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Research article

An averaging principle for stochastic evolution equations with jumps and random time delays

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Abstract: This paper investigates an averaging principle for stochastic evolution equations with jumps and random time delays modulated by two-time-scale Markov switching processes in which both fast and slow components co-exist. We prove that there exists a limit process (averaged equation) being substantially simpler than that of the original one.

Keywords: averaging principle; stochastic evolution equations; jumps; random time delays; two-time-scale Markov switching processes.
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1. Introduction

The theory of evolution equations has attracted researchers' great interest stimulated by its numerous practical applications in the areas such as physics, population dynamics, electrical engineering, medicine biology, ecology [1, 7]. In fact, much evidence has been gathered that Poisson jumps are ubiquitous in many fields of science. By now, it is well established that stochastic systems driven by Poisson jumps are more suitable for capturing sudden bursty fluctuations, some large moves and unpredictable events than classical diffusion models [2, 4, 13]. Therefore, it is necessary to study stochastic evolution equation (SEEs) driven by jumps. For example, in the areas of population dynamics, Rathinasamy, Yin and Yasodha [20] discussed a stochastic age-dependent population equation with Markovian switching

$$\begin{cases} \frac{\partial P}{\partial t} + \frac{\partial P}{\partial a} = -\mu(t,a)P + f(r(t),P) + g(r(t),P)\frac{dW(t)}{dt} + h(r(t),P)\frac{dN(t)}{dt}, & (t,a) \in Q, \\ P(0,a) = P_0(a), r(0) = r_0, \quad a \in [0,\tilde{A}], \\ P(t,0) = \int_0^{\tilde{A}} \beta(t,a)P(t,a)da, \quad t \in [0,T], \end{cases} \tag{1.1}$$

where $T > 0, \tilde{A} > 0, Q := (0,T) \times (0,\tilde{A}), P = P(t,a)$ is the population density of age a at time t, $r(t) \geq 0$ is a continuous-time Markov chain, $\mu(r(t),a)$ denotes the mortality rate of age a at time t, $\beta(t,a)$ denotes the fertility rate of females of age a at time t and $f(\cdot,\cdot)$ denotes the effects of the external environment on the population system. $g(\cdot,\cdot)$ is a diffusion coefficient, W is a Brownian motion. $h(\cdot,\cdot)$ is a jump coefficient, N is a scalar Poisson process with intensity $\tilde{\lambda}$.

This paper considers SEEs with jumps and random time delays modulated by Markov chains. To illustrate, consider the following SEEs with random time delays

$$dX(t) = (AX(t) + f(X(t), X(t - r(t))))dt + \int_U h(X(t), z)\tilde{N}(dt, dz), \tag{1.2}$$

where $U \in \mathcal{B}_\sigma(\mathbb{K} - \{0\})$ which will be defined in next section, $r(t) \geq 0$ is a continuous-time Markov chain in a finite state space $\mathbb{S} := \{r_1, \ldots, r_n\}$ with generator $\tilde{\mathbb{Q}} = (\tilde{q}_{ij}) \in \mathbb{R}^{n\times n}$. Without loss of generality, assume that $r_1 < r_2 < \cdots < r_n$. Recall that the generator $\tilde{\mathbb{Q}}$ is weakly irreducible, if the system of equation

$$\begin{cases} \tilde{\nu}\tilde{\mathbb{Q}} = 0, \\ \sum_{i=1}^{n} \tilde{\nu}_i = 1, \end{cases} \tag{1.3}$$

has a unique solution $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2, \ldots, \tilde{\nu}_n)$ satisfying $\tilde{\nu}_i \geqslant 0$ for all $i = 1, \ldots, n$. The solution $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2, \ldots, \tilde{\nu}_n)$ is termed as the stationary measure.

This system (1.2) is a switching system among the following n fixed delay subsystems

$$dX_i(t) = (AX_i(t) + f(X_i(t), X_i(t - r_i)))dt + \int_U h(X_i(t), z)\tilde{N}(dt, dz), \quad i = 1, 2 \cdots n. \tag{1.4}$$

The random switching is governed by the Markov chain $r(t) \geq 0$. In our setup, the Markov chain has both fast and slow motions and involves strong and weak interactions. To reflect the fast and slow motions of the Markov chain, we introduce a small parameter $\varepsilon > 0$ and rewrite the Markov chain $r(t)$ as $r^\varepsilon(t)$ and the generator $\tilde{\mathbb{Q}}$ as $\mathbb{Q}^\varepsilon$. Thus, the Markov chain displays two-time-scales. That is, a usual running time t and a stretched (fast) time t/ε. Suppose that the generator of the Markov chain is given by

$$\mathbb{Q}^\varepsilon := \frac{\mathbb{Q}}{\varepsilon} + \widehat{\mathbb{Q}},$$

where both $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$ are generators of suitable continuous-time Markov chains such that $\mathbb{Q}/\varepsilon$ and $\widehat{\mathbb{Q}}$ represent the fast-varying and slowly-changing parts, respectively. Throughout the paper, we assume that $\mathbb{Q}$ is weakly irreducible in the sense defined in (1.3). Denote the stationary measure associated with $\mathbb{Q}$ as $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$.

