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Research article

Multidimensional stability of V-shaped traveling fronts in bistable reaction-diffusion equations with nonlinear convection

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Abstract: This paper is concerned with the multidimensional stability of V-shaped traveling fronts for a reaction-diffusion equation with nonlinear convection term in $\mathbb{R}^n$ ($n \geq 3$). We consider two cases for initial perturbations: one is that the initial perturbations decay at space infinity and another one is that the initial perturbations do not necessarily decay at space infinity. In the first case, we show that the V-shaped traveling fronts are asymptotically stable. In the second case, we first show that the V-shaped traveling fronts are also asymptotically stable under some further assumptions. At the same time, we also show that there exists a solution that oscillates permanently between two V-shaped traveling fronts, which means that the traveling fronts are not asymptotically stable under general bounded perturbations.

Keywords: reaction-diffusion equation; nonlinear convection; V-shaped traveling front; multidimensional stability
Mathematics Subject Classification: 35K57, 35C07, 35B35, 35B40

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1. Introduction

In this paper, we study the multidimensional asymptotic stability of two-dimensional V-shaped traveling fronts of the following reaction-diffusion equation with nonlinear convection term:

$$\begin{cases} \frac{\partial u}{\partial t} + (g(u))_z = \Delta u + f(u), \ x \in \mathbb{R}^{n-2}, \ y \in \mathbb{R}, \ z \in \mathbb{R}, \ t > 0, \\ u(x, y, z, 0) = u_0(x, y, z), \ x \in \mathbb{R}^{n-2}, \ y \in \mathbb{R}, \ z \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_{n-2}^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ and $n \geq 3$. We assume that the initial value $u_0 \in BUC^1(\mathbb{R}^n)$. In (1.1), $(g(u))_z$ is a nonlinear convection term and the function $f \in C^2(\mathbb{R})$ is the reaction term. Suppose f satisfies the following assumption:

(F) (i) $f(0) = f(1) = 0,\ f'(0) < 0,\ f'(1) < 0$;
(ii) $\{r \in [0, 1] : f(r) = 0\} = \{0, \lambda, 1\}$ with $f'(\lambda) > 0$;
(iii) $\int_0^1 f(r)\mathrm{d}r > 0$;
(iv) $f(r) < 0,\ f'(r) < 0$ for $r > 1$; $f(r) > 0,\ f'(r) < 0$ for $r < 0$.

The assumption (F) implies that the reaction term f is of bistable type. A typical example of such f is the cubic function $f(u) = u(u - a)(1 - u)$, where $a \in (0, \frac{1}{2})$ is a given number. We also assume the flux g satisfies the following assumption:

(G) $g(r) \in C^{2+\gamma_0}(\mathbb{R})$, $\gamma_0 \in (0, 1)$; $g''(r) \leq 0$ for $r \in [0, 1]$.

Examples of such g are $g(u) = -\rho u^2$ and $g(u) = \rho u(1 - u)$, where $\rho > 0$ is a constant. From the assumption (G), we deduce that there exist positive constants l_1 and l_2 such that

$$|g'(r)| \leq l_1,\ |g''(r)| \leq l_2 \ \text{ for all } r \in [-1, 2]. \tag{1.2}$$

For each $\theta \in [0, \pi)$, it follows from Crooks and Toland [1] that there exist a function $U_\theta(\cdot) \in C^2(\mathbb{R})$ and a constant c_θ satisfying

$$\begin{cases} -U_\theta'' + (c_\theta + g'(U_\theta)\sin\theta)\, U_\theta' - f(U_\theta) = 0,\ U_\theta'(X) > 0,\ X \in \mathbb{R}, \\ U_\theta(-\infty) = 0,\ U_\theta(+\infty) = 1. \end{cases} \tag{1.3}$$

Let $\vec{e}_\pm = (0, \cdots, 0, \pm\cos\theta, \sin\theta) \in S^{n-1}$. Then the functions

$$U_\theta\left((x, y, z) \cdot \vec{e}_+ + c_\theta t\right) = U_\theta\left(y\cos\theta + z\sin\theta + c_\theta t\right)$$

and

$$U_\theta\left((x, y, z) \cdot \vec{e}_- + c_\theta t\right) = U_\theta\left(-y\cos\theta + z\sin\theta + c_\theta t\right)$$

are planar traveling fronts of (1.1) with wave speed c_θ along the directions $\vec{e}_+$ and $\vec{e}_-$, respectively. In particular, the profile function $U_\theta(\cdot)$ is unique up to a translation and the wave speed c_θ is unique. It is clear that $c_0 > 0$ due to the assumption (F). However, due to the convection term, it is not necessary that $c_\theta > 0$ for $\theta \in (0, \pi)$. For more results on traveling wave front of (1.1) with various nonlinearity f, we refer to [2–9].

Recently, we studied nonplanar traveling fronts of (1.1) with $n = 2$. Assume that (F) and (G) hold. Fix $\theta \in (0, \frac{\pi}{2})$ satisfying the following assumption:

(C) $c_\theta + g'(r)\sin\theta > 0$ for any $r \in [0, 1]$.

In our recent paper [10], we proved the existence of V-shaped traveling fronts to (1.1) in $\mathbb{R}^2$, namely, there exists a function $V(\cdot, \cdot) \in C^2\left(\mathbb{R}^2\right)$ satisfying

$$\begin{cases} -V_{yy} - V_{zz} + (s_\theta + g'(V))\, V_z - f(V) = 0, \quad (y, z) \in \mathbb{R}^2, \\ V(-y, z) = V(y, z),\ \frac{\partial}{\partial z}V(y, z) > 0, \quad \forall\ (y, z) \in \mathbb{R}^2, \\ \frac{\partial}{\partial y}V(y, z) > 0, \quad \forall\ (y, z) \in (0, +\infty) \times \mathbb{R}, \\ V(y, z) > U_\theta\left(|y|\cos\theta + z\sin\theta\right), \quad \forall\ (y, z) \in \mathbb{R}^2, \\ \lim_{R\to+\infty} \sup_{y^2+z^2>R^2} \left|V(y, z) - U_\theta\left(|y|\cos\theta + z\sin\theta\right)\right| = 0, \end{cases} \tag{1.4}$$

where $s_\theta = \frac{c_\theta}{\sin\theta}$. In addition, for any initial value $u_0(y,z) \in BUC^1(\mathbb{R}^2)$ with $u_0 \geq U_\theta(|y|\cos\theta + z\sin\theta)$ and

$$\lim_{R\to\infty} \sup_{y^2+z^2\geq R^2} |u_0(y,z) - U_\theta(|y|\cos\theta + z\sin\theta)| = 0, \tag{1.5}$$

the solution $u(y,z,t;u_0)$ of (1.1) satisfies

$$\lim_{t\to\infty} \sup_{(y,z)\in\mathbb{R}^2} |u(y,z,t;u_0) - V(y, z + s_\theta t)| = 0. \tag{1.6}$$

Here we would like to point out that the last results have been obtained when the convection term is absent, see [11]. In fact, nonplanar traveling fronts of (1.1) without convection have been extensively studied in recent years, see [12–25]. For reaction-diffusion system and time periodic reaction-diffusion equation, we refer to [26–34]. For more results on non-planar traveling wave solutions of reaction-diffusion equations, we refer to [35–38].

