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Research article

Geometric properties of a certain class of multivalent analytic functions associated with the second-order differential subordination

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Abstract: We investigate some geometric properties of the class $Q_n(A, B, \alpha)$ which is defined by the second-order differential subordination and find the sharp lower bound on $|z| = r < 1$ for the following functional:

$$\mathrm{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\}$$

over the class $Q_n(A, B, 0)$.

Keywords: analytic functions; univalent functions; second-order differential subordination; sharp bound; Schwarz lemma; Carathéodory inequality

Mathematics Subject Classification: Primary 30C45; Secondary 30C80

1. Introduction

Throughout this paper, we assume that

$$n \in \mathbb{N},\ p \in \mathbb{N} \setminus \{1\},\ -1 \leq B < 1,\ B < A \ \text{ and } \alpha > 0. \tag{1.1}$$

Let $\mathcal{A}_n(p)$ be the class of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k}z^{p+k} \tag{1.2}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

For functions $f(z)$ and $g(z)$ analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$ and write $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that

$$|w(z)| \leq |z| \quad \text{and} \quad f(z) = g\big(w(z)\big) \quad (z \in \mathbb{U}).$$

If $g(z)$ is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition. A function $f(z) \in \mathcal{A}_n(p)$ is said to be in the class $Q_n(A, B, \alpha)$ if it satisfies the following second-order differential subordination:

$$(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z) \prec p\frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}). \tag{1.3}$$

Recently, several authors (see, for example, [1–8, 10–15, 17] and the references cited therein) introduced and studied various subclasses of multivalent analytic functions. Some properties such as distortion bounds, inclusion relations and coefficient estimates are investigated. In this paper we obtain inclusion relation, sharp bounds on $\mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right)$, $\mathrm{Re}\left(\frac{f(z)}{z^p}\right)$, $|f(z)|$ and coefficient estimates for functions $f(z)$ belonging to the class $Q_n(A, B, \alpha)$. Furthermore, we investigate a new problem, that is, to find

$$\min_{|z|=r<1} \mathrm{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\},$$

where $f(z)$ varies in the class:

$$Q_n(A, B, 0) = \left\{f(z) \in A_n(p) : \frac{f'(z)}{z^{p-1}} \prec p\frac{1+Az}{1+Bz} \quad (z \in \mathbb{U})\right\}. \tag{1.4}$$

We need the following lemma in order to derive our main results for the class $Q_n(A, B, \alpha)$.

Lemma. (see [9]) *Let the function $g(z)$ be analytic in $\mathbb{U}$. Suppose also that the function $h(z)$ is analytic and convex univalent in $\mathbb{U}$ with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu}zg'(z) \prec h(z),$$

where $\mathrm{Re}\mu \geq 0$ and $\mu \neq 0$, then $g(z) \prec h(z)$.

2. Geometric properties of functions in class $Q_n(A, B, \alpha)$

Theorem 1. *Let $0 < \alpha_1 < \alpha_2$. Then $Q_n(A, B, \alpha_2) \subset Q_n(A, B, \alpha_1)$.*

Proof. Suppose that

$$g(z) = z^{1-p}f'(z) \tag{2.1}$$

for $f(z) \in \mathcal{Q}_n(A, B, \alpha_2)$. Then the function $g(z)$ is analytic in $\mathbb{U}$ with $g(0) = p$. By using (1.3) and (2.1), we have

$$\begin{aligned}(1-\alpha_2)z^{1-p}f'(z) + \frac{\alpha_2}{p-1}z^{2-p}f''(z) &= g(z) + \frac{\alpha_2}{p-1}zg'(z)\\ &\prec p\frac{1+Az}{1+Bz}.\end{aligned} \tag{2.2}$$

An application of the above Lemma yields

$$g(z) \prec p\frac{1+Az}{1+Bz}. \tag{2.3}$$

By noting that $0 < \frac{\alpha_1}{\alpha_2} < 1$ and that the function $\frac{1+Az}{1+Bz}$ is convex univalent in $\mathbb{U}$, it follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned}&(1-\alpha_1)z^{1-p}f'(z) + \frac{\alpha_1}{p-1}z^{2-p}f''(z)\\ &= \frac{\alpha_1}{\alpha_2}\left((1-\alpha_2)z^{1-p}f'(z) + \frac{\alpha_2}{p-1}z^{2-p}f''(z)\right) + \left(1-\frac{\alpha_1}{\alpha_2}\right)g(z)\\ &\prec p\frac{1+Az}{1+Bz}.\end{aligned}$$

This shows that $f(z) \in \mathcal{Q}_n(A, B, \alpha_1)$. The proof of Theorem 1 is completed.