Now, we consider the following SEEs driven by jumps with random time delays modulated by two-time-scale Markov switching processes, that is, we can rewrite Eq.(1.2) as

$$dX^\varepsilon(t) = (AX^\varepsilon(t) + f(X^\varepsilon(t), X^\varepsilon(t - r^\varepsilon(t))))dt + \int_U h(X^\varepsilon(t), z)\tilde{N}(dt, dz), \quad t \in [0, T], \tag{1.5}$$

where $X_0 = \psi(\cdot) \in D^b_{\mathcal{F}_0}([-r_n, 0]; \mathbb{H})$, $\psi(\cdot)$ satisfies

$$\mathbb{E}[\sup_{-r_n \leq s \leq 0} |\psi(s)|^2_{\mathbb{H}}] < +\infty, \tag{1.6}$$

and the Lipschitz condition, namely, there exists a constant $K_1 > 0$ such that

$$|\psi(t_1) - \psi(t_2)| \le K_1 |t_1 - t_2|. \tag{1.7}$$

An interesting question is: Whether or when the averaging principle still holds for the overall system consisting of n fixed delay subsystems, can an averaged system be obtained? Such a question is the main concern of this paper. It is well known that there is an extensive literature on averaging principles for singularly perturbed stochastic (partial) differential equations [5, 6, 10, 16, 22, 24, 25]. Cerrai and Freidlin [6] presented an averaged result for stochastic parabolic equations. Bréhier [5] derived explicit convergence rates in strong and weak convergence for averaging of stochastic parabolic equations. Pei et al. [14, 17] obtained an averaging principle for fast-slow stochastic partial differential equations (SPDEs) in which the slow variable is driven by a fractional Brownian motion (fBm) and the fast variable is driven by an additive Brownian motion. Xu, Miao and Liu [23] obtained an averaging principle for two-time-scale SPDEs driven by Poisson random measures in the sense of mean-square. Xu and Liu [22] proved a stochastic averaging principle for two-time-scale jump-diffusion SDEs under the non-Lipschitz coefficients. Pei, Xu and Wu [15] studied the existence, uniqueness and averaging principles for two-time-scales hyperbolic-parabolic equations driven by Poisson random measures. Bao, Yin and Yuan [3] obtained averaging principles for SPDEs driven by α-stable noises with two-time-scale Markov switching. As for random delays cases, Wu, Yin, and Wang [21] considered a class of nonlinear systems with random time delays modeled by a two-time-scale Markov chain. Pei, Xu, Yin and Zhang [19] examined averaging principles for functional SPDEs driven by a fBm modulated by two-time-scale Markov switching process. Pei, Xu and Yin [18] considered SPDEs driven by fBms with random time delays modulated by two-time-scale Markov switching processes.

Therefore, it is quite natural to ask whether or when an averaging principle for SEEs driven by jumps with random time delays still holds. On the one hand, to the authors' knowledge, the averaging principle for SEEs driven by jumps with random time delays modulated by two-time-scale Markov switching processes has not been considered. Therefore, it is necessary to obtain the averaging principle for SEEs driven by jumps with random time delays. On the other hand, since introducing randomly switching time delays, it is much more difficult to make system analysis and control design. However, in our setup, we can employ a two-time scale framework to describe the dynamic relationships of the switching time delay and the underlying system dynamics. Thus, it will be very useful to consider delay-dependent stability by the two-time-scale approach whose main advantage is that the stability can be examined by an "average" system as a bridge. Thus, in this paper, my aim is to obtain the averaging principle for SEEs driven by jumps with random time delays. Then, in forthcoming work, it will be possible to consider delay-dependent stability involving jumps.

The rest of the paper is arranged as follows. Section 2 presents preliminary results that are needed in the subsequent sections. In Section 3, we established an averaging principle for the case of SEEs driven by jumps with random time delays.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets). Let $\mathbb{H}, \mathbb{K}$ be two separable Hilbert spaces and we denote by $\langle \cdot \rangle_{\mathbb{H}}, \langle \cdot \rangle_{\mathbb{K}}$ their inner products and by $\|\cdot\|_{\mathbb{H}}, \|\cdot\|_{\mathbb{K}}$ their norms, respectively. To proceed, given $\tau > 0$ and $D_\tau(\mathbb{H}) := D([-\tau, 0]; \mathbb{H})$ denotes the family of all right-continuous functions with left-hand limits φ from $[-\tau, 0]$ to $\mathbb{H}$ with the notation $\|\varphi\|_{D_\tau} := \sup_{-\tau \le s \le 0} \|\varphi(s)\|_{\mathbb{H}}$. We use $D^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{H})$ to denote the family of all almost surely bounded $\mathcal{F}_0$-measurable, $D([-\tau, 0]; \mathbb{H})$-valued random variables. Define X_t by $X_t(\theta) = X(t + \theta), \theta \in [-\tau, 0]$. Let $\mathcal{B}_\sigma(\mathbb{H})$ denote the Borel σ-algebra of $\mathbb{H}$.

Let $p(t), t \ge 0$ be a $\mathbb{K}$-valued, σ-finite stationary Poisson point process with characteristic measure ν on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \ge 0}, \mathbb{P})$. The counting random measure N_p is defined by

$$N_p((t_1, t_2] \times \Lambda)(\omega) := \sum_{t_1 < s \le t_2} I_\Lambda(p(s)),$$

for any $\Lambda \in \mathcal{B}_\sigma(\mathbb{K})$ is the Poisson random measure associated to the Poisson point process $p(t)$. Let $\nu(\cdot) = \mathbb{E}N((0, 1] \times \cdot)$. Then we defined the compensated Poisson measure $\tilde{N}$ associated to the Poisson point process $p(t)$ by

$$\tilde{N}(dt, dz) := N_p(dt, dz) - \nu(dz)dt,$$

ν is a σ-finite measure and is called the Lévy measure.

