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*Research article*

# Free boundary problem pricing defaultable corporate bonds with multiple credit rating migration risk and stochastic interest rate

**Zhehao Huang**[1], **Yingting Miao**[2] **and Zhenzhen Wang**[3,*]

[1] Guangzhou Institute of International Finance, Guangzhou University, Guangzhou, P. R.China
[2] Department of Mathematics, City University of Hong Kong, Hong Kong, P. R.China
[3] School of Mathematics, Sun Yat-sen University, Guangzhou, P. R.China

* **Correspondence:** Email: wangzhzh29@mail.sysu.edu.cn.

**Abstract:** In this paper, valuation for a defaultable corporate bond subject to multiple credit rating migration risk and stochastic volatility of interest rate is addressed in the structure framework through a free boundary problem, which is derived by a series of transformations. The existence, uniqueness and regularity of solution to the free boundary problem are obtained to verify the rationality of the bond pricing model. Furthermore, we show that the solution of the free boundary problem is convergent to a close form steady status, which may provide some information on the developing characteristics of the bond price. As the coexistence of stochastic interest rate and defaulting boundary, this convergence is achieved through an auxiliary free boundary problem and a Lyapunov argument. Interestingly, the converged steady status can be explicitly solved, which is not the case in the existing literatures on multiple credit rating migration. Finally, we present an explicit formula for valuating this defaultable bond with multiple credit rating migration risk and stochastic interest rate.

**Keywords:** credit rating migration risk; stochastic interest rate; free boundary problem; ssymptotic behavior; corporate bond pricing
**Mathematics Subject Classification:** 91G40, 91G30, 35R35

## 1. Introduction

The globalization of financial markets has been developing rapidly, which requires more on credit risk management. Credit risk, which is attracting more and more attention from people in academics and practices, refers to not only default risk but also credit rating migration risk. The credit rating migration risk is playing a more and more significant role in financial markets and risk management, especially after the outbreak and spread of 2008 financial crisis. In particular, the credit rating migration risk makes difference to the corporate bond pricing.

There are two traditional models for default risk, involving the structural models and reduced form models. In the reduced form models, the default event is depicted and captured by introducing an exogenous variant. The default time is modeled by a stochastic default intensity in this approach, see [7, 14, 17] and so forth. The assumption on the structural models is that if the below bound of some insolvency threshold is met by corporate value, default occurs. In the model proposed by Merton [28], a default event may only occur at the maturity. Subsequently, Black and Cox [2] extended Merton's model to a first-passage-time model, where default may occur at any time before the debt maturity, see also [1, 3, 18, 25] and so forth.

With regard to literatures working on credit rating migration, a commonly adopted approach is the Markov chain, which is captured by transition intensity matrix coming from general statistic data, see [5, 8, 9] and so forth. The framework of reduced form then can be directly developed for dynamic credit rating migration process, see [6, 15, 30] and so forth. However, the Markov chain ignores the role played by the corporate value when modelling credit rating migration. In fact, the corporate value is an important factor in the credit rating migration and should be taken into consideration. Accordingly, from the corporate perspective, Liang et al. [20] started to model and analyze credit rating migration risk by structural model based on Merton's model. They set a predetermined migration threshold to divide the corporate value into high and low rating regions, where the corporate value follows different stochastic processes. However in practice, the threshold dividing credit ratings is usually not predetermined. To solve this problem, Hu et al. [11] improved the model proposed by Liang et al. [20]. They determined the migration boundary by the dynamic proportion between corporate debt and corporate value, which results in a free boundary problem. Subsequently, Liang et al. [21] incorporated a risk discount factor, which measures the sensibility of credit rating migration to the proportion, into the model and showed that an asymptotic traveling wave solution exists in the free boundary problem. Problem on credit rating migration in switching macro regions can also be referred to Wu and Liang [36], while credit contingent interest rate swap with credit rating migration can be seen in Liang and Zou [23].

In particular, the aforementioned works [11, 20, 21, 23, 36] only take two credit ratings into consideration in credit rating migration problem. The credit region is divided into the high rating region and the low rating region, which results in only one free boundary in the corresponding free boundary problem. However, in practice, we should notice that there are usually more than two credit ratings used when accessing the corporate credit levels. The Standard & Poor's, an international rating agency, downgraded the long-term sovereign credit rating of Greece from A- to BBB+ on the evening of December 16, 2009. This verifies the fact and inspires us to consider multiple credit ratings in migration problems. Wu and Liang [34] provided some numeric results for multiple credit rating migration problem. Wang et al. [32] presented some theoretical results by showing that the asymptotic traveling wave solution obtained in Liang et al. [21] persists in the free boundary problem with multiple free boundaries. By considering stochastic interest rate in reality [19, 24, 29], Yin et al. [37] improved the model by replacing the constant interest rate with a stochastic version. This improved model covers the previous works where only two credit ratings are involved with an interest-dependent volatility [22] or multiple credit ratings migration with constant interest rate [32,34]. Then Huang et al. [12] continued to study the bond pricing model with multiple credit rating migration and stochastic interest rate. They contributed to establishing the asymptotic traveling wave solution in the time-heterogeneous free boundary problem with multiple free boundaries.

The aforementioned models for credit rating migration in the structure framework are based on the Merton's model, i.e., default may only occur at the maturity. However, in practice, default may occur at any time up to maturity [4, 26, 31]. Wu et al. [35] relaxed the default restriction in the credit rating migration model by setting a predetermined threshold capturing the first-passage time when the default occurs. Once the corporate value falls below the threshold at any time, the default occurs. This results in a free boundary problem subject to a new boundary condition. Again we have to notice that the credit region in their credit rating migration model pricing a defaultable corporate bond is still divided into the high and low rating regions and meanwhile, a constant interest rate is also considered. Hence, motivated by these existing works on the effect of credit rating migration risk when valuating a corporate bond, in this paper, we devote to studying a pricing model for a defaultable corporate bond with both multiple credit rating migration risk and stochastic interest rate. Our work extends the existing works [12, 37], where the multiplicity of credit rating and stochasticity of interest rate are involved in their models, by inserting the default risk. Meanwhile, we improve the results of Wu et al. [35], who considered default risk in pricing the corporate bond with only one credit rating migration boundary and constant interest rate, to fit the effects of multiple credit ratings and stochastic interest rate.

The difficulties in analysis are generated from the joint effects and mutual restrictions among the multiplicity of credit ratings, stochastic volatility of interest rate and presence of default boundary. Besides that the free boundary problem turns into a initial-boundary problem, in particular, it is perplexed by not only a time-dependent and discontinuous coefficient but also a time-dependent process arisen in the problem. In addition to an irreducible barrier boundary, these indeed cause some troubles in deriving necessary estimates and then proving the existence and uniqueness of solution to the free boundary problem. Another contribution is the asymptotic behavior of solution to the free boundary problem. We prove that the solution converges to some steady status, which is the spatially homogeneous solution of an auxiliary free boundary problem, whose coefficients are the long time limits of the time-dependent coefficients in the original free boundary problem. This convergence, which shows us the developing tendency of solution, is established by two steps. In the first step, it is shown that the solution of the original free boundary problem converges to the solution of the auxiliary free boundary problem with time tending to infinity, while in the second step, it is shown that the solution of the auxiliary free boundary problem converges to the steady status. Moreover, the steady status can be solved explicitly. Thus, we present an explicit formula to valuate the defaultable corporate bond with multiple credit rating migration risk and stochastic volatility of interest rate.

The paper is organized as follows. In Section 2, the pricing model is constructed. In Section 3, the model is reduced into a free boundary problem with initial condition and boundary conditions involving default boundary and migrating boundaries. In Section 4, an approximated free boundary problem is analyzed and some preliminary lemmas for uniform estimates are collected. In Section 5, through the approximated free boundary problem, the existence and uniqueness of solution to the free boundary problem are obtained. In Section 6, it is proved that the solution is convergent to a steady status by a Lyapunov argument. Then we conclude the paper by presenting an explicit pricing formula for the defaultable corporate bond in Section 7.

## 2. The baseline pricing model

In this section, we set up the baseline pricing model for a defaultable corporate bond subject to multiple credit rating migration risk and stochastic interest rate. Some necessary assumptions are put forward as follows.