Theorem 2. *Let $f(z) \in \mathcal{Q}_n(A, B, \alpha)$. Then, for $|z| = r < 1$,*

$$\mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right) \geq p\left(1-(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}r^{nm}}{\alpha nm+p-1}\right), \tag{2.4}$$

$$\mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right) > p\left(1-(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}}{\alpha nm+p-1}\right), \tag{2.5}$$

$$\mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right) \leq p\left(1+(p-1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}r^{nm}}{\alpha nm+p-1}\right) \tag{2.6}$$

and

$$\mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right) < p\left(1+(p-1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}}{\alpha nm+p-1}\right) \quad (B \neq -1). \tag{2.7}$$

All the bounds are sharp for the function $f_n(z)$ given by

$$f_n(z) = z^p + p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}z^{nm+p}}{(nm+p)(\alpha nm+p-1)} \quad (z \in \mathbb{U}). \tag{2.8}$$

Proof. It is known that for $|\xi| \leq \sigma$ $(\sigma < 1)$ that

$$\left|\frac{1+A\xi}{1+B\xi} - \frac{1-AB\sigma^2}{1-B^2\sigma^2}\right| \leq \frac{(A-B)\sigma}{1-B^2\sigma^2} \tag{2.9}$$

and

$$\frac{1-A\sigma}{1-B\sigma} \leq \mathrm{Re}\left(\frac{1+A\xi}{1+B\xi}\right) \leq \frac{1+A\sigma}{1+B\sigma}. \tag{2.10}$$

Let $f(z) \in \mathcal{Q}_n(A,B,\alpha)$. Then we can write

$$(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z) = p\frac{1+Aw(z)}{1+Bw(z)} \quad (z \in \mathbb{U}), \tag{2.11}$$

where $w(z) = w_n z^n + w_{n+1}z^{n+1} + \cdots$ is analytic and $|w(z)| < 1$ for $z \in \mathbb{U}$. By the Schwarz lemma, we know that $|w(z)| \leq |z|^n$ $(z \in \mathbb{U})$. It follows from (2.11) that

$$\frac{(1-\alpha)(p-1)}{\alpha}z^{\frac{(1-\alpha)(p-1)}{\alpha}-1}f'(z) + z^{\frac{(1-\alpha)(p-1)}{\alpha}}f''(z) = \frac{p(p-1)}{\alpha}z^{\frac{p-1}{\alpha}-1}\left(\frac{1+Aw(z)}{1+Bw(z)}\right),$$

which implies that

$$\left(z^{\frac{(1-\alpha)(p-1)}{\alpha}}f'(z)\right)' = \frac{p(p-1)}{\alpha}z^{\frac{p-1}{\alpha}-1}\left(\frac{1+Aw(z)}{1+Bw(z)}\right).$$

After integration we arrive at

$$\begin{aligned} f'(z) &= \frac{p(p-1)}{\alpha}z^{-\frac{(1-\alpha)(p-1)}{\alpha}}\int_0^z \xi^{\frac{p-1}{\alpha}-1}\left(\frac{1+Aw(\xi)}{1+Bw(\xi)}\right)d\xi \\ &= \frac{p(p-1)}{\alpha}z^{p-1}\int_0^1 t^{\frac{p-1}{\alpha}-1}\left(\frac{1+Aw(tz)}{1+Bw(tz)}\right)dt. \end{aligned} \tag{2.12}$$

Since

$$|w(tz)| \leq t^n r^n \quad (|z| = r < 1;\ 0 \leq t \leq 1),$$

we get from (2.12) and left-hand inequality in (2.10) that, for $|z| = r < 1$,

$$\begin{aligned} \mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right) &\geq \frac{p(p-1)}{\alpha}\int_0^1 t^{\frac{p-1}{\alpha}-1}\left(\frac{1-At^n r^n}{1-Bt^n r^n}\right)dt \\ &= p - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}r^{nm}}{\alpha nm + p - 1}, \end{aligned} \tag{2.13}$$

and, for $z \in \mathbb{U}$,

$$\begin{aligned} \mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right) &> \frac{p(p-1)}{\alpha}\int_0^1 t^{\frac{p-1}{\alpha}-1}\left(\frac{1-At^n}{1-Bt^n}\right)dt \\ &= p - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}}{\alpha nm + p - 1}. \end{aligned}$$

Similarly, by using (2.12) and the right-hand inequality in (2.10), we have (2.6) and (2.7) (with $B \neq -1$).

