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Research article

On distributional finite continuous Radon transform in certain spaces

Nitu Gupta and V. R. Lakshmi Gorty*

SVKM's NMIMS University, MPSTME, V. L. Mehta Road, Vile Parle (W), Mumbai, Maharashtra, 400056, India

* **Correspondence:** Email: vr.lakshmigorty@nmims.edu; Tel: +912242334000, +912242334084.

Abstract: The classical finite continuous Radon transform is extended to generalized functions on certain spaces. The inversion formula by the kernel method is shown in a weak distributional sense. In the concluding section, its application in Mathematical Physics is discussed.

Keywords: finite continuous Radon transform; testing spaces; inversion; kernel method
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1. Introduction

The continuous Radon transform was defined by [7] as

$$\hat{f}(p,\ \phi) = R\{f(x,\ y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,\ y)\ \delta(p - x\cos\theta - y\sin\theta)\ dx\, dy \tag{1.1}$$

where $R\{f(x,\ y)\}$ is an integral of f over the line $L_{(p,\ \theta)}$ for $p = x\cos\theta - y\sin\theta$.

Johann Radon in [2] demonstrated the generalizations involving the reconstruction of overall hyperplanes. Orthogonal series representation of generalized function was studied by Pathak [1]. In [14] author developed expansions in distributions. The author in [12] studied a series of orthogonal functions in a distributional sense. [11] applied generalized functions in harmonic analysis. A fast butterfly algorithm for generalized Radon transform has been studied by [13]. The author described a novel method for 3-D model content based on generalized Radon transform as in [9].

In [6], the finite continuous Radon transform of a function $f(x,\ y)$ defined in the interval $[-k,\ k] \times [-l,\ l]$ was introduced by Lakshmi Gorty and Nitu Gupta as

$$Rf(H(V_{m,\,n},\ p)) = \frac{1}{8kl}\int_{-l}^{l}\int_{-k}^{k} f(x,\ y)\left[1 + 2\cos\pi\left(p - \frac{m}{k}x - \frac{n}{l}y\ \right)\right] dx\, dy. \tag{1.2}$$

or

$$Rf\left(H\left(V_{m,\,n},\;p\right)\right)=\frac{1}{8kl}\int_{-l}^{l}\int_{-k}^{k}f\left(x,\;y\right)\delta\left(\pi\left(p-V_{m,\,n}\cdot\left(x,\;y\right)\right)\right)dx\,dy, \tag{1.3}$$

for every $m,\;n=1,\;2,\;3,\cdots$ where $V_{m,\,n}\cdot\left(x,\;y\right)=\frac{m}{k}x+\frac{n}{l}y$.

Theorem 1.1. *From (1.1), (1.2) and [6], assume* $f\left(t_1,\;t_2\right)$ *as a function defined and absolutely integrable on a rectangle* $\{(t_1,\;t_2):-k<x<k,\;-l<y<-l\}$ *then*

$$c_{m,\,n}=\frac{1}{8kl}\int_{-l}^{l}\int_{-k}^{k}f\left(t_1,t_2\right)\delta\left(\pi\left(p-V_{m,\,n}\cdot\left(t_1,\;t_2\right)\right)\right)dt_1dt_2.$$

If $f\left(t_1,\;t_2\right)$ *is a bounded variation in* $[-k_1,\;k_1]\times[-l_1,\;l_1]$, $[-k<-k_1<k_1<k]$, $[-l<-l_1<l_1<l]$ *and if* $(t_1,\;t_2)\in[-k_1,\;k_1]\times[-l_1,\;l_1]$, *then the series* $\sum\limits_{m,\,n=1}^{\infty}c_{m,\,n}\left[1+2\cos\pi\left(p-\frac{m}{k}x-\frac{n}{l}y\right)\right]$, *converges to* $\frac{1}{2}\left[f\left(-k,\;-l\right)+f\left(k,\;l\right)\right]$.

This paper aims to extend classical finite continuous Radon transform [6] generalized functions on certain spaces. The inversion formula due to the kernel method in a weak distributional sense is analyzed. The technique employed in developing the transform is applied to solve certain partial differential equations in Mathematical Physics. The earlier studies show the infinite range of Radon transform, whereas the finite range of continuous Radon transform in a distributional sense is dealt with for the first time in this text. The advantage of our study with distributional finite continuous Radon transform over Fourier transform enables the projection at an angle θ.

In this text, notation and terminology is from [15] and the interval is considered as $I=[-k_1,\;k_1]\times[-l_1,\;l_1]$.

From [6], we get

$$\Delta_{x,\,y,\,\theta}=\left[\left(\sin^2\theta\right)\Delta_x-\left(\cos^2\theta\right)\Delta_y\right] \tag{1.4}$$

where $\Delta_x=D_x^2$ and $\Delta_y=D_y^2$; $l,\,k$ are real constants and $D_x=\frac{d}{dx}$ and $D_y=\frac{d}{dy}$.

