\cdot + iy_0)}(\xi, y_0)|^2 d\xi)^{\frac{1}{2}}$$

$$\le (\sup_{\xi \le 0} \frac{(1 - ae^{2y\xi})^m}{e^{y\xi - y_0\xi}})(\int_{-\infty}^{0} |g\widehat{(\cdot + iy_0)}(\xi, y_0)|^2 d\xi)^{\frac{1}{2}}$$

$$\le (\sup_{\xi \le 0} \frac{(1 - ae^{2y\xi})^m}{e^{y\xi - y_0\xi}})(\int_{-\infty}^{+\infty} |g\widehat{(\cdot + iy_0)}(\xi, y_0)|^2 d\xi)^{\frac{1}{2}}$$

$$\le (\sup_{\xi \le 0} \frac{(1 - ae^{2y\xi})^m}{e^{y\xi - y_0\xi}})E$$

$$\le (\frac{y_0 - y}{a(y_0 - y) + 2amy})^{\frac{y_0 - y}{2y}} E$$

$$\le (\frac{y_0 - y}{2may})^{\frac{y_0 - y}{2y}} E \le (\frac{y_0 - y}{(m + 1)ay})^{\frac{y_0 - y}{2y}} E.$$

Then

$$\|g^m\widehat{(\cdot + iy)}(\xi, y) - g\widehat{(\cdot + iy)}(\xi, y)\| \le (\frac{y_0 - y}{(m + 1)ay})^{\frac{y_0 - y}{2y}} E. \tag{2.12}$$

Due to $m = [\frac{E}{\delta}]^{\frac{2y}{y_0}}$, we obtain

$$m \le (\frac{E}{\delta})^{\frac{2y}{y_0}} \le m + 1. \tag{2.13}$$

Combining (2.11), (2.12) and (2.13), we obtain

$$\|g^{m,\delta}(\cdot + iy)(\xi, y) - g(\cdot + iy)(\xi, y)\| \le [(\frac{y_0 - y}{ay})^{\frac{y_0 - y}{2y}} + \sqrt{a}]E^{\frac{y}{y_0}}\delta^{1 - \frac{y}{y_0}} + \delta.$$

3. The convergence error estimate with an a posteriori parameter choice rule

In the following text, the Morozov's discrepancy principle is used to select regularization parameter m. We choose m to satisfy the first occurrence of

$$\|\hat{K}g^{m,\delta}\widehat{(\cdot + iy)}(\xi, y) - \hat{g}^\delta(\xi)\| \le r\delta, \tag{3.1}$$

where $\|\hat{g}^\delta(\xi)\| > r\delta$, and $r > 1$ is a constant.

Lemma 3.1. Suppose $\rho(m) = \|\hat{K}g^{m,\delta}\widehat{(\cdot + iy)}(\xi, y) - \hat{g}^\delta(\xi)\|$, we can deduce the following properties

(a) $\rho(m)$ is a continuous function;

(b) $\lim\limits_{m \to 0} \rho(m) = \|g^\delta\|$;

(c) $\lim\limits_{m \to \infty} \rho(m) = 0$;

(d) $\rho(m)$ is a strictly decreasing function.

Lemma 3.2. If m is chosen by (3.1), then we can obtain

$$m \le C_2 \left(\frac{E}{(r-1)\delta} \right)^{\frac{2y}{y_0}}, \tag{3.2}$$

here $C_2 = \frac{y_0}{ay}$.

Proof. Because $(1 - ae^{2y\xi}) \le 1$ and $\frac{1}{2} < \gamma < 1$, so we have $[(1 - ae^{2y\xi})^{m-1}]^\gamma \le 1$. According to (3.1), (2.6), (1.10) and lemma 2.1, we obtain

$$r\delta \le \left(\int_{-\infty}^{+\infty} |\hat{K}R_{m-1}\hat{g}^\delta(\xi) - \hat{g}^\delta(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

$$= \left(\int_{-\infty}^{0} |[(1 - ae^{2y\xi})^{m-1}]^\gamma \hat{g}^\delta(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

$$\le \left(\int_{-\infty}^{0} |\hat{g}^\delta(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

$$\le \left(\int_{-\infty}^{0} |(\hat{g}^\delta(\xi) - \hat{g}(\xi))|^2 d\xi \right)^{\frac{1}{2}} + \left(\int_{-\infty}^{0} |(1 - ae^{2y\xi})^{m-1}\hat{g}(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

$$\le \delta + \left(\int_{-\infty}^{0} |(1 - ae^{2y\xi})^{m-1}e^{y_0\xi}g\widehat{(\cdot + iy_0)}(\xi, y_0)|^2 d\xi \right)^{\frac{1}{2}}$$