Furthermore, for the function $f_n(z)$ given by (2.8), we find that $f_n(z) \in \mathcal{A}_n(p)$,

$$f_n'(z) = pz^{p-1} + p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}z^{nm+p-1}}{\alpha nm + p - 1} \tag{2.14}$$

and

$$(1-\alpha)z^{1-p}f_n'(z) + \frac{\alpha}{p-1}z^{2-p}f_n''(z) = p + p(A-B)\sum_{m=1}^{\infty}(-B)^{m-1}z^{nm} = p\frac{1+Az^n}{1+Bz^n}.$$

Hence $f_n(z) \in Q_n(A, B, \alpha)$ and, from (2.14), we conclude that the inequalities (2.4) to (2.7) are sharp. The proof of Theorem 2 is completed.

Corollary. *Let $f(z) \in Q_n(A, B, \alpha)$. If*

$$(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}}{\alpha nm + p - 1} \le 1, \tag{2.15}$$

then $f(z)$ is p-valent close-to-convex in $\mathbb{U}$.
Proof. Let $f(z) \in Q_n(A, B, \alpha)$ and (2.15) be satisfied. Then, by using (2.5) in Theorem 2, we see that

$$\mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right) > 0 \quad (z \in \mathbb{U}).$$

This shows that $f(z)$ is p-valent close-to-convex in $\mathbb{U}$. The proof of the corollary is completed.

Theorem 3. *Let $f(z) \in Q_n(A, B, \alpha)$. Then, for $|z| = r < 1$,*

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right) \ge 1 - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}r^{nm}}{(nm+p)(\alpha nm + p - 1)}, \tag{2.16}$$

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right) \le 1 + p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}r^{nm}}{(nm+p)(\alpha nm + p - 1)} \tag{2.17}$$

and

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right) > 1 - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}}{(nm+p)(\alpha nm + p - 1)}. \tag{2.18}$$

All of the above bounds are sharp.
Proof. It is obvious that

$$\begin{aligned} f(z) &= \int_0^z f'(\xi)d\xi = z\int_0^1 f'(tz)dt \\ &= z^p\int_0^1 t^{p-1}\frac{f'(tz)}{(tz)^{p-1}}dt \quad (z \in \mathbb{U}). \end{aligned} \tag{2.19}$$

Making use of (2.4) in Theorem 2, it follows from (2.19) that

$$\begin{aligned}\operatorname{Re}\left(\frac{f(z)}{z^p}\right) &= \int_0^1 t^{p-1}\operatorname{Re}\left(\frac{f'(tz)}{(tz)^{p-1}}\right)dt\\ &\geq \int_0^1 t^{p-1}\left(p - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}(rt)^{nm}}{\alpha nm + p - 1}\right)dt\\ &= 1 - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}r^{nm}}{(nm+p)(\alpha nm + p - 1)},\end{aligned}$$

which gives (2.16).

Similarly, we deduce from (2.6) in Theorem 2 and (2.19) that (2.17) holds true.

Also, with the help of (2.13), we find that

$$\begin{aligned}\operatorname{Re}\left(\frac{f'(tz)}{(tz)^{p-1}}\right) &\geq \frac{p(p-1)}{\alpha}\int_0^1 u^{\frac{p-1}{\alpha}-1}\left(\frac{1-A(utr)^n}{1-B(utr)^n}\right)du\\ &> p - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}t^{nm}}{\alpha nm + p - 1} \quad (|z| = r < 1;\ 0 < t \leq 1).\end{aligned}$$

From this and (2.19), we obtain (2.18).

Furthermore, it is easy to see that the inequalities (2.16), (2.17) and (2.18) are sharp for the function $f_n(z)$ given by (2.8). Now the proof of Theorem 3 is completed.