The following operational formula is easily computable.

$$\begin{aligned}&\Delta_{x,\,y,\,\theta}\delta\left(\pi\left(p-V_{m,\,n}\cdot\left(x,\;y\right)\right)\right)\\&=\left[\left(-\lambda_m^2\right)^k\sin^2\theta\cos^{2k}\theta-\left(-\lambda_n^2\right)^k\cos^2\theta\sin^{2k}\theta\right]\delta\left(\pi\left(p-V_{m,\,n}\cdot\left(x,\;y\right)\right)\right)\end{aligned} \tag{1.5}$$

for every $k=0,\;1,\;2,\cdots$.

2. The testing function space $V_p\left(I\right)$ and it's dual

Let p be a real number, with $1<p<\infty$; $\phi\left(x,\;y\right)$ be an infinitesimally differentiable complex-valued function in I in the given space $V_P\left(I\right)$.

$$\gamma_k^p\left(\phi\right)=\sup_I\left|\Delta_{x,\,y,\,\theta}\,\phi\left(x,\;y\right)\right|<\infty. \tag{2.1}$$

The collection of seminorms $\left(\gamma_k^p\right)$ generates the topology of linear space $V_p(I)$. Thus $V_p(I)$ is a countably multinormed space analogous to [5].

Theorem 2.1. *$V_p(I)$ is complete and a Frchet space.*

Proof. It can be proved with an argument similar to the one used by [8, 15, p. 253]. Let J denotes an arbitrary compact subset of I. Let $(x_1,\ y_1)$ be any fixed point in I and $z = (x,\ y)$ be a variable point in I.

Also, let D_x^{-1} and D_y^{-1} be the integral operators: $D_x^{-1} = \int_{x_1}^{x} \cdots dx$ and $D_y^{-1} = \int_{y_1}^{y} \cdots dy$ respectively. Let $\{\phi_{m,\,n}\}$ be a Cauchy sequence in $V_p(I)$. For each non-negative integer k, it follows from (2.1) that, $\Delta_{x,\,y,\,\theta}\ \phi_{m,\,n}(x,\ y)$ converges uniformly in J as $m,\ n \to \infty$.

If $k = 0$, then $\{\phi_{m,\,n}\}$ converges uniformly in J as $m,\ n \to \infty$.

If $k = 1$, then

$$\Delta_{x,\,y,\,\theta}\phi_{m,\,n}(x,\ y) = \left[\left(\sin^2\theta\right)\Delta_x - \left(\cos^2\theta\right)\Delta_y\right]\phi_{m,\,n}(x,\ y). \tag{2.2}$$

From (2.2), we can easily obtain,

$$\begin{aligned} &D_x^{-1}\left[\left(\sin^2\theta\right)\Delta_x\phi_{m,\,n}(x,\ y)\right] \\ &= \left(\sin^2\theta\right)D_x^{-1}D_x^2\left(\phi_{m,\,n}(x,\ y)\right) \\ &= \left(\sin^2\theta\right)D_x\left(\phi_{m,\,n}(x,\ y)\right) - \left(\sin^2\theta\right)D_{x_1}\left(\phi_{m,\,n}(x_1,\ y)\right) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &D_x^{-1}\sin^{-2}\theta D_x^{-1}\left[\left(\sin^2\theta\right)\Delta_x\phi_{m,\,n}(x,\ y)\right] \\ &= \phi_{m,\,n}(x,\ y) - \phi_{m,\,n}(x_1,\ y) - (x - x_1)\,D_{x_1}\left(\phi_{m,\,n}(x_1,\ y)\right). \end{aligned} \tag{2.4}$$

Similarly from (2.2), it can be obtained as

$$\begin{aligned} &D_y^{-1}\left[\left(\cos^2\theta\right)\Delta_y\phi_{m,\,n}(x,\ y)\right] \\ &= \left(\cos^2\theta\right)D_y^{-1}D_y^2\left(\phi_{m,\,n}(x,\ y)\right) \\ &= \left(\cos^2\theta\right)D_y\left(\phi_{m,\,n}(x,\ y)\right) - \left(\cos^2\theta\right)D_{y_1}\left(\phi_{m,\,n}(x,\ y_1)\right) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &D_y^{-1}\cos^{-2}\theta D_y^{-1}\left[\left(\cos^2\theta\right)\Delta_y\phi_{m,\,n}(x,\ y)\right] \\ &= \phi_{m,\,n}(x,\ y) - \phi_{m,\,n}(x,\ y_1) - (y - y_1)\,D_{y_1}\left(\phi_{m,\,n}(x,\ y_1)\right). \end{aligned} \tag{2.6}$$

The left-hand side of (2.3)–(2.6) converges uniformly in J.

