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Research article

Refined inequalities of perturbed Ostrowski type for higher-order absolutely continuous functions and applications

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Abstract: First of all, we establish an identity for higher-order differentiable functions. Then, we prove some integral inequalities for mappings that have continuous derivatives up to the order $n-1$ with $n \geq 1$ and whose n-th derivatives are the element of L_1, L_r, and L_∞. In addition, estimates of new composite quadrature rules are examined. Finally, natural applications for exponential and logarithmic functions are given.

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1. Introduction

Since 1938, the year of its publication, Ostrowski inequality [22] has attracted attention of many researchers. This inequality has become a cornerstone in mathematics owing to abundance of its application in numeric integration methods, some particular means and many other fields. Especially in the last 25 years, many authors have investigated refinements, counterparts and generalizations of Ostrowki inequality. They also established new results connected to Ostrowski inequality under various assumptions of the functions. For instance, Dragomir et al. pointed out some generalizations of Ostrowski inequality and applied them to some special means and numerical integration in [11]. What's more, it is presented an original result for twice differentiable functions by Cerone et al. in [8]. In [10], Dragomir and Barnett also found an inequality different from result given in [8] by using mappings whose second derivatives are limited. Sarikaya and Set proved new inequalities for mappings whose second derivative are the element of L_1, L_p and L_∞ by using a new Montgomery type identity in [23]. As well as the cases when first and second derivatives are required, there are problems in which higher-order derivatives are required. Therefore, many researchers worked on inequalities involving higher-order differentiable functions. For example, Cerone et al. deduced Ostrowski type results by using functions whose higher-order derivatives are bounded in [9]. Furthermore, in [1, 19, 25, 28], authors established generalized Ostrowski type and related inequalities for higher-order differentiable functions on L_p and L_∞. Recently, perturbed integral inequalities have become a widely used generalization method. Dragomir deduced novel perturbed type inequalities for absolutely continuous mappings in [12, 13], and some other researchers obtained similar inequalities to Dragomir's results for twice differentiable functions in [7, 15]. Morever, Erden and Sarikaya provided some perturbed results for functions whose higher-order derivatives have different properties such as convex, bounded, bounded variation and Lipschitzian in [16, 17].

Another method of generalizing inequalities is to use three or five-step kernels. A large number of companion of these recent inequalities, based on quadratic kernel function with three and five sections, for various functions are established. As an example, the authors yielded certain companions of Ostrowski type for bounded variation mappings in [5, 6]. In addition, some mathematicians examined three companion versions of perturbed inequalities under the diverse assumptions of the mappings in [14, 24]. By means of the quadratic kernel function with three pieces, some companions of Perturbes conclusions of Ostrowski type for the case when $f'' \in L_1$, $f'' \in L_2$ and $f''' \in L_2$ are derived by Liu et al. in [20]. Afterwards, Qayyum et al. generalized these inequalities for five-step quadratic kernel and gave some applications for quadrature rules and CDF of a random variable in [26]. Moreover in [27] they extended similar companions of Ostrowski type to higher-order differentiable functions for the case when $f^{(n+1)} \in L_1$, $f^{(n)} \in L_2$ and $f^{(n+1)} \in L_2$.

In 1938, Ostrowski in [22] presented, in the following theorem, his famous integral inequality, known as Ostrowski inequality.

Theorem 1. *Suppose that $\psi : [\alpha,\beta] \to \mathbb{R}$ is a differentiable mapping on (α,β) whose derivative $\psi' : (\alpha,\beta) \to \mathbb{R}$ is bounded on (α,β), i.e. $\|\psi'\|_\infty := \sup_{s\in(\alpha,\beta)} |\psi'(s)| < \infty$. Then, we have the inequality*

$$\left|\psi(y) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \psi(s)ds\right| \le \left[\frac{1}{4} + \frac{\left(y - \frac{a+b}{2}\right)^2}{(\beta-\alpha)^2}\right](b-a)\|\psi'\|_\infty, \tag{1.1}$$

for all $y \in [\alpha,\beta]$. The constant $\frac{1}{4}$ is the best possible.

Dragomir in [12] provided the following identity to obtain new perturbed inequalities for absolutely continuous mappings.

Theorem 2. *Assume that $\psi : [\alpha,\beta] \to \mathbb{C}$ is an absolutely continuous on $[\alpha,\beta]$ and $y \in [\alpha,\beta]$. Then, for any $\gamma_1(y)$ and $\gamma_2(y)$ complex numbers, we have*

$$\begin{aligned}
&\frac{1}{\beta-\alpha}\int_a^y (s-a)\left[\psi'(s)-\gamma_1(y)\right]ds + \frac{1}{\beta-\alpha}\int_y^b (s-b)\left[\psi'(s)-\gamma_2(y)\right]ds \\
=&\psi(y) + \frac{1}{2(\beta-\alpha)}\left[(b-y)\gamma_2(y) - (y-a)^2\gamma_1(y)\right] - \frac{1}{\beta-\alpha}\int_a^b \psi(s)ds
\end{aligned}$$

where the integrals presented in the left side of the above expression are evaluated in the Lebesgue sense.

In [16], Erden obtained the following equality to prove some perturbed inequalities of Ostrowski type for higher degree differentiable mappings.