### 2.1. Assumptions

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. Suppose that the corporation issues a defaultable bond, which is a contingent claim of its value on the space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $S$ denote the corporate value in the risk neutral world. It satisfies the Black-Scholes model

$$dS = r(t)S\,dt + \sigma(t)S\,dW_t,$$

where $r$ is the time-varying interest rate and $\sigma$ is the heterogeneous volatility with respect to credit ratings. $W_t$ is the Brownian motion generating the filtration $\{\mathscr{F}_t\}$. The credit is divided into $n$ ratings. In different credit ratings, the corporation shows different volatilities of its value. We denote the volatility in the $i$'th rating by $\sigma(t) = \sigma_i$, $i = 1, 2, \cdots, n$, and in addition, they satisfy

$$0 < \sigma_1 < \sigma_2 < \cdots < \sigma_{n-1} < \sigma_n < \infty,$$

which means that in the highest credit rating, the corporation shows the smallest volatility $\sigma_1$ and in the lowest credit rating, it shows the largest volatility $\sigma_n$. The stochastic interest rate $r$ is supposed to satisfy the Vasicek model [27, 33]

$$dr = a(t)(\theta(t) - r)dt + \sigma_r(t)dW_t^r,$$

which is widely popular in financial application, where the parameters $a$, $\theta$, $\sigma_r$ are supposed to be positive constants in this paper. $\sigma_r$ is the volatility of the interest rate. $\theta$ is considered as the central location or the long-term value. $a$ determines the speed of adjustment.

We suppose that the corporation issues only one defaultable bond with face value $F$. The effect of corporate value on the bond value is focused on and the discount value of bond is considered. Denote by $\phi_t$ the discount value of bond at time $t$. The corporation exhibits two risks, the default risk and credit rating migration risk. The corporation can default before maturity time $T$. The default time $\tau_d$ is the first moment when the corporate value falls below the threshold $K$, namely that

$$\tau_d = \inf\{t > 0 | S_0 > K, S_t \leq K\},$$

where $K < F \cdot D(t, T)$, where $0 < D(t, T) < 1$ is the discount function. Once the corporation defaults, the investors will get what is left. Hence, $\phi_t(K) = K$ and at the maturity time $T$, the investors can get $\phi_T = \min\{S_T, F\}$. On the other hand, the credit regions are determined by the leverage $\gamma(t) = \phi_t/S_t$. Denote the thresholds of leverage $\gamma(t)$ by $\gamma_i$, $i = 1, 2, \cdots, n-1$, and they satisfy

$$0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{n-2} < \gamma_{n-1} < 1.$$

The credit rating migration times are the first moments when the corporate credit rating is upgraded or downgraded. They are defined as follows:

$$\tau_1 = \inf\{t > 0 | \phi_0/S_0 < \gamma_1, \phi_t/S_t \geq \gamma_1\},$$

$$\tau_n = \inf\{t > 0 | \phi_0/S_0 > \gamma_{n-1}, \phi_t/S_t \leq \gamma_{n-1}\},$$

$$\tau_{i,i+1} = \inf\{t > 0 | \gamma_{i-1} < \phi_0/S_0 < \gamma_i, \phi_t/S_t \geq \gamma_i\},\ i = 2, 3, \cdots, n-1,$$

$$\tau_{i,i-1} = \inf\{t > 0 | \gamma_{i-1} < \phi_0/S_0 < \gamma_i, \phi_t/S_t \leq \gamma_{i-1}\},\ i = 2, 3, \cdots, n-1.$$

$\tau_1$ is the first moment that the corporation degrades from the highest credit rating. $\tau_n$ is the first moment that the corporation upgrades from the lowest credit rating. $\tau_{i,i+1}$ is the first moment that the corporation jumps up into the $i+1$'th credit rating from the $i$'th credit rating, while $\tau_{i,i-1}$ is the first moment that the corporation jumps down into the $i-1$'th credit rating from the $i$'th credit rating.

### 2.2. Cash flow

Once the credit rating migrates before the maturity $T$, a virtual substitute termination happens, namely that the bond is virtually terminated and substituted by a new one with a new credit rating [35]. Thus, there is a virtual cash flow of the bond. Denote the bond values in different credit ratings by $\phi_i(t, S)$, $i = 1, 2, \cdots, n$. Then they are the conditional expectations as follows

$$\phi_1(t, S) = \mathbb{E}_{t,S}[h_1(t, T)|\phi_1(t, S) < \gamma_1 S],$$

where

$$\begin{aligned} h_1(t,T) =& e^{-\int_t^T r(s)ds} \min\{S_T, F\} \cdot \chi(\min\{\tau_1, \tau_d\} \geq T) \\ &+ e^{-\int_t^{\tau_1} r(s)ds} \phi_2(\tau_1, S_{\tau_1}) \cdot \chi(t < \tau_1 < \min\{\tau_d, T\}) \\ &+ e^{-\int_t^{\tau_d} r(s)ds} K \cdot \chi(t < \tau_d < \min\{\tau_1, T\}), \end{aligned}$$

and for $i = 2, 3, \cdots, n-1$,

$$\phi_i(t, S) = \mathbb{E}_{t,S}[h_i(t, T)|\gamma_{i-1} S < \phi_i(t, S) < \gamma_i S],$$

where

$$\begin{aligned} h_i(t,T) =& e^{-\int_t^T r(s)ds} \min\{S_T, F\} \cdot \chi(\min\{\tau_{i,i+1}, \tau_{i,i-1}, \tau_d\} \geq T) \\ &+ e^{-\int_t^{\tau_{i,i+1}} r(s)ds} \phi_{i+1}(\tau_{i,i+1}, S_{\tau_{i,i+1}}) \cdot \chi(t < \tau_{i,i+1} < \min\{\tau_{i,i-1}, \tau_d, T\}) \\ &+ e^{-\int_t^{\tau_{i,i-1}} r(s)ds} \phi_{i-1}(\tau_{i,i-1}, S_{\tau_{i,i-1}}) \cdot \chi(t < \tau_{i,i-1} < \min\{\tau_{i,i+1}, \tau_d, T\}) \\ &+ e^{-\int_t^{\tau_d} r(s)ds} K \cdot \chi(t < \tau_d < \min\{\tau_{i,i+1}, \tau_{i,i-1}, T\}) \end{aligned}$$

and

$$\phi_n(t, S) = \mathbb{E}_{t,S}[h_n(t, T)|\phi_n(t, S) > \gamma_{n-1} S],$$

where

$$\begin{aligned} h_n(t,T) =& e^{-\int_t^T r(s)ds} \min\{S_T, F\} \cdot \chi(\min\{\tau_n, \tau_d\} \geq T) \\ &+ e^{-\int_t^{\tau_n} r(s)ds} \phi_{n-1}(\tau_n, S_{\tau_n}) \cdot \chi(t < \tau_n < \min\{\tau_d, T\}) \\ &+ e^{-\int_t^{\tau_d} r(s)ds} K \cdot \chi(t < \tau_d < \min\{\tau_n, T\}), \end{aligned}$$

where $\chi$ is the indicative function, satisfying $\chi = 1$ if the event happens and otherwise, $\chi = 0$.

### 2.3. PDE problem

Suppose that the correlation between the interest rate and the corporate value is given by $dW_t^r \cdot dW_t = \rho t$, $-1 \leq \rho \leq 1$. By the Feynman-Kac formula, we can derive that $\phi_i$, $i = 1, 2, \cdots, n$, are functions of time $t$, interest rate $r$ and value $S$. They satisfy the following PDE in their regions

$$\frac{\partial \phi_1}{\partial t} + \frac{\sigma_1^2 S^2}{2} \frac{\partial^2 \phi_1}{\partial S^2} + \sigma_r \sigma_1 \rho S \frac{\partial^2 \phi_1}{\partial S \partial r} + rS \frac{\partial \phi_1}{\partial S} + \frac{\partial^2 \phi_1}{\partial r^2} + a(\theta - r) \frac{\partial \phi_1}{\partial r} - r\phi_1 = 0,\ \phi_1 < \gamma_1 S, \tag{2.1}$$

and for $i = 2, 3, \cdots, n-1$,

$$\frac{\partial \phi_i}{\partial t} + \frac{\sigma_i^2 S^2}{2}\frac{\partial^2 \phi_i}{\partial S^2} + \sigma_r\sigma_i\rho S\frac{\partial^2 \phi_i}{\partial S\,\partial r} + rS\frac{\partial \phi_i}{\partial S} + \frac{\partial^2 \phi_i}{\partial r^2} + a(\theta - r)\frac{\partial \phi_i}{\partial r} - r\phi_i = 0,\ \gamma_{i-1}S < \phi_i < \gamma_i S, \quad (2.2)$$