Then, it follows from these expressions that $D_x\phi_{m,n}(x, y)$, $D_x^2\phi_{m,n}(x, y)$, $D_y\phi_{m,n}(x, y)$ and $D_y^2\phi_{m,n}(x, y)$ also converges, uniformly in J.

A simple induction process shows that $D_x^k\phi_{m,n}(x, y)$ and $D_y^k\phi_{m,n}(x, y)$ converges uniformly in J as $m, n \to \infty$ for each non-negative integer k. Thus, there exists an infinitesimally differentiable function $\phi(x, y)$ defined in I such that $D_x^k\phi_{m,n}(x, y) \to D_x^k\phi(x, y)$ and $D_y^k\phi_{m,n}(x, y) \to D_y^k\phi(x, y)$ as $m, n \to \infty$.

Finally, it is obvious that $\phi \in V_p(I)$ and $\phi(x, y)$ is the limit of the sequence $\phi_{m,n}(x, y)$ in this space. $V_p'(I)$ is the dual space of $V_p(I)$. We assign to $V_p'(I)$ the usual weak convergence. Thus the proof of the theorem by [15, p. 253]. □

Properties analogous to [8] and [15] are:

Property 2.1. $\Delta_{x,y,\theta}\phi(x, y) = \sum_{j=0}^{2k}\left[\left(\sin^2\theta\right)\Delta_x^{2k-j} - \left(\cos^2\theta\right)\Delta_y^{2k-j}\right]\phi(x, y)$.

Property 2.2. $|\langle f, \phi\rangle| \le K \max_{0\le k\le s} \gamma_k^p(\phi)$ *for every* $\phi \in V_p(I)$ *for a positive constant K and a non-negative integer s.*

Property 2.3. *Let* $f(x, y)$ *be defined in I such that* $\int_{-l}^{l}\int_{-k}^{k} |f(x, y)|\, dx\, dy$ *exists.*

Then $|\langle f, \phi\rangle| = \int_{-l}^{l}\int_{-k}^{k} f(x, y)\phi(x, y)\, dx\, dy$, *for* $f(x, y)$ *generating a regular generalized function in* $V_p'(I)$.

Property 2.4. *For each* $m, n = 1, 2, 3, \cdots$, *the function* $\delta(\pi(p - V_{m,n}\cdot(x, y)))$, $-k < x < k$, $-l < y < -l$ *is a member of* $V_p(I)$. *Using (1.5), we verify the same:*

$$\begin{aligned}&\gamma_k^p(\delta(\pi(p - V_{m,n}\cdot(x, y))))\\ &= \sup_I \left|\left[\left(-\lambda_m^2\right)^k \sin^2\theta\cos^{2k}\theta - \left(-\lambda_n^2\right)^k \cos^2\theta\sin^{2k}\theta\right]\delta(\pi(p - V_{m,n}\cdot(x, y)))\right| < \infty,\end{aligned}$$

for each $k = 0, 1, 2, \cdots$.

3. Inversion theorem

Let $f(x, y) = f(X)$, where $X = (x, y)$. The finite continuous Radon transform $Rf(H(V_{m,n}, p))$ of f is defined by (1.2) as follows:

$$Rf(H(V_{m,n}, p)) = \frac{1}{8lk}\int_I f(X)\delta(\pi(p - V_{m,n}\cdot X))\, dX. \tag{3.1}$$

The inner product from (3.1) can be written as:

$$Rf(H(V_{m,n}, p)) = \left\langle f(X), \frac{1}{8lk}\delta(\pi(p - V_{m,n}\cdot X))\right\rangle. \tag{3.2}$$

where $\delta(\pi(p - V_{m,n}\cdot X)) \in V_p(I)$ for every $m, n = 1, 2, 3, \cdots$.

We now list some properties involving $S_{M,\,N}(\tau,\ X)$ as in [3] where

$$S_{M,\,N}(\tau,\ X)=\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{1}{8lk}\delta\left(\pi\left(p-V_{m,\,n}\cdot(\tau)\right)\right)\delta\left(\pi\left(p-V_{m,\,n}\cdot(X)\right)\right)$$

for $M,\ N\in I^{+}$; $\forall\quad M\neq N$ and $X\in I$ which we shall need in the sequel.

Property 3.1. *Let $f\in V_{p}'(I)$. For $\tau=(t_1,\ t_2)\in I$ follows:*

$$\int_{-l}^{l}\int_{-k}^{k}\langle f(\tau),\ S_{M,\,N}(\tau,\ X)\rangle\,\phi(x,\ y)\,dx\,dy=\left\langle f(\tau),\ \int_{-l}^{l}\int_{-k}^{k}S_{M,N}(\tau,\ X)\,\phi(x,\ y)\,dx\,dy\right\rangle. \tag{3.3}$$

Property 3.2. *Let $\alpha,\ \beta\in\mathbb{R}$ for $\alpha\in(-k,\ k)$ and $\beta\in(-l,\ l)$ respectively.*
Then $\lim\limits_{M,\ N\to\infty}\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha}S_{M,\,N}(\tau,\ X)\,\phi(x,\ y)\,dx\,dy=1,\ \ (t_1,\ t_2)\in(-\alpha,\ \alpha)\times(-\beta,\ \beta)$.
Hence follows as a consequence of theorem 1.1, when $f(t_1,\ t_2)=1$.