Theorem 4. *Let $f(z) \in Q_n(A, B, \alpha)$ and $AB \leq 1$. Then, for $|z| = r < 1$,*

$$|f(z)| \leq r^p + p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}r^{nm+p}}{(nm+p)(\alpha nm + p - 1)} \tag{2.20}$$

and

$$|f(z)| < 1 + p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}}{(nm+p)(\alpha nm + p - 1)}. \tag{2.21}$$

The above bounds are sharp.

Proof. Since $AB \leq 1$, it follows from (2.9) that

$$\left|\frac{1+A\xi}{1+B\xi}\right| \leq \left|\frac{1-AB\sigma^2}{1-B^2\sigma^2}\right| + \frac{(A-B)\sigma}{1-B^2\sigma^2} = \frac{1+A\sigma}{1+B\sigma} \quad (|\xi| \leq \sigma < 1). \tag{2.22}$$

By virtue of (2.12) and (2.22), we have, for $|z| = r < 1$,

$$\begin{aligned}\left|\frac{f'(uz)}{(uz)^{p-1}}\right| &\leq \frac{p(p-1)}{\alpha}\int_0^1 t^{\frac{p-1}{\alpha}-1}\left|\frac{1+Aw(utz)}{1+Bw(utz)}\right|dt\\ &\leq \frac{p(p-1)}{\alpha}\int_0^1 t^{\frac{p-1}{\alpha}-1}\left(\frac{1+A(utr)^n}{1+B(utr)^n}\right)dt \qquad (2.23)\\ &< \frac{p(p-1)}{\alpha}\int_0^1 t^{\frac{p-1}{\alpha}-1}\left(\frac{1+Au^nt^n}{1+Bu^nt^n}\right)dt. \qquad (2.24)\end{aligned}$$

By noting that

$$|f(z)| \leq r^p \int_0^1 u^{p-1} \left| \frac{f'(uz)}{(uz)^{p-1}} \right| du,$$

we deduce from (2.23) and (2.24) that the desired inequalities hold true.

The bounds in (2.20) and (2.21) are sharp with the extremal function $f_n(z)$ given by (2.8). The proof of Theorem 4 is completed.

Theorem 5. *Let* $f(z) \in \mathcal{Q}_1(A, B, \alpha)$ *and*

$$g(z) \in \mathcal{Q}_1(A_0, B_0, \alpha_0) \quad (-1 \leq B_0 < 1;\ B_0 < A_0;\ \alpha_0 > 0).$$

If

$$p(p-1)(A_0 - B_0) \sum_{m=1}^{\infty} \frac{B_0^{m-1}}{(m+p)(\alpha_0 m + p - 1)} \leq \frac{1}{2}, \tag{2.25}$$

then $(f * g)(z) \in \mathcal{Q}_1(A, B, \alpha)$, *where the symbol* $*$ *denotes the familiar Hadamard product of two analytic functions in* $\mathbb{U}$.

Proof. Since $g(z) \in \mathcal{Q}_1(A_0, B_0, \alpha_0)$, we find from the inequality (2.18) in Theorem 3 and (2.25) that

$$\mathrm{Re}\left(\frac{g(z)}{z^p}\right) > 1 - p(p-1)(A_0 - B_0) \sum_{m=1}^{\infty} \frac{B_0^{m-1}}{(m+p)(\alpha_0 m + p - 1)} \geq \frac{1}{2} \quad (z \in \mathbb{U}).$$

Thus the function $\frac{g(z)}{z^p}$ has the following Herglotz representation:

$$\frac{g(z)}{z^p} = \int_{|x|=1} \frac{d\mu(x)}{1 - xz} \quad (z \in \mathbb{U}), \tag{2.26}$$

where $\mu(x)$ is a probability measure on the unit circle $|x| = 1$ and $\int_{|x|=1} d\mu(x) = 1$.