and

$$\frac{\partial \phi_n}{\partial t} + \frac{\sigma_n^2 S^2}{2}\frac{\partial^2 \phi_n}{\partial S^2} + \sigma_r\sigma_n\rho S\frac{\partial^2 \phi_n}{\partial S\,\partial r} + rS\frac{\partial \phi_n}{\partial S} + \frac{\partial^2 \phi_n}{\partial r^2} + a(\theta - r)\frac{\partial \phi_n}{\partial r} - r\phi_n = 0,\ \phi_n > \gamma_{n-1}S, \quad (2.3)$$

with terminal conditions

$$\phi_i(T, r, S) = \min\{S, F\},\ i = 1, 2, \cdots, n, \quad (2.4)$$

and boundary condition

$$\phi_n(t, r, K) = K. \quad (2.5)$$

The bond value is continuous when it passes a rating threshold, i.e., $\phi_i = \phi_{i+1}$ on the rating migration boundaries, where $i = 1, 2, \cdots, n-1$. Also, if we construct a risk free portfolio $\pi$ by longing a bond and shorting $\Delta$ amount asset value $S$, i.e., $\pi = \phi - \Delta S$ and such that $d\pi = r\pi$, this portfolio is also continuous when it passes the rating migration boundaries, namely that $\pi_i = \pi_{i+1}$ or $\Delta_i = \Delta_{i+1}$ on the rating migration boundaries, where $i = 1, 2, \cdots, n-1$. By Black-Scholes theory [16], it is equivalent to

$$\frac{\partial \phi_i}{\partial S} = \frac{\partial \phi_{i+1}}{\partial S} \text{ on the rating migration boundary, } i = 1, 2, \cdots, n-1. \quad (2.6)$$

Denote by $P(t, r)$ the value of a guaranteed zero-coupon bond with face value 1 at the maturity $t = T$, where the interest rate follows the Vasicek model. By the Feynman-Kac formula, $P(t, r)$ satisfies the following PDE

$$\frac{\partial P}{\partial t} + \frac{\sigma_r^2}{2}\frac{\partial^2 P}{\partial r^2} + a(\theta - r)\frac{\partial P}{\partial r} - rP = 0,\ r > 0,\ 0 < t < T,$$

with terminal condition $P(T, r) = 1$, whose explicit solution is solved as $P(t, r) = e^{A(T-t)}$ [13], where

$$A(T-t) = \frac{1}{a^2}(B^2(T-t) - (T-t))\left(a^2\theta - \frac{\sigma_r^2}{2}\right) - \frac{\sigma_r^2}{4a}B^2(T-t) - rB(T-t)$$

and

$$B(T-t) = \frac{1}{a}(1 - e^{-a(T-t)}).$$

Take transformations

$$y = \frac{S}{P(t, r)},\ \psi_i(t, y) = \frac{\phi_i(t, r, S)}{P(t, r)},\ i = 1, 2, \cdots, n.$$

Then $\psi_i$, $i = 1, 2, \cdots, n$ satisfy

$$\frac{\partial \psi_1}{\partial t} + \frac{\hat{\sigma}_1^2 y^2}{2}\frac{\partial^2 \psi_1}{\partial y^2} = 0,\ \psi_1 < \gamma_1 y, \quad (2.7)$$

and for $i = 2, 3, \cdots, n-1$,

$$\frac{\partial \psi_i}{\partial t} + \frac{\hat{\sigma}_i^2 y^2}{2}\frac{\partial^2 \psi_i}{\partial y^2} = 0,\ \gamma_{i-1}y < \psi_i < \gamma_i y, \quad (2.8)$$

and

$$\frac{\partial \psi_n}{\partial t} + \frac{\hat{\sigma}_n^2 y^2}{2}\frac{\partial^2 \psi_n}{\partial y^2} = 0,\ \psi_n > \gamma_{n-1} y, \tag{2.9}$$

where

$$\hat{\sigma}_i^2 = \sigma_i^2 + 2\rho\sigma_i\sigma_r B(T-t) + \sigma_r^2 B^2(T-t).$$

The terminal conditions are given as

$$\psi_i(T, y) = \min\{y, F\},\ i = 1, 2, \cdots, n, \tag{2.10}$$

and the boundary conditions are

$$\psi_n(t, Ke^{-A(T-t)}) = Ke^{-A(T-t)} \tag{2.11}$$

and

$$\psi_i = \psi_{i+1},\ \frac{\partial \psi_i}{\partial y} = \frac{\partial \psi_{i+1}}{\partial y},\ \text{on the rating migration boundary} \tag{2.12}$$

for $i = 1, 2, \cdots, n-1$.

## 3. Free boundary problem

We introduce the standard transformation of variable $x = \log y$, remaining $T - t$ as $t$, and define

$$\varphi(t, x) = e^{-x}\psi_i(T-t, e^x) \text{ in the } i\text{'th rating region.}$$

Using (2.12), we drive the following equation from (2.7)-(2.9) as

$$\frac{\partial \varphi}{\partial t} - \frac{\hat{\sigma}^2}{2}\frac{\partial^2 \varphi}{\partial x^2} - \frac{\hat{\sigma}^2}{2}\frac{\partial \varphi}{\partial x} = 0,\ \log K - A(t) < x < \infty,\ t > 0, \tag{3.1}$$

where $\hat{\sigma} = \hat{\sigma}_1$ as $\varphi < \gamma_1$, $\hat{\sigma} = \hat{\sigma}_i$ as $\gamma_{i-1} < \varphi < \gamma_i$ for $i = 1, 2, \cdots, n-1$, $\hat{\sigma} = \hat{\sigma}_n$ as $\varphi > \gamma_{n-1}$,

$$\hat{\sigma}_i^2(t) = \sigma_i^2 + 2\rho\sigma_i\sigma_r B(t) + \sigma_r^2 B^2(t),\ i = 1, 2, \cdots, n. \tag{3.2}$$

Meanwhile denote by

$$\bar{\sigma}_i^2 = \sigma_i^2 + \frac{2\rho\sigma_i\sigma_r}{a} + \frac{\sigma_r^2}{a^2},\ i = 1, 2, \cdots, n, \tag{3.3}$$

the limits of $\hat{\sigma}_i^2(t)$ as time tends to infinity, $i = 1, 2, \cdots, n$. Without loss of generality, suppose that $F = 1$ and there holds $K < 1$. Then (3.1) is supplemented with the initial condition

$$\varphi(0, x) = \min\{1, e^{-x}\}. \tag{3.4}$$

and boundary condition

$$\varphi(t, s(t)) = 1, \tag{3.5}$$

where $s(t) = \log K - A(t)$. Take $u(t, x) = \varphi(t, x + s(t))$. Then $u$ satisfies

$$\frac{\partial u}{\partial t} - \frac{\hat{\sigma}^2}{2}\frac{\partial^2 u}{\partial x^2} - \Big(\frac{\hat{\sigma}^2}{2} + \dot{s}(t)\Big)\frac{\partial u}{\partial x} = 0,\ 0 < x < \infty,\ t > 0, \tag{3.6}$$

where
$$\dot{s}(t) = \left(\frac{\sigma_r^2}{2a^2} - \theta\right)(2B(t)e^{-at} - 1) + \frac{\sigma_r^2}{2a}B(t)e^{-at} + re^{-at}$$
and $\hat{\sigma} = \hat{\sigma}_1$ as $u < \gamma_1$, $\hat{\sigma} = \hat{\sigma}_i$ as $\gamma_{i-1} < u < \gamma_i$ for $i = 1, 2, \cdots, n-1$, $\hat{\sigma} = \hat{\sigma}_n$ as $u > \gamma_{n-1}$, with the initial condition
$$u(0, x) = \min\{1, e^{-(x+s(0))}\} = \min\{1, e^{-(x+\log K)}\} \tag{3.7}$$
and boundary condition
$$u(t, 0) = 1. \tag{3.8}$$
The domain will be divided into $n$ rating regions $Q_i$, $i = 1, 2, \cdots, n$. We will prove that the domain can be separated by $n-1$ free boundaries $x = \lambda_i(t)$, $i = 1, 2, \cdots, n-1$. These boundaries are a prior unknown since they should be solved by equations
$$u(t, \lambda_i(t)) = \gamma_i,\ i = 1, 2, \cdots, n-1, \tag{3.9}$$
where $u$ is also a priori unknown. Since we have assumed that (3.1) is valid cross the free boundaries, (2.12) implies that for $i = 1, 2, \cdots, n-1$,
$$u(t, \lambda_i(t)-) = u(t, \lambda_i(t)+) = \gamma_i,\ \frac{\partial u}{\partial x}(t, \lambda_i(t)-) = \frac{\partial u}{\partial x}(t, \lambda_i(t)+). \tag{3.10}$$
In the work [35] and [12], where the former model is subject to constant interest rate and the later one is subject to stochastic interest rate but without default boundary, the process $\dot{s}(t)$ in (3.6) is replaced by a constant. The presence of $\dot{s}(t)$ indeed leads to some technical differences in deriving the estimates in the following argument. We can rewrite the formula of $\dot{s}(t)$ as
$$\dot{s}(t) = (\beta + r)e^{-at} - \beta e^{-2at} - \frac{a}{2}\left(\beta - \frac{\sigma_r^2}{2a^2}\right),$$
where
$$\beta = \frac{\sigma_r^2}{a^3} - \frac{2\theta}{a} + \frac{\sigma_r^2}{2a^2}.$$
It is not difficult to analyze and derive that one of the following conditions holds, then there holds $\dot{s}(t) \geq 0$:
$$0 \leq 2a^2\beta \leq \sigma_r^2; \tag{3.11}$$
or
$$\beta \leq -r,\ a(\beta + r)^2 + \sigma_r^2\beta \leq 2a^2\beta^2; \tag{3.12}$$
or
$$-r < \beta < 0. \tag{3.13}$$