It is clear that the asymptotic stability of the V-shaped traveling front $V(y,z) \in C(\mathbb{R}^2)$ of (1.1) was established in $\mathbb{R}^2$, see [10]. In this paper, we further consider the multidimensional stability of the V-shaped traveling front $V(y,z)$, namely, we consider the stability of $V(y,z)$ in $\mathbb{R}^n$ with $n \geq 3$. It should be pointed out that the multidimensional stability of planar traveling fronts of (1.1) without convection has been studied by many authors, see [39–43] for the Allen-Cahn equation. In [39, 40, 43], it is assumed that the initial perturbations are sufficiently small and decay to zero at space infinity. In [42] the authors proved the asymptotic stability under any (possibly large) initial perturbations that decay to zero at space infinity. Furthermore, they proved the asymptotic stability of planar traveling fronts for almost periodic perturbation. Moreover, the existence of a solution that oscillates permanently between two planar waves was shown, which implies that planar traveling fronts are not asymptotically stable under general perturbations. Motano and Nara [41] studied how a planar front behaves when arbitrarily large (but bounded) perturbation is given near the front region. They showed that the planar front is asymptotically stable in $L^\infty(\mathbb{R}^n)$ under spatially ergodic perturbations, which include quasi-periodic and almost periodic ones as special cases. Roquejoffre and Roussier-Michon [44] considered the large time behavior of planar traveling fronts and showed that the dynamics of planar fronts are similar to that of the heat equation. More recently, Sheng *et al.* [28] and Cheng and Yuan [45] studied the multidimensional stability of V-shaped traveling fronts and pyramidal traveling fronts of the Allen-Cahn equation by using the method of [42], respectively.

Following (1.4) and (1.5), we know that the asymptotic stability of V-shaped traveling fronts $V(y,z)$ of (1.1) in [10] was established in $\mathbb{R}^2$ under the initial value $u_0(y,z)$ satisfying that $u_0(y,z) - V(y,z)$ decays to zero as $|y| + |z| \to \infty$. In this paper we use the method of [42] to study the stability of V-shaped traveling fronts $V(y,z)$ of (1.1) under the initial value $u_0(x,y,z)$ in $\mathbb{R}^n$ with $n \geq 3$, where $x \in \mathbb{R}^{n-2}$, $y \in \mathbb{R}$ and $z \in \mathbb{R}$. In contrast to that in [10], in this paper we deal with the following two cases:

Case A: The initial perturbation $u_0(x,y,z) - V(y,z)$ decays to zero as $|x| + |y| + |z| \to \infty$.

Case B: The initial perturbation $u_0(x,y,z) - V(y,z)$ does not necessarily decay to zero as $|x|+|y|+|z| \to \infty$.

In the remainder of this paper we always assume that (F) and (G) hold and $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfies the assumption (C). Let $(U_\theta(\cdot), c_\theta)$ be defined by (1.3) and let $s_\theta = \frac{c_\theta}{\sin\theta}$. Let $V(y,z)$ be defined by (1.4). For the sake of convenience, in the sequel we always denote $(U_\theta(\cdot), c_\theta)$ and s_θ by $(U(\cdot), c)$ and s respectively. In the following we first consider the asymptotic stability of $V(y,z)$ in Case A.

Theorem 1.1. *Let $n \geq 3$. Assume that the initial value $u_0(x, y, z) \in BUC^1(\mathbb{R}^n)$ and satisfies*

$$\lim_{R\to\infty} \sup_{|x|+|y|+|z|\geq R} |u_0(x, y, z) - V(y, z)| = 0.$$

Then the solution $u(x, y, z, t)$ of (1.1) *satisfies*

$$\lim_{t\to\infty} \sup_{(x,y,z)\in\mathbb{R}^n} |u(x, y, z, t) - V(y, z + st)| = 0. \tag{1.7}$$

Theorem 1.1 implies that the V-shaped traveling front $V(y, z + st)$ is asymptotically stable under any initial perturbations that decay to zero as $|x| + |y| + |z| \to \infty$. The following theorem gives the convergence rate for (1.7) when the initial perturbation belongs to L^1 in a certain sense.

Theorem 1.2. *Let $n \geq 3$. Assume that the initial value $u_0(x, y, z)$ of* (1.1) *is given by*

$$u_0(x, y, z) = V\left(y, z + v_0(x)\right), \tag{1.8}$$

for some function $v_0 \in L^1(\mathbb{R}^{n-2}) \cap L^\infty(\mathbb{R}^{n-2}) \cap BUC^1(\mathbb{R}^{n-2})$. Then the solution $u(x, y, z, t)$ of (1.1) *satisfies*

$$\sup_{(x,y,z)\in\mathbb{R}^n} |u(x, y, z, t) - V(y, z + st)| \leq Ct^{-\frac{n-2}{2}},\ t > 0, \tag{1.9}$$

where $C > 0$ is a constant depending on $f,\ g,\ \|v_0\|_{L^1(\mathbb{R}^{n-2})}$ and $\|v_0\|_{L^\infty(\mathbb{R}^{n-2})}$.

The following proposition shows that the convergence rate (1.9) is optimal in some sense.

Proposition 1.3. *Let $n \geq 3$. Let u_0 be defined as in* (1.8) *and assume that v_0 either satisfies $v_0 \geq 0,\ v_0 \not\equiv 0$ or $v_0 \leq 0,\ v_0 \not\equiv 0$. Then there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1(1 + t)^{-\frac{n-2}{2}} \leq \sup_{(x,y,z)\in\mathbb{R}^n} |u(x, y, z, t) - V(y, z + st)| \leq C_2 t^{-\frac{n-2}{2}},\ t \geq 0. \tag{1.10}$$

Remark 1.4. Theorem 1.1 and 1.2 show that the V-shaped traveling front is not only asymptotically stable, but also algebraically stable under certain perturbations. Furthermore, Proposition 1.3 also implies that this convergence rate is optimal in some sense, that is , the convergence rate is not faster than $O\left(t^{-\frac{n-2}{2}}\right)$.

Next, we state our results in Case B. Firstly, we show that the V-shaped traveling front is also asymptotically stable if the initial value $u_0(x, y, z)$ satisfies some certain assumptions in Case B.

Theorem 1.5. *Let $n \geq 3$. Suppose that the initial value $u_0(x, y, z) \in BUC^1(\mathbb{R}^n)$ of* (1.1) *satisfies*

$$V(y, z) \leq u_0(x, y, z) \leq \hat{u}_0\left(y, z\right), \quad \forall (x, y, z) \in \mathbb{R}^3,$$

where $\hat{u}_0\left(y, z\right) \in BUC^1(\mathbb{R}^n)$ satisfies

$$\lim_{R\to\infty} \sup_{y^2+z^2\geq R^2} |\hat{u}_0(y, z) - U_\theta\left(|y| \cos\theta + z \sin\theta\right)| = 0. \tag{1.11}$$

Then the solution $u(x, y, z, t)$ of (1.1) *satisfies*

$$\lim_{t\to\infty} \sup_{(x,y,z)\in\mathbb{R}^n} |u(x, y, z, t) - V(y, z + st)| = 0. \tag{1.12}$$

Secondly, we present a result on the existence of a solution of (1.1) that oscillates permanently between two V-shaped traveling fronts.

Theorem 1.6. *Let $n = 3$. Then for any $\delta > 0$, there exists a bounded function $v_0^*(x) \in BUC^1(\mathbb{R})$ with $\|v_0^*\|_{L^\infty(\mathbb{R})} = \delta$ such that the solution $u(x,y,z,t)$ of (1.1) with $u(x,y,z,0) = V\left(y, z + v_0^*(x)\right)$ satisfies*

$$\lim_{m\to\infty} \sup_{|x|\leq m!-1,(y,z)\in\mathbb{R}^2} |u(x,y,z,t_m) - V(y, z + st_m + (-1)^m\delta)| = 0,$$

where $t_m = m(m!)^2/4$.