For $f(z) \in \mathcal{Q}_1(A, B, \alpha)$, we have

$$z^{1-p}(f * g)'(z) = (z^{1-p} f'(z)) * (z^{-p} g(z))$$

and

$$z^{2-p}(f * g)''(z) = (z^{2-p} f''(z)) * (z^{-p} g(z)).$$

Thus

$$\begin{aligned}
&(1-\alpha) z^{1-p}(f * g)'(z) + \frac{\alpha}{p-1} z^{2-p}(f * g)''(z) \\
&= (1-\alpha)\Big((z^{1-p} f'(z)) * (z^{-p} g(z))\Big) + \frac{\alpha}{p-1}\Big((z^{2-p} f''(z)) * (z^{-p} g(z))\Big) \\
&= h(z) * \frac{g(z)}{z^p},
\end{aligned} \tag{2.27}$$

where

$$h(z) = (1-\alpha) z^{1-p} f'(z) + \frac{\alpha}{p-1} z^{2-p} f''(z) \prec p \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}). \tag{2.28}$$

In view of the fact that the function $\frac{1+Az}{1+Bz}$ is convex univalent in $\mathbb{U}$, it follows from (2.26) to (2.28) that

$$(1-\alpha)z^{1-p}(f*g)'(z)+\frac{\alpha}{p-1}z^{2-p}(f*g)''(z)=\int_{|x|=1}h(xz)d\mu(x)\prec p\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}).$$

This shows that $(f*g)(z)\in\mathcal{Q}_1(A,B,\alpha)$. The proof of Theorem 5 is completed.

Theorem 6. *Let*

$$f(z)=z^p+\sum_{k=n}^{\infty}a_{p+k}z^{p+k}\in\mathcal{Q}_n(A,B,\alpha).\tag{2.29}$$

Then

$$|a_{p+k}|\leq\frac{p(p-1)(A-B)}{(p+k)(\alpha k+p-1)}\quad(k\geq n).\tag{2.30}$$

The result is sharp for each $k\geq n$.

Proof. It is known that, if

$$\varphi(z)=\sum_{j=1}^{\infty}b_jz^j\prec\psi(z)\quad(z\in\mathbb{U}),$$

where $\varphi(z)$ is analytic in $\mathbb{U}$ and $\psi(z)=z+\cdots$ is analytic and convex univalent in $\mathbb{U}$, then $|b_j|\leq 1$ $(j\in\mathbb{N})$.

By using (2.29), we have

$$\begin{aligned}\frac{(1-\alpha)z^{1-p}f'(z)+\frac{\alpha}{p-1}z^{2-p}f''(z)-p}{p(A-B)}&=\frac{1}{p(p-1)(A-B)}\sum_{k=n}^{\infty}(p+k)(\alpha k+p-1)a_{p+k}z^k\\&\prec\frac{z}{1+Bz}\quad(z\in\mathbb{U}).\end{aligned}\tag{2.31}$$

In view of the fact that the function $\frac{z}{1+Bz}$ is analytic and convex univalent in $\mathbb{U}$, it follows from (2.31) that

$$\frac{(p+k)(\alpha k+p-1)}{p(p-1)(A-B)}|a_{p+k}|\leq 1\quad(k\geq n),$$

which gives (2.30).

Next we consider the function $f_k(z)$ given by

$$f_k(z)=z^p+p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}z^{km+p}}{(km+p)(\alpha km+p-1)}\quad(z\in\mathbb{U};\ k\geq n).$$

Since

$$(1-\alpha)z^{1-p}f_k'(z)+\frac{\alpha}{p-1}z^{2-p}f_k''(z)=p\frac{1+Az^k}{1+Bz^k}\prec p\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U})$$

and

$$f_k(z)=z^p+\frac{p(p-1)(A-B)}{(p+k)(\alpha k+p-1)}z^{p+k}+\cdots$$

for each $k\geq n$, the proof of Theorem 6 is completed.

Theorem 7. *Let $f(z) \in \mathcal{Q}_n(A, B, 0)$. Then, for $|z| = r < 1$,*

(i) if $M_n(A, B, \alpha, r) \geq 0$, we have

$$\begin{aligned}&\mathrm{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\}\\ &\geq \frac{p[p-1-((p-1)(A+B)+\alpha n(A-B))r^n + (p-1)ABr^{2n}]}{(p-1)(1-Br^n)^2};\end{aligned} \tag{2.32}$$

(ii) if $M_n(A, B, \alpha, r) \leq 0$, we have

$$\mathrm{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\} \geq \frac{p(4\alpha^2 K_A K_B - L_n^2)}{4\alpha(p-1)(A-B)r^{n-1}(1-r^2)K_B}, \tag{2.33}$$

where

$$\begin{cases} K_A = 1 - A^2r^{2n} - nAr^{n-1}(1-r^2),\\ K_B = 1 - B^2r^{2n} - nBr^{n-1}(1-r^2),\\ L_n = 2\alpha(1-ABr^{2n}) - \alpha n(A+B)r^{n-1}(1-r^2) - (p-1)(A-B)r^{n-1}(1-r^2),\\ M_n(A, B, \alpha, r) = 2\alpha K_B(1-Ar^n) - L_n(1-Br^n). \end{cases} \tag{2.34}$$

The above results are sharp.