Assume that $\{S(t), t \ge 0\}$ is an analytic semigroup with its infinitesimal generator A, then it is possible under some circumstances to define the fractional power $-A$ for any $\theta \in (0, 1)$ which is a closed linear operator with its domain $D((-A)^\theta)$. In this work, we always use the same symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises. Then, we assume that the operator A satisfies the following condition.

(H1) A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \ge 0\}$ in $\mathbb{H}$ such that the following inequality holds:

$$\|S(t)\| \le Me^{-\lambda t}, \quad t \ge 0,$$

for some constants $M \ge 1$ and $\lambda > 0$.

Lemma 2.1. *Suppose that (H1) holds. Then for any $\theta \in (0, 1)$, the following equality holds:*

$$S(t)(-A)^\theta x = (-A)^\theta S(t)x, \quad x \in D((-A)^{-\theta}),$$

and there exists a positive constant $M_\theta > 0$ such that

$$\begin{aligned} \|(-A)^\theta S(t)\| &\le M_\theta t^{-\theta} e^{-\lambda t}, \quad t > 0, \\ \|(S(t) - \mathbf{I})(-A)^{-\theta}\| &\le M_\theta t^\theta, \quad t > 0, \end{aligned}$$

where $\mathbf{I}$ *is the identity operator.*

3. Stochastic evolution equations with jumps and random time delays

Throughout this paper, we assume that the following conditions are satisfied.

(H2) There exist constants $K_2, K_3 > 0$, for any $x_1, y_1, x_2, y_2 \in \mathbb{H}$ and $q \ge 2$, such that

$$\begin{aligned} &|f(x_1, y_1) - f(x_2, y_2)|^2_{\mathbb{H}} \le K_2(|x_1 - x_2|^2_{\mathbb{H}} + |y_1 - y_2|^2_{\mathbb{H}}), \\ &\int_U |h(x_1, z) - h(x_2, z)|^q_{\mathbb{H}} \nu(dz) \le K_3 |x_1 - x_2|^q_{\mathbb{H}}. \end{aligned}$$

(H3) There exist constants $K_4, K_5 > 0$, for any $x, y \in \mathbb{H}$ and $q \geq 2$, such that

$$\begin{aligned} |f(x,y)|_{\mathbb{H}}^2 &\leq K_4(1+|x|_{\mathbb{H}}^2+|y|_{\mathbb{H}}^2), \\ \int_U |h(x,z)|_{\mathbb{H}}^q \nu(dz) &\leq K_5(1+|x|_{\mathbb{H}}^q). \end{aligned}$$

To proceed, we show that as $\varepsilon \to 0$, X^ε to Eq.(1.5) has a limit $\bar{X}$ which satisfies the following SEEs that an averaged system

$$d\bar{X}(t) = \Big(A\bar{X}(t) + \sum_{i=1}^{n} f(\bar{X}(t), \bar{X}(t-r_i))v_i\Big)dt + \int_U h(\bar{X}(t), z)\tilde{N}(dt, dz), \tag{3.1}$$

in the mild sense, with initial value $\bar{X}_0 = \psi(\cdot) \in D^b_{\mathcal{F}_0}([-r_n, 0]; \mathbb{H})$. Here, as $\varepsilon \to 0$, $\sum_{i=1}^n f(\bar{X}(t), \bar{X}(t-r_i))v_i$ is the averaged drift term of $f(X^\varepsilon(t), X^\varepsilon(t - r^\varepsilon(t)))$ in (1.5) based on the the stationary measure v associated with the generator $\mathbb{Q}$.

Following the discretization techniques inspired by Khasminskii in [10], we formulate our main result of averaging principle.

Theorem 3.1. *Let the condition* (H1)–(H3) *hold and assume further that* $\psi(0) \in \mathcal{D}((-A)^\theta)$ *with* $\theta \in (0, \frac{1}{2})$. *Then, for sufficiently small* $\varepsilon \in (0,1)$, *we have*

$$\mathbb{E}[\sup_{t\in[0,T]} |X^\varepsilon(t) - \bar{X}(t)|_{\mathbb{H}}^{2p}] \leq C_{p,n}((\varepsilon^\rho)^{2p\theta} + \varepsilon^{\frac{\beta-\rho}{2}}) \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

where $C_{p,n} > 0$ *is a constant which is independent of* ε *and* $p \in [1, \frac{1}{2\theta})$, $0 < \rho < \beta < 1$.

In this section, we firstly prove the existence and uniqueness of mild solution to Eq. (1.5). Then, we give some estimations on the solution process. Finally, we prove the averaging principe result (Theorem 3.1).

3.1. The existence and uniqueness of mild solution and some estimations on the solution process

Definition 3.2. (cf. [11] for example) An $\mathbb{H}$-valued process $X^\varepsilon(t), t \in [0,T]$ is called a mild solution of Eq (1.5), if

1. $X^\varepsilon(t)$ is $\mathcal{F}_t$-adapted and cadlag;
2. $X^\varepsilon(t)$ satisfies the following integral equation:

$$\begin{aligned} X^\varepsilon(t) &= S(t)\psi(0) + \int_0^t S(t-s)f(X^\varepsilon(s), X^\varepsilon(s - r^\varepsilon(s)))ds \\ &\quad + \int_0^t \int_U S(t-s)h(\bar{X}(s), z)\tilde{N}(ds, dz), \quad t \in [0,T], \end{aligned} \tag{3.2}$$

where $r^\varepsilon(t)$ is a two-time-scales Markov chain taking values in $\mathbb{S}$.