Lemma 1. *Supposing that $\psi : [\alpha,\beta] \to \mathbb{C}$ is an $n-$time differentiable function on (α,β) and $y \in [\alpha,\beta]$. Then, for any $\gamma_1(y)$ and $\gamma_2(y)$ complex numbers, one has the identity*

$$\begin{aligned}
&\int\limits_{\alpha}^{y} \frac{(s-\alpha)^n}{n!}\left[\psi^{(n)}(s)-\gamma_1(y)\right]ds + \int\limits_{y}^{\beta} \frac{(s-\beta)^n}{n!}\left[\psi^{(n)}(s)-\gamma_2(y)\right]ds \\
&=\sum_{k=0}^{n-1} \frac{(-1)^{n+1}(\beta-y)^{k+1}+(-1)^{n+1-k}(y-\alpha)^{k+1}}{(k+1)!}\psi^{(k)}(y) \qquad (1.2)\\
&\quad-\gamma_1(y)\frac{(y-\alpha)^{n+1}}{(n+1)!}-(-1)^n\gamma_2(y)\frac{(\beta-y)^{n+1}}{(n+1)!}+(-1)^n\int\limits_{\alpha}^{\beta}\psi(s)ds,
\end{aligned}$$

where the integrals given in the left side of (1.2) are calculated in the Lebesgue sense.

An equality for higher-order differentiable mappings are first developed in this work. Afterwards, new perturbed inequalities for function whose higher-order derivatives are absolutely continuous are established, and sufficiency of the newly obtained composite quadrature rules is observed. Finally, some applications for exponential and logarithmic functions are given. At this point is worth to remark that recently a class of "absolutely continuous functions" of n variables was introduced by J. Maly (see [21]) and subsequently modified by some mathematicians like S. Hencl (see [18]) and D. Bongiorno (see [2]–[4]). Therefore, an interesting open problem could be to establish new perturbed inequalities by these new classes of absolutely continuous functions.

2. An identity for high degree differentiable functions

In this section, it is obtained an equality for $n-$times differentiable functions. This identity and its particular cases will be used to prove our main results.

Lemma 2. *Assume that $\psi : [a,b] \to \mathbb{R}$ is an $n-$times differentiable function such that (n-1)-th derivative of ψ is absolutely continuous on $[a,b]$, for $n \in \mathbb{N}$. Then, for any $\mu_j(x)$, $j = 1,2,3$ real numbers and all $x \in \left[a, \frac{a+b}{2}\right]$, we have the identity*

$$\begin{aligned}
&\int\limits_{a}^{x} \frac{(t-a)^n}{n!}\left[\psi^{(n)}(t)-\mu_1(x)\right]dt + \int\limits_{x}^{a+b-x} \frac{1}{n!}\left(t-\frac{a+b}{2}\right)^n\left[\psi^{(n)}(t)-\mu_2(x)\right]dt \\
&+\int\limits_{a+b-x}^{b} \frac{(t-b)^n}{n!}\left[\psi^{(n)}(t)-\mu_3(x)\right]dt \qquad (2.1)\\
&=S(\psi : n,x)-R(n,x)+(-1)^n\int\limits_{a}^{b}\psi(t)dt
\end{aligned}$$

where $S(\psi : n, x)$ and $R(n, x)$ are defined by

$$S(\psi : n, x) = \sum_{k=0}^{n-1} \frac{(-1)^{n+1}\left[\psi^{(k)}(a+b-x)+(-1)^k\psi^{(k)}(x)\right]}{(k+1)!}\left[(x-a)^{k+1} + (-1)^k\left(\frac{a+b}{2}-x\right)^{k+1}\right] \tag{2.2}$$

and

$$R(n, x) = \left[\mu_1(x) + (-1)^n \mu_3(x)\right]\frac{(x-a)^{n+1}}{(n+1)!} + \left[1 + (-1)^n\right]\frac{\mu_2(x)}{(n+1)!}\left(\frac{a+b}{2}-x\right)^{n+1}.$$

Proof. Applying the integration by parts n times to the integrals in the left of (2.1) for Lebesgue formula and then arranging the results by using fundamental analysis operations, we can readily write the equality (2.1). □

Now, we give particular cases of (2.1) in the following Corollary.

Corollary 1. *Under the assumptions of Lemma 2 with $\mu_j(x) = \mu_j$, $j = 1, 2, 3$*
i) if we take $x = \frac{a+b}{2}$, then we have

$$\int_a^{\frac{a+b}{2}} \frac{(t-a)^n}{n!}\left[\psi^{(n)}(t) - \mu_1\right]dt + + \int_{\frac{a+b}{2}}^{b} \frac{(t-b)^n}{n!}\left[\psi^{(n)}(t) - \mu_3\right]dt$$
$$= \sum_{k=0}^{n-1} \frac{(-1)^{n+1}\left[1 + (-1)^k\right](b-a)^{k+1}}{(k+1)!2^{k+1}}\psi^{(k)}\left(\frac{a+b}{2}\right)$$
$$- \left[\lambda_1 + (-1)^n \mu_3\right]\frac{(b-a)^{n+1}}{2^{n+1}(n+1)!} + (-1)^n \int_a^b \psi(t)dt$$

which also is a special case of identity (1.2).
ii) if we choose $x = \frac{3a+b}{4}$, then we get

$$\int_a^{\frac{3a+b}{4}} \frac{(t-a)^n}{n!}\left[\psi^{(n)}(t) - \mu_1\right]dt + \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \frac{1}{n!}\left(t - \frac{a+b}{2}\right)^n\left[\psi^{(n)}(t) - \mu_2\right]dt$$
$$+ \int_{\frac{a+3b}{4}}^{b} \frac{(t-b)^n}{n!}\left[\psi^{(n)}(t) - \mu_3\right]dt$$
$$= \sum_{k=0}^{n-1} \frac{(-1)^{n+1}\left[1 + (-1)^k\right](b-a)^{k+1}}{(k+1)!4^{k+1}}\left[\mu^{(k)}\left(\frac{a+3b}{4}\right) + (-1)^k \psi^{(k)}\left(\frac{3a+b}{4}\right)\right]$$
$$- \frac{(b-a)^{n+1}}{4^{n+1}(n+1)!}\left\{\left[\mu_1 + (-1)^n \mu_3\right] + \left[1 + (-1)^n\right]\mu_2\right\} + (-1)^n \int_a^b \psi(t)dt.$$