Remark 1.7. A typical example of initial functions satisfying the requirement of Theorem 1.5 is that $u_0(x,y,z) = V(y, z + \psi(z)v(x))$ in $(x,y,z) \in \mathbb{R}^3$, where the functions $v(x)$ and $\psi(z)$ respectively satisfy the following conditions:

(i) $v(x) \in BUC^1(\mathbb{R}^{n-2})$ is a periodic function in $\mathbb{R}^{n-2}$ and $v(x) > 0$;

(ii) $\psi(z) \in C^1(\mathbb{R})$, $\psi(z) > 0$ and $\lim_{|z|\to\infty} \psi(z) = 0$.

Since $v(x) > 0$ is a periodic function, there exists a positive constant $v_{\max}$ such that $v_{\max} = \max_{x\in\mathbb{R}^{n-2}} v(x)$. Take $\hat{u}_0(y,z) = V(y, z + \psi(z)v_{\max})$. Then $V(y,z) \leq u_0(x,y,z) \leq \hat{u}_0(y,z)$ and (1.11) holds. In addition, it is clear that the initial function $u_0(x,y,z)$ does not decay when $|x| + |y| + |z| \to \infty$. Theorem 1.5 shows that even if in Case B, the V-shaped traveling front is still asymptotically stable under some further assumptions on the initial value. On the other hand, Theorem 1.6 implies that even very small perturbations to the V-shaped traveling front $V(y, z + st)$ can give rise to permanent oscillation, hence the V-shaped traveling front is not asymptotically stable under general bounded perturbations. It can be viewed as a counter-example to the asymptotic stability of V-shaped traveling fronts in Case B. From the viewpoint of dynamical system, Theorem 1.6 gives two ω-limit points of the solution $u(x,y,z,t)$ in the $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$-topology. From the above discussion, we conclude that the V-shaped traveling fronts may be stable or unstable in Case B.

The rest of the paper is organized as follows. In Section 2, we study Case A, namely, we prove Theorem 1.1 and 1.2. In the proofs, we use the moving coordinated with speed s so that the V-shaped traveling fronts can be viewed as stationary states. Setting

$$u(x,y,z,t) = w(x,y,\chi,t),\ \chi = z + st,$$

the Eq (1.1) can be rewritten as

$$\begin{cases} w_t = \Delta w - (g'(w) + s)\, w_\chi + f(w),\ x \in \mathbb{R}^{n-2},\ y \in \mathbb{R},\ \chi \in \mathbb{R},\ t > 0, \\ w(x,y,\chi,0) = u_0(x,y,\chi),\ x \in \mathbb{R}^{n-2},\ y \in \mathbb{R},\ \chi \in \mathbb{R}, \end{cases}$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_{n-2}^2 + \partial^2/\partial y^2 + \partial^2/\partial \chi^2$. For convenience, we denote $w(x,y,\chi,t)$ as $u(x,y,z,t)$ and consider the problem of the form

$$\begin{cases} u_t = \Delta u - (g'(u) + s)\, u_z + f(u),\ x \in \mathbb{R}^{n-2},\ y \in \mathbb{R},\ z \in \mathbb{R},\ t > 0, \\ u(x,y,z,0) = u_0(x,y,z),\ x \in \mathbb{R}^{n-2},\ y \in \mathbb{R},\ z \in \mathbb{R}. \end{cases} \tag{1.13}$$

The global existence of a unique solution $u(x,z,t;u_0)$ of the Eq (1.13) follows from [46, Theorem 7.1.2, Propositions 7.1.9 and 7.1.10 and Remark 7.1.12] and the assumptions (F) and (G), see also [2, Proposition A.3 and Theorem A.7]. In particular, $u(t;u_0)(\cdot) \in C^1\left((0,\infty), BUC(\mathbb{R}^2)\right) \cap C\left((0,\infty), BUC^2(\mathbb{R}^2)\right) \cap C\left([0,\infty), BUC^1(\mathbb{R}^2)\right)$, where $u(t;u_0)(x,z) := u(x,z,t;u_0)$. It is clear that, for each $\xi \in \mathbb{R}$, the function $V(y, z+\xi)$ is a stationary solution for problem (1.13). In this section, we also show that the convergence rate in Theorem 1.1 and Theorem 1.2 is optimal in some sense, namely, we prove Proposition 1.3. In Section 3, we consider Case B. Firstly, we prove that the V-shaped traveling fronts are asymptotically stable under certain assumptions of the initial value, namely, we prove Theorems 1.5. Secondly, using the supersolutions and subsolutions constructed in Section 2, we prove that the V-shaped traveling fronts are not asymptotically stable, namely, we prove Theorem 1.6.

2. Stability of V-shaped fronts in Case A

In this section, we consider Case A and prove the asymptotic stability of V-shaped traveling fronts under perturbations that decay to zero as $|x|+|y|+|z| \to \infty$. We first state some known results of the curvature flow problem in [47], see also [42].

The mean curvature flow for a graphical surface $\Psi(x,t)$ on $\mathbb{R}^{n-2}$ is given by the Cauchy problem of the form

$$\begin{cases} \frac{\Psi_t}{\sqrt{1+|\nabla\Psi|^2}} = \operatorname{div}\left(\frac{\nabla\Psi}{\sqrt{1+|\nabla\Psi|^2}}\right), \ x \in \mathbb{R}^{n-2}, \ t>0, \\ \Psi(x,0) = \Psi_0(x), \ x \in \mathbb{R}^{n-2}. \end{cases} \tag{2.1}$$

If the first and second derivatives of Ψ with respect to x are all bounded on $\mathbb{R}^{n-2}$, then we take some large constant $k>0$ such that

$$\begin{aligned} 0 =& \Psi_t - \sqrt{1+|\nabla\Psi|^2}\operatorname{div}\left(\frac{\nabla\Psi}{\sqrt{1+|\nabla\Psi|^2}}\right) \\ =& \Psi_t - \Delta\Psi + \sum_{i,j=1}^{n-2} \frac{\Psi_{x_i}\Psi_{x_j}\Psi_{x_ix_j}}{1+|\nabla\Psi|^2} \\ \geq& \Psi_t - \Delta\Psi - k|\nabla\Psi|^2. \end{aligned}$$

Clearly, $\Psi(x,t)$ is a subsolution of the following Cauchy problem:

$$\begin{cases} v_t^+ = \Delta v^+ + k|\nabla v^+|^2, \ x \in \mathbb{R}^{n-2}, \ t>0, \\ v^+(x,0) = u_0(x), \ x \in \mathbb{R}^{n-2}. \end{cases}$$

Taking $w(x,t) = e^{kv^+(x,t)}$, we have

$$\begin{cases} w_t = \Delta w, \ x \in \mathbb{R}^{n-2}, \ t>0, \\ w(x,0) = e^{ku_0(x)}, \ x \in \mathbb{R}^{n-2}. \end{cases}$$

Then, from the solution of a standard heat equation, the explicit expression for $v^+(x,t)$ is given by

$$v^+(x,t) = \frac{1}{k}\ln\left(\int_{\mathbb{R}^{n-2}} \Gamma(x-\eta,t)e^{ku_0(\eta)}\mathrm{d}\eta\right), \tag{2.2}$$

where $\Gamma(\xi, \tau)$ is the heat kernel given by

$$\Gamma(\xi, \tau) = \frac{1}{(4\pi\tau)^{\frac{n-2}{2}}} e^{-\frac{|\xi|^2}{4\tau}}.$$

Consequently, the expression (2.2) gives an upper estimate for $\Psi(x, t)$ of (2.1). By considering the equation

$$\begin{cases} v_t^- = \Delta v^- - k|\nabla v^-|^2, \ x \in \mathbb{R}^{n-2}, \ t > 0, \\ v^-(x, 0) = u_0(x), \ x \in \mathbb{R}^{n-2}, \end{cases}$$

we also give a lower estimate for $\Psi(x, t)$ of (2.1).