$$\le \delta + (\sup_{\xi < 0} e^{y_0\xi}(1 - ae^{2y\xi})^{m-1}) \left(\int_{-\infty}^{0} |g\widehat{(\cdot + iy_0)}(\xi, y_0)|^2 d\xi \right)^{\frac{1}{2}}$$

$$\le \delta + (\sup_{\xi < 0} e^{y_0\xi}(1 - ae^{2y\xi})^{m-1}) \left(\int_{-\infty}^{+\infty} |g\widehat{(\cdot + iy_0)}(\xi, y_0)|^2 d\xi \right)^{\frac{1}{2}}$$

$$\le \delta + (\sup_{\xi < 0} e^{y_0\xi}(1 - ae^{2y\xi})^{m-1})E$$

$$\le \delta + E \left(\frac{y_0}{2a(m-1)y + ay_0} \right)^{\frac{y_0}{2y}}$$

$$\le \delta + E \left(\frac{y_0}{amy} \right)^{\frac{y_0}{2y}}.$$

So

$$m \le \frac{y_0}{ay} \left(\frac{E}{(r-1)\delta} \right)^{\frac{2y}{y_0}}.$$

Lemma 3.3. Using (1.2) and (3.1), we can obtain

$$\|\hat{K}(g^m\widehat{(\cdot + iy)}(\xi, y) - g\widehat{(\cdot + iy)}(\xi, y))\| \le (r+1)\delta. \tag{3.3}$$

Proof. Because $(1 - ae^{2y\xi}) \le 1$ and $\frac{1}{2} < \gamma < 1$, so we have $[(1 - ae^{2y\xi})^m]^\gamma \le 1$. Therefore,we obtain

$$
\begin{aligned}
&\|\hat{K}(g^m\widehat{(\cdot + iy)}(\xi, y) - g\widehat{(\cdot + iy)}(\xi, y))\| \\
&= (\int_{-\infty}^{+\infty} |\hat{K}(g^m\widehat{(\cdot + iy)}(\xi, y) - g\widehat{(\cdot + iy)}(\xi, y))|^2 d\xi)^{\frac{1}{2}} \\
&= (\int_{-\infty}^{0} |[(1 - ae^{2y\xi})^m]^\gamma \hat{g}(\xi)|^2 d\xi)^{\frac{1}{2}} \\
&\le (\int_{-\infty}^{0} |\hat{g}(\xi)|^2 d\xi)^{\frac{1}{2}} \\
&\le (\int_{-\infty}^{0} |(\hat{g}(\xi) - \hat{g}^\delta(\xi))|^2 d\xi)^{\frac{1}{2}} \\
&+ (\int_{-\infty}^{0} |(1 - ae^{2y\xi})^m \hat{g}^\delta(\xi)|^2 d\xi)^{\frac{1}{2}} \\
&\le (\int_{-\infty}^{+\infty} |\hat{g}(\xi) - \hat{g}^\delta(\xi)|^2 d\xi)^{\frac{1}{2}} + r\delta \\
&\le \delta + r\delta = (1 + r)\delta.
\end{aligned}
$$

Theorem 3.4. Assuming that the a priori condition (1.3) and the noise assumption (1.2) hold. If the regularization parameter $m = m(\delta)$ satisfies the iteration stopping criterion (3.1), we can obtain the following convergence error estimates:

$$
\|g^{m,\delta}(\cdot + iy)(\xi, y) - g(\cdot + iy)(\xi, y)\| \le C_3 E^{\frac{y}{y_0}} \delta^{1-\frac{y}{y_0}} + \delta, \tag{3.4}
$$

where $C_3 = [(r + 1)^{1-\frac{y}{y_0}} + \sqrt{\frac{y_0}{2y}}(r - 1)^{\frac{-y}{y_0}}]$.

Proof. By Parseval's theorem and triangle equation

$$
\begin{aligned}
&\|g^{m,\delta}(\cdot + iy)(\xi, y) - g(\cdot + iy)(\xi, y)\| \\
&= \|g^{m,\delta}\widehat{(\cdot + iy)}(\xi, y) - g\widehat{(\cdot + iy)}(\xi, y)\|, \\
&\le \|g^{m,\delta}\widehat{(\cdot + iy)}(\xi, y) - g^m\widehat{(\cdot + iy)}(\xi, y)\|, \\
&+ \|g^m\widehat{(\cdot + iy)}(\xi, y) - g\widehat{(\cdot + iy)}(\xi, y)\|, \\
&= I_1 + I_2.
\end{aligned}
\tag{3.5}
$$