Proof. Equality in (2.32) occurs for $z = 0$. Thus we assume that $0 < |z| = r < 1$.

For $f(z) \in \mathcal{Q}_n(A, B, 0)$, we can write

$$\frac{f'(z)}{pz^{p-1}} = \frac{1 + Az^n\varphi(z)}{1 + Bz^n\varphi(z)} \quad (z \in \mathbb{U}), \tag{2.35}$$

where $\varphi(z)$ is analytic and $|\varphi(z)| \leq 1$ in $\mathbb{U}$. It follows from (2.35) that

$$\begin{aligned}&(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\\ &= \frac{f'(z)}{z^{p-1}} + \frac{\alpha p(A-B)(nz^n\varphi(z) + z^{n+1}\varphi'(z))}{(p-1)(1+Bz^n\varphi(z))^2}\\ &= \frac{f'(z)}{z^{p-1}} + \frac{\alpha np}{(p-1)(A-B)}\left(\frac{f'(z)}{pz^{p-1}} - 1\right)\left(A - B\frac{f'(z)}{pz^{p-1}}\right) + \frac{\alpha p(A-B)z^{n+1}\varphi'(z)}{(p-1)(1+Bz^n\varphi(z))^2}.\end{aligned} \tag{2.36}$$

By using the Carathéodory inequality:

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - r^2},$$

we obtain

$$\begin{aligned}\mathrm{Re}\left\{\frac{z^{n+1}\varphi'(z)}{(1+Bz^n\varphi(z))^2}\right\} &\geq -\frac{r^{n+1}(1-|\varphi(z)|^2)}{(1-r^2)|1+Bz^n\varphi(z)|^2}\\ &= -\frac{r^{2n}|A - \frac{Bf'(z)}{pz^{p-1}}|^2 - |\frac{f'(z)}{pz^{p-1}} - 1|^2}{(A-B)^2r^{n-1}(1-r^2)}.\end{aligned} \tag{2.37}$$

Put $\frac{f'(z)}{pz^{p-1}} = u + iv \quad (u, v \in \mathbb{R})$. Then (2.36) and (2.37), together, yield

$$\begin{aligned}
&\mathrm{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\} \geq p\left(1 + \frac{\alpha n(A+B)}{(p-1)(A-B)}\right)u - \frac{\alpha npA}{(p-1)(A-B)} \\
&\quad - \frac{\alpha npB}{(p-1)(A-B)}(u^2 - v^2) - \frac{\alpha p[r^{2n}((A-Bu)^2 + (Bv)^2) - ((u-1)^2 + v^2)]}{(p-1)(A-B)r^{n-1}(1-r^2)} \\
&= p\left(1 + \frac{\alpha n(A+B)}{(p-1)(A-B)}\right)u - \frac{\alpha np}{(p-1)(A-B)}(A + Bu^2) - \frac{\alpha p(r^{2n}(A-Bu)^2 - (u-1)^2)}{(p-1)(A-B)r^{n-1}(1-r^2)} \\
&\quad + \frac{\alpha p}{(p-1)(A-B)}\left(nB + \frac{1 - B^2r^{2n}}{r^{n-1}(1-r^2)}\right)v^2. \qquad (2.38)
\end{aligned}$$

We note that

$$\begin{aligned}
\frac{1-B^2r^{2n}}{r^{n-1}(1-r^2)} &\geq \frac{1-r^{2n}}{r^{n-1}(1-r^2)} = \frac{1}{r^{n-1}}\left(1 + r^2 + r^4 + \cdots + r^{2(n-2)} + r^{2(n-1)}\right) \\
&= \frac{1}{2r^{n-1}}\left[(1 + r^{2(n-1)}) + (r^2 + r^{2(n-2)}) + \cdots + (r^{2(n-1)} + 1)\right] \\
&\geq n \geq -nB. \qquad (2.39)
\end{aligned}$$