The following lemma gives the large time behavior of the solutions of

$$v_t^\pm = \Delta v^\pm \pm k \left|\nabla v^\pm\right|^2, \ x \in \mathbb{R}^{n-2}, \ t > 0.$$

Lemma 2.1. (See [42, Lemma 2.4 and Remark 2.5]) *Let $k > 0$ be any constant. Let $v^\pm(x, t)$ be solutions to the Cauchy problems:*

$$\begin{cases} v_t^\pm = \Delta v^\pm \pm k \left|\nabla v^\pm\right|^2, \ x \in \mathbb{R}^{n-2}, \ t > 0, \\ v^\pm(x, 0) = v_0(x), \ x \in \mathbb{R}^{n-2}. \end{cases}$$

Suppose that $v_0(x)$ is bounded and continuous on $\mathbb{R}^{n-2}$ and satisfies $\lim_{|x|\to\infty} |v_0(x)| = 0$. Then the solutions $v^\pm(x, t)$ satisfy

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^{n-2}} \left|v^\pm(x, t)\right| = 0,$$

respectively. Furthermore, if $v_0 \in L^1(\mathbb{R}^{n-2})$, then we have

$$\sup_{x\in\mathbb{R}^{n-2}} |v^\pm(x, t)| \le \frac{1}{k} \left\|e^{kv_0} - 1\right\|_{L^1(\mathbb{R}^{n-2})} \cdot t^{-\frac{n-2}{2}}, \ t > 0.$$

Before constructing a series of supersolutions and subsolutions, we first give some auxiliary lemmas.

Lemma 2.2. (See [10, Lemma 3.2 and Theorem 2.3]) *For any $\delta \in (0, \frac{1}{2})$, there exists a positive constant $\beta := \beta(\delta)$ such that*

$$V_z(y, z) \ge \beta \ \text{for} \ \delta \le V(y, z) \le 1 - \delta.$$

In addition, we have

$$\lim_{R\to+\infty} \sup_{|z+m_*|y||\ge R} V_z = 0,$$

where $m_* = \sqrt{s^2 - c^2}\Big/ c$.

Now, we introduce a lemma which plays a key role in constructing supersolutions and subsolutions.

Lemma 2.3. *There exists a constant $k > 0$ such that*

$$-kV_z(y, z) \le V_{zz}(y, z) \le kV_z(y, z), \quad \forall (y, z) \in \mathbb{R}^2.$$

Proof. It follows from the first equation of (1.4) that

$$\Delta V_z - (s + g'(V))\, V_{zz} + f'(V)V_z - g''(V)V_z^2 = 0, \quad \forall (y, z) \in \mathbb{R}^2. \tag{2.3}$$

If we write $W = V_z$, then (2.3) becomes

$$\Delta W - (s + g'(V))\, W_z + (f'(V) - g''(V)W)\, W = 0. \tag{2.4}$$

Setting

$$b(y, z) = -(s + g'(V)),\ c(y, z) = f'(V) - g''(V)W,$$

then (2.4) can be rewritten as

$$\Delta W + bW_z + cW = 0.$$

From Lemma 2.2, there exists a constant $K > 0$ such that

$$0 < W(x, z) = V_z(x, z) \leq K, \quad \forall\ (x, z) \in \mathbb{R}^2.$$

By the interior L^p estimates for the second derivatives of elliptic equations (see [48, Theorem 9.11]) and the embedding theorem (see [48, Theorem 7.26]), there exists a constant $\Lambda > 0$ such that

$$\|W\|_{C^{1+\alpha}(\mathbb{R}^2)} \leq \Lambda$$

for some constant $\alpha \in (0, 1]$. Using the above estimate, the assumptions (F) and (G), and the Schauder interior estimates for the second derivatives of elliptic equations (see [48, Theorem 6.2]), we have that for some $0 < \alpha \leq 1$ and $C_0 > 0$, there are

$$|W|^*_{2,\alpha;B_2(y_0,z_0)} \leq C_0|W|_{0;B_2(y_0,z_0)}, \quad \forall\ (y_0, z_0) \in \mathbb{R}^2, \tag{2.5}$$

where $B_r(y_0, z_0)$ is a ball of radius r in $\mathbb{R}^2$ with origin (y_0, z_0). On the other hand, since $B_1(y_0, z_0) \subset\subset B_2(y_0, z_0)$ and $\operatorname{dist}(B_1(y_0, z_0), \partial B_2(y_0, z_0)) = 1$, we have

$$|W|_{m,\alpha;B_1(y_0,z_0)} \leq |W|^*_{m,\alpha;B_2(y_0,z_0)}, \quad \forall\ (y_0, z_0) \in \mathbb{R}^2. \tag{2.6}$$

Here we refer the definitions of the norms $|\cdot|^*_{m,\alpha;\Omega}$, $|\cdot|_{m,\alpha;\Omega}$ and $|\cdot|_{m;\Omega}$ to [48] and [28]. Combining (2.5) and (2.6) yields

$$|W|_{2,\alpha;B_1(y_0,z_0)} \leq C_0|W|_{0;B_2(y_0,z_0)}, \quad \forall\ (y_0, z_0) \in \mathbb{R}^2.$$

Since $W(x, z) > 0$, the Harnack-type inequality [48, Theorem 2.5] implies that there exists another constant $C_1 > 0$ such that

$$|W|_{2,\alpha;B_1(y_0,z_0)} \leq C_0C_1W(y_0, z_0), \quad \forall\ (y_0, z_0) \in \mathbb{R}^2.$$

Taking

$$k := C_0C_1,$$

we obtain that

$$|W_z(y_0, z_0)| \leq kW(y_0, z_0), \quad \forall\ (y_0, z_0) \in \mathbb{R}^2,$$

that is

$$|V_{zz}| \leq kV_z \quad \text{in } \mathbb{R}^2.$$

This completes the proof of Lemma 2.3. □

Now, we will show that the functions $V(y, z + v^{\pm}(x, t))$ are a supersolution and a subsolution of (1.13), respectively. In what follows, Δ_x and ∇_x denote the $(n-2)$-dimensional Laplacian and the $(n-2)$-dimensional gradient, respectively.