In addition to these results, it can be obtained main equality established by Dragomir in [14] if $n = 1$ is taken in (2.1). Also, should we choose $n = 2$ in the identity (2.1), we have the equality that is proved by Sarikaya et al. in [24]. So, it can be easily deduced great numbers of inequalities which is derived from our main results obtained by using the equality (2.1).

3. Inequalities for absolutely continuous derivatives

It is examined the case when $\psi^{(n+1)} \in L_\infty [a, b]$ in the following Theorem.

Theorem 3. *Assuming that $\psi : [a, b] \to \mathbb{R}$ is an $n-$times differentiable function such that n-th derivative of ψ is absolutely continuous on $[a, b]$, for $n \in \mathbb{N}$. If the $(n+1)$.th derivative $\psi^{(n+1)}$ is bounded on (a, b), then, for any $x \in \left[a, \frac{a+b}{2}\right]$, one possesses*

$$\begin{aligned}
&\left|S(\psi : n, x) - \frac{[1+(-1)^n]}{(n+1)!}\left[(x-a)^{n+1} + \left(\frac{a+b}{2} - x\right)^{n+1}\right]\psi^{(n)}(x)\right. \\
&\left. + (-1)^n \int_a^b \psi(t)dt\right| \\
\leq &\frac{(x-a)^{n+2}}{(n+2)!}\left\|\psi^{(n+1)}\right\|_{[a,x],\infty} + \frac{2}{(n+1)!}\left(\frac{a+b}{2} - x\right)^{n+2}\left\|\psi^{(n+1)}\right\|_{[x,b],\infty} \\
&+ .\left[\frac{(b-a)(x-a)^{n+1}}{(n+1)!} - \frac{2n+3}{(n+2)!}(x-a)^{n+2}\right]\left\|\psi^{(n+1)}\right\|_{[x,b],\infty}
\end{aligned} \tag{3.1}$$

where $S(\psi : n, x)$ is defined as in (2.2).

Proof. Writing $\psi^{(n)}(x)$ instead of $\mu_1(x)$, $\mu_2(x)$, $\mu_3(x)$ in (2.1) and later taking modulus in both sides of the equality (2.1), we have the inequality

$$\begin{aligned}
&\left|S(\psi : n, x) - \frac{[1+(-1)^n]}{(n+1)!}\left[(x-a)^{n+1} + \left(\frac{a+b}{2} - x\right)^{n+1}\right]\psi^{(n)}(x)\right. \\
&\left. + (-1)^n \int_a^b \psi(t)dt\right| \\
\leq &\int_a^x \frac{(t-a)^n}{n!}\left|\psi^{(n)}(t) - \psi^{(n)}(x)\right| dt + \int_x^{a+b-x} \frac{1}{n!}\left|t - \frac{a+b}{2}\right|^n \left|\psi^{(n)}(t) - \psi^{(n)}(x)\right| dt \\
&+ \int_{a+b-x}^b \frac{(b-t)^n}{n!}\left|\psi^{(n)}(t) - \psi^{(n)}(x)\right| dt.
\end{aligned} \tag{3.2}$$

Due to the fact that

$$\left|\psi^{(n)}(t) - \psi^{(n)}(x)\right| = \left|\int_x^t \psi^{(n+1)}(\xi)d\xi\right|,$$

it follows that

$$\int_a^x \frac{(t-a)^n}{n!} \int_t^x \left|\psi^{(n+1)}(\xi)\right| d\xi dt \leq \int_a^x \frac{(t-a)^n}{n!} (x-t) \left\|\psi^{(n+1)}\right\|_{[t,x],\infty} dt$$

$$\leq \left\|\psi^{(n+1)}\right\|_{[a,x],\infty} \int_a^x \frac{(t-a)^n}{n!} (x-t)\, dt$$

$$= \frac{(x-a)^{n+2}}{(n+2)!} \left\|\psi^{(n+1)}\right\|_{[a,x],\infty}.$$

If we similarly calculate the other integrals in the right of (3.2), and later we add all these inequalities, we can easily deduced desired inequality (3.1). □

Now, we establish a new inequality similar to the inequality (3.1) by using a different method in the following Theorem.