## 4. Approximated problem with some uniform estimates

### *4.1. Approximation*

Let $H(\xi)$ be the Heaviside function, namely that $H(\xi) = 0$ for $\xi < 0$ and $H(\xi) = 1$ for $\xi \geq 0$. Then we can rewrite the volatility $\hat{\sigma}$ in (3.6) as
$$\hat{\sigma} = \hat{\sigma}_1 + \sum_{i=1}^{n-1}(\hat{\sigma}_{i+1} - \hat{\sigma}_i)H(u - \gamma_i).$$

We approximate $H(\xi)$ by a $C^\infty$ function $H_\epsilon(\xi)$ satisfying

$$H_\epsilon(\xi) = 0 \text{ for } \xi < -\epsilon,\ H_\epsilon(\xi) = 1 \text{ for } \xi > 0,\ H'_\epsilon(\xi) \geq 0 \text{ for } -\infty < \xi < \infty.$$

Consider the approximated free boundary problem

$$\mathscr{L}^\epsilon[u_\epsilon] \equiv \frac{\partial u_\epsilon}{\partial t} - \frac{\hat{\sigma}_\epsilon^2}{2}\frac{\partial^2 u_\epsilon}{\partial x^2} - \Big(\frac{\hat{\sigma}_\epsilon^2}{2} + \dot{s}(t)\Big)\frac{\partial u_\epsilon}{\partial x} = 0,\ 0 < x < \infty,\ t > 0, \tag{4.1}$$

with initial condition

$$u_\epsilon(0, x) = \min\{1, e^{-(x+\log K)}\} \tag{4.2}$$

and

$$u_\epsilon(t, 0) = 1, \tag{4.3}$$

where

$$\hat{\sigma}_\epsilon = \hat{\sigma}_1 + \sum_{i=1}^{n-1}(\hat{\sigma}_{i+1} - \hat{\sigma}_i)H_\epsilon(u_\epsilon - \gamma_i).$$

Problem (4.1)-(4.3) admits a unique classical solution $u_\epsilon$. Now we proceed to derive some estimates for $u_\epsilon$.

### 4.2. *Estimates for approximated problem*

Some uniform estimates are presented in this section, which are sufficient to obtain the existence and uniqueness of solution to problem (3.6)-(3.10).

**Lemma 4.1.** *Let $u_\epsilon$ be the solution of problem (4.1)-(4.3). Suppose that one of the conditions (3.11)-(3.13) holds. Then there holds*

$$0 \leq u_\epsilon \leq \min\{1, e^{-(x+\log K)}\},\ 0 < x < \infty,\ t > 0.$$

*Proof.* It is easy to verify that 0 is the lower solution of $u_\epsilon$ and meanwhile, $e^{-(x+\log K)}$ and 1 are upper solutions. The result is a direct application of comparison principle. □

**Lemma 4.2.** *Let $u_\epsilon$ be the solution of problem (4.1)-(4.3). Then there exists a constant $C > 0$, independent of $\epsilon$, such that*

$$-C \leq \frac{\partial u_\epsilon}{\partial x} \leq 0,\ 0 < x < \infty,\ t > 0.$$

*Proof.* It is easy to see that $\hat{\sigma}_\epsilon^2$ can be written as

$$\hat{\sigma}_\epsilon^2 = \hat{\sigma}_1^2 + \sum_{i=1}^{n-1}(\hat{\sigma}_{i+1}^2 - \hat{\sigma}_i^2)H_\epsilon(u_\epsilon - \gamma_i).$$

Differentiating (4.1) with respect to $x$ gives

$$\mathscr{L}_1^\epsilon\Big[\frac{\partial u_\epsilon}{\partial x}\Big] \triangleq \mathscr{L}^\epsilon\Big[\frac{\partial u_\epsilon}{\partial x}\Big] - \frac{1}{2}\sum_{i=1}^{n-1}(\hat{\sigma}_{i+1}^2 - \hat{\sigma}_i^2)H'_\epsilon(u_\epsilon - \gamma_i)\Big(\frac{\partial^2 u_\epsilon}{\partial x^2} + \frac{\partial u_\epsilon}{\partial x}\Big)\frac{\partial u_\epsilon}{\partial x} = 0.$$

It is known that $\frac{\partial u_\epsilon}{\partial x}(0,x)=0$ for $0<x<-\log K$ and $\frac{\partial u_\epsilon}{\partial x}(0,x)=-e^{-(x+\log K)}\le 0$ for $x>-\log K$. Since

$$\frac{u_\epsilon(t,x)-u_\epsilon(t,0)}{x}\le 0,$$

then letting $x\to 0$, it holds that $\frac{\partial u_\epsilon}{\partial x}(t,0)\le 0$. Thus it follows by maximum principle that there holds $\frac{\partial u_\epsilon}{\partial x}\le 0$.

On the other hand, since $\dot{s}(t)$ is uniformly bounded, take an appropriate value $C>0$, such that

$$\mathscr{L}^\epsilon[e^{-Cx}]=\Big(-\frac{\hat{\sigma}_\epsilon^2}{2}C^2+\Big(\frac{\hat{\sigma}_\epsilon^2}{2}+\dot{s}(t)\Big)C\Big)e^{-Cx}<0.$$

Clearly, $e^{-Cx}|_{x=0}=1$ and $e^{-Cx}\le\min\{1,e^{-(x+\log K)}\}$ for $C$ sufficiently large. Then there hold $u_\epsilon(t,x)\ge e^{-Cx}$ and

$$\frac{u_\epsilon(t,x)-u_\epsilon(t,0)}{x}\ge\frac{e^{-Cx}-1}{x}.$$

Letting $x\to 0$, we have $\frac{\partial u_\epsilon}{\partial x}(t,0)\ge -C$. Clearly, there holds

$$\mathscr{L}_1^\epsilon[-C]=-\frac{C^2}{2}\sum_{i=1}^{n-1}(\hat{\sigma}_{i+1}^2-\hat{\sigma}_i^2)H'_\epsilon(u_\epsilon-\gamma_i)\le 0,$$

as $\hat{\sigma}_{i+1}^2>\hat{\sigma}_i^2$ for $i=1,2,\cdots,n-1$. It follows by the comparison principle that there holds $\frac{\partial u_\epsilon}{\partial x}\ge -C$. □

**Lemma 4.3.** *Let $u_\epsilon$ be the solution of problem (4.1)-(4.3). Suppose that one of the conditions (3.11)-(3.13) holds. Then there exist constants $C_1$, $C_2$, $C_3$ and $C_4$, independent of $\epsilon$, such that*

$$-C_3-\frac{C_2}{\sqrt{t}}\exp\Big(-\frac{C_1}{t}|x+\log K|^2\Big)\le\frac{\partial u_\epsilon}{\partial t}\le C_4,\ 0<x<\infty,\ t>0.$$