Using (2.11) and lemma 3.2, we can approximate I_1

$$
I_1 := \|g^{m,\delta}\widehat{(\cdot + iy)}(\xi, y) - g^m\widehat{(\cdot + iy)}(\xi, y)\| \tag{3.6}
$$

$$
\le \sqrt{am}\delta + \delta \tag{3.7}
$$

$$
\le \sqrt{a}\sqrt{\frac{y_0}{2ay}}(\frac{E}{(r - 1)\delta})^{\frac{y}{y_0}}\delta + \delta \tag{3.8}
$$

$$
= \sqrt{\frac{y_0}{2y}}(\frac{1}{r - 1})^{\frac{y}{y_0}} E^{\frac{y}{y_0}} \delta^{1-\frac{y}{y_0}} + \delta. \tag{3.9}
$$

Because $0 < 1 - (1 - ae^{2y\xi})^m < 1$ and $\frac{1}{2} < \gamma < 1$, so $[1 - (1 - ae^{2y\xi})^m]^\gamma > [1 - (1 - ae^{2y\xi})^m]$. Futher, we can get $1 - [1 - (1 - ae^{2y\xi})^m]^\gamma < 1 - [1 - (1 - ae^{2y\xi})^m] = (1 - ae^{2y\xi})^m$. Using Hölder inequality and (1.3), (2.6), lemma 3.3, we can estimate I_2

$$
\begin{aligned}
I_2 &:= \|g^m\widehat{(\cdot + iy)}(\xi, y) - g\widehat{(\cdot + iy)}(\xi, y)\| \\
&= \left(\int_{-\infty}^{+\infty} |g^m\widehat{(\cdot + iy)}(\xi, y) - g\widehat{(\cdot + iy)}(\xi, y)|^2 d\xi \right)^{\frac{1}{2}} \\
&= \left(\int_{-\infty}^{0} \left| \frac{[1 - [1 - (1 - ae^{2y\xi})^m]^\gamma]}{e^{y\xi}} \hat{g}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \\
&\leq \left(\int_{-\infty}^{0} \left| \frac{[1 - (1 - (1 - ae^{2y\xi})^m)]}{e^{y\xi}} \hat{g}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \\
&= \left(\int_{-\infty}^{0} \left| \frac{(1 - ae^{2y\xi})^m}{e^{y\xi}} \hat{g}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \\
&= \left(\int_{-\infty}^{0} \left| \frac{1}{e^{y\xi}} \hat{g}^{\frac{y}{y_0}}(\xi)(1 - ae^{2y\xi})^m \hat{g}^{\frac{y_0-y}{y_0}}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \\
&\leq \left[\left(\int_{-\infty}^{0} \left| \frac{1}{e^{y\xi}} \hat{g}^{\frac{y}{y_0}}(\xi) \right|^{\frac{2y_0}{y}} d\xi \right)^{\frac{y}{y_0}} \cdot \left(\int_{-\infty}^{0} |(1 - ae^{2y\xi})^m \hat{g}^{\frac{y_0-y}{y_0}}(\xi)|^{\frac{2y_0}{y_0-y}} d\xi \right)^{\frac{y_0-y}{y_0}} \right]^{\frac{1}{2}} \\
&= \left[\left(\int_{-\infty}^{0} |e^{-2y_0\xi} \hat{g}^2(\xi)| d\xi \right)^{\frac{y}{y_0}} \cdot \left(\int_{-\infty}^{0} |\hat{g}^2(\xi)(1 - ae^{2y\xi})^{\frac{2my_0}{y_0-y}}| d\xi \right)^{\frac{y_0-y}{y_0}} \right]^{\frac{1}{2}} \\
&\leq \left[\left(\int_{-\infty}^{+\infty} |e^{-2y_0\xi} \hat{g}^2(\xi)| d\xi \right)^{\frac{y}{y_0}} \cdot \left(\int_{-\infty}^{0} |\hat{g}^2(\xi)(1 - ae^{2y\xi})^{2m}| d\xi \right)^{\frac{y_0-y}{y_0}} \right]^{\frac{1}{2}} \\
&\leq E^{\frac{y}{y_0}} [(r + 1)\delta]^{1 - \frac{y}{y_0}}.
\end{aligned}
$$

Finally, based on the above two estimates, the a posteriori estimate is obtained as follows

$$
\|g^{m,\delta}\widehat{(\cdot + iy)}(\xi, y) - g\widehat{(\cdot + iy)}(\xi, y)\| \leq \left[(r + 1)^{1 - \frac{y}{y_0}} + \sqrt{\frac{y_0}{2y}} (r - 1)^{\frac{-y}{y_0}} \right] E^{\frac{y}{y_0}} \delta^{1 - \frac{y}{y_0}} + \delta.
$$

4. Numerical implementation and numerical examples

In this section, we illustrate the effectiveness of the fractional Landweber iterative regularization method for solving this problem through different examples. Examples are the same as those in [5]. For the sake of calculation, we fix $y_0 = 1$ and the domain is

$$
\Omega := \{z = x + iy \in \mathbb{C} | |x| \leq 10, 0 < y < y_0, y_0 > 0\}. \tag{4.1}
$$