Theorem 4. *Supposing that $\psi : [a,b] \to \mathbb{R}$ is an $n-$times differentiable function such that n-th derivative of ψ is absolutely continuous on $[a,b]$, for $n \in \mathbb{N}$. If $\psi^{(n+1)} \in L_r([a,b])$ for $r > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$, then, for any $x \in \left[a, \frac{a+b}{2}\right]$, one has*

$$\begin{aligned} &\left| S(\psi : n, x) - \frac{[1+(-1)^n]}{(n+1)!} \left[(x-a)^{n+1} + \left(\frac{a+b}{2} - x \right)^{n+1} \right] \psi^{(n)}(x) \right. \\ &\left. + (-1)^n \int_a^b \psi(t) dt \right| \\ &\leq \frac{(x-a)^{n+1+\frac{1}{s}}}{n!} B\left(n+1, 1+\frac{1}{s}\right) \left\|\psi^{(n+1)}\right\|_{[a,x],r} \\ &+ \frac{1}{n!} \left(\frac{a+b}{2} - x \right)^{n+1+\frac{1}{s}} \left\|\psi^{(n+1)}\right\|_{[x,b],r} \\ &\times \left[B(n+1, 1+\frac{1}{s}) + \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^{n+1-k+\frac{1}{s}}}{n+1-k+\frac{1}{s}} \right] \\ &+ \frac{1}{n!} (b-x)^{n+1+\frac{1}{s}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1 - \left(1 - \frac{x-a}{b-x}\right)^{k+1+\frac{1}{s}}}{k+1+\frac{1}{s}} \left\|\psi^{(n+1)}\right\|_{[x,b],r} \end{aligned} \tag{3.3}$$

where $S(\psi : n, x)$ is defined as in (2.2), and $B(\cdot,\cdot)$ is a beta function.

Proof. We consider the inequality (3.2). Using Holder's inequality, which is applied to integrals, for the first integral in the right of (3.2), we have

$$\int_a^x \frac{(t-a)^n}{n!} \int_t^x \left|\psi^{(n+1)}(\xi)\right| d\xi dt \leq \int_a^x \frac{(t-a)^n}{n!} (x-t)^{\frac{1}{s}} \left\|\psi^{(n+1)}\right\|_{[t,x],r} dt$$

$$\leq \left\|\psi^{(n+1)}\right\|_{[a,x],r} \int_a^x \frac{(t-a)^n}{n!} (x-t)^{\frac{1}{s}}\, dt.$$

Applying the change of the variable $u = \frac{x-t}{x-a}$ to the above integral and from $du = \frac{-dt}{x-a}$, we conclude that

$$\int_a^x \frac{(t-a)^n}{n!} \int_t^x \left|\psi^{(n+1)}(\xi)\right| d\xi dt \leq \frac{(x-a)^{n+1+\frac{1}{s}}}{n!} B\left(n+1, 1+\frac{1}{s}\right) \left\|\psi^{(n+1)}\right\|_{[a,x],r}.$$

In a similar way, if we use Holder's inequality for the second integrals in the right of (3.2), we have

$$\begin{aligned}
&\int_x^{a+b-x} \frac{1}{n!} \left|t - \frac{a+b}{2}\right|^n \int_x^t \left|\psi^{(n+1)}(\xi)\right| d\xi dt \\
\leq &\left\|\psi^{(n+1)}\right\|_{[x,b],r} \int_x^{\frac{a+b}{2}} \frac{1}{n!} \left(\frac{a+b}{2} - t\right)^n (t-x)^{\frac{1}{s}} dt \\
+ &\left\|\psi^{(n+1)}\right\|_{[x,b],r} \int_{\frac{a+b}{2}}^{a+b-x} \frac{1}{n!} \left(t - \frac{a+b}{2}\right)^n (t-x)^{\frac{1}{s}} dt
\end{aligned}$$

For we to calculate both of these integrals, we should apply the change of the variable

$$u = \frac{t-x}{\frac{a+b}{2} - x} \qquad du = \frac{dt}{\frac{a+b}{2} - x}.$$

Herewith, it is deduced that

$$\begin{aligned}
&\int_x^{a+b-x} \frac{1}{n!} \left|t - \frac{a+b}{2}\right|^n \int_x^t \left|\psi^{(n+1)}(\xi)\right| d\xi dt \\
\leq &\frac{1}{n!} \left(\frac{a+b}{2} - x\right)^{n+1+\frac{1}{s}} \left\|\psi^{(n+1)}\right\|_{[x,b],r} \\
\times &\left[B(n+1, 1+\frac{1}{s}) + \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^{n+1-k+\frac{1}{s}}}{n+1-k+\frac{1}{s}}\right].
\end{aligned}$$

For the last integral in the right of (3.2), if we use the change of the variable $u = \frac{t-x}{b-x}$ after applying Hölder's inequality to this integral, then one has

$$\begin{aligned}
&\int_{a+b-x}^{b} \frac{(b-t)^n}{n!} \int_x^t \left|\psi^{(n+1)}(\xi)\right| d\xi dt \\
\leq &\frac{1}{n!} (b-x)^{n+1+\frac{1}{s}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1 - \left(1 - \frac{x-a}{b-x}\right)^{k+1+\frac{1}{s}}}{k+1+\frac{1}{s}} \left\|\psi^{(n+1)}\right\|_{[x,b],r}.
\end{aligned}$$

Thus, desired inequality (3.3) can be found by combining all these integrals and the Theorem is proved. □

Theorem 5. *Assuming that $\psi : [a,b] \to \mathbb{R}$ is an $n-$times differentiable function such that n-th derivative of ψ is absolutely continuous on $[a,b]$, for $n \in \mathbb{N}$. If $\psi^{(n+1)} \in L_1([a,b])$, then, for any $x \in \left[a, \frac{a+b}{2}\right]$, we possess*

$$\begin{aligned}
&\left| S(\psi : n, x) - \frac{[1+(-1)^n]}{(n+1)!}\left[(x-a)^{n+1} + \left(\frac{a+b}{2} - x\right)^{n+1}\right]\psi^{(n)}(x) \right. \\
&\left. + (-1)^n \int_a^b \psi(t)dt \right| \\
\leq & \frac{(x-a)^{n+1}}{(n+1)!}\left\|\psi^{(n+1)}\right\|_{[a,x],1} + \frac{(x-a)^{n+1}}{(n+1)!}\left\|\psi^{(n+1)}\right\|_{[x,b],1} \\
&+ \frac{2}{(n+1)!}\left(\frac{a+b}{2} - x\right)^{n+1}\left\|\psi^{(n+1)}\right\|_{[x,b],1}
\end{aligned} \tag{3.4}$$

where $S(\psi : n, x)$ is defined as in (2.2).