Lemma 2.4. *(Supersolutions and Subsolutions). Let $u(x, y, z, t)$ be the solution of* (1.13) *with the initial value $u_0(x, y, z) \in BUC^1(\mathbb{R}^n)$. Suppose that the functions $v^+(x, t)$ and $v^-(x, t)$ solving the following problems*

$$\begin{cases} \frac{\partial}{\partial t} v^+(x,t) = \Delta_x v^+ + k|\nabla_x v^+|^2, \ x \in \mathbb{R}^{n-2}, \ t > 0, \\ v^+(x, 0) = v_0^+(x), \ x \in \mathbb{R}^{n-2}, \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial t} v^-(x,t) = \Delta_x v^- - k|\nabla_x v^-|^2, \ x \in \mathbb{R}^{n-2}, \ t > 0, \\ v^-(x, 0) = v_0^-(x), \ x \in \mathbb{R}^{n-2}, \end{cases}$$

respectively. Where $k > 0$ is the constant defined in Lemma 2.3. If

$$V(y, z + v_0^-(x)) \le u_0(x, y, z) \le V(y, z + v_0^+(x)), \ (x, y, z) \in \mathbb{R}^n,$$

then $u^+(x, y, z, t) := V(y, z + v^+(x, t))$ and $u^-(x, y, z, t) := V(y, z + v^-(x, t))$ are a supersolution and a subsolution of (1.13)*, respectively.*

Proof. We only show that $u^+(x, y, z, t)$ is a supersolution of (1.13), since the subsolution can be proved in a similar way.

Set

$$L[u] := u_t - \Delta u + (g'(u) + s) u_z - f(u).$$

Using the equality $-V_{yy} - V_{zz} + (g'(V) + s) V_z - f(V) = 0$ and Lemma 2.3, we have

$$\begin{aligned} L[u^+] =& u_t^+ - \Delta u^+ + (g'(u^+) + s) u_z^+ - f(u^+) \\ =& v_t^+ V_z - \sum_{i=1}^{n-2} \left(v_{x_i x_i}^+ V_z + \left(v_{x_i}^+\right)^2 V_{zz} \right) - V_{yy} - V_{zz} + (g'(V) + s) V_z - f(V) \\ =& v_t^+ V_z - \Delta_x v^+ V_z - \left|\nabla_x v^+\right|^2 V_{zz} \\ =& (kV_z - V_{zz}) \left|\nabla_x v^+\right|^2 \ge 0. \end{aligned}$$

This completes the proof. □

Now we prove Theorem 1.2.

Proof of Theorem 1.2. Let the function $v^+(x, t)$ as in Lemma 2.4. Then we have

$$u(x, y, z, t) \le V(y, z + v^+(x, t)) \le V(y, z) + \|V_z\|_{L^\infty(\mathbb{R}^2)} \cdot \sup_{x \in \mathbb{R}^{n-2}} \left|v^+(x, t)\right|.$$

From Lemma 2.1, we have

$$u(x, y, z, t) - V(y, z) \le \|V_z\|_{L^\infty(\mathbb{R}^2)} \cdot \sup_{x \in \mathbb{R}^{n-2}} \left|v^+(x, t)\right| \le Ct^{-\frac{n-2}{2}},$$

for some positive constant C. Similarly, we also obtain

$$u(x, y, z, t) - V(y, z) \ge -Ct^{-\frac{n-2}{2}}.$$

This completes the proof. □

Next, we give a proof of Proposition 1.3. Our argument is based again on Lemma 2.4.

Proof of Proposition 1.3. We only consider the case that $v_0 \leq 0,\ v_0 \not\equiv 0$. The right hand side inequality of (1.10) immediately follows from Theorem 1.2. To prove the left hand side inequality of (1.10), from Lemma 2.4, it suffices to show that the solution $v(x,t)$ of the problem

$$\begin{cases} v_t = \Delta_x v + k\,|\nabla_x v|^2\,,\ x \in \mathbb{R}^{n-2},\ t > 0, \\ v(x,0) = v_0(x),\ x \in \mathbb{R}^{n-2} \end{cases}$$

satisfies $v(0,t) \leq -C(1+t)^{-\frac{n-2}{2}}$ for some constant $C > 0$. Indeed, Lemma 2.4 gives that

$$\begin{aligned} u(0,0,0,t) \leq & V(0, v(0,t)) \\ \leq & V(0,0) + \min_{z \in \left[-\|v_0\|_{L^\infty(\mathbb{R}^{n-2})},0\right]} |V_z(0,z)| \cdot v(0,t) \\ \leq & V(0,0) - C'(1+t)^{-\frac{n-2}{2}},\ t \geq 0, \end{aligned}$$

where C' is a positive constant.

Similar to (2.2), the explicit expression for $v(x,t)$ is given by

$$v(x,t) = \frac{1}{k} \ln\left(\int_{\mathbb{R}^{n-2}} \Gamma(x-\eta,t) e^{k v_0(\eta)} \mathrm{d}\eta \right),$$

where $\Gamma(\xi,\tau)$ is the heat kernel on $\mathbb{R}^{n-2}$. Since $v_0 \leq 0$ and $v_0 \not\equiv 0$, there exists a constant $\delta > 0$ and a nonempty open set $D \subset \mathbb{R}^{n-2}$ such that $v_0 \leq -\delta$ for $x \in D$. Then we have

$$\begin{aligned} v(x,t) \leq & \frac{1}{k} \ln\left(1 - \int_D \Gamma(x-\eta,t)\left(1 - e^{-k\delta}\right) \mathrm{d}\eta \right) \\ \leq & \frac{1}{k} \ln\left(1 - |D| \left(1 - e^{-k\delta}\right) \cdot \min_{\eta \in D} \Gamma(x-\eta,t) \right) \\ \leq & -\frac{|D|}{k} \left(1 - e^{-k\delta}\right) \cdot \min_{\eta \in D} \Gamma(x-\eta,t), \end{aligned}$$

which implies $v(0,t) \leq -C(1+t)^{-\frac{n-2}{2}}$. This completes the proof. □

Now, we construct some new types of supersolutions and subsolutions.

Lemma 2.5. *Let $k > 0$ be defined as in Lemma 2.3. Then there exist constants $\delta_0 > 0$, $\beta > 0$ and $\sigma \geq 1$ such that, for any $\delta \in (0, \delta_0]$ and any functions $v^\pm(x,t)$ satisfying*

$$v_t^\pm = \Delta_x v^\pm \pm k \left|\nabla_x v^\pm\right|^2,$$

the functions defined by

$$\tilde{u}(x,y,z,t) := V\left(y, z + v^+(x,t) + \sigma\delta\left(1 - e^{-\beta t}\right)\right) + \delta e^{-\beta t},$$

$$\hat{u}(x,y,z,t) := V\left(y, z + v^-(x,t) - \sigma\delta\left(1 - e^{-\beta t}\right)\right) - \delta e^{-\beta t}$$

are a supersolution and a subsolution of (1.13)*, respectively.*

Proof. Using the equality $-V_{yy} - V_{zz} + (s + g'(V))\, V_z - f(V) = 0$ and Lemma 2.3, we have

$$
\begin{aligned}
L[\tilde{u}] =& \tilde{u}_t - \Delta\tilde{u} + (s + g'(\tilde{u}))\, \tilde{u}_z - f(\tilde{u}) \\
=& V_z v_t^+ + \sigma\delta\beta e^{-\beta t} V_z - \delta\beta e^{-\beta t} - \Delta_x v^+ V_z - \left|\nabla_x v^+\right|^2 V_{zz} - V_{yy} - V_{zz} \\
&+ \left(s + g'\left(V + \delta e^{-\beta t}\right)\right) V_z - f\left(V + \delta e^{-\beta t}\right) \\
=& \left(v_t^+ - \Delta_x v^+\right) V_z - \left|\nabla_x v^+\right|^2 V_{zz} + \sigma\delta\beta e^{-\beta t} V_z - \delta\beta e^{-\beta t} \\
&+ \left(g'\left(V + \delta e^{-\beta t}\right) - g'(V)\right) V_z + f(V) - f(V + \delta e^{-\beta t}) \\
=& \left(kV_z - V_{zz}\right) \left|\nabla_x v^+\right|^2 + \sigma\delta\beta e^{-\beta t} V_z - \delta\beta e^{-\beta t} \\
&+ \delta e^{-\beta t} \int_0^1 g''\left(V + \theta\delta e^{-\beta t}\right) \mathrm{d}\theta V_z - \delta e^{-\beta t} \int_0^1 f'\left(V + \theta\delta e^{-\beta t}\right) \mathrm{d}\theta \\
\geq& \delta e^{-\beta t} \left(\left(\sigma\beta + \int_0^1 g''\left(V + \theta\delta e^{-\beta t}\right) \mathrm{d}\theta\right) V_z - \beta - \int_0^1 f'\left(V + \theta\delta e^{-\beta t}\right) \mathrm{d}\theta\right).
\end{aligned}
$$