*Proof.* Differentiating (4.1) with respect to $t$ gives

$$\mathscr{L}^\epsilon\Big[\frac{\partial u_\epsilon}{\partial t}\Big]-\frac{1}{2}\frac{\partial\hat{\sigma}_\epsilon^2}{\partial t}\frac{\partial u_\epsilon}{\partial t}+\frac{\dot{s}(t)}{2}\frac{\partial\hat{\sigma}_\epsilon^2}{\partial t}\frac{\partial u_\epsilon}{\partial x}-\ddot{s}(t)\frac{\partial u_\epsilon}{\partial x}=0,$$

where

$$\ddot{s}(t)=2a\beta e^{-2at}-a(\beta+r)e^{-at}.$$

Since

$$\frac{\partial\hat{\sigma}_\epsilon^2}{\partial t}=h_1(t)+h_2(t)\frac{\partial u_\epsilon}{\partial t},$$

where

$$h_1(t)=\frac{\partial\hat{\sigma}_1^2}{\partial t}+\sum_{i=1}^{n-1}\Big(\frac{\partial\hat{\sigma}_{i+1}^2}{\partial t}-\frac{\partial\hat{\sigma}_i^2}{\partial t}\Big)H_\epsilon(u_\epsilon-\gamma_i),$$

and

$$h_2(t)=\sum_{i=1}^{n-1}(\hat{\sigma}_{i+1}^2-\hat{\sigma}_i^2)H'_\epsilon(u_\epsilon-\gamma_i),$$

we can write

$$\mathscr{L}^{\epsilon}\Big[\frac{\partial u_\epsilon}{\partial t}\Big] = \frac{h_2(t)}{2}\Big(\frac{\partial u_\epsilon}{\partial t}\Big)^2 + \frac{1}{2}\Big(h_1(t) - \dot{s}(t)h_2(t)\frac{\partial u_\epsilon}{\partial x}\Big)\frac{\partial u_\epsilon}{\partial t} + \Big(\ddot{s}(t) - \frac{\dot{s}(t)h_1(t)}{2}\Big)\frac{\partial u_\epsilon}{\partial x}. \tag{4.4}$$

According to the formulas of $\hat{\sigma}_i^2$, $i = 1, 2, \cdots, n$, we have $h_1(t) \le \frac{\partial \hat{\sigma}_n^2}{\partial t}$. On the other hand, there exists $\widetilde{\lambda}_i^\epsilon(t)$ such that $H'_\epsilon(u_\epsilon(t, \widetilde{\lambda}_i^\epsilon(t)) - \gamma_i)$ attains its maximum and

$$h_2(t) \le \tilde{h}_2(t) \triangleq \max_{1\le i\le n-1}(\hat{\sigma}_{i+1}^2(t) - \hat{\sigma}_i^2(t))H'_\epsilon(u_\epsilon(t, \widetilde{\lambda}_i^\epsilon(t)) - \gamma_i).$$

Denote by $y(t)$ the solution of the following ODE

$$y'(t) = \tilde{h}_2(t)y^2(t) + \frac{1}{2}\Big(\frac{\partial \hat{\sigma}_n^2}{\partial t} + C\dot{s}(t)\tilde{h}_2(t)\Big)y(t) + C\Big(|\ddot{s}(t)| + \frac{\dot{s}(t)}{2}\frac{\partial \hat{\sigma}_n^2}{\partial t}\Big),\ y(0) = y_0, \tag{4.5}$$

where the constant $C$ is given as the one in Lemma 4.2. At $x = -\log K$, $\frac{\partial^2 u_\epsilon}{\partial x^2}(0, x)$ produces a Dirac measure of density $-1$. Thus $\frac{\partial u_\epsilon}{\partial t}(0, x) \le 0$ in the distribution sense. In addition, since the second order compatibility condition is satisfied at $(0, 0)$, we have $\frac{\partial u_\epsilon}{\partial t}$ is continuous at $(0, 0)$. Meanwhile, $\frac{\partial u_\epsilon}{\partial t}(0, t) = 0$. By further approximating the initial data with smooth function if necessary, there holds by the comparison principle that

$$\frac{\partial u_\epsilon}{\partial t}(t, x) \le y(t),\ 0 < x < \infty,\ t > 0,$$

if we set $y_0 = 0$. The ODE (4.5) can be solved formally as

$$y(t) = q(t)\exp\Big(\int_0^t p(s)ds\Big),$$

where

$$p(t) = \tilde{h}_2(t)y(t) + \frac{1}{2}\Big(\frac{\partial \hat{\sigma}_n^2}{\partial t} + C\dot{s}(t)\tilde{h}_2(t)\Big),$$

and

$$q(t) = C\int_0^t \Big(|\ddot{s}(r)| + \frac{\dot{s}(r)}{2}\frac{\partial \hat{\sigma}_n^2}{\partial r}\Big)\exp\Big(-\int_0^r p(\tau)d\tau\Big)dr.$$

Since $\frac{\partial \hat{\sigma}_n^2}{\partial t} \to 0$ as $t \to \infty$ and $H'_\epsilon$ converges to the Dirac measure as $\epsilon \to 0$, this implies that $\exp(\int_0^t p(s)ds)$ is uniformly bounded with respect to $t$. With regard to $q(t)$, in addition to $\ddot{s}(t) \to 0$ as $t \to \infty$, it is known that $q(t)$ is also uniformly bounded. Hence, we conclude that $y(t)$ is uniformly bounded with respect to $t$.

On the other hand, since $u_\epsilon(0, 0) = 1 > \gamma_{n-1}$, and by Hölder continuity of solution, there exists a $\rho > 0$, independent of $\epsilon$, such that

$$u_\epsilon(t, x) > \frac{1 + \gamma_{n-1}}{2}$$

for $|x| \le \rho$, $0 \le t \le \rho^2$. Thus for sufficiently small $\epsilon < \frac{1}{2}(1 - \gamma_{n-1})$, $\hat{\sigma}_\epsilon = \hat{\sigma}_n$ for $|x + \log K| \le \rho$, $0 \le t \le \rho^2$. It follows from the standard parabolic estimates [10] that

$$\frac{\partial u_\epsilon}{\partial t} \ge -C_2 - \frac{C_2}{\sqrt{t}}\exp\Big(-\frac{C_1}{t}|x + \log K|^2\Big)$$

for $|x+\log K| < \frac{\rho}{2}$, $0 < t \le \frac{\rho^2}{4}$. Note that (4.4) can be rewritten as

$$\mathscr{L}^{\epsilon}\Big[\frac{\partial u_\epsilon}{\partial t}\Big] = \frac{h_2(t)}{2}\Big(\frac{\partial u_\epsilon}{\partial t} - \dot{s}(t)\frac{\partial u_\epsilon}{\partial x}\Big)\frac{\partial u_\epsilon}{\partial t} + \frac{h_1(t)}{2}\frac{\partial u_\epsilon}{\partial t} + \Big(\ddot{s}(t) - \frac{\dot{s}(t)h_1(t)}{2}\Big)\frac{\partial u_\epsilon}{\partial x}.$$

As $\dot{s}(t)$ is uniformly bounded, we take a sufficiently large constant $C_3$ such that $C_3 \ge \sup_{t\ge 0}|\dot{s}(t)|C$, where constant $C$ is the one given in Lemma 4.2. Then if $\frac{\partial u_\epsilon}{\partial t} < -C_3$, there holds

$$\mathscr{L}^{\epsilon}\Big[\frac{\partial u_\epsilon}{\partial t}\Big] \ge \frac{1}{2}\frac{\partial \hat{\sigma}_n^2}{\partial t}\frac{\partial u_\epsilon}{\partial t} - \Big(|\ddot{s}(t)| + \frac{\dot{s}(t)}{2}\frac{\partial \hat{\sigma}_n^2}{\partial t}\Big)C.$$

Denote by $z(t)$ the solution of the following ODE

$$z'(t) = \frac{1}{2}\frac{\partial \hat{\sigma}_n^2}{\partial t}z(t) - \Big(|\ddot{s}(t)| + \frac{\dot{s}(t)}{2}\frac{\partial \hat{\sigma}_n^2}{\partial t}\Big)C,\ z(0) = z_0,$$

which can be solved as

$$z(t) = b(t)\exp\Big(\frac{1}{2}\int_0^t \frac{\partial \hat{\sigma}_n^2}{\partial s}ds\Big),$$

where

$$b(t) = z_0 - C\int_0^t \Big(|\ddot{s}(r)| + \frac{\dot{s}(r)}{2}\frac{\partial \hat{\sigma}_n^2}{\partial r}\Big)\exp\Big(-\frac{1}{2}\int_0^r \frac{\partial \hat{\sigma}_n^2}{\partial \tau}d\tau\Big)dr.$$

As $\frac{\partial \hat{\sigma}_n^2}{\partial t}$ and $\ddot{s}(t)$ tends to 0 as $t \to \infty$, $z(t)$ is uniformly bounded. Moreover, $z(t)$ is also decreasing if $z_0 \le 0$. Take the initial data $|z_0|$ and constant $C_3$ sufficiently large such that

$$C_3 \ge \sup_{t\ge 0}|z(t)| \ge C_2 + \frac{C_2}{\sqrt{t}}\exp\Big(-\frac{C_1}{t}|x+\log K|^2\Big)$$

on the boundary

$$\Big\{|x+\log K| = \frac{\rho}{2},\ 0 < t < \frac{\rho^2}{4}\Big\}\bigcup\Big\{|x+\log K| < \frac{\rho}{2},\ t = \frac{\rho^2}{4}\Big\}.$$