Proof. Considering the first integral in the right of (3.2), we find that

$$\begin{aligned}
\int_a^x \frac{(t-a)^n}{n!} \int_t^x \left|\psi^{(n+1)}(\xi)\right| d\xi dt \leq & \int_a^x \frac{(t-a)^n}{n!} \left\|\psi^{(n+1)}\right\|_{[t,x],1} dt \\
\leq & \left\|\psi^{(n+1)}\right\|_{[a,x],1} \int_a^x \frac{(t-a)^n}{n!} dt \\
= & \frac{(x-a)^{n+1}}{(n+1)!}\left\|\psi^{(n+1)}\right\|_{[a,x],1}.
\end{aligned}$$

If the other two integrals given in the right of (3.2) are observed in a similar way, then the inequality given the Theorem can be deduced. The proof is thus completed. □

Corollary 2. *Let f and x be defined as in Theorem 3. If we use the fact that*

$$\max\{\Upsilon, \Phi\} = \frac{\Upsilon + \Phi}{2} + \left|\frac{\Upsilon - \Phi}{2}\right|$$

and the property of maximum $\max\{a^n, b^n\} = (\max\{a, b\})^n$ *for $a, b > 0$ and $n \in \mathbb{N}$ in the right hand side of (3.4), then one has the inequality*

$$\begin{aligned}
&\left| S(\psi : n, x) - \frac{[1+(-1)^n]}{(n+1)!}\left[(x-a)^{n+1} + \left(\frac{a+b}{2} - x\right)^{n+1}\right]\psi^{(n)}(x) \right. \\
&\left. + (-1)^n \int_a^b \psi(t)dt \right| \\
\leq & \frac{1}{(n+1)!}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^{n+1}\left\|\psi^{(n+1)}\right\|_{[a,b],1}.
\end{aligned}$$

As well as these main results given in this section, if we choose $x = \frac{a+b}{2}$ or $x = \frac{3a+b}{4}$ in the inequalities (3.1), (3.3), and (3.4), then we can deduce new Ostrowski type inequalities. Furthermore, taking $n = 1$ or $n = 2$ in inequalities presented in this section, we find some results obtained in the earlier works. Because these results can be readily obtained by using elementary analysis operations, the examination of these cases is left to reader.

4. Applications for quadrature rules

We handle applications of the inequalities presented in the former part so as to find the assessment of composite quadrature rules. To put it in different way, the new approaches to the estimation of quadrature formula for perturbed inequalities obtained by using mappings whose higher-order derivatives are bounded are examined.

First of all, we observe relation between our new quadrature rules and exact value of the integrals of the functions that will be presented. Then, we define new quadrature rules which are deduced in investigating upper bound in Theorem 3.

$$\begin{aligned}\int_a^b \psi(t)dt \approx Q_{n,1}(\psi) = &\sum_{k=0}^{n-1} \frac{(-1)^k \left[\psi^{(k)}(b) + (-1)^k \psi^{(k)}(a)\right]}{2^{k+1}(k+1)!}(b-a)^{k+1} \\ &+ \frac{[1+(-1)^n]}{2^{n+1}(n+1)!}(b-a)^{n+1}\psi^{(n)}(a),\end{aligned}$$

$$\begin{aligned}\int_a^b \psi(t)dt \approx Q_{n,2}(\psi) = &\sum_{k=0}^{n-1} \frac{\left[1+(-1)^k\right]\psi^{(k)}\left(\frac{a+b}{2}\right)}{2^{k+1}(k+1)!}(b-a)^{k+1} \\ &+ \frac{[1+(-1)^n]}{2^{n+1}(n+1)!}(b-a)^{n+1}\psi^{(n)}\left(\frac{a+b}{2}\right)\end{aligned}$$

and

$$\begin{aligned}\int_a^b \psi(t)dt \approx Q_{n,3}(\psi) = &\sum_{k=0}^{n-1} \frac{\left[1+(-1)^k\right]\left[\psi^{(k)}\left(\frac{a+3b}{4}\right) + (-1)^k \psi^{(k)}\left(\frac{3a+b}{4}\right)\right]}{(k+1)!}\left(\frac{b-a}{4}\right)^{k+1} \\ &+ 2\frac{[1+(-1)^n]}{(n+1)!}\left(\frac{b-a}{4}\right)^{n+1}\psi^{(n)}\left(\frac{3a+b}{4}\right).\end{aligned}$$

Now, we observe approaches to the estimation of $Q_{n,1}(\psi)$, $Q_{n,2}(\psi)$ and $Q_{n,3}(\psi)$ quadrature formulas for the functions $\psi_1(x) = x^4+2x^2+1$, $\psi_2(x) = cosx-x$, $\psi_3(x) = e^x \sin x$, $\psi_4(x) = e^{x^2}$, $\psi_5(x) = \sin x - x^3$, and $\psi_6(x) = \ln(x^2+1)\sin(x^2+1)$.