Theorem 3.1 (**Inversion theorem**). *If $Rf\left(H\left(V_{m,\,n},\ p\right)\right)$ is distributional finite Radon transform of f from (3.1), then*

$$f(x,\ y)=\lim_{N,\ M\to\infty}\sum_{n=1}^{N}\sum_{m=1}^{M}Rf\left(H\left(V_{m,\,n},\ p\right)\right)\delta(u\cdot X), \tag{3.4}$$

converges in $D'(I)$.

Proof. Let $\phi(x,\ y)\in D(I)$. Assume the support of $\phi(x,\ y)\in[-\alpha,\ \alpha]\times[-\beta,\ \beta]$ where $-k<-\alpha<\alpha<k$, $-l<-\beta<\beta<l$. From (3.4) it is equivalent in proving that

$$\langle S_{M,\,N}(\tau,\ X),\ \phi(X)\rangle\to\langle f(\tau),\ \phi(\tau)\rangle \tag{3.5}$$

as $M,\ N\to\infty$.
Thus, it is represented as:

$$\left\langle\sum_{n=1}^{N}\sum_{m=1}^{M}Rf\left(H\left(V_{m,\,n},\ p\right)\right)\delta\left(\pi\left(p-V_{m,\,n}\cdot X\right)\right),\ \phi(X)\right\rangle \tag{3.6}$$

$$=\int_{-b}^{b}\int_{-a}^{a}\sum_{n=1}^{N}\sum_{m=1}^{M}Rf\left(H\left(V_{m,n},p\right)\right)\delta\left(\pi\left(p-V_{m,\,n}\cdot X\right)\right)\phi(X)\,dx\,dy \tag{3.7}$$

$$=\int_{-b}^{b}\int_{-a}^{a}\sum_{n=1}^{N}\sum_{m=1}^{M}\left\langle f(\tau),\ \frac{1}{8lk}\delta\left(\pi\left(p-V_{m,\,n}\cdot\tau\right)\right)\right\rangle\delta\left(\pi\left(p-V_{m,\,n}\cdot X\right)\right)\phi(X)\,dx\,dy \tag{3.8}$$

$$=\int_{-b}^{b}\int_{-a}^{a}\left\langle f(\tau),\ \sum_{n=1}^{N}\sum_{m=1}^{M}\frac{1}{8lk}\delta\left(\pi\left(p-V_{m,\,n}\cdot\tau\right)\right)\delta\left(\pi\left(p-V_{m,\,n}\cdot X\right)\right)\right\rangle\phi(X)\,dx\,dy \tag{3.9}$$

$$=\int_{-b}^{b}\int_{-a}^{a}\langle f(\tau),\ S_{M,\,N}(\tau,\ X)\rangle\,\phi(X)\,dx\,dy \tag{3.10}$$

$$= \left\langle f(\tau),\ \int_{-b}^{b}\int_{-a}^{a} S_{M,N}(\tau,\ X)\,\phi(X)\,dx\,dy \right\rangle \tag{3.11}$$

$$\to \langle f(\tau),\ \phi(\tau)\rangle. \tag{3.12}$$

It can be observed that $\sum_{n=1}^{N}\sum_{m=1}^{M} \frac{1}{8lk}\delta(\pi(p - V_{m,n}\cdot\tau))\,\delta(\pi(p - V_{m,n}\cdot X))$ is locally integrable over $[-k,\ k]\times[-l,\ l]$ justifying (3.6) equals (3.7).

To prove (3.11) converges to (3.12), we need to prove that for each $k = 0,\ 1,\ 2,\ \cdots$.

$$\left[\Delta_{x,y,\theta}\right]_t^k = \left[\left(\sin^2\theta\right)\Delta_{t_1}^k - \left(\cos^2\theta\right)\Delta_{t_2}^k\right]\left[\int_{-a}^{a}\int_{-b}^{b} S_{M,N}(\tau,\ X)\,\phi(X)\,dx\,dy - \phi(\tau)\right] \to 0 \tag{3.13}$$

as $M,\ N \to \infty$ uniformly $\forall\quad t = (t_1,\ t_2) \in [-k,\ k]\times[-l,\ l]$. □

Thus (1.3) gives:

$$\left[\left(\sin^2\theta\right)\Delta_{t_1} - \left(\cos^2\theta\right)\Delta_{t_2}\right] S_{M,N}(\tau,\ X) = \left[\left(\sin^2\theta\right)\Delta_{x} - \left(\cos^2\theta\right)\Delta_{y}\right] S_{M,N}(\tau,\ X). \tag{3.14}$$