By the classical method [11, 12], these assumptions (H1)–(H3) can guarantee the existence and uniqueness of the mild solution to Eq (1.5).

By Definition 3.2, the integral form of Eq (3.1) can be written as

$$\begin{aligned}\bar{X}(t) &= S(t)\psi(0) + \sum_{i=1}^{n}\int_0^t S(t-s)f(\bar{X}(s),\bar{X}(s-r_i))v_i ds \\ &\quad + \int_0^t\int_U S(t-s)h(\bar{X}(s),z)\tilde{N}(ds,dz), \quad t\in[0,T], \\ \bar{X}(s) &= \psi(s), \quad s\in[-r_n,0], \quad r_n>0, \end{aligned} \tag{3.3}$$

where $r_i \in \mathbb{S}, i = 1, 2, \ldots, n$.

Lemma 3.3. *Let* (H1)–(H3) *hold. Then, for* $p \geq 1$*, we have*

$$\mathbb{E}[\sup_{t\in[-r_n,T]}(|\bar{X}(t)|_{\mathbb{H}}^{2p} + |X^{\varepsilon}(t)|_{\mathbb{H}}^{2p})] \leq C_{p,n},$$

where $C_{p,n} > 0$ *is a constant which is independent of* ε*.*

Proof: Taking expectation and employing the elementary inequality, we get from Eq (3.3) that

$$\begin{aligned}\mathbb{E}[\sup_{t\in[-r_n,T]}|\bar{X}(t)|_{\mathbb{H}}^{2p}] &\leq C_p\mathbb{E}[\sup_{-r_n\leq t\leq 0}|\psi(t)|_{\mathbb{H}}^{2p}] + C_p\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\sum_{i=1}^{n}\int_0^t S(t-s)f(\bar{X}(s),\bar{X}(s-r_i))v_i ds\Big|_{\mathbb{H}}^{2p}\Big] \\ &\quad + C_p\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\int_U S(t-s)h(X^{\varepsilon}(s),z)\tilde{N}(ds,dz)\Big|_{\mathbb{H}}^{2p}\Big] \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Firstly, for I_1, by (1.6), it is easy to know there exists a constant $C_{p,n} > 0$ such that

$$I_1 \leq C_{p,n}.$$

Next, for I_2, we have

$$\begin{aligned} I_2 &\leq C_{p,n}\sum_{i=1}^{n}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t S(t-s)f(\bar{X}(s),\bar{X}(s-r_i))v_i ds\Big|_{\mathbb{H}}^{2p}\Big] \\ &\leq C_{p,n}\sum_{i=1}^{n}\int_0^T \mathbb{E}[|f(\bar{X}(s),\bar{X}(s-r_i))|_{\mathbb{H}}^{2p}]ds \\ &\leq C_{p,n}\sum_{i=1}^{n}\int_0^T (1+\mathbb{E}[|\bar{X}(s)|_{\mathbb{H}}^{2p} + |\bar{X}(s-r_i)|_{\mathbb{H}}^{2p}])ds \\ &\leq C_{p,n} + C_{p,n}\int_0^T \mathbb{E}[\sup_{s_1\in[-r_n,s]}|\bar{X}(s_1)|_{\mathbb{H}}^{2p}]ds.\end{aligned}$$

Then, for I_3, by [8, Lemma 3.1], we have

$$\begin{aligned} I_3 &= C_p\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\int_U S(t-s)h(\bar{X}(s),z)\tilde{N}(ds,dz)\Big|_{\mathbb{H}}^{2p}\Big] \\ &\leq C_p\mathbb{E}\Big[\Big(\int_0^T\int_U |h(\bar{X}(s),z)|_{\mathbb{H}}^2 \nu(dz)ds\Big)^p\Big] + C_p\mathbb{E}\Big[\int_0^t\int_U |h(\bar{X}(s),z)|_{\mathbb{H}}^{2p}\nu(dz)ds\Big]\end{aligned}$$

$$\leq \quad C_p + C_p \int_0^T \mathbb{E}[\sup_{s_1\in[-r_n,s]} |\bar{X}(s_1)|_{\mathbb{H}}^{2p}]ds.$$

Therefore, by Gromwall's inequality, we have

$$\mathbb{E}[\sup_{t\in[-r_n,T]} |\bar{X}(t)|_{\mathbb{H}}^{2p}] \leq C_{p,n},$$

where $C_{p,n} > 0$ is a constant.

Finally, with the same proof, one obtains bounds on X^ε, uniformly with respect to ε, i.e.

$$\mathbb{E}[\sup_{t\in[-r_n,T]} |X^\varepsilon(t)|_{\mathbb{H}}^{2p}] \leq C_{p,n},$$

where $C_{p,n} > 0$ is a constant which is independent of ε. This completes the proof. □

Lemma 3.4. *Let* (H1)–(H3) *hold and assume further that* $\psi(0) \in \mathcal{D}((-A)^\theta)$ *with* $\theta \in (0, \frac{1}{2})$. *Then, for* $0 \leq t < t+\delta \leq T, \delta \in (0,1)$, $p \in [1, \frac{1}{2\theta})$, *we have*

$$\sup_{t\in[0,T]} \mathbb{E}[|\bar{X}(t+\delta) - \bar{X}(t)|_{\mathbb{H}}^{2p}] = C_{p,n}\delta^{2p\theta}.$$