From the Table 1, we see that $Q_{n,1}(\psi)$ and $Q_{n,2}(\psi)$ quadrature rules give the precise value of the integral of ψ_1 for $n = 4$. This situation shows that $Q_{n,1}(\psi)$ and $Q_{n,2}(\psi)$ will present the exact value of the integrals of any functions which are polynomial of degree m for $n = m$.

Table 1. Error estimations for the functions $\psi_1,\ \psi_2,\ \psi_3$.

	$\psi(x)$	$[a,b]$	$\int_a^b \psi(x)dx$	n	$Q_{n,1}(\psi)$	$Q_{n,2}(\psi)$
	ψ_1	$[0,1]$	$1,866667$	4	$1,866667$	$1,866667$
Error:					0	0
	ψ_2	$\left[0,\frac{\pi}{2}\right]$	$-0,233701$	10	$-0,233701$	$-0,233701$
Error:					0	0
	ψ_3	$[0,1]$	$0,909331$	6	$0,909338$	$0,909331$
Error:					$7.0356E-6$	$1.373E-7$
	ψ_3	$[0,1]$	$0,909331$	15	$0,909331$	$0,909331$
Error:					0	0

Liu et al. offered some quadrature rules deduced by using twice differentiable functions in [20]. While they find the errors 5.1×10^{-4} and 6.5×10^{-5} for the function ψ_2 in their numerical results, we can calculate the same function with the error 0 by using $Q_{n,1}(\psi)$ and $Q_{n,2}(\psi)$. So, we understand that the results obtained using higher-order differentiable mappings are more useful. Furthermore, $Q_{n,1}(\psi)$ provide an error of the order 10^{-6} with the integral of ψ_3 for $n = 6$, whereas $Q_{n,2}(\psi)$ report an error of the order 10^{-7} with the integral of ψ_3 for $n = 6$. However, we capture the exact precise of the integral of ψ_3 for $n = 15$. These cases show that if we increase the order of n, then the exact values of the integrals are approximated.

From the Table 2, it is seen which results are deduced on different interval for the function ψ_4. We also examined the cases when $n = 11$ and $n = 20$ on the same intervals in the Table 2.

Table 2. Error estimations for the function ψ_4.

	$\psi(x)$	$[a,b]$	$\int_a^b \psi(x)dx$	n	$Q_{n,1}(\psi)$	$Q_{n,2}(\psi)$
	ψ_4	$[0,1]$	$1,462652$	11	$1,462658$	$1,462652$
Error:					$5.8789E-6$	$1.888E-7$
	ψ_4	$[0,1]$	$1,462652$	20	$1,462652$	$1,462652$
Error:					0	0
	ψ_4	$[1,2]$	$14,989976$	11	$14,99375$	$14,98989$
Error:					$3.7732E-3$	$8.42317E-5$
	ψ_4	$[1,2]$	$14,989976$	20	$14,989976$	$14,989976$
Error:					$8.4E-9$	0

We examine which results are obtained for the different order derivatives in the above table. As shown in all the table, $Q_{n,2}(\psi)$ generally give better results than $Q_{n,1}(\psi)$. However, $Q_{n,1}(\psi)$ report better outcomes for the function ψ_6 while $Q_{n,2}(\psi)$ give more effective results for the function ψ_5. So, we can conclude that a quadrature rule is not always more useful than the others. What is more, $Q_{n,3}(\psi)$ offers worse conclusions than the other two rules as shown in Table 3. If more effective outcomes are required, then it can be applied numerical integration methods which are used to find closer estimates by dividing the intervals into thinner pieces.

Table 3. Error estimations for the functions $\psi_5,\ \psi_6$.

	$\psi(x)$	$[a,b]$	$\int_a^b \psi(x)dx$	n	$Q_{n,1}(\psi)$	$Q_{n,2}(\psi)$	$Q_{n,3}(\psi)$
	ψ_5	$[2,3]$	$-15,676154$	6	$-15,676155$	$-15,676154$	$-15,675991$
Error:					$1.141E-6$	$6.4E-9$	$1.6344E-4$
	ψ_5	$[2,3]$	$-15,676154$	9	$-15,676154$	$-15,676154$	$-15,675992$
Error:					$1E-10$	0	$1.6272E-4$
	ψ_6	$[-1,1]$	$0,509386$	15	$0,509028$	$0,508178$	$0,441388$
Error:					$3.5805E-4$	$1.2076E-3$	$6.7997E-2$
	ψ_6	$[-1,1]$	$0,509386$	30	$0,509386$	$0,509249$	$0,441388$
Error:					$9E-10$	$1.3736E-4$	$6.6799E-2$