On the other hand, from (F), there exists a constant $\delta_0 (0 < \delta_0 < \frac{1}{8})$ such that

$$
- f'(r) \geq k_0 > 0 \ \text{ for } \ r \in [-2\delta_0, 2\delta_0] \cup [1 - 2\delta_0, 1 + 2\delta_0], \tag{2.7}
$$

where

$$
k_0 := \frac{1}{2} \min\{-f'(0), -f'(1)\} > 0.
$$

Let $\beta_3 > 0$ be defined by Lemma 2.2 with δ_0. For $\delta_0 \leq V\left(y, z + v^+(x,t) + \sigma\delta\left(1 - e^{-\beta t}\right)\right) \leq 1 - \delta_0$, from (1.2) and Lemma 2.2, we have

$$
\begin{aligned}
&\left(\sigma\beta + \int_0^1 g''\left(V + \theta\delta e^{-\beta t}\right) \mathrm{d}\theta\right) V_z - \beta - \int_0^1 f'\left(V + \theta\delta e^{-\beta t}\right) \mathrm{d}\theta \\
\geq& (\sigma\beta - l_2)\beta_3 - \beta - M \geq 0
\end{aligned}
$$

if we take

$$
\sigma > \frac{\beta + M}{\beta\beta_3} + \frac{l_2}{\beta},
$$

where

$$
M := \sup_{-1 \leq r \leq 2} |f'(r)|. \tag{2.8}
$$

For $V\left(y, z + v^+(x,t) + \sigma\delta\left(1 - e^{-\beta t}\right)\right) > 1 - \delta_0$ or $V\left(y, z + v^+(x,t) + \sigma\delta\left(1 - e^{-\beta t}\right)\right) < \delta_0$, following from Lemma 2.2 and (2.7) we have

$$
\left(\sigma\beta + \int_0^1 g''\left(V + \theta\delta e^{-\beta t}\right) \mathrm{d}\theta\right) V_z - \beta - \int_0^1 f'\left(V + \theta\delta e^{-\beta t}\right) \mathrm{d}\theta \geq k_0 - \beta \geq 0
$$

if we take

$$
\sigma > \frac{l_2}{\beta} \text{ and } 0 < \beta < k_0.
$$

Thus, if we take β small and σ large as

$$0 < \beta < k_0,\ \sigma > \max\left\{1, \frac{M+\beta}{\beta\beta_3} + \frac{l_2}{\beta}\right\},$$

we obtain $L[\tilde{u}] \geq 0$. Similarly, we can obtain $L[\hat{u}] \leq 0$. This completes the proof. □

To prove Theorem 1.1, we need another auxiliary lemma.

Lemma 2.6. *Assume that the initial value $u_0(x, y, z) \in BUC^1(\mathbb{R}^n)$ satisfies*

$$\lim_{R\to\infty} \sup_{|x|+|y|+|z|\geq R} |u_0(x, y, z) - V(y, z)| = 0. \tag{2.9}$$

Then, for any fixed $T > 0$, the solution $u(x, y, z, t)$ of (1.13) *satisfies*

$$\lim_{R\to\infty} \sup_{|x|+|y|+|z|\geq R} |u(x, y, z, T) - V(y, z)| = 0.$$

Proof. Define a function $w(x, y, z, t)$ by

$$\omega(x, y, z, t) := u(x, y, z, t) - V(y, z).$$

Then $\omega(x, y, z, t)$ solves the following Cauchy problem

$$\begin{cases} \omega_t = \Delta\omega - (s + g'(\omega + V))\,\omega_z + (f'(V + \theta_1\omega) - g''(V + \theta_2\omega)V_z)\,\omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^{n-2},\ (y, z) \in \mathbb{R}^2,\ t > 0, \\ \omega(x, y, z, 0) = u_0(x, y, z) - V(y, z),\ x \in \mathbb{R}^{n-2},\ (y, z) \in \mathbb{R}^2, \end{cases} \tag{2.10}$$

where $\theta_i(x, y, z, t)$ is the functions that satisfy $0 \leq \theta_i(x, y, z, t) \leq 1$, $i = 1, 2$.

In order to prove the lemma, it suffices to consider the case where $\omega(x, y, z, 0) \geq 0$ and the case where $\omega(x, y, z, 0) \leq 0$, since the general case follows easily from these special cases and the comparison principle. In the following, we always assume that $\omega(x, y, z, 0) \geq 0$.

Letting

$$g_1(x, y, z, t) = -\left(s + g'(\omega + V)\right),\ g_2(x, y, z, t) = f'(V + \theta_1\omega) - g''(V + \theta_2\omega)V_z,$$

then (2.10) can be rewritten as

$$\begin{cases} \omega_t = \Delta\omega + g_1(x, y, z, t)\omega_z + g_2(x, y, z, t)\omega,\ x \in \mathbb{R}^{n-2},\ (y, z) \in \mathbb{R}^2,\ t > 0, \\ \omega(x, y, z, 0) = u_0(x, y, z) - V(y, z),\ x \in \mathbb{R}^{n-2},\ (y, z) \in \mathbb{R}^2. \end{cases}$$

Because of $\omega(x, y, z, 0) \geq 0$, the maximum principle gives $\omega(x, y, z, t) \geq 0$. Then, by assumptions (F) and (G), there exists a constant $M_* > 0$ such that

$$\omega_t = \Delta\omega + g_1(x, y, z, t)\omega_z + g_2(x, y, z, t)\omega \leq \Delta\omega + g_1(x, y, z, t)\omega_z + M_*\omega.$$

Since $g \in C^{2+\gamma_0}(\mathbb{R})$ and $g_1(x, y, z, t)$ is bounded continuous in $\mathbb{R}^n \times \mathbb{R}^+$, Friedman [49, Chapter 9, Theorem 2] implies that the fundamental solution $\Gamma(x, y, z, \zeta, \eta_1, \eta_2, t, \tau)$ of the problem

$$\begin{cases} \tilde{\omega}_t = \Delta\tilde{\omega} + g_1(x, y, z, t)\tilde{\omega}_z + M_*\tilde{\omega},\ x \in \mathbb{R}^{n-2},\ (y, z) \in \mathbb{R}^2,\ t > 0, \\ \tilde{\omega}(x, y, z, 0) = u_0(x, y, z) - V(y, z),\ x \in \mathbb{R}^{n-2},\ (y, z) \in \mathbb{R}^2 \end{cases} \tag{2.11}$$

exists and satisfies

$$\Gamma(x,y,z,\zeta,\eta_1,\eta_2,t,\tau) \le \frac{c_1}{(t-\tau)^{\frac{n}{2}}} e^{-c_2 \frac{|x-\zeta|^2+(y-\eta_1)^2+(z-\eta_2)^2}{t-\tau}} \text{ for } 0 \le \tau < t \le T,$$

where c_1, c_2 are positive constants depending only on T. A solution of problem (2.11) can be expressed as

$$\tilde{\omega}(x,y,z,t) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^{n-2}} \Gamma(x,y,z,\zeta,\eta_1,\eta_2,t,0)\,(u_0(\zeta,\eta_1,\eta_2) - V(\eta_1,\eta_2))\,\mathrm{d}\zeta\mathrm{d}\eta_1\mathrm{d}\eta_2.$$