The data $g(x)$ has error, which is expressed as follows

$$
g^\delta = g + \varepsilon randn(size(g)), \tag{4.2}
$$

where "randn($\cdot$)" generates a random number that obeys the standard normal distribution. Use the following equation to express the noise level

$$
\delta = \|g^\delta - g\|_{L^2} := \sqrt{\frac{1}{M + 1} \sum_{n=1}^{M+1} |g^\delta(n) - g(n)|^2}. \tag{4.3}
$$

Under numerical calculation, we select $M = 100$, $p = 2$ and the a priori bound is $E = \|g\|_{L^2}$. The way to calculate the approximate solution is fast Fourier transform. We present the error between $g(x + iy)$ and $g^{m,\delta}(x + iy)$ under the means of L_2 norm,

$$e_a(g(x + iy)) := \|g^{m,\delta}(x + iy) - g(x + iy)\|. \tag{4.4}$$

Example 1. Take function

$$g(z) = e^{-z^2} = e^{-(x+iy)^2} = e^{y^2-x^2}(\cos(2xy) - i\sin(2xy)),$$

with $g(x) = e^{-x^2}$, $\mathrm{Re}g(z) = e^{y^2-x^2}\cos(2xy)$, $\mathrm{Im}g(z) = e^{y^2-x^2}\sin(2xy)$.
Example 2. Take function

$$g(z) = \cos(z) = \cos(x + iy) = \cosh(y)\cos(x) - i\sinh(y)\sin(x),$$

with $g(x) = \cos(x)$, $\mathrm{Re}g(z) = \cosh(y)\cos(x)$, $\mathrm{Im}g(z) = -\sinh(y)\sin(x)$.

Figure 1 shows the comparison of the real parts of the exact solution and the approximate solution at $y = 0.5$ and $y = 0.9$ for different noise levels $\varepsilon = 0.1, 0.001$ for example 1. Figure 2 shows the comparison of the imaginary parts of the exact solution and the approximate solution at $y = 0.5$ and $y = 0.9$ for different noise levels $\varepsilon = 0.1, 0.001$ for example 1.
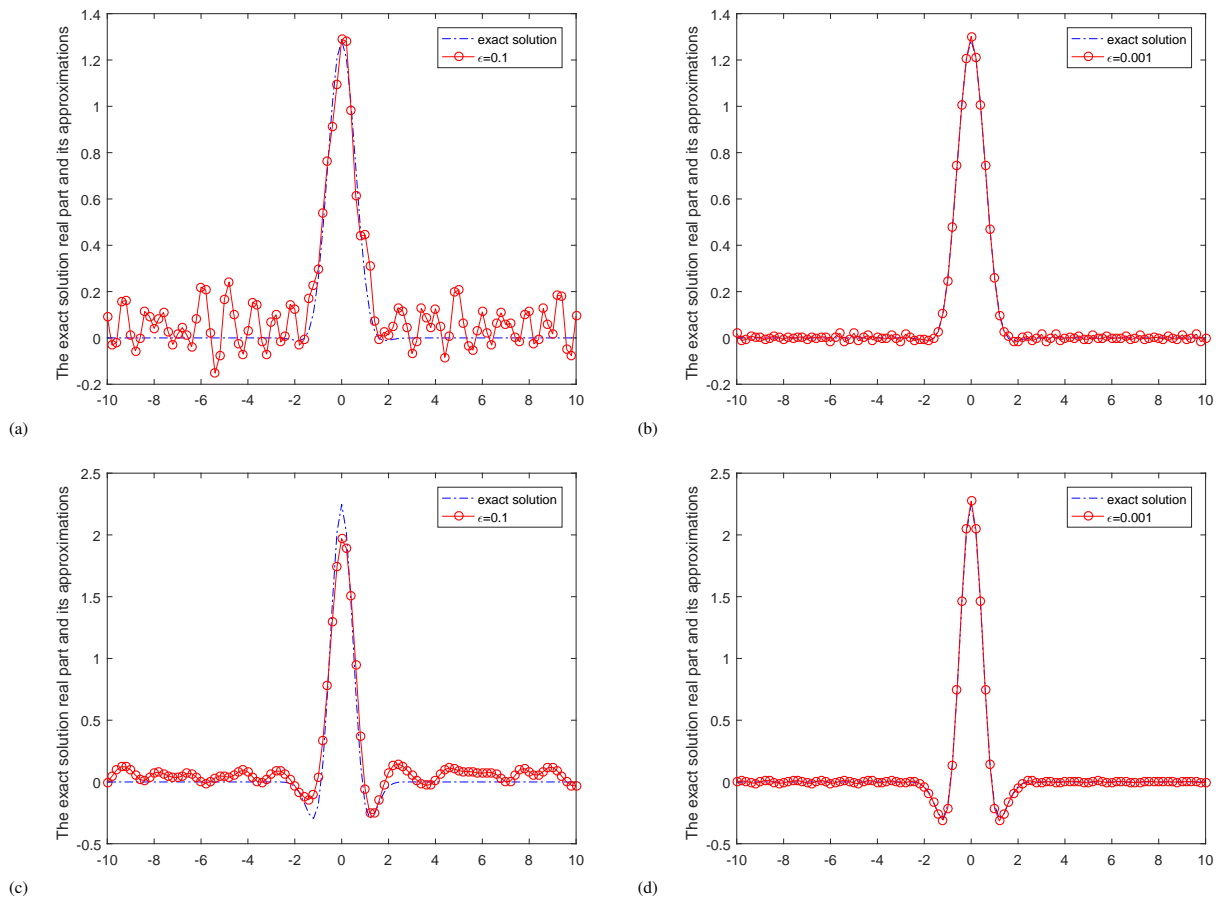