Let $I_d : a = x_0 < x_1 < ... < x_{d-1} < x_d = b$ be a partition of the interval $[a,b]$, and $m_p = x_{p+1} - x_p$ for $(p = 0, ..., d-1)$. Define the perturbed quadrature rules

$$\begin{aligned} Q_{n,1}(\psi, I_d) &= \sum_{k=0}^{n-1} \sum_{p=0}^{d-1} \frac{(-1)^k \left[\psi^{(k)}\left(x_{p+1}\right) + (-1)^k \psi^{(k)}\left(x_p\right)\right]}{2^{k+1}(k+1)!} \left(m_p\right)^{k+1} \\ &\quad + \frac{[1+(-1)^n]}{2^{n+1}(n+1)!} \sum_{p=0}^{d-1} \left(m_p\right)^{n+1} \psi^{(n)}(x_p), \end{aligned} \tag{4.1}$$

$$\begin{aligned} Q_{n,2}(\psi, I_d) &= \sum_{k=0}^{n-1} \sum_{p=0}^{d-1} \frac{\left[1+(-1)^k\right] \psi^{(k)}\left(\frac{x_p+x_{p+1}}{2}\right)}{2^{k+1}(k+1)!} \left(m_p\right)^{k+1} \\ &\quad + \frac{[1+(-1)^n]}{2^{n+1}(n+1)!} \sum_{p=0}^{d-1} \psi^{(n)}\left(\frac{x_p+x_{p+1}}{2}\right)\left(m_p\right)^{n+1} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &Q_{n,3}(\psi, I_d) \\ &= \sum_{k=0}^{n-1} \sum_{p=0}^{d-1} \frac{\left[1+(-1)^k\right]\left[\psi^{(k)}\left(\frac{x_p+3x_{p+1}}{4}\right) + (-1)^k \psi^{(k)}\left(\frac{3x_p+x_{p+1}}{4}\right)\right]}{2^{2k+2}(k+1)!} \left(m_p\right)^{k+1} \\ &\quad + \frac{[1+(-1)^n]}{2^{2n+1}(n+1)!} \sum_{p=0}^{d-1} \left(m_p\right)^{n+1} \psi^{(n)}\left(\frac{3x_p+x_{p+1}}{4}\right). \end{aligned}$$

Theorem 6. *Let $\psi : F \to \mathbb{R}$ be a n time differentiable function on F° and $[a,b] \subset F^\circ$. If the nth derivative $\psi^{(n)}$ is absolutely continuous on $[a,b]$, then we have*

$$\int_a^b \psi(t)dt = Q_{n,1}(\psi, I_d) + R_{n,1}(\psi, I_d),$$

where $Q_{n,1}(\psi, I_d)$ is defined by formula (4.1), and the remainder $R_{n,1}(\psi, I_d)$ satisfies the estimations

$$\left|R_{n,1}(\psi, I_d)\right| \leq \begin{cases} \frac{1}{2^{n+1}(n+1)!}\|\psi^{(n+1)}\|_{[a,b],\infty} \sum\limits_{p=0}^{d-1} \left(m_p\right)^{n+2} \\ \frac{A(n,q)}{2^{n+1+\frac{1}{s}} n!}\|\psi^{(n+1)}\|_{[a,b],r} \sum\limits_{p=0}^{d-1} \left(m_p\right)^{n+1+\frac{1}{s}} \\ \frac{1}{2^{n}(n+1)!}\|\psi^{(n+1)}\|_{[a,b],1} \sum\limits_{p=0}^{d-1} \left(m_p\right)^{n+1} \end{cases}$$

where $\frac{1}{r} + \frac{1}{s} = 1$ and $A(n, s)$ is defined by

$$A(n, s) = \left[B(n + 1, 1 + \frac{1}{s}) + \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^{n+1+\frac{1}{s}-k}}{n + 1 + \frac{1}{s} - k}\right].$$

Proof. Choosing $x = a$ in the inequalities (3.1), (3.3) and (3.4). If we apply the resulting inequalities to the intervals $\left[x_p, x_{p+1}\right]$ for $p = 0, 1, ..., d - 1$ and then sum over p from 0 to $d - 1$, from the triangle inequality, we reach the estimations (6). □

Remark 1. Supposing that all the assumptions of Theorem 3 hold. Then we have

$$\int_a^b \psi(t)dt = Q_{n,2}(\psi, I_d) + R_{n,2}(\psi, I_d)$$

and

$$\int_a^b \psi(t)dt = Q_{n,3}(\psi, I_d) + R_{n,3}(\psi, I_d)$$

where $Q_{n,2}(\psi, I_d)$ and $Q_{n,3}(\psi, I_d)$ are defined as in (4.2) and (4.3) respectively. Taking $x = \frac{a+b}{2}$ and $x = \frac{3a+b}{4}$ in the inequalities (3.1), (3.3) and (3.4), and later applying similar processes in the proof of the above theorem to the resulting inequalities, then new estimations provided by the remainders $R_{n,2}(\psi, I_d)$ and $R_{n,3}(\psi, I_d)$ can be obtained.

5. Applications for some elementary functions

For convenience, we give the following notations $B_k(x)$ and $C_n(x)$ that will be used throughout this section in order to simplify the details of presentations. $B_k(x)$ and $C_n(x)$ are defined by

$$B_k(x) = \left[(x - a)^{k+1} + (-1)^k \left(\frac{a + b}{2} - x\right)^{k+1}\right]$$

and

$$C_n(x) = \left[(x - a)^{n+1} + \left(\frac{a + b}{2} - x\right)^{n+1}\right].$$

Initially, we think the exponential mapping $\psi(t) = e^t$ with $t \in \mathbb{R}$, then, for $a \leq x \leq \frac{a+b}{2}$, one possesses

$$\left\|\psi^{(n+1)}\right\|_{[a,x],\infty} = e^x \quad , \quad \left\|\psi^{(n+1)}\right\|_{[x,b],\infty} = e^b \tag{5.1}$$

and

$$\left\|\psi^{(n+1)}\right\|_{[a,x],r} = \left(\frac{e^{rx} - e^{ra}}{r}\right)^{\frac{1}{r}} \quad , \quad \left\|\psi^{(n+1)}\right\|_{[x,b],r} = \left(\frac{e^{rb} - e^{rx}}{r}\right)^{\frac{1}{r}} \tag{5.2}$$

for $r \geq 1$. In this case, by the inequality (3.1) and the equalities in (5.1), one has the inequality