Then we obtain the estimate

$$\begin{aligned}
0 \le& \omega(x,y,z,t) \le \tilde{\omega}(x,y,z,t)\\
\le& \int_{\mathbb{R}^2}\int_{\mathbb{R}^{n-2}} \Gamma(x,y,z,\zeta,\eta_1,\eta_2,t,\tau)\,(u_0(\zeta,\eta_1,\eta_2) - V(\eta_1,\eta_2))\,\mathrm{d}\zeta\mathrm{d}\eta_1\mathrm{d}\eta_2\\
\le& c_1 \int_{\mathbb{R}^2}\int_{\mathbb{R}^{n-2}} e^{-c_2\left(|X|^2+Y^2+Z^2\right)}\\
&\times \Big(u_0(x+\sqrt{t}X, y+\sqrt{t}Y, z+\sqrt{t}Z) - V(y+\sqrt{t}Y, z+\sqrt{t}Z)\Big)\mathrm{d}X\mathrm{d}Y\mathrm{d}Z.
\end{aligned}$$

Since (2.9), then for any fixed $T > 0$, we have

$$\lim_{R\to\infty} \sup_{|x|+|y|+|z|\ge R} |u(x,y,z,T) - V(y,z)| = 0.$$

Similarly, we can treat the case that $\omega(x,y,x,0) \le 0$. This completes the proof. □

Next, we end this section by proving Theorem 1.1.

Proof of Theorem 1.1. We only show a lower estimate, since an upper estimate is obtained similarly. Taking constants $k > 0$ as in Lemma 2.3 and $\sigma \ge 1$ as in Lemma 2.5. Let a constant $\varepsilon > 0$ be arbitrarily fixed. Define a constant $\hat{\varepsilon} := \varepsilon/\left(2\|V_z\|_{L^\infty(\mathbb{R}^2)} + 1\right)$. Since $f(r) > 0$ for $r < 0$ by the assumption (F), we have that

$$\liminf_{t\to+\infty} u(x,y,z,t) = 0$$

by the comparison principle. Consequently, there exists a constant $T_1 > 0$ such that

$$u(x,y,z,T_1) \ge -\frac{\hat{\varepsilon}}{2\sigma},\ (x,y,z) \in \mathbb{R}^n.$$

Furthermore, Lemma 2.6 implies that there exists a constant $R > 0$ that satisfies

$$\sup_{|x|+|y|+|z|\ge R} |u(x,y,z,T_1) - V(y,z)| \le \frac{\hat{\varepsilon}}{\sigma}.$$

Then we can take a function $v_0(x) \ge 0$ that satisfies $\lim_{|x|\to\infty} v_0(x) = 0$ and

$$u(x,y,z,T_1) \ge V(y, z - v_0(x)) - \frac{\hat{\varepsilon}}{\sigma},\ (x,y,z) \in \mathbb{R}^n.$$

Assume that $v(x,t)$ solves the following Cauchy problem:

$$\begin{cases} v_t = \Delta_x v - k|\nabla_x v|^2, \ x \in \mathbb{R}^{n-2}, \ t > 0, \\ v(x,0) = v_0(x), \ x \in \mathbb{R}^{n-2}. \end{cases}$$

Then Lemma 2.1 implies that there exists a constant $T_2 > 0$ for which $v(x,t) \leq \hat{\varepsilon}$ holds true for $t \geq T_2$. Finally, by using the comparison principle and the solution constructed in Lemma 2.5, we obtain

$$\begin{aligned} u(x,y,z,t) \geq & V\left(y, z - v(x, t - T_1) - \hat{\varepsilon}\left(1 - e^{-\beta(t-T_1)}\right)\right) - \frac{\hat{\varepsilon}}{\sigma} e^{-\beta(t-T_1)} \\ \geq & V(y, z - 2\hat{\varepsilon}) - \hat{\varepsilon} \\ \geq & V(y,z) - \left(2\|V_z\|_{L^\infty(\mathbb{R}^2)} + 1\right)\hat{\varepsilon} \\ \geq & V(y,z) - \varepsilon \end{aligned}$$

for $t \geq T_1 + T_2$. This completes the proof of Theorem 1.1. □

3. Stability and unstability of V-shaped fronts in Case B

In this section, we consider Case B and give proofs of Theorems 1.5 and 1.6.

Proof of Theorem 1.5. Consider the following two-dimensional problem

$$\begin{cases} \frac{\partial u}{\partial t} + (g(u))_z = u_{yy} + u_{zz} + f(u), \quad y \in \mathbb{R}, \ z \in \mathbb{R}, \ t > 0, \\ u(y,z,0) = \hat{u}_0(y,z), \quad y \in \mathbb{R}, \ z \in \mathbb{R}. \end{cases} \tag{3.1}$$

Denote the solution by $u(y,z,t;\hat{u}_0)$. Following from (1.6) (see also [10]), we have that

$$\lim_{t\to\infty} \sup_{(y,z)\in\mathbb{R}^n} |u(y,z,t;\hat{u}_0) - V(y, z+st)| = 0. \tag{3.2}$$

It is clear that $u(y,z,t;\hat{u}_0)$ is also a solution of (1.1) with initial value $u(x,y,z,0) = \hat{u}_0(y,z)$. Since $V(y,z) \leq u_0(x,y,z) \leq \hat{u}_0(y,z)$ in $(x,y,z) \in \mathbb{R}^3$, it follows from the comparison principle that

$$V(y, z+st) \leq u(x,y,z,t;u_0) \leq u(y,z,t;\hat{u}_0) \quad \forall (x,y,z) \in \mathbb{R}^3, \ t > 0, \tag{3.3}$$

where $u(x,y,z,t;u_0)$ denotes the solution of (1.1). Finally, using (3.2) and (3.3) we obtain

$$\lim_{t\to\infty} \sup_{(x,y,z)\in\mathbb{R}^n} |u(x,y,z,t;u_0) - V(y, z+st)| = 0.$$

This completes the proof. □

We now give the proof of Theorem 1.6. This theorem implies that the V-shaped traveling fronts are not necessarily asymptotically stable if the initial perturbations are not decay to zero as $|x|+|y|+|z| \to \infty$ even they are very small. By using supersolutions and subsolutions constructed in the previous section and the following lemma, we construct a sequence of supersolutions and subsolutions that push the solution back and forth in the z-direction, then forcing the solution to oscillate permanently with non-decaying amplitude.