Since the order of differentiation and integration in (3.13) is interchangeable, we can write

$$\begin{aligned}
&\Delta_{t,\theta}^k\left[\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} S_{M,N}(\tau,\ X)\,\phi(X)\,dx\,dy\right] \\
&= \int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} \left[\left(\sin^2\theta\right)\Delta_{t_1} - \left(\cos^2\theta\right)\Delta_{t_2}\right] S_{M,N}(\tau,\ X)\,\phi(X)\,dx\,dy \\
&= \int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} \left[\left(\sin^2\theta\right)\Delta_{x} - \left(\cos^2\theta\right)\Delta_{y}\right] S_{M,N}(\tau,\ X)\,\phi(X)\,dx\,dy, (from\quad (3.14)) \\
&= \int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} S_{M,N}(\tau,\ X)\left[\left(\sin^2\theta\right)\Delta_{x} - \left(\cos^2\theta\right)\Delta_{y}\right]\phi(X)\,dx\,dy.
\end{aligned}$$

Integration by parts and operating by $\Delta_{t,\theta}^k$ successively, it can be shown as:

$$\begin{aligned}
&\Delta_{t,\theta}^k\left[\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} S_{M,N}(\tau,\ X)\,\phi(X)\,dx\,dy\right] \\
&= \int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} S_{M,N}(\tau,\ X)\left[\left(\sin^2\theta\right)\Delta_{x}^k - \left(\cos^2\theta\right)\Delta_{y}^k\right]\phi(X)\,dx\,dy.
\end{aligned}$$

From property 3.2, we have

$$\begin{aligned}
&\Delta_{t,\,\theta}^{k}\left[\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} S_{M,\,N}(\tau,\ X)\,\phi(X)\,dx\,dy-\phi(\tau)\right]\\
&=\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} S_{M,\,N}(\tau,\ X)\left\{\begin{array}{c}\left[\left(\sin^{2}\theta\right)\Delta_{x}^{k}-\left(\cos^{2}\theta\right)\Delta_{y}^{k}\right]\phi(X)\\ -\left[\left(\sin^{2}\theta\right)\Delta_{t_1}^{k}-\left(\cos^{2}\theta\right)\Delta_{t_2}^{k}\right]\phi(\tau)\end{array}\right\}dx\,dy\\
&=\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} S_{M,\,N}(\tau,\ X)\,[\psi(X)-\psi(\tau)]\,dx\,dy
\end{aligned}$$

where $\psi(X)=\Delta_{x,\,y,\,\theta}\phi(X)$ is a member of $D'(I)$ with support contained in $(-\alpha,\ \alpha)\times(-\beta,\ \beta)$.
Thus the proof.

Theorem 3.2 (Uniqueness theorem). *Let $f,\ g\in V_p{}'(I)$ and the generalized finite continuous Radon transform of f and g be $Rf(H(V_{m,\,n},\ p))$ and $Rg(H(V_{m,\,n},\ p))$ respectively, as defined by (1.2). If $Rf(H(V_{m,\,n},\ p))=Rg(H(V_{m,\,n},\ p))$, then $f=g$ in the sense of equality in $D'(I)$.*

The obvious proof follows using (3.4).

Example 3.1. *Consider a Dirac delta function $\delta(\pi(V_{m,\,n}\cdot(\tau)-k_0))$ for $k_0\in I$. Since $\delta(\pi(V_{m,\,n}\cdot(\tau)-k_0))\in E'(I)$ and $E'(I)$ is a subspace of $V_p{}'(I)$, therefore $\delta(\pi(V_{m,\,n}\cdot(\tau)-k_0))\in V_p{}'(I)$. The generalized finite continuous Radon transform of $\delta(\pi(V_{m,\,n}\cdot(\tau)-k_0))$ is given as*

$$\begin{aligned}
&R\delta(H(V_{m,\,n},\ p))\\
&=\langle\delta(\pi(V_{m,\,n}\cdot(\tau)-k_0)),\ \delta(\pi(p-V_{m,\,n}\cdot(\tau)))\rangle\\
&=\frac{1}{8lk}\delta(\pi(p-V_{m,\,n}\cdot(k_0)))
\end{aligned}$$

$\forall\ m,\ n=1,\ 2,\ 3,\ \cdots$ *as in [4, p. 25] and [10, p. 260].*
Now for any $\phi(X)\in D'(I)$,

$$\begin{aligned}
&\left\langle\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{1}{8lk}\delta(\pi(p-V_{m,\,n}\cdot(K)))\,\delta(\pi(p-V_{m,\,n}\cdot(X))),\ \phi(X)\right\rangle\\
&=\int_{-l}^{l}\int_{-k}^{k}\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{1}{8lk}\delta(\pi(p-V_{m,\,n}\cdot(K)))\,\delta(\pi(p-V_{m,\,n}\cdot(X)))\,\phi(X)\,dx\,dy\\
&=\int_{-l}^{l}\int_{-k}^{k} S_{M,\,N}(K,\ X)\,\phi(X)\,dx\,dy\\
&\to\phi(K)
\end{aligned}$$

where $\phi(K)=\langle S_{M,\,N}(K,\ X),\ \phi(\tau)\rangle$. Hence the proof.