We claim that the region

$$\Big\{\frac{\partial u_\epsilon}{\partial t} < -C_3\Big\}\Big\backslash\Big\{|x+\log K| < \frac{\rho}{2}, 0 < t \le \frac{\rho^2}{4}\Big\}$$

is an empty set. If not, on the parabolic boundary of this region, we clearly have $\frac{\partial u_\epsilon}{\partial t} \ge -C_3$, which implies by the comparison principle that

$$\frac{\partial u_\epsilon}{\partial t} \ge z(t) \ge -C_3$$

in this region. This is a contradiction. □

**Remark 4.4.** *In the work [35], it was proved that $\frac{\partial u_\epsilon}{\partial t} \le 0$, which is different from the result shown in Lemma 4.3. However, although we get a similar result to Lemma 5.4 in [12], the proof is very different and more technical. This is due to the joint effect of stochastic interest rate and default boundary.*

**Corollary 4.5.** *Let $u_\epsilon$ be the solution of problem (4.1)-(4.3). Suppose that one of the conditions (3.11)-(3.13) holds. Then there exist constants $C_1$, $C_2$, $C_3$ and $C_4$, independents of $\epsilon$, such that*

$$-C_3 - \frac{C_2}{\sqrt{t}}\exp\Big(-\frac{C_1}{t}|x+\log K|^2\Big) \le \frac{\partial^2 u_\epsilon}{\partial x^2} \le C_4,\ 0 < x < \infty,\ t > t_0.$$

Denote by $\lambda_i^\epsilon$, $i = 1, 2, \cdots, n-1$ the approximated free boundaries, which are the solutions of equations

$$u_\epsilon(t, \lambda_i^\epsilon(t)) = \gamma_i,\ i = 1, 2, \cdots, n-1. \tag{4.6}$$

Then we have the following estimates for the approximated free boundaries.

**Lemma 4.6.** *Let $\lambda_i^\epsilon$, $i = 1, 2, \cdots, n-1$, be the approximated free boundaries defined in (4.6). Suppose that one of the conditions (3.11)-(3.13) holds. Then there exist constants $C_1$, $C_2$, independent of $\epsilon$, such that*

$$C_1 \le \lambda_{n-1}^\epsilon(t) \le \lambda_{n-2}^\epsilon(t) \le \cdots \le \lambda_2^\epsilon(t) \le \lambda_1^\epsilon(t) \le C_2.$$

*Proof.* Since

$$u_\epsilon(t, \lambda_i^\epsilon(t)) = \gamma_i < \gamma_{i+1} = u_\epsilon(t, \lambda_{i+1}^\epsilon(t)),$$

which implies that $\lambda_i^\epsilon(t) \ge \lambda_{i+1}^\epsilon(t)$ by Lemma 4.2. From Lemma 4.1, we have

$$u_\epsilon(t, x) \le e^{-(x+\log K)},$$

which implies that

$$u_\epsilon(t, x) < \gamma_1 \text{ for } x > -\log \gamma_1 K.$$

This means that region $\{x > -\log \gamma_1 K\}$ is in the highest rating region and hence

$$\lambda_1^\epsilon(t) \le C_2 \triangleq -\log \gamma_1 K.$$

Denote by $m = \sup_{t\ge 0} \dot{s}(t)$ and

$$v(x) = \frac{1+\gamma_{n-1}}{2}\exp\Big(-\Big(1 + \frac{2m}{\sigma_1^2}\Big)x\Big).$$

Then $v(0) = \frac{1}{2}(1+\gamma_{n-1}) < 1 = u_\epsilon(t, 0)$. We can see that $v(x) \le v(0) < 1$ and

$$v(x)e^{x+\log K} = \frac{1+\gamma_{n-1}}{2}\exp\Big(-\frac{2m}{\sigma_1^2}x + \log K\Big) < 1,$$

which implies that

$$v(x) < \min\{1, e^{-(x+\log K)}\} = u_\epsilon(0, x).$$

In addition, we have

$$\mathscr{L}^\epsilon[v] = \frac{1+\gamma_{n-1}}{2}\exp\Big(-\Big(1+\frac{2m}{\sigma_1^2}\Big)x\Big)\Big(1+\frac{2m}{\sigma_1^2}\Big)\Big(\dot{s}(t) - \frac{m\hat{\sigma}_\epsilon^2}{\sigma_1^2}\Big) \le 0.$$

By the comparison principle, we have $v(x) \le u_\epsilon(t, x)$, which implies that

$$u_\epsilon(t, x) \ge \frac{1+\gamma_{n-1}}{2}\exp\Big(-\Big(1+\frac{2m}{\sigma_1^2}\Big)x\Big) > \gamma_{n-1}$$

for

$$x < C_1 \triangleq \frac{\sigma_1^2}{\sigma_1^2 + 2m} \log \frac{1 + \gamma_{n-1}}{2\gamma_{n-1}}.$$

This means that region $\{x < C_1\}$ is in the lowest rating region and hence $\lambda_{n-1}^\epsilon(t) \ge C_1$. □

**Lemma 4.7.** *Let $\lambda_i^\epsilon$, $i = 1, 2, \cdots, n-1$, be the approximated free boundaries defined in (4.6). Suppose that one of the conditions (3.11)-(3.13) holds. Then there exists constant $C$ independent of $\epsilon$, such that*

$$-C \le \frac{d\lambda_i^\epsilon}{dt} \le C,\ 0 < t < T,\ i = 1, 2, \cdots, n-1.$$

*Proof.* Clearly, there holds

$$\frac{d\lambda_i^\epsilon}{dt} = -\frac{\partial u_\epsilon}{\partial t}(t, \lambda_i^\epsilon(t)) \Big/ \frac{\partial u_\epsilon}{\partial x}(t, \lambda_i^\epsilon(t)),\ i = 1, 2, \cdots, n-1.$$

Since $\lambda_i^\epsilon(0) = -\log \gamma_i - \log K$, $i = 1, 2, \cdots, n-1$, by Lemma 4.3, there is a constant $\rho > 0$ independent of $\epsilon$ such that

$$\lambda_i^\epsilon(t) + \log K \ge \rho \text{ for } 0 \le t \le \rho^2,\ i = 1, 2, \cdots, n-1.$$

It follows from Lemma 4.3 that

$$-C_0 \le \frac{\partial u_\epsilon}{\partial t}(t, \lambda_i^\epsilon(t)) \le C_0,\ i = 1, 2, \cdots, n-1,$$

where $C_0$ is a constant independent of $\epsilon$. To finish the proof, it is sufficient to prove that

$$-\frac{\partial u_\epsilon}{\partial x}(t, \lambda_i^\epsilon(t)) \ge C^*$$

for some positive constant $C^*$ independent of $\epsilon$. As shown in Lemma 4.2, we have

$$\mathscr{L}_1^\epsilon\Big[-\frac{\partial u_\epsilon}{\partial x}\Big] = \mathscr{L}^\epsilon\Big[-\frac{\partial u_\epsilon}{\partial x}\Big] + \frac{1}{2}\sum_{i=1}^{n-1}(\hat\sigma_{i+1}^2 - \hat\sigma_i^2)H_\epsilon'(u_\epsilon - \gamma_i)\Big(\frac{\partial^2 u_\epsilon}{\partial x^2} + \frac{\partial u_\epsilon}{\partial x}\Big)\frac{\partial u_\epsilon}{\partial x} = 0.$$

In addition, there also holds $-\frac{\partial u_\epsilon}{\partial x}(0, x) = 0$ for $0 < x < -\log K$ and $-\frac{\partial u_\epsilon}{\partial x}(0, x) = e^{-(x+\log K)}$ for $x > -\log K$, and $-\frac{\partial u_\epsilon}{\partial x}(t, 0) \ge 0$ for $t > 0$. By Lemmas 4.3 and 4.6, there exists constant $R > 0$ independent of $\epsilon$, such that

$$\frac{2}{R} \le \lambda_i^\epsilon(t) \le R - 1 \text{ for } 0 < t \le T,\ i = 1, 2, \cdots, n-1,$$

and

$$\lambda_i^\epsilon(t) + \log K \ge \rho \text{ for } 0 \le t \le \rho^2,\ i = 1, 2, \cdots, n-1.$$