Lemma 3.1. (See [42, Lemma 3.1 and Lemma 3.2]). *Let $k > 0$ be defined as in Lemma 2.3 and $v^{\pm}(x,t)$ be the solutions to the problem*

$$\begin{cases} v_t^{\pm} = v_{xx}^{\pm} \pm kv^{\pm}, \ x \in \mathbb{R}, \ t > 0, \\ v(x,0) = v_0^{\pm}(x), \ x \in \mathbb{R}, \end{cases}$$

respectively. Suppose that the initial value $v_0^{\pm}(x)$ are bounded on $\mathbb{R}$ and satisfy

$$\begin{aligned} &v_0^{+}(x) \le \delta \, for \; x \in \mathbb{R}, \\ &v_0^{+}(x) \le -\delta \, for \; |x| \in [m!+1, (m+1)!-1] \end{aligned}$$

and

$$\begin{aligned} &v_0^{-}(x) \ge -\delta \, for \; x \in \mathbb{R}, \\ &v_0^{-}(x) \ge \delta \, for \; |x| \in [m!+1, (m+1)!-1] \end{aligned}$$

for some constant $\delta > 0$ and some integer $m \ge 2$, respectively. Then there exists a constant $C > 0$ depending only on δ and k such that

$$\sup_{|x| \le m!-1} v^{+}(x,T) \le -\delta + C \int_{|\zeta| \in \left[0, \frac{2}{\sqrt{m}}\right] \cup [\sqrt{m}, \infty)} e^{-\zeta^2} \mathrm{d}\zeta$$

and

$$\sup_{|x| \le m!-1} v^{-}(x,T) \ge \delta - C \int_{|\zeta| \in \left[0, \frac{2}{\sqrt{m}}\right] \cup [\sqrt{m}, \infty)} e^{-\zeta^2} \mathrm{d}\zeta,$$

respectively, where $T = m(m!)^2/4$.

Proof of Theorem 1.6. Set

$$I_m = [m!+1, (m+1)!-1], \ \tilde{I}_m = [0, m!] \cup [(m+1)!, \infty).$$

Define two sequences of smooth functions $\left\{v_{0,i}^{\pm}(x)\right\}_{i=1,2,\cdots}$ satisfying

$$|v_{0,i}^{+}(x)| \le \delta, \ x \in \mathbb{R} \ \text{ and } \ v_{0,i}^{+}(x) = \begin{cases} -\delta, & |x| \in I_{2i}, \\ \delta, & |x| \in \tilde{I}_{2i} \end{cases}$$

and

$$|v_{0,i}^{-}(x)| \le \delta, \ x \in \mathbb{R} \ \text{ and } \ v_{0,i}^{-}(x) = \begin{cases} \delta, & |x| \in I_{2i+1}, \\ -\delta, & |x| \in \tilde{I}_{2i+1}, \end{cases}$$

respectively. We also take a function $v_0^*(x) \in C^{\infty}(\mathbb{R})$ to satisfy

$$v_{0,i}^{-}(x) \le v_0^*(x) \le v_{0,i}^{+}(x) \ \text{ for all } \ i \ge 1.$$

Let $u^*(x,y,z,t)$ be the solution to (1.13) with $u^*(x,y,z,0) = V\left(y, z + v_0^*(x)\right)$ and $v_i^{+}(x,t)$ be the solution to the following Cauchy problem:

$$\begin{cases} (v_i^{+})_t = (v_i^{+})_{xx} + k\left(\left(v_i^{+}\right)_x\right)^2, \ x \in \mathbb{R}, \ t > 0, \\ v_i^{+}(x,0) = v_{0,i}^{+}(x), \ x \in \mathbb{R}. \end{cases}$$

From $-\delta \le v_0^*(x) \le v_{0,i}^+(x)$, we have

$$V(y,z-\delta) \le V\left(y,z+v_0^*(x)\right) \le V\left(y,z+v_{0,i}^+(x)\right),$$

then Lemma 2.4 implies that

$$V(y,z-\delta) \le u^*(x,y,z,t) \le V\left(y,z+v_i^+(x,t)\right).$$

Thus, it follows from Lemma 3.1 that

$$\begin{aligned} V(y,z-\delta) \le & \sup_{|x|\le(2i)!-1} u^*\left(x,y,z,t_{2i}\right) \\ \le & \sup_{|x|\le(2i)!-1} V\left(y,z+v_{0,i}^+\left(x,t_{2i}\right)\right) \\ \le & V(y,z-\delta) + \|V_z\|_{L^\infty(\mathbb{R}^2)} \cdot C \int_{|\zeta|\in\left[0,\frac{2}{\sqrt{2i}}\right]\cup\left[\sqrt{2i},\infty\right)} e^{-\zeta^2} \mathrm{d}\zeta, \end{aligned}$$

where $t_{2i} = (2i)\left((2i)!\right)^2/4$. This implies that

$$\lim_{i\to\infty} \sup_{(y,z)\in\mathbb{R}^2} \sup_{|x|\le(2i)!-1} |u^*(x,y,z,t_{2i}) - V(y,z-\delta)| = 0. \tag{3.4}$$

Similarly, again by using Lemma 3.1 and the equalities $v_{0,i}^-(x) \le v_0^*(x) \le \delta$ for $i = 1,2,3,\cdots$, we get

$$\begin{aligned} V(y,z+\delta) \ge & \sup_{|x|\le(2i+1)!-1} u^*\left(x,y,z,t_{2i+1}\right) \\ \ge & \sup_{|x|\le(2i+1)!-1} V\left(y,z+v_{0,i}^-\left(x,t_{2i+1}\right)\right) \\ \ge & V(y,z+\delta) - \|V_z\|_{L^\infty(\mathbb{R}^2)} \cdot C \int_{|\zeta|\in\left[0,\frac{2}{\sqrt{2i+1}}\right]\cup\left[\sqrt{2i+1},\infty\right)} e^{-\zeta^2} \mathrm{d}\zeta, \end{aligned}$$

where $t_{2i+1} = (2i+1)\left((2i+1)!\right)^2/4$. Then, we have

$$\lim_{i\to\infty} \sup_{(y,z)\in\mathbb{R}^2} \sup_{|x|\le(2i+1)!-1} |u^*(x,y,z,t_{2i+1}) - V(y,z+\delta)| = 0. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain the expected result. This completes the proof. □

4. Discussion

Recently, the study on multidimensional traveling fronts for scalar reaction-diffusion equations has attracted much attention. For example, V-formed curved fronts for two-dimensional spaces (see [10–12, 15–17, 20, 28, 33]), cylindrically symmetric traveling fronts (see [14, 30, 36]) and traveling fronts with pyramidal shapes (see [21,23–25,35,37,45]) in higher-dimensional spaces. For reaction-diffusion system, we refer to [26, 27, 29, 31, 32].

In [10], under the assumptions (F) and (G), we establish the existence and stability of V-shaped traveling fronts $V(y,z)$ of (1.1) in $\mathbb{R}^2$ for every direction $\theta \in (0,\pi/2)$ satisfying (C). On the basis of [10], we establish the multidimensional asymptotic stability of V-shaped traveling fronts $V(y, z)$ in $\mathbb{R}^n (n \geqslant 3)$ in this article. Here we would like to mention that the main method of this paper comes from Matano *et al.* [42] and Sheng *et al.* [28]. However, in order to overcome the difficulties caused by nonlinear convection, we have to choose a suitable space for the initial data u_0, namely $u_0 \in C^1(\mathbb{R}^2)$. Moreover, there seems to be no research on the multidimensional asymptotic stability of the nonplanar traveling fronts for a reaction-diffusion equation with nonlinear convection, even if one-dimensional traveling fronts (or planar traveling fronts).

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Conflict of interest

The authors declare no conflict of interest.

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