4. Operational theory

For arbitrary $\phi(X) \in V_p(I)$ and $f \in V_p'(I)$, define a generalized operator $\Delta^*_{x,y,\theta}$ on $V_p'(I)$ for adjoint operator of $\Delta_{x,y,\theta}$ on $V_p(I)$. It may be noted from [6] that, $\Delta_{x,y,\theta}$ is a self-adjoint operator.

$$\left\langle \Delta^*_{x,y,\theta} f(X),\ \phi(X) \right\rangle = \left\langle f(X),\ \Delta_{x,y,\theta}\phi(X) \right\rangle. \tag{4.1}$$

Since $\phi(X) \to \Delta_{x,y,\theta}\phi(X)$ is linear and continuous mapping; $\Delta_{x,y,\theta}\phi(X) \in V_p(I)$ when $\phi(X) \in V_p(I)$. Implies $\Delta^*_{x,y,\theta}$ is linear and continuous on $V'_p(I)$.

For any integer k, the method of induction gives:

$$\left(\Delta^*_{x,y,\theta}\right)^k = \left\langle f(X), \left[\left(\sin^2\theta\right)(\Delta_x)^k - \left(\cos^2\theta\right)\left(\Delta_y\right)^k\right]\phi(X)\right\rangle \tag{4.2}$$

and $\left(\Delta^*_{x,y,\theta}\right)^k$ is linear and continuous on $V'_p(I)$.

Therefore

$$\begin{aligned}&\left\langle \left(\Delta^*_{x,y,\theta}\right)^k f(X), \frac{1}{8lk}\delta\left(\pi\left(p - V_{m,n}\cdot(x,\ y)\right)\right)\right\rangle \\ &= \frac{1}{8lk}\left[\left(-\lambda_m^2\right)^k \sin^2\theta\cos^{2k}\theta - \left(-\lambda_n^2\right)^k\cos^2\theta\sin^{2k}\theta\right]\left\langle f(X), \delta\left(\pi\left(p - V_{m,n}\cdot(x,\ y)\right)\right)\right\rangle.\end{aligned}$$

Implies

$$\begin{aligned}&R\left\{\left(\Delta^*_{x,y,\theta}\right)^k f\right\}\left(H\left(V_{m,n},\ p\right)\right) \\ &= \left[\left(-\lambda_m^2\right)^k \sin^2\theta\cos^{2k}\theta - \left(-\lambda_n^2\right)^k\cos^2\theta\sin^{2k}\theta\right] Rf\left(H\left(V_{m,n},\ p\right)\right)\end{aligned} \tag{4.3}$$

$\forall\ \ m,\ n = 1,\ 2,\ 3,\ \cdots$ which gives an operational formula.

From (4.1) and by self-adjoint operator property, we get

$$\Delta^*_{x,y,\theta} f = \Delta_{x,y,\theta} f \tag{4.4}$$

so that $\Delta^*_{x,y,\theta}$ can be replaced by $\Delta_{x,y,\theta}$ in (4.3).

Let P be a polynomial, where g is a given member of $V_p'(I)$, then

$$P\left(\Delta^*_{x,y,\theta}\right)u = g, \tag{4.5}$$

for every $X \in (-k,\ k) \times (-l,\ l)$.

Note that $P\left[\left(-\lambda_m^2\right)^k \sin^2\theta\ \cos^{2k}\theta - \left(-\lambda_n^2\right)^k\cos^2\theta\ \sin^{2k}\theta\right] \neq 0,\ \forall\ \ m,\ n = 1,\ 2,\ 3,\ \cdots$ where u is an unknown variable, generalized in $V_p'(I)$.

Applying generalized finite Radon transformation to (4.5) and using (4.4) follows:

$$P\left[\left(-\lambda_m^2\right)\sin^2\theta\ \cos^2\theta - \left(-\lambda_n^2\right)\cos^2\theta\ \sin^2\theta\right] Ru\left(H\left(V_{m,n},\ p\right)\right) = Rg\left(H\left(V_{m,n},\ p\right)\right)$$

for every $m,\ n = 1,\ 2,\ 3,\ \cdots$.