Figure 1. For example 1: (a) and (b) Real part at $y = 0.5, \varepsilon = 0.1, 0.001$, (c) and (d) Real part at $y = 0.9, \varepsilon = 0.1, 0.001$.
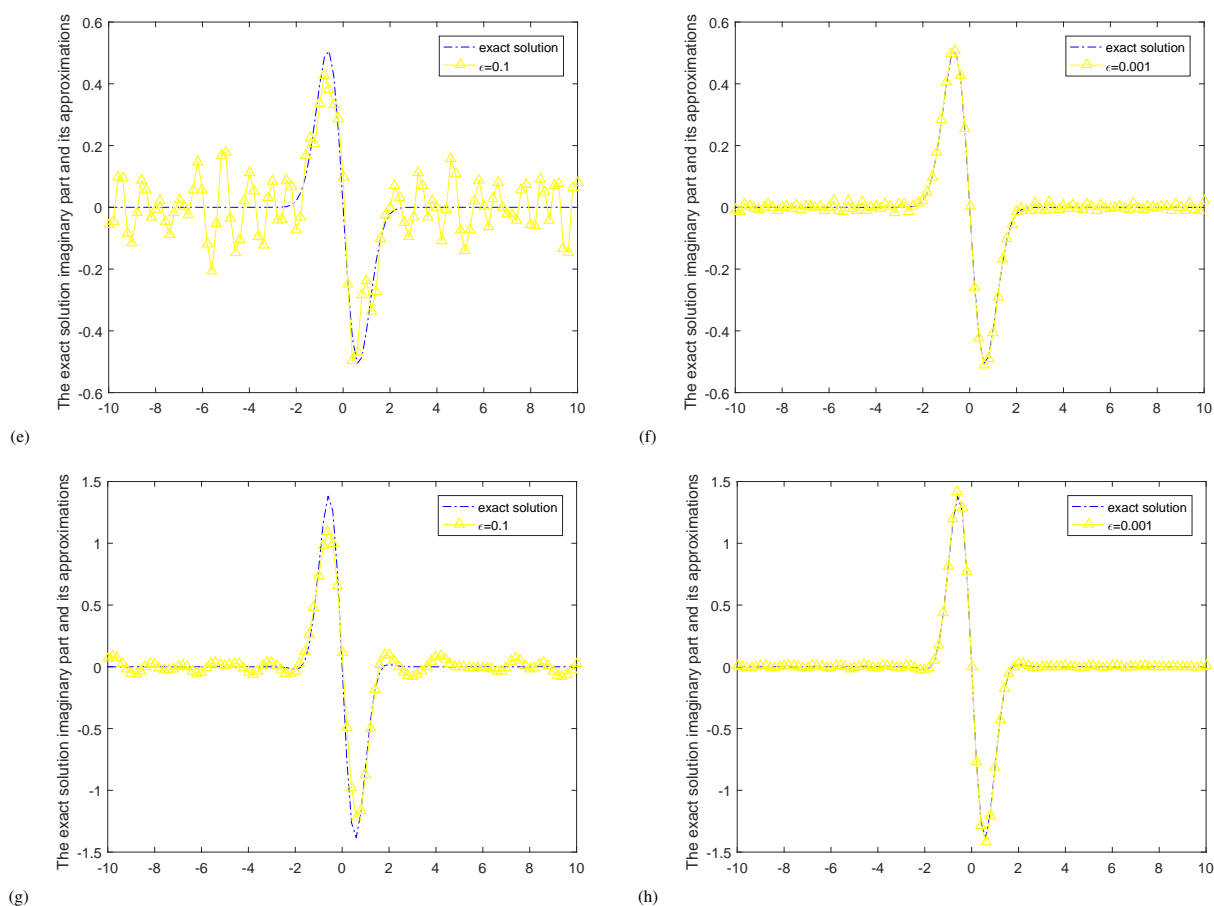
Figure 2. For example 1: (e) and (f) Imaginary part at $y = 0.5, \varepsilon = 0.1, 0.001$, (g) and (h) Imaginary part at $y = 0.9, \varepsilon = 0.1, 0.001$.