Proof: From Eq (1.5) and Eq (3.1), one has

$$\begin{aligned}
\mathbb{E}[|\bar{X}(t+\delta) - \bar{X}(t)|_{\mathbb{H}}^{2p}] \quad &\leq \quad C_p\mathbb{E}[|(S(\delta) - \mathbf{I})S(t)\psi(0)|_{\mathbb{H}}^{2p}] \\
&\quad + C_p\mathbb{E}\Big[\Big|\sum_{i=1}^n \int_0^{t+\delta} S(t+\delta-s)f(\bar{X}(s), \bar{X}(s-r_i))v_i ds \\
&\quad - \sum_{i=1}^n \int_0^t S(t-s)f(\bar{X}(s), \bar{X}(s-r_i))v_i ds\Big|_{\mathbb{H}}^{2p}\Big] \\
&\quad + C_p\mathbb{E}\Big[\Big|\int_0^{t+\delta}\int_U S(t+\delta-s)h(\bar{X}(s), z)\tilde{N}(ds, dz) \\
&\quad - \int_0^t \int_U S(t-s)h(\bar{X}(s), z)\tilde{N}(ds, dz)\Big|_{\mathbb{H}}^{2p}\Big] \\
&=: \quad J_1 + J_2 + J_3.
\end{aligned}$$

Firstly, for J_1, by Lemma 2.1 and the condition $\psi(0) \in \mathcal{D}((-A)^\theta)$, one has

$$\begin{aligned}
J_1 \quad &\leq \quad C_p\|(S(\delta) - \mathbf{I})(-A)^{-\theta}\|^{2p}\|S(t)\|^{2p}\mathbb{E}[|(-A)^\theta\psi(0)|_{\mathbb{H}}^{2p}] \\
&\leq \quad C_p\delta^{2p\theta}.
\end{aligned} \tag{3.4}$$

Then, for J_2, we have

$$\begin{aligned}
J_2 \quad &\leq \quad C_{p,n}\sum_{i=1}^n \mathbb{E}\Big[\Big|\int_0^{t+\delta} S(t+\delta-s)f(\bar{X}(s), \bar{X}(s-r_i))ds - \int_0^t S(t-s)f(\bar{X}(s), \bar{X}(s-r_i))ds\Big|_{\mathbb{H}}^{2p}\Big] \\
&\leq \quad C_{p,n}\sum_{i=1}^n \mathbb{E}\Big[\Big|\int_0^t (S(\delta) - \mathbf{I})S(t-s)f(\bar{X}(s), \bar{X}(s-r_i))ds\Big|_{\mathbb{H}}^{2p}\Big] \\
&\quad + C_{p,n}\sum_{i=1}^n \mathbb{E}\Big[\Big|\int_t^{t+\delta} S(t+\delta-s)f(\bar{X}(s), \bar{X}(s-r_i))ds\Big|_{\mathbb{H}}^{2p}\Big]
\end{aligned}$$

$$=: \quad J_{21} + J_{22}.$$

Therefore, by Lemma 2.1 and Lemma 3.3, we have

$$\begin{aligned}
J_{21} &\le C_{p,n}\Big(\int_0^t (\|(S(\delta)-\mathbf{I})(-A)^{-\theta}\|\|S(t-s)(-A)^{\theta}\|)ds\Big)^{2p-1} \\
&\quad \times \sum_{i=1}^{n}\int_0^t (\|(S(\delta)-\mathbf{I})(-A)^{-\theta}\|\|S(t-s)(-A)^{\theta}\|)\mathbb{E}[|f(\bar{X}(s),\bar{X}(s-r_i))|_{\mathbb{H}}^{2p}]ds, \\
&\le C_{p,n}M_{\theta}^{2p}\delta^{(2p-1)\theta}\sum_{i=1}^{n}\int_0^t (t-s)^{-\theta}\delta^{\theta}(1+\mathbb{E}[|\bar{X}(s)|_{\mathbb{H}}^{2p}]+\mathbb{E}[|\bar{X}(s-r_i)|_{\mathbb{H}}^{2p}])ds \\
&\le C_{p,n}M_{\theta}^{2p}\delta^{2p\theta},
\end{aligned} \tag{3.5}$$

For J_{22}, one has

$$\begin{aligned}
J_{22} &\le C_{p,n}\sum_{i=1}^{n}\mathbb{E}\Big[\Big|\int_t^{t+\delta} S(t+\delta-s)f(\bar{X}(s),\bar{X}(s-r_i))v_i ds\Big|_{\mathbb{H}}^{2p}\Big] \\
&\le C_{p,n}\delta^{2p-1}\sum_{i=1}^{n}\int_t^{t+\delta}\mathbb{E}[|f(\bar{X}(s),\bar{X}(s-r_i))|_{\mathbb{H}}^{2p}]ds \\
&\le C_{p,n}\delta^{2p-1}\int_t^{t+\delta}(1+\mathbb{E}[\sup_{s_1\in[-r_n,s]}|\bar{X}(s_1)|_{\mathbb{H}}^{2p}])ds \\
&\le C_{p,n}\delta^{2p}.
\end{aligned}$$