Therefore

$$Ru\left(H\left(V_{m,\,n},\ p\right)\right)=\frac{Rg\left(H\left(V_{m,\,n},\ p\right)\right)}{P\left[\left(-\lambda_m^2\right)\sin^2\theta\,\cos^2\theta-\left(-\lambda_n^2\right)\cos^2\theta\,\sin^2\theta\right]}. \tag{4.6}$$

Applying inversion theorem 3.1 to (4.6), we get

$$u\left(x,\ y\right)=\lim_{M,\,N\to\infty}\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{Rg\left(H\left(V_{m,\,n},\ p\right)\right)}{P\left[\left(-\lambda_m^2\right)\sin^2\theta\,\cos^2\theta-\left(-\lambda_n^2\right)\cos^2\theta\,\sin^2\theta\right]}\delta\left(u\cdot X\right) \tag{4.7}$$

with equality in the sense of $D'\left(I\right)$, which is a solution to (4.5). This solution is a restriction of $u\in {V_p}'(I)$ to $D\left(I\right)$ analogous to [8].

Hence the solution of a distributional differential Eq (4.5) is given by (4.7).

5. Application

In this section, an application of generalized finite continuous Radon transform using boundary conditions is demonstrated. The Dirichlet's problem is used to find a function $v\left(x,\ y,\ z\right)$ on the domain $R\left\{\left(x,\ y,\ z\right):-\pi<x<\pi,\ -\pi<y<\pi,\ -\pi<z<\pi\right\}$, where $V\left(x,\ y,\ z\right)$ satisfies the following differential equation

$$\left(\sin^2\theta\right)\frac{\partial^2 v}{\partial x^2}+\left(\cos^2\theta\right)\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0, \tag{5.1}$$

with the following boundary conditions:

(i) As $x\to-\pi,\ y\to-\pi$ or $x\to\pi,\ y\to\pi$ $v\left(x,\ y,\ z\right)$ converges in sense of $D'\left(I\right)$ to zero on $Z\le z<\infty$ for each $Z>0$.

(ii) As $z\to\infty,\ v\left(x,\ y,\ z\right)$ converges in sense of $D'\left(I\right)$ to zero.

(iii) As $z\to 0^+,\ v\left(x,\ y,\ z\right)$ converges in sense of $D'\left(I\right)$ to $f\left(x,\ y\right)\in {V_p}'\left(I\right)$.

Now (5.1) can be written as

$$\left[\left(\sin^2\theta\right)\Delta_x-\left(\cos^2\theta\right)\Delta_y\right]v+\frac{\partial^2 v}{\partial z^2}=0. \tag{5.2}$$

Applying generalized finite Radon transform to (5.2), we obtain

$$\left(\lambda_n^2-\lambda_m^2\right)\sin^2\left(2\theta\right)Rv\left(H\left(V_{m,\,n},\ p\right)\right)+\frac{\partial^2 Rv\left(H\left(V_{m,\,n},\ p\right)\right)}{\partial z^2}=0,$$

for every $m,\ n=1,\ 2,\ 3,\ \cdots$ where

$$Rv\left(H\left(V_{m,\,n},\ p\right)\right)=\left\langle v\left(X,\ z\right),\frac{1}{8lk}\delta\left(\pi\left(p-V_{m,\,n}\cdot\left(X\right)\right)\right)\right\rangle.$$

Thus

$$Rv\left(H\left(V_{m,\,n},\ p\right)\right)=A\left(V_{m,\,n},\ p\right)e^{-\left(\sqrt{\lambda_m-\lambda_n}\sin 2\theta\right)z}+B\left(V_{m,\,n},\ p\right)e^{\left(\sqrt{\lambda_m-\lambda_n}\sin 2\theta\right)z} \tag{5.3}$$

where $A\left(V_{m,\,n},\ p\right)$ and $B\left(V_{m,\,n},\ p\right)$ are constants.

From boundary conditions (ii), (iii) and considering $\lim_{z\to\infty} Rv\left(H\left(V_{m,n},\ p\right)\right) = 0$ and $\lim_{z\to 0^+} Rv\left(H\left(V_{m,n},\ p\right)\right) = F(u)$ respectively gives $B\left(V_{m,n},\ p\right) = 0$ and $A\left(V_{m,n},\ p\right) = F\left(V_{m,n},\ p\right)$ in (5.3).
Thus

$$Rv\left(H\left(V_{m,n},\ p\right)\right) = F\left(V_{m,n},\ p\right) e^{-\left(\sqrt{\lambda_m - \lambda_n}\sin 2\theta\right) z}. \tag{5.4}$$

Now applying inversion theorem 3.1 to (5.4), we get

$$v(x,\ y,\ z) = \lim_{M,\ N\to\infty} \sum_{n=1}^{N}\sum_{m=1}^{M} F\left(V_{m,n},\ p\right) e^{-\left(\sqrt{\lambda_m-\lambda_n}\sin 2\theta\right)z}\delta\left(\pi\left(p - V_{m,n}\cdot(x,\ y)\right)\right).$$