Figure 3 shows the comparison of the real parts of the exact solution and the approximate solution at $y = 0.1$ and $y = 0.5$ for different noise levels $\varepsilon = 0.01, 0.001$ for example 2. Figure 4 shows the comparison of the imaginary parts of the exact solution and the approximate solution at $y = 0.1$ for different noise levels $\varepsilon = 0.01, 0.001$ and $y = 0.5$ for different noise levels $\varepsilon = 0.001$ for example 2.

From Figures 1–4, we can find that the smaller the ε is, the better the computed approximation is. And the bigger the y is, the worse the computed approximation is. The fitting effect of example 2 is better than that of example 1.
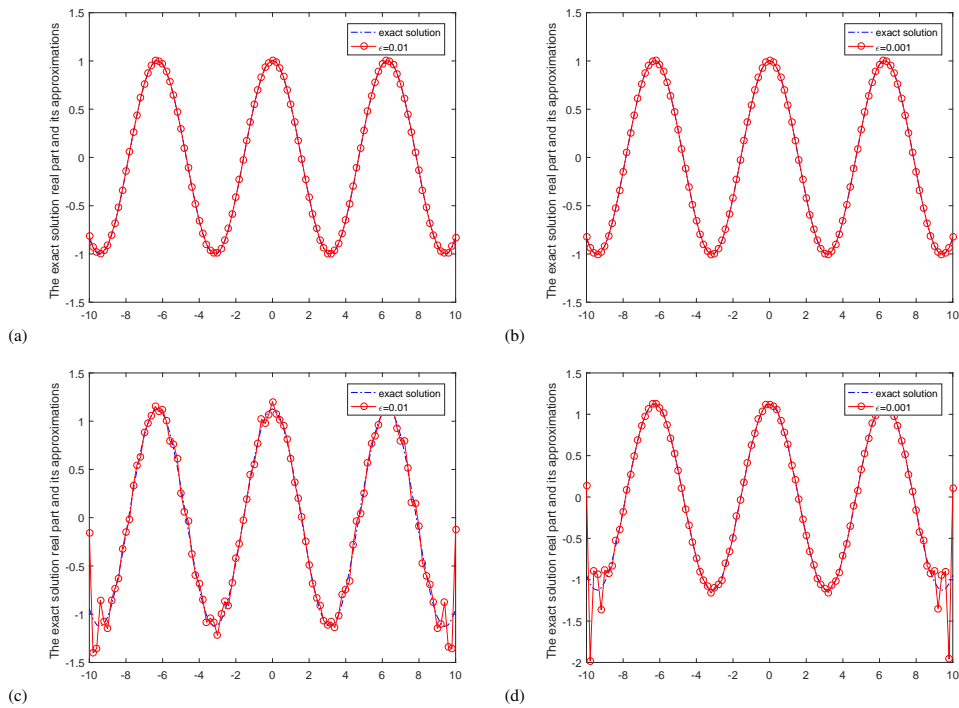
Figure 3. For example 2: (a) and (b) Real part at $y = 0.1, \varepsilon = 0.01, 0.001$, (c) abd (d) Real part at $y = 0.5, \varepsilon = 0.01, 0.001$.
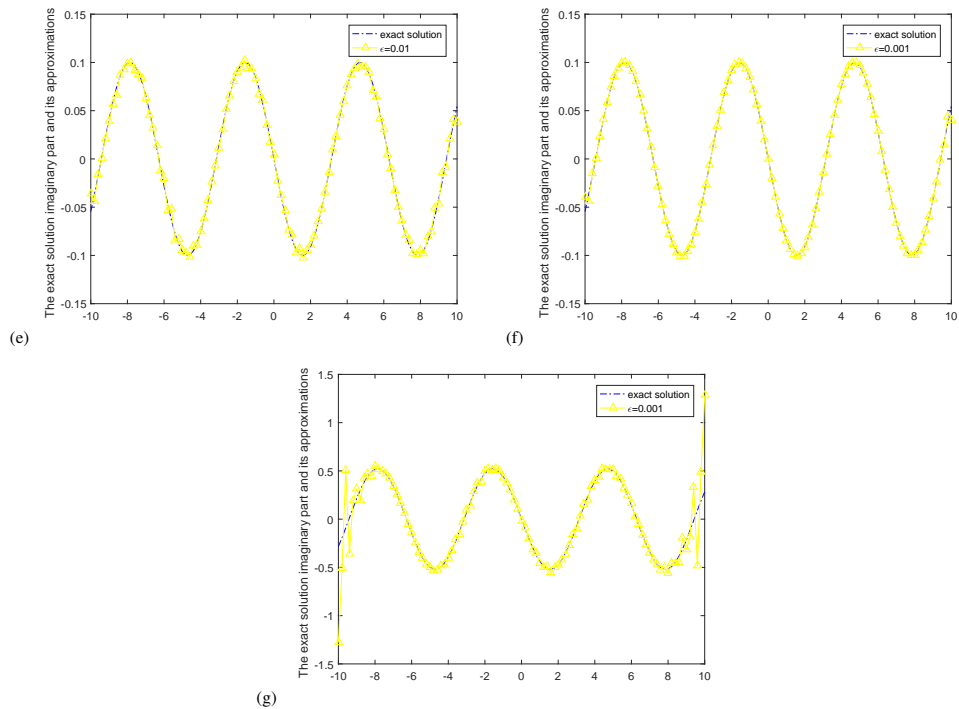