Consider the region

$$\Omega^* = \Big\{\frac{\rho}{2} - \log K < x < R,\ 0 < t < \rho^2\Big\} \bigcup \Big\{\frac{1}{R} \le x \le R,\ \rho^2 \le t \le T\Big\}. \tag{4.7}$$

The parabolic boundary of this region $\Omega^*$ consists of five line segments. On the initial line segment $\{t = 0,\ \frac{\rho}{2} - \log K \le x \le R\}$, there holds that $-\frac{\partial u_\epsilon}{\partial x}(0, x) = e^{-(x+\log K)}$. The remaining four parabolic boundaries

$\{0 \le t \le T,\ x = R\} \cup \{0 \le t \le \rho^2,\ x = \frac{\rho}{2} - \log K\} \cup \{t = \rho^2,\ \frac{1}{R} \le x \le \frac{\rho}{2} - \log K\} \cup \{\rho^2 \le t \le T,\ x = \frac{1}{R}\}$ are completely and uniformly within the highest or lowest rating region (independent of $\epsilon$). Thus by compactness and the strong maximum principle, on these four boundaries, it holds that $-\frac{\partial u_\epsilon}{\partial x} \ge \bar{C} > 0$ for some $\bar{C}$ independent of $\epsilon$. It follows that

$$-\frac{\partial u_\epsilon}{\partial x} \ge \min\{1, \bar{C}\} \equiv C^* \text{ on } \Omega^*, \tag{4.8}$$

which completes the proof of the lemma. □

## 5. Existence and uniqueness

Lemmas 4.1-4.3 and Corollary 4.5 provide uniform estimates for approximated solution $u_\epsilon$. By taking a limit $\epsilon \to 0$ (along a subsequence if necessary), we derive the existence of solution to problem (3.6)-(3.10). Lemmas 4.6-4.7 show that there are uniform estimates in $C^1([0, T])$ for the approximated free boundaries $\lambda_i^\epsilon$, $i = 1, 2, \cdots, n-1$. Therefore, the limits of $\lambda_i^\epsilon$ as $\epsilon \to 0$ exist, which are denoted by $\lambda_i$, $i = 1, 2, \cdots, n-1$. These $\lambda_i$, $i = 1, 2, \cdots, n-1$, are the free boundaries of the original problem.

**Theorem 5.1.** *The free boundary problem (3.6)-(3.10) admits a solution* $(u, \lambda_i, i = 1, 2, \cdots, n-1)$ *with*

$$u \in W^{1,2}_\infty([0, T] \times (0, \infty) \setminus \overline{Q_{t_0}}) \bigcap W^{0,1}_\infty([0, T] \times (0, \infty))$$

*for any* $t_0 > 0$*, where*

$$Q_{t_0} = (0, t_0^2) \times (-t_0 - \log K, t_0 - \log K)$$

*and* $\lambda_i \in W^1([0, T])$*,* $i = 1, 2, \cdots, n-1$*.*

By the classical parabolic theory, it is also clear that the solution is in $\cap_{i=1}^n C^\infty(\Omega_i)$, where

$$\Omega_1 = \{(t, x) : x > \lambda_1(t),\ 0 < t \le T\},$$

and for $i = 2, 3, \cdots, n-1$

$$\Omega_i = \{(t, x) : \lambda_i(t) < x < \lambda_{i-1}(t),\ 0 < t \le T\},$$

and

$$\Omega_n = \{(t, x) : x < \lambda_{n-1}(t),\ 0 < t \le T\}.$$

Now we prove the uniqueness of solution to the problem (3.6)-(3.10).

**Theorem 5.2.** *The solution* $(u, \lambda_i, i = 1, 2, \cdots, n-1)$ *of the problem (3.6)-(3.10) with*

$$u \in W^{1,2}_\infty([0, T] \times (0, \infty) \setminus \overline{Q_{t_0}}) \bigcap W^{0,1}_\infty([0, T] \times (0, \infty))$$

*and* $\lambda_i \in C([0, T])$ *is unique.*

*Proof.* Suppose that $(u, \lambda_i, i = 1, 2, \cdots, n-1)$ and $(\tilde{u}, \tilde{\lambda}_i, i = 1, 2, \cdots, n-1)$ are two solutions of the problem (3.6)-(3.10). Then $u(t, \lambda_i(t)) = \tilde{u}(t, \tilde{\lambda}_i(t)) = \gamma_i$, $i = 1, 2, \cdots, n-1$ and

$$u(t, \lambda_i(t)) - \tilde{u}(t, \lambda_i(t)) = \tilde{u}(t, \tilde{\lambda}_i(t)) - \tilde{u}(t, \lambda_i(t)),\ i = 1, 2, \cdots, n-1.$$

Besides, at $t = 0$, $\lambda_i(0) = \tilde{\lambda}_i(0) = -\log K - \log\gamma_i$, $i = 1, 2, \cdots, n-1$. As shown in Lemma 4.7, there exists a constant $C^* > 0$, such that $u_x < -C^*$ and $\tilde{u}_x < -C^*$ on the region $\Omega^*$ defined by (4.7). Then by the implicit function theorem, there exists $\rho > 0$, such that when $0 < t < \rho$,

$$|\lambda_i(t) - \tilde{\lambda}_i(t)| \leq C \max_{0<x<\infty} |u(t, x) - \tilde{u}(t, x)|,\ i = 1, 2, \cdots, n-1, \tag{5.1}$$

where $C$ is a positive constant, whose value may change line on line but makes no difference. Let $w = u - \tilde{u}$ and denote by $\hat{\sigma}$ and $\tilde{\hat{\sigma}}$ the corresponding coefficients, then $w$ satisfies

$$\frac{1}{\hat{\sigma}^2}\frac{\partial w}{\partial t} - \frac{1}{2}\frac{\partial^2 w}{\partial x^2} - \Big(\frac{\dot{s}(t)}{\hat{\sigma}^2} + \frac{1}{2}\Big)\frac{\partial w}{\partial x} = \Big(\frac{1}{\tilde{\hat{\sigma}}^2} - \frac{1}{\hat{\sigma}^2}\Big)\Big(\frac{\partial \tilde{u}}{\partial t} - \dot{s}(t)\frac{\partial \tilde{u}}{\partial x}\Big). \tag{5.2}$$

$u$, $\tilde{u}$ and their derivatives decay exponentially fast to 0 as $x \to \infty$. Multiplying (5.2) by $w$ on both sides and integrating $x$ from 0 to $\infty$ gives

$$\int_0^\infty \Big(\frac{w}{\hat{\sigma}^2}\frac{\partial w}{\partial t} - \frac{w}{2}\frac{\partial^2 w}{\partial x^2} - \Big(\frac{\dot{s}(t)}{\hat{\sigma}^2} + \frac{1}{2}\Big)w\frac{\partial w}{\partial x}\Big)dx = \int_0^\infty \Big(\frac{1}{\tilde{\hat{\sigma}}^2} - \frac{1}{\hat{\sigma}^2}\Big)\Big(\frac{\partial \tilde{u}}{\partial t} - \dot{s}(t)\frac{\partial \tilde{u}}{\partial x}\Big)w dx. \tag{5.3}$$

Since

$$\frac{1}{\tilde{\hat{\sigma}}^2} - \frac{1}{\hat{\sigma}^2} \equiv 0 \text{ for } x \notin \bigcup_{i=1}^{n-1}[\lambda_i(t) \wedge \tilde{\lambda}_i(t), \lambda_i(t) \vee \tilde{\lambda}_i(t)],$$

and $\frac{\partial \tilde{u}}{\partial t}$ and $\frac{\partial \tilde{u}}{\partial x}$ are uniformly bounded outside the region $\overline{Q_{t_0}}$, we conclude that they are bounded for $x \in \cup_{i=1}^{n-1}[\lambda_i(t) \wedge \tilde{\lambda}_i(t), \lambda_i(t) \vee \tilde{\lambda}_i(t)]$. Since $w$ decays exponentially to 0 as $x \to \infty$, for any $t > 0$, there exists $x_0 < \infty$ such that

$$\max_{0<x<\infty} w^2(t, x) = w^2(t, x_0).$$

Take

$$\overline{w} = \frac{1}{\epsilon}\int_{x_0}^{x_0+\epsilon} w(t, x)dx = w(t, x^*),$$

for some $x^* \in (x_0, x_0 + \epsilon)$. Then there holds

$$\begin{aligned}
\max_{0<x<\infty} |w(t, x)|^2 \leq& 2|w(t, x_0) - \overline{w}|^2 + 2|\overline{w}|^2 \\
=& 2\Big(\int_{x_0}^{x^*} \frac{\partial w}{\partial x}dx\Big)^2 + \frac{2}{\epsilon^2}\Big(\int_{x_0}^{x_0+\epsilon} w dx\Big)^2 \\
\leq& 2\epsilon\int_{x_0}^{x_0+\epsilon} \Big(\frac{\partial w}{\partial x}\Big)^2 dx + \frac{2}{\epsilon}\int_{x_0}^{x_0+\epsilon} w^2 dx \\
\leq& 2\epsilon\int_0^\infty \Big(\frac{\partial w}{\partial x}\Big)^2 dx + \frac{2}{\epsilon}\int_0^\infty w^2 dx.
\end{aligned} \tag{5.4}$$