Combining (2.38) and (2.39), we have

$$\begin{aligned}
\mathrm{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\} &\geq p\left(1 + \frac{\alpha n(A+B)}{(p-1)(A-B)}\right)u - \frac{\alpha np}{(p-1)(A-B)}(A + Bu^2) \\
&\quad + \frac{\alpha p((u-1)^2 - r^{2n}(A-Bu)^2)}{(p-1)(A-B)r^{n-1}(1-r^2)} \\
&=: \psi_n(u). \qquad (2.40)
\end{aligned}$$

Also, (2.10) and (2.35) imply that

$$\frac{1-Ar^n}{1-Br^n} \leq u = \mathrm{Re}\left(\frac{f'(z)}{pz^{p-1}}\right) \leq \frac{1+Ar^n}{1+Br^n}.$$

We now calculate the minimum value of $\psi_n(u)$ on the segment $\left[\frac{1-Ar^n}{1-Br^n}, \frac{1+Ar^n}{1+Br^n}\right]$. Obviously, we get

$$\psi_n'(u) = p\left(1 + \frac{\alpha n(A+B)}{(p-1)(A-B)}\right) - \frac{2\alpha npB}{(p-1)(A-B)}u + \frac{2\alpha p((1-B^2r^{2n})u - (1-ABr^{2n}))}{(p-1)(A-B)r^{n-1}(1-r^2)},$$

$$\psi_n''(u) = \frac{2\alpha p}{(p-1)(A-B)}\left(\frac{1-B^2r^{2n}}{r^{n-1}(1-r^2)} - nB\right) \geq \frac{2\alpha np(1-B)}{(p-1)(A-B)} > 0 \quad \text{(see (2.36))} \qquad (2.41)$$

and $\psi_n'(u) = 0$ if and only if

$$\begin{aligned}
u = u_n &= \frac{2\alpha(1-ABr^{2n}) - \alpha n(A+B)r^{n-1}(1-r^2) - (p-1)(A-B)r^{n-1}(1-r^2)}{2\alpha(1-B^2r^{2n} - nBr^{n-1}(1-r^2))} \\
&= \frac{L_n}{2\alpha K_B} \quad \text{(see (2.31))}. \qquad (2.42)
\end{aligned}$$

Since

$$
\begin{aligned}
&2\alpha K_B(1+Ar^n) - L_n(1+Br^n)\\
&\qquad = 2\alpha\left[(1+Ar^n)(1-B^2r^{2n}) - (1+Br^n)(1-ABr^{2n})\right]\\
&\qquad\quad + \alpha n r^{n-1}(1-r^2)\left[(A+B)(1+Br^n) - 2B(1+Ar^n)\right] + (p-1)(A-B)r^{n-1}(1-r^2)(1+Br^n)\\
&\qquad = 2\alpha(A-B)r^n(1+Br^n) + \alpha n(A-B)r^{n-1}(1-r^2)(1-Br^n) + (p-1)(A-B)r^{n-1}(1-r^2)(1+Br^n)\\
&\qquad > 0,
\end{aligned}
$$

we see that

$$
u_n < \frac{1+Ar^n}{1+Br^n}. \tag{2.43}
$$

But u_n is not always greater than $\frac{1-Ar^n}{1-Br^n}$. The following two cases arise.

(i) $u_n \le \frac{1-Ar^n}{1-Br^n}$, that is, $M_n(A,B,\alpha,r) \ge 0$ (see (2.34)). In view of $\psi_n'(u_n) = 0$ and (2.41), the function $\psi_n(u)$ is increasing on the segment $\left[\frac{1-Ar^n}{1-Br^n}, \frac{1+Ar^n}{1+Br^n}\right]$. Therefore, we deduce from (2.40) that, if $M_n(A,B,\alpha,r) \ge 0$, then

$$
\begin{aligned}
&\mathrm{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\} \ge \psi_n\left(\frac{1-Ar^n}{1-Br^n}\right)\\
&\quad = p\left(1 + \frac{\alpha n(A+B)}{(p-1)(A-B)}\right)\left(\frac{1-Ar^n}{1-Br^n}\right) - \frac{\alpha n p}{(p-1)(A-B)}\left(A + B\left(\frac{1-Ar^n}{1-Br^n}\right)^2\right)\\
&\quad = p\frac{1-Ar^n}{1-Br^n} - \frac{\alpha n p}{(p-1)(A-B)}\left(1 - \frac{1-Ar^n}{1-Br^n}\right)\left(A - B\frac{1-Ar^n}{1-Br^n}\right)\\
&\quad = \frac{p[p-1-((p-1)(A+B)+\alpha n(A-B))r^n + (p-1)ABr^{2n}]}{(p-1)(1-Br^n)^2}.
\end{aligned}
$$

This proves (2.32).