Then, for J_3, by [8, Lemma 3.1], Lemma 2.1 and Lemma 3.3, we have

$$\begin{aligned}
J_3 &= C_p\mathbb{E}\Big[\Big|\int_0^{t+\delta}\int_U S(t+\delta-s)h(\bar{X}(s),z)\tilde{N}(ds,dz) - \int_0^t\int_U S(t-s)h(\bar{X}(s),z)\tilde{N}(ds,dz)\Big|_{\mathbb{H}}^{2p}\Big] \\
&\le C_p\mathbb{E}\Big[\Big|\int_0^t\int_U (S(t+\delta-s)-S(t-s))h(\bar{X}(s),z)\tilde{N}(ds,dz)\Big|_{\mathbb{H}}^{2p}\Big] \\
&\quad + C_p\mathbb{E}\Big[\Big|\int_t^{t+\delta}\int_U S(t+\delta-s)h(\bar{X}(s),z)\tilde{N}(ds,dz)\Big|_{\mathbb{H}}^{2p}\Big] \\
&\le C_p\mathbb{E}\Big(\int_0^t \|(S(\delta)-\mathbf{I})S(t-s)\|^2\int_U |h(\bar{X}(s),z)|_{\mathbb{H}}^2\nu(dz)ds\Big)^p \\
&\quad + C_p\int_0^t \|(S(\delta)-\mathbf{I})S(t-s)\|^{2p}\int_U \mathbb{E}[|h(\bar{X}(s),z)|_{\mathbb{H}}^{2p}]\nu(dz)ds \\
&\quad + C_p\mathbb{E}\Big[\Big(\int_t^{t+\delta}\|S(t+\delta-s)\|^2\int_U |h(\bar{X}(s),z)|_{\mathbb{H}}^2\nu(dz)ds\Big)^p\Big] \\
&\quad + C_p\int_t^{t+\delta}\|S(t+\delta-s)\|^{2p}\int_U \mathbb{E}[|h(\bar{X}(s),z)|_{\mathbb{H}}^{2p}]\nu(dz)ds \\
&\le C_{p,n}\delta^{2p\theta}.
\end{aligned}$$

Here, we need the condition $p \in [1, \frac{1}{2\theta})$.

Finally, putting above results together, it yields that

$$\sup_{t\in[0,T]}\mathbb{E}[|\bar{X}(t+\delta)-\bar{X}(t)|_{\mathbb{H}}^{2p}] = C_{p,n}\delta^{2p\theta}.$$

This completes the proof. □

3.2. *Proof of main result*

By Hölder's inequality and (H2), it follows from (1.5) and (3.1) that

$$\begin{aligned}
\mathbb{E}[\sup_{t\in[0,T]}|X^{\varepsilon}(t)-\bar{X}(t)|_{\mathbb{H}}^{2p}] &\le C_{p,n}\sum_{i=1}^{n}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t S(t-s)(f(X^{\varepsilon}(s),X^{\varepsilon}(s-r_i))\\
&\quad -f(\bar{X}(s),\bar{X}(s-r_i)))\mathbf{I}_{\{r^{\varepsilon}(s)=r_i\}}ds\Big|_{\mathbb{H}}^{2p}\Big]\\
&\quad +C_{p,n}\sum_{i=1}^{n}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t S(t-s)f(\bar{X}(s),\bar{X}(s-r_i))\{\mathbf{I}_{\{r^{\varepsilon}(s)=r_i\}}-\nu_i\}ds\Big|_{\mathbb{H}}^{2p}\Big]\\
&\quad +C_p\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\int_U S(t-s)h(X^{\varepsilon}(s),z)\tilde{N}(ds,dz)\\
&\quad -\int_0^t\int_U S(t-s)h(\bar{X}(s),z)\tilde{N}(ds,dz)\Big|_{\mathbb{H}}^{2p}\Big]\\
&=: C_{p,n}(\Psi_1+\sum_{i=1}^{n}\Psi_{2i})+C_p\Psi_3.
\end{aligned}$$

Now, let us evaluate Ψ_1,Ψ_3, by (H2), we have

$$\begin{aligned}
\Psi_1 &= \sum_{i=1}^{n}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t S(t-s)(f(X^{\varepsilon}(s),X^{\varepsilon}(s-r_i))-f(\bar{X}(s),\bar{X}(s-r_i)))\mathbf{I}_{\{r^{\varepsilon}(s)=r_i\}}ds\Big|_{\mathbb{H}}^{2p}\Big]\\
&\le C_p\sum_{i=1}^{n}\int_0^T\mathbb{E}[|f(X^{\varepsilon}(s),X^{\varepsilon}(s-r_i))-f(\bar{X}(s),\bar{X}(s-r_i))|_{\mathbb{H}}^{2p}]ds\\
&\le C_p\sum_{i=1}^{n}\int_0^T\mathbb{E}[|X^{\varepsilon}(s)-\bar{X}(s)|_{\mathbb{H}}^{2p}+|X^{\varepsilon}(s-r_i)-\bar{X}(s-r_i)|_{\mathbb{H}}^{2p}]ds\\
&\le C_{p,n}\int_0^T\mathbb{E}[|X^{\varepsilon}(s)-\bar{X}(s)|_{\mathbb{H}}^{2p}]ds.
\end{aligned}$$

For Ψ_3, using [8, Lemma 3.1], we get

$$\begin{aligned}
\Psi_3 &\le C_p\mathbb{E}\Big[\Big(\int_0^T\int_U|h(X^{\varepsilon}(s),z)-h(\bar{X}(s),z)|_{\mathbb{H}}^{2}\nu(dz)ds\Big)^p\Big]\\
&\quad +C_p\mathbb{E}\Big[\int_0^T\int_U|h(X^{\varepsilon}(s),z)-h(\bar{X}(s),z)|_{\mathbb{H}}^{2p}\nu(dz)ds\Big]\\
&\le C_p\int_0^T\mathbb{E}[|X^{\varepsilon}(s)-\bar{X}(s)|_{\mathbb{H}}^{2p}]ds.
\end{aligned}$$