It follows that

$$
\begin{aligned}
&\int_0^\infty \Big(\frac{1}{\tilde{\hat{\sigma}}^2}-\frac{1}{\hat{\sigma}^2}\Big)\Big(\frac{\partial\tilde{u}}{\partial t}-\dot{s}(t)\frac{\partial\tilde{u}}{\partial x}\Big)wdx\\
\le& C\max_{0<x<\infty}|w(t,x)|\sum_{i=1}^{n-1}\int_{\lambda_i(t)\wedge\tilde{\lambda}_i(t)}^{\lambda_i(t)\vee\tilde{\lambda}_i(t)}\Big|\frac{1}{\tilde{\hat{\sigma}}^2}-\frac{1}{\hat{\sigma}^2}\Big|dx\\
\le& C\max_{0<x<\infty}|w(t,x)|\sum_{i=1}^{n-1}|\lambda_i(t)-\tilde{\lambda}_i(t)|\\
\le& C\max_{0<x<\infty}|w(t,x)|^2 \text{ (by (5.1))}\\
\le&\frac{C}{\epsilon}\int_0^\infty w^2dx+\epsilon\int_0^\infty\Big(\frac{\partial w}{\partial x}\Big)^2dx.
\end{aligned}
\tag{5.5}
$$

We now proceed to estimate the left side of (5.3). First, we have

$$
\begin{aligned}
\int_0^{+\infty}\frac{w}{\hat{\sigma}^2}\frac{\partial w}{\partial t}dx=&\int_0^{\lambda_{n-1}}\frac{w}{\hat{\sigma}_n^2}\frac{\partial w}{\partial t}dx+\sum_{i=1}^{n-2}\int_{\lambda_{i+1}}^{\lambda_i}\frac{w}{\hat{\sigma}_{i+1}^2}\frac{\partial w}{\partial t}dx+\int_{\lambda_1}^\infty\frac{w}{\hat{\sigma}_1^2}\frac{\partial w}{\partial t}dx\\
=&g_1'(t)+g_2(t)+\sum_{i=1}^{n-1}\Big(\frac{1}{\hat{\sigma}_i^2}-\frac{1}{\hat{\sigma}_{i+1}^2}\Big)\lambda_i'(t)\frac{w^2(t,\lambda_i(t))}{2}\\
\ge&g_1'(t)+\sum_{i=1}^{n-1}\Big(\frac{1}{\hat{\sigma}_i^2}-\frac{1}{\hat{\sigma}_{i+1}^2}\Big)\lambda_i'(t)\frac{w^2(t,\lambda_i(t))}{2},
\end{aligned}
\tag{5.6}
$$

where

$$
g_1(t)=\int_0^{\lambda_{n-1}}\frac{w^2}{2\hat{\sigma}_n^2}dx+\sum_{i=1}^{n-2}\int_{\lambda_{i+1}}^{\lambda_i}\frac{w^2}{2\hat{\sigma}_{i+1}^2}dx+\int_{\lambda_1}^\infty\frac{w^2}{2\hat{\sigma}_1^2}dx,
$$

and

$$
g_2(t)=\int_0^{\lambda_{n-1}}\frac{w^2}{2\hat{\sigma}_n^4}\frac{\partial\hat{\sigma}_n^2}{\partial t}dx+\sum_{i=1}^{n-2}\int_{\lambda_{i+1}}^{\lambda_i}\frac{w^2}{2\hat{\sigma}_{i+1}^4}\frac{\partial\hat{\sigma}_{i+1}^2}{\partial t}dx+\int_{\lambda_1}^\infty\frac{w^2}{2\hat{\sigma}_1^4}\frac{\partial\hat{\sigma}_1^2}{\partial t}dx.
$$

Second, we have

$$
\int_0^\infty -\frac{w}{2}\frac{\partial^2 w}{\partial x^2}dx=\frac{1}{2}\int_0^\infty\Big(\frac{\partial w}{\partial x}\Big)^2dx \tag{5.7}
$$

and

$$
\int_0^\infty\Big(\frac{\dot{s}(t)}{\hat{\sigma}^2}+\frac{1}{2}\Big)w\frac{\partial w}{\partial x}dx\le\epsilon\int_0^\infty\Big(\frac{\partial w}{\partial x}\Big)^2dx+\frac{C}{\epsilon}\int_0^\infty w^2dx, \tag{5.8}
$$

as $\dot{s}(t)$ is uniformly bounded. Combining the above inequalities (5.5)-(5.8), taking into account (5.3), we drive

$$
\begin{aligned}
g_1'(t)\le&\epsilon\int_0^\infty\Big(\frac{\partial w}{\partial x}\Big)^2dx+\frac{C}{\epsilon}\int_0^\infty w^2dx+\sum_{i=1}^{n-1}\Big(\frac{1}{\hat{\sigma}_{i+1}^2}-\frac{1}{\hat{\sigma}_i^2}\Big)\lambda_i'(t)\frac{w^2(t,\lambda_i(t))}{2}\\
\le&\epsilon\int_0^\infty\Big(\frac{\partial w}{\partial x}\Big)^2dx+\frac{C}{\epsilon}\int_0^\infty w^2dx+C\max_{0\le x\le\infty}|w(t,x)|^2\\
\le&C\epsilon\int_0^\infty\Big(\frac{\partial w}{\partial x}\Big)^2dx+\frac{C}{\epsilon}\int_0^\infty w^2dx.
\end{aligned}
$$

It is easy to see that there exists a constant $C_0 > 0$ such that

$$C_0 \int_0^\infty w^2 dx \le g_1(t) \le C\epsilon \int_0^t \int_0^\infty \left(\frac{\partial w}{\partial x}\right)^2 dxds + \frac{C}{\epsilon} \int_0^t \int_0^\infty w^2 dxds.$$

Then for sufficiently small $\epsilon$, by applying the Gronwall's inequality, we conclude that $w \equiv 0$. This proves the uniqueness for $0 \le t \le \rho$. A close examination of the proof indicates that the uniqueness result can be extended to any time interval, where $\frac{\partial u}{\partial x}$ is strictly negative, which is already verified in (4.8). □

## 6. Asymptotic behavior

### 6.1. Steady status

Denote by $\psi$ the solution of the following static problem

$$\frac{\bar{\bar{\sigma}}^2}{2}\frac{\partial^2 \psi}{\partial x^2} + \left(\frac{\bar{\bar{\sigma}}^2}{2} + \theta - \frac{\sigma_r^2}{2a^2}\right)\frac{\partial \psi}{\partial x} = 0, \tag{6.1}$$

with boundary conditions

$$\psi(0) = 1, \ \lim_{x\to\infty} \psi(x) = 0, \tag{6.2}$$

$$\psi(\lambda_i^*) = \gamma_i, \ \frac{\partial \psi}{\partial x}(\lambda_i^*+) = \frac{\partial \psi}{\partial x}(\lambda_i^*-), \ i = 1, 2, \cdots, n-1, \tag{6.3}$$

where $\bar{\bar{\sigma}} = \bar{\bar{\sigma}}_1$ as $\psi < \gamma_1$, $\bar{\bar{\sigma}} = \bar{\bar{\sigma}}_i$ as $\gamma_{i-1} < \psi < \gamma_i$ for $i = 1, 2, \cdots, n-1$, $\bar{\bar{\sigma}} = \bar{\bar{\sigma}}_n$ as $\psi > \gamma_{n-1}$, and $\bar{\bar{\sigma}}_i$, $i = 1, 2, \cdots, n$, are given in (3.3). We suppose that in the $i$'th rating region, $\psi$ admits the following form

$$\psi(x) = p_i + q_i \exp(k_i x), \ i = 1, 2, \cdots, n, \tag{6.4}$$

where $p_i$, $q_i$ and $k_i$ are undetermined constants. Substituting (6.4) into (6.1) in the corresponding rating region gives

$$k_i = \frac{\sigma_r^2}{a^2\bar{\bar{\sigma}}_i^2} - \frac{2\theta}{\bar{\bar{\sigma}}_i^2} - 1, \ i = 1