Figure 4. For example 2: (e) and (f) Imaginary part at $y = 0.1, \varepsilon = 0.01, 0.001$, (g) Imaginary part at $y = 0.5, \varepsilon = 0.001$.

Table 1 shows the error results for different y and ε in example 1. We take $\gamma = 0.55$ and $\gamma = 1$ for comparison. According to the data in Table 1, we can see that the smaller the γ, the smaller the error result. The larger y is, the smaller the error result is, which is consistent with the error estimation result obtained in Section 3.

Table 1. Comparison of the error corresponding to Example 1 for various values of ε, γ and y.

ε			0.01	0.001	0.0001
$y = 0.3$	$\gamma = 0.55$	Re	0.3185	0.0923	0.0755
		Im	0.3111	0.0929	0.0788
	$\gamma = 1$	Re	0.9560	0.7388	0.6420
		Im	0.9406	0.7352	0.6383
$y = 0.5$	$\gamma = 0.55$	Re	0.2306	0.0831	0.0371
		Im	0.2216	0.0857	0.0392
	$\gamma = 1$	Re	0.3525	0.2821	0.2171
		Im	0.3480	0.2760	0.1920
$y = 0.9$	$\gamma = 0.55$	Re	0.1718	0.0570	0.0068
		Im	0.1851	0.0568	0.0100
	$\gamma = 1$	Re	0.2874	0.1184	0.0874
		Im	0.2659	0.1141	0.0916

Table 2 shows the results of example 1 for different iteration steps m for y and ε. We take $\gamma = 0.55$ and $\gamma = 1$ for comparison. From the data in Table 2, we can see that the smaller the γ, the smaller the iteration steps m. When y is larger, the number of iteration steps m is larger, which means that the better the image fitting effect, the greater the value of iteration steps m.

In addition, when $\gamma = 0.55$, the regularization method involved is fractional Landweber iterative regularization method. When $\gamma = 1$, it is the standard Landweber iterative regularization method. It can be seen from Tables 1–3 that the results of fractional Landweber iterative regularization method are significantly less than those of Landweber iterative regularization method in terms of error and iteration steps m. Therefore, the fractional Landweber iterative regularization method is more effective than the Landweber iterative regularization method.

Table 2. Comparison of the iterative steps m corresponding to Example 1 for various values of ε, γ and y.

ε		0.01	0.001	0.0001
$y = 0.3$	$\gamma = 0.55$	9	17	23
	$\gamma = 1$	21	38	44
$y = 0.5$	$\gamma = 0.55$	16	32	175
	$\gamma = 1$	32	61	223
$y = 0.9$	$\gamma = 0.55$	37	56	218
	$\gamma = 1$	65	89	334

5. Conclusion

In this paper, we use the fractional Landweber regularization method to solve the problem of analytic continuation on strip domain. We not only give the a priori regularization parameter choice rule, but also we give the a posteriori regularization parameter choice rule. Under these two parameter selection rules, the corresponding convergence error estimates are obtained respectively. For the ill-posed problem discussed in this paper, besides the fractional Landweber iterative regularization method used in this paper, there are also other regularization methods, such as Tikhonov regularization method, quasi-boundary regularization method and so on. In these methods, the process of obtaining a priori and a posteriori error convergence estimates is similar, but in the results obtained, Tikhonov regularization method will produce saturation effect, while the fractional Landweber iterative regularization method will not produce saturation effect. Numerical examples also show this regularization method is effective.

Acknowledgements

Conflict of interest

The authors declare that they have no competing interests.

References

1. M. M. Lavrent'ev, V. G. Romanov, S. P. Shishatskiĭ, *Ill-posed problems of mathematical physics and analysis*, Providence: American Mathematical Society, 1986.

2. J. Franklin, Analytic continuation by the fast Fourier transform, *SIAM J. Sci. Stat. Comput.*, **11** (1990), 112–122.