Also

$$\begin{aligned}&\langle v(X,\ z),\ \phi(X)\rangle\\ &= \int_I \lim_{M,\ N\to\infty} \sum_{n=1}^{N}\sum_{m=1}^{M} F\left(V_{m,n},\ p\right) e^{-\left(\sqrt{\lambda_m-\lambda_n}\sin 2\theta\right)z}\delta\left(\pi\left(p - V_{m,n}\cdot(x,\ y)\right)\right) dX.\end{aligned}$$

Therefore $v(X,\ z)$ can be represented as a classical function:

$$v(x,\ y,\ z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} F\left(V_{m,n},\ p\right) e^{-\left(\sqrt{\lambda_m-\lambda_n}\ \sin 2\theta\right)z}\delta\left(\pi\left(p - V_{m,n}\cdot(x,\ y)\right)\right). \tag{5.5}$$

Here we note that $\lambda_m = \frac{m\pi}{k\cos\theta}$, $\lambda_n = \frac{n\pi}{l\sin\theta}$, as $m,\ n\to\infty$, $F\left(V_{m,n},\ p\right) = O\left(u^s\right)$ for some s which is a non-negative integer.

Also, we observe that $\left(\lambda_m - \lambda_n\right)^{1/2} = \left[\left(\frac{m}{k\cos\theta} - \frac{n}{l\ \sin\theta}\right)\pi\right]^{1/2}$ as $m,\ n\to\infty$, the factor $e^{-\left(\sqrt{\lambda_m-\lambda_n}\ \sin 2\theta\right) z}$, ensuring (5.5) converges uniformly on every plane in $(x,\ y,\ z)$ of the form $Z \le z < \infty\ (Z > 0)$. Thus, we can use the operator $\left(\sin^2\theta\right)\frac{\partial^2}{\partial x^2} + \left(\cos^2\theta\right)\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ under the summation in (5.5). Since $e^{-\left(\sqrt{\lambda_m-\lambda_n}\sin 2\theta\right)z}\delta\left(\pi\left(p - V_{m,n}\cdot(x,\ y)\right)\right)$ satisfies (5.1), so does v.

Further, we get

$$\begin{aligned}&\lim_{z\to 0^+}\langle v(X,\ z),\ \phi(X)\rangle\\ &= \lim_{z\to 0^+}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} F\left(V_{m,n},\ p\right) e^{-\left(\sqrt{\lambda_m-\lambda_n}\sin 2\theta\right)z}\left\langle\delta\left(\pi\left(p - V_{m,n}\cdot(x,\ y)\right)\right),\ \phi(X)\right\rangle.\end{aligned} \tag{5.6}$$

$$\lim_{z\to 0^+}\langle v(X,\ z),\ \phi(X)\rangle = \int_I \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} F\left(V_{m,n},\ p\right)\delta\left(\pi\left(p - V_{m,n}\cdot(x,\ y)\right)\right)\phi(X)\, dX. \tag{5.7}$$

The step (5.6) is straightforward, which gives

$$\lim_{z\to 0^+}\langle v(X,\ z),\ \phi(X)\rangle = \langle f,\ \phi\rangle. \tag{5.8}$$

Since (5.7) is convergent on $V_p(I)$ and $Z \le z < \infty$ for each $Z > 0$ from (5.5), we can write [5] as

$$|v(x,\ y,\ z)| \le \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|F\left(V_{m,n},\ p\right)\right| e^{-\left(\sqrt{\lambda_m-\lambda_n}\sin 2\theta\right)z}\left|\delta\left(\pi\left(p - V_{m,n}\cdot(x,\ y)\right)\right)\right|. \tag{5.9}$$

Now we can see that the series converges uniformly on $-\infty < X < \infty$.

Considering the limits $x \to -\pi, \ y \to -\pi$ or $x \to \pi, \ y \to \pi$ under the summation sign in (5.9), verifies the boundary condition (i).

Similarly

$$|v(x, \ y, \ z)| \leq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|F(V_{m,n}, \ p)\right| e^{-\left(\sqrt{\lambda_m-\lambda_n} \ \sin 2\theta\right) z}\left|\delta\left(\pi\left(p - V_{m,n}\cdot(x, \ y)\right)\right)\right| \to 0$$

as $z \to \infty$, thus verifies the boundary condition (ii).

6. Conclusion

In this study, the classical finite continuous Radon transform has been extended to generalized functions on certain spaces. The inversion formula due to the kernel method in a weak distributional sense is also established. Application from Mathematical Physics is demonstrated to solve Dirichlet's problem in the concluding section.

7. Future Work

The inverse generalized finite continuous Radon transform can be studied for local tomography, related medical and other imaging technologies. 3-D Model search and retrieval can be extended using generalized finite continuous Radon transform. Researchers can also develop applications involving angle θ in the engineering field.

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Conflict of interest

Authors hereby declare that they have no competing interests. The authors also declare no conflicts of interest in this paper.

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