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+1}\left[e^{a+b-x} + (-1)^k e^x\right]}{(k+1)!} B_k(x)\right.\\
&\left.-\frac{[1+(-1)^n]}{(n+1)!} C_n(x) e^x + (-1)^n \left(e^b - e^a\right)\right|\\
\leq &\frac{(x-a)^{n+2}}{(n+2)!} e^x + \frac{2}{(n+1)!}\left(\frac{a+b}{2} - x\right)^{n+2} e^b\\
&+ \left[\frac{(b-a)(x-a)^{n+1}}{(n+1)!} - \frac{2n+3}{(n+2)!}(x-a)^{n+2}\right] e^b
\end{aligned}$$

for $a \leq x \leq \frac{a+b}{2}$. Also, by the inequality (3.3) and the equalities in (5.2), we conclude that

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+1}\left[e^{a+b-x} + (-1)^k e^x\right]}{(k+1)!} B_k(x)\right.\\
&\left.-\frac{[1+(-1)^n]}{(n+1)!} C_n(x) e^x + (-1)^n \left(e^b - e^a\right)\right|\\
\leq &\frac{(x-a)^{n+1+\frac{1}{s}}}{n!} B\left(n+1, 1+\frac{1}{s}\right)\left(\frac{e^{rx} - e^{ra}}{r}\right)^{\frac{1}{r}}\\
&+ A(n,s)\frac{1}{n!}\left(\frac{a+b}{2} - x\right)^{n+1+\frac{1}{s}}\left(\frac{e^{rb} - e^{rx}}{r}\right)^{\frac{1}{r}}\\
&+ \frac{1}{n!}\left(\frac{e^{rb} - e^{rx}}{r}\right)^{\frac{1}{r}} (b-x)^{n+1+\frac{1}{s}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1 - \left(1 - \frac{x-a}{b-x}\right)^{k+1+\frac{1}{s}}}{k+1+\frac{1}{s}}
\end{aligned}$$

where $A(n,s)$ is defined as in (6).

Now, we deal with another function which is of large interest. Let $\psi(t) = \ln t,\ t > 0$. Utilizing the inequality (3.1), because of

$$\psi^{(k)}(t) = \frac{(-1)^{k-1}(k-1)!}{t^k} \quad k \geq 1 \text{ and } t > 0$$

and

$$\left\|\psi^{(n+1)}\right\|_{[a,x],\infty} = \frac{n!}{a^{n+1}} \qquad \left\|\psi^{(n+1)}\right\|_{[x,b],\infty} = \frac{n!}{x^{n+1}},$$

we have the logarithmic inequality

$$
\begin{aligned}
&\Bigg|(-1)^{n+1}\ln x(a+b-x) \\
&+\sum_{k=1}^{n-1}\frac{(-1)^n\left[(a+b-x)^k+(-1)^k x^k\right]}{k\,(k+1)\,x^k(a+b-x)^k}B_k(x) \\
&+\frac{[1+(-1)^n]}{n\,(n+1)\,x^n}C_n(x)+(-1)^n\left[\ln\frac{b^b}{a^a}-(b-a)\right]\Bigg| \\
\le&\frac{(x-a)^{n+2}}{(n+2)!}\frac{n!}{a^{n+1}}+\frac{2}{(n+1)!}\left(\frac{a+b}{2}-x\right)^{n+2}\frac{n!}{x^{n+1}} \\
&+\left[\frac{(b-a)\,(x-a)^{n+1}}{(n+1)!}-\frac{2n+3}{(n+2)!}\,(x-a)^{n+2}\right]\frac{n!}{x^{n+1}}
\end{aligned}
$$

for any $0 < a \le x \le \frac{a+b}{2} < \infty$. Similarly, it may be obtained new logarithmic inequalities by using the inequalities (3.3) and (3.4). For these results, we need to know that

$$
\left\|\psi^{(n+1)}\right\|_{[a,x],r}=\frac{n!\left(x^{(n+1)r-1}-a^{(n+1)r-1}\right)^{\frac{1}{r}}}{[(n+1)r-1]^{\frac{1}{r}}\,x^{n+1-\frac{1}{r}}a^{n+1-\frac{1}{r}}}
$$

$$
\left\|\psi^{(n+1)}\right\|_{[x,b],r}=\frac{n!\left(b^{(n+1)r-1}-x^{(n+1)r-1}\right)^{\frac{1}{r}}}{[(n+1)r-1]^{\frac{1}{r}}\,b^{n+1-\frac{1}{r}}x^{n+1-\frac{1}{r}}}
$$

for $r \ge 1$.

In addition to all these results, by choosing $x = \frac{a+b}{2}$ in the inequalities given in this section, it can be stated the results that provide some simple estimates for the exponential and logarithm at the midpoint.

6. Conclusions

In this work, we establish some integral inequalities for higher-order differentiable functions. Some applications of the inequalities developed in this paper are also given. In order to validate that their generalized behavior, we show the relation of our results with previously published ones. In future works, authors can obtain similar inequalities by using the different classes of functions.

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Conflict of interest

The authors declare no conflict of interest.

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