Now, let $\lfloor s\rfloor:=\lfloor\frac{s}{\varepsilon^{\rho}}\rfloor\varepsilon^{\rho}, 0<\varepsilon^{\rho}<1,\rho>0$ with $\lfloor\frac{s}{\varepsilon^{\rho}}\rfloor$ denoting the integer part of $\frac{s}{\varepsilon^{\rho}}$. Then, one has

$$\begin{aligned}
\Psi_{2i} &\le C_p\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t S(t-s)(\mathbf{I}-S(s-\lfloor s\rfloor))f(\bar{X}(s),\bar{X}(s-r_i))\mathbf{I}_{\{r^{\varepsilon}(s)=r_i\}}ds\Big|_{\mathbb{H}}^{2p}\Big]\\
&\quad +C_p\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t S(t-\lfloor s\rfloor)[f(\bar{X}(s),\bar{X}(s-r_i))-f(\bar{X}(\lfloor s\rfloor),\bar{X}(\lfloor s\rfloor-r_i))]\mathbf{I}_{\{r^{\varepsilon}(s)=r_i\}}ds\Big|_{\mathbb{H}}^{2p}\Big]\\
&\quad +C_p\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t(S(t-\lfloor s\rfloor)-S(t-s))f(\bar{X}(\lfloor s\rfloor),\bar{X}(\lfloor s\rfloor-r_i))]\nu_i ds\Big|_{\mathbb{H}}^{2p}\Big]
\end{aligned}$$

$$
\begin{aligned}
&+C_p\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t S(t-s)v_i[f(\bar{X}(\lfloor s\rfloor),\bar{X}(\lfloor s\rfloor-r_i))]-f(\bar{X}(s),\bar{X}(s-r_i))]ds\Big|_{\mathbb{H}}^{2p}\Big]\\
&+C_p\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t S(t-\lfloor s\rfloor)f(\bar{X}(\lfloor s\rfloor),\bar{X}(\lfloor s\rfloor-r_i))\{\mathbf{I}_{\{r^\varepsilon(s)=r_i\}}-v_i\}ds\Big|_{\mathbb{H}}^{2p}\Big]\\
=:&\ \sum_{k=1}^{5}\Pi_{ki}(t).
\end{aligned}
$$

Next, let $t_j := j\varepsilon^\rho$, $j = 0,1,...,\lfloor\frac{t}{\varepsilon^\rho}\rfloor$ and $t_{\lfloor\frac{t}{\varepsilon^\rho}\rfloor+1} := t, \rho > 0$. Then, similarly to the proof of J_{21}, for $\Pi_{1i}(t)$ and $\Pi_{3i}(t)$, using Hölder's inequality, we have

$$\Pi_{1i}(t)+\Pi_{3i}(t)\le C_{p,n}(\varepsilon^\rho)^{2p\theta}.$$

For $\Pi_{2i}(t)$ and $\Pi_{4i}(t)$, by (H2) and Lemma 3.4, we have

$$
\begin{aligned}
\Pi_{2i}(t) &\le C_p\mathbb{E}\Big[\sup_{t\in[0,T]}\int_0^t \|S(t-\lfloor s\rfloor)\|^{2p}|f(\bar{X}(s),\bar{X}(s-r_i))-f(\bar{X}(\lfloor s\rfloor),\bar{X}(\lfloor s\rfloor-r_i))|_{\mathbb{H}}^{2p}ds\Big]\\
&\le C_p\mathbb{E}\Big[\sup_{t\in[0,T]}\int_0^t (|\bar{X}(\lfloor s\rfloor)-\bar{X}(s)|_{\mathbb{H}}^{2p}+|\bar{X}(\lfloor s\rfloor-r_i)-\bar{X}(s-r_i)|_{\mathbb{H}}^{2p})ds\Big]\\
&\le C_p\int_0^T \mathbb{E}[|\bar{X}(s_j)-\bar{X}(s)|_{\mathbb{H}}^{2p}]ds+C_p\int_0^T \mathbb{E}[|\bar{X}(s-r_i)-\bar{X}(s_j-r_i)|_{\mathbb{H}}^{2p}]ds.
\end{aligned}
$$

Then, by the initial condition on $\psi(\cdot)$ (1.7) and Lemma 3.4, we obtain

$$\Pi_{2i}(t)+\Pi_{4i}(t)\le C_{p,n}(\varepsilon^\rho)^{2p\theta}.$$

To proceed, let us estimate $\Pi_{5i}(t)$. By Hölder's inequality, (H3), Lemma 3.3 and the boundedness of $|\mathbf{I}_{\{r^\varepsilon(s)=r_i\}}-v_i|$, we have