3. I. S. Stefanescu, On the stable analytic continuation with a condition of uniform boundedness, *J. Math. Phys.*, **27** (1986), 2657–2686.

4. A. G. Ramm, The ground-penetrating radar problem III, *J. Inverse Ill-Pose. Probl.*, **8** (2000), 23–30.

5. C. L. Fu, F. F. Dou, X. L. Feng, Z. Qian, A simple regularization method for stable analytic continuation, *Inverse Probl.*, **24** (2008), 1–15.

6. Z. Q. Zhang, Y. J. Ma, A modified kernel method for numerical analytic continuation, *Inverse Probl. Sci. Eng.*, **21** (2013), 840–853.

7. X. T. Xiong, L. Zhu, M. Li, Regularization methods for a problem of analytic continuation, *Math. Comput. Simulat.*, **82** (2012), 332–345.

8. H. Cheng, C. L. Fu, X. L. Feng, An optimal filtering method for stable analytic continuation, *J. Comput. Appl. Math.*, **236** (2012), 2582–2589.

9. Y. Zhang, B. Hofmann, On fractional astmptotical regularization of linear ill-posed problems in Hilbert Spaces, *Fract. Calc. Appl. Anal.*, **22** (2019), 699–721.

10. H. Egger, A. Neubauer, Preconditioning Landweber iteration in Hilbert scales, *Numer. Math.*, **101** (2005), 643–662.

11. R. Gong, B. Hofmann, Y. Zhang, A new class of accelerated regularization methods, with application to bioluminescence tomography, *Inverse Probl.*, **36** (2020), 055013.

12. Z. L. Deng, C. L. Fu, X. L. Feng, Y. X. Zhang, A mollification regularization method for stable analytic continuation, *Math. Comput. Simulat.*, **81** (2011), 1593–1608.

13. Y. X. Zhang, C. L. Fu, L. Yan, Approximate inverse method for stable analytic continuation in a strip domain, *J. Comput. Appl. Math.*, **235** (2011), 2979–2992.

14. H. Cheng, C. L. Fu, Y. X. Zhang, An iteration method for stable analytic continuation, *Appl. Math. Comput.*, **233** (2014), 203–213.

15. X. L. Feng, W. T. Ning, A wavelet regularization method for solving numerical analytic continuation, *Int. J. Comput. Math.*, **92** (2015), 1025–1038.

16. C. L. Fu, Z. L. Deng, X. L. Feng, F. F. Dou, A modified Tikhonov regularization for stable analytic continuation, *SIAM J. Numer. Anal.*, **47** (2009), 2982–3000.

17. X. T. Xiong, Q. Cheng, A modified Lavrentiev iterative regularization method for analytic continuation, *J. Comput. Appl. Math.*, **327** (2018), 127–140.

18. H. W. Engl, M. Hanke, A. Neubauer, *Regularization of inverse problems*, Boston: Kluwer Academic, 1996.

19. F. Yang, X. Liu, X. X. Li, C. Y. Ma, Landweber iterative regularization method for identifying the unknown source of the time-fractional diffusion equation, *Adv. Differ. Equ.*, **2017** (2017), 388–402.

20. F. Yang, X. Liu, X. X. Li, Landweber iterative regularization method for identifying the unknown source of the modified Helmholtz equation, *Bound. Value Probl.*, **2017** (2017), 91–106.

21. F. Yang, Y. P. Ren, X. X. Li, D. G. Li, Landweber iterative method for identifying a space-dependent source for the time-fractional diffusion equation, *Bound. Value Probl.*, **2017** (2017), 1–19.

22. F. Yang, Y. P. Ren, X. X. Li, Landweber iteration regularization method for identifying unknown source on a columnar symmetric domain, *Inverse Probl. Sci. Eng.*, **26** (2018), 1109–1129.

23. F. Yang, Y. Zhang, X. X. Li, Landweber iterative method for identifying the initial value problem of the time-space fractional diffusion-wave equation, *Numer. Algorithms*, **83** (2020), 1509–1530.

24. F. Yang, N. Wang, X. X. Li, Landweber iterative method for an inverse source problem of time-fractional diffusion-wave equation on spherically symmtric domain, *J. Appl. Anal. Comput.*, **10** (2020), 514–529.

25. X. T. Xiong, X. M. Xue, Z. Qian, A modified iterative regularization method for ill-posed problems, *Appl. Numer. Math.*, **122** (2017), 108–128.

26. E. Klann, R. Ramlau, Regularization by fractional filter methods and data smoothing, *Inverse Probl.*, **24** (2008), 045005.

27. P. Mathé, M. T. Nair, B. Hofman, Regularization of linear ill-posed problems involving multiplication operators, *Appl. Anal.*, DOI: 10.1080/00036811.2020.1758308.

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