Next we consider the function $f(z)$ given by

$$
f(z) = p\int_0^z t^{p-1}\frac{1-At^n}{1-Bt^n}dt \in \mathcal{Q}_n(A,B,0).
$$

It is easy to find that

$$
(1-\alpha)r^{1-p}f'(r) + \frac{\alpha}{p-1}r^{2-p}f''(r) = \frac{p[p-1-((p-1)(A+B)+\alpha n(A-B))r^n + (p-1)ABr^{2n}]}{(p-1)(1-Br^n)^2},
$$

which shows that the inequality (2.32) is sharp.

(ii) $u_n \ge \frac{1-Ar^n}{1-Br^n}$, that is, $M_n(A,B,\alpha,r) \le 0$. In this case, we easily see that

$$
\mathrm{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\} \ge \psi_n(u_n). \tag{2.44}
$$

In view of (2.34), $\psi_n(u)$ in (2.40) can be written as follows:

$$\psi_n(u) = \frac{p(\alpha K_B u^2 - L_n u + \alpha K_A)}{(p-1)(A-B)r^{n-1}(1-r^2)}. \tag{2.45}$$

Therefore, if $M_n(A, B, \alpha, r) \le 0$, then it follows from (2.42), (2.44) and (2.45) that

$$\begin{aligned}\mathrm{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\} &\ge \frac{p(\alpha K_B u_n^2 - L_n u_n + \alpha K_A)}{(p-1)(A-B)r^{n-1}(1-r^2)} \\ &= \frac{p(4\alpha^2 K_A K_B - L_n^2)}{4\alpha(p-1)(A-B)r^{n-1}(1-r^2)K_B}.\end{aligned}$$

To show that the inequality (2.33) is sharp, we take

$$f(z) = p\int_0^z t^{p-1}\frac{1+At^n\varphi(t)}{1+Bt^n\varphi(t)}dt \quad \text{and} \quad \varphi(z) = -\frac{z-c_n}{1-c_n z} \quad (z \in \mathbb{U}),$$

where $c_n \in \mathbb{R}$ is determined by

$$\frac{f'(r)}{pr^{p-1}} = \frac{1+Ar^n\varphi(r)}{1+Br^n\varphi(r)} = u_n \in \left[\frac{1-Ar^n}{1-Br^n}, \frac{1+Ar^n}{1+Br^n}\right).$$

Clearly, $-1 \le \varphi(r) < 1$, $-1 \le c_n < 1$, $|\varphi(z)| \le 1$ $(z \in \mathbb{U})$, and so $f(z) \in \mathcal{Q}_n(A, B, 0)$. Since

$$\varphi'(r) = -\frac{1-c_n^2}{(1-c_n r)^2} = -\frac{1-|\varphi(r)|^2}{1-r^2},$$

from the above argument we obtain that

$$(1-\alpha)r^{1-p}f'(r) + \frac{\alpha}{p-1}r^{2-p}f''(r) = \psi_n(u_n).$$

The proof of Theorem 7 is completed.

3. Conclusions

In our present investigation, we have introduced and studied some geometric properties of the class $\mathcal{Q}_n(A, B, \alpha)$ which is defined by using the principle of second-order differential subordination. For this function class, we have derived the sharp lower bound on $|z| = r < 1$ for the following functional:

$$\mathrm{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\}$$

over the class $\mathcal{Q}_n(A, B, 0)$. We have also obtained other properties of the function class $\mathcal{Q}_n(A, B, \alpha)$.

For the benefit and motivation of the interested readers, we have chosen to include a number of recent developments of the related subject of the widespread usages of the basic (or q-) calculus in Geometric Function Theory of Complex Analysis (see, for example, [11,14,16,18]), of which the citation [11] happens to be a survey-cum-expository review article on this important subject.

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Conflict of interest

The authors declare no conflicts of interest.

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