$$
\begin{aligned}
\Pi_{5i}(t) &\le \frac{C_p}{(\varepsilon^\rho)^{(2p-1)}}\mathbb{E}\Big[\sup_{t\in[0,T]}\sum_{j=0}^{\lfloor\frac{t}{\varepsilon^\rho}\rfloor}\Big|\int_{t_j}^{t_{j+1}} S(t-t_j)f(\bar{X}(t_j),\bar{X}(t_j-r_i))\{\mathbf{I}_{\{r^\varepsilon(s)=r_i\}}-v_i\}ds\Big|_{\mathbb{H}}^{2p}\Big]\\
&\le \frac{C_p}{(\varepsilon^\rho)^{(2p-1)}}\sup_{t\in[0,T]}\Big(\sum_{j=0}^{\lfloor\frac{t}{\varepsilon^\rho}\rfloor}e^{-2p\lambda(t-t_j)}\Big)\Big(\mathbb{E}\Big[\sup_{t\in[0,T]}|f(\bar{X}(t),\bar{X}(t-r_i)|_{\mathbb{H}}^{4p})\Big]\Big)^{\frac{1}{2}}\\
&\quad\times\max_{0\le j\le\lfloor\frac{T}{\varepsilon^\rho}\rfloor}\Big(\mathbb{E}\Big[\Big|\int_{t_j}^{t_{j+1}}\{\mathbf{I}_{\{r^\varepsilon(s)=r_i\}}-v_i\}ds\Big|^{4p}\Big]\Big)^{\frac{1}{2}}\\
&\le \frac{C_p}{(\varepsilon^\rho)^{(2p-1)}}\sup_{t\in[0,T]}\Big(\sum_{j=0}^{\lfloor\frac{t}{\varepsilon^\rho}\rfloor}e^{-2p\lambda(t-t_j)}\Big)(1+\mathbb{E}[\sup_{t\in[0,T]}|\bar{X}(t)|_{\mathbb{H}}^{4p}]+\mathbb{E}[\sup_{t\in[0,T]}|\bar{X}(t-r_i)|_{\mathbb{H}}^{4p}])^{\frac{1}{2}}\\
&\quad\times\max_{0\le j\le\lfloor\frac{T}{\varepsilon^\rho}\rfloor}\Big(\mathbb{E}\Big[\Big|\int_{t_j}^{t_{j+1}}\{\mathbf{I}_{\{r^\varepsilon(s)=r_i\}}-v_i\}ds\Big|^{4p}\Big]\Big)^{\frac{1}{2}}\\
&\le C_p(e^{2p\lambda\varepsilon^\rho}-1)^{-1}\max_{0\le j\le\lfloor\frac{T}{\varepsilon^\rho}\rfloor}\Big(\mathbb{E}\Big[\Big|\int_{t_j}^{t_{j+1}}\{\mathbf{I}_{\{r^\varepsilon(s)=r_i\}}-v_i\}ds\Big|^{2}\Big]\Big)^{\frac{1}{2}}.
\end{aligned}
\tag{3.6}
$$

To show (3.6), we adopt an argument similar to that of Yin and Zhang [26, Theorem 7.2, page 170] and Bao, Yin and Yuan [3, Proof of Theorem 2.1, (2.24), pp. 653]. Let

$$\eta^\varepsilon(u) := \frac{1}{2}\mathbb{E}\Big[\Big|\int_{t_j}^{u}\{\mathbf{1}_{\{r^\varepsilon(s)=i\}} - v_i\}ds\Big|^2\Big], \quad u \in [t_j, t_{j+1}].$$

Then, it is easy to see from the chain rule that

$$\frac{d\eta^\varepsilon(u)}{du} = \mathbb{E}\Big[\int_{t_j}^{u}(\mathbf{1}_{\{r^\varepsilon(u)=i\}} - v_i)(\mathbf{1}_{\{r^\varepsilon(s)=i\}} - v_i)ds\Big], \quad u \in [t_j, t_{j+1}].$$

Let $\bar{t}_k := k\varepsilon^\beta, k = 0, 1, \dots, \lfloor (u - t_j)/\varepsilon^\beta \rfloor + 1$, where $\bar{t}_0 := t_j$ and $\bar{t}_{\lfloor (u-t_j)/\varepsilon^\beta \rfloor + 1} =: u$. Thus, by [3, Proof of Theorem 2.1, (2.24), pp. 653], we have

$$\mathbb{E}\Big[\Big|\int_{t_j}^{t_{j+1}}\{\mathbf{I}_{\{r^\varepsilon(s)=r_i\}} - v_i\}ds\Big|^2\Big] = O(\varepsilon^{\rho+\beta}),$$

where $\beta \in (\rho, 1)$. Next, by Taylor's formula, it is easy to know $(e^{2p\lambda_1\varepsilon^\rho} - 1) = O(\varepsilon^\rho)$ for sufficiently small $\varepsilon^\rho \in (0, 1)$. Thus, we get

$$\Pi_{5i}(t) \le C_p\varepsilon^{\frac{\beta-\rho}{2}}.$$

Finally, by Gronwall's inequality, we have

$$\mathbb{E}[\sup_{t\in[0,T]} |X^\varepsilon(t) - \bar{X}(t)|_{\mathbb{H}}^{2p}] \le C_{p,n}((\varepsilon^\rho)^{2p\theta} + \varepsilon^{\frac{\beta-\rho}{2}}),$$

with $\beta \in (\rho, 1)$. Thus, we have

$$\lim_{\varepsilon\to 0}\mathbb{E}[\sup_{t\in[0,T]} |X^\varepsilon(t) - \bar{X}(t)|_{\mathbb{H}}^{2p}] = 0,$$

This completes the proof. □

4. Conclusions

In this paper, by focusing on SEEs with jumps and random time delays modulated by two-time-scale Markov switching processes in which both fast and slow components co-exist, we establish an averaging principle under suitable conditions. For future research, delay-dependent stability involving two-time-scale structure and jumps is both interesting and important. It deserves to be further investigated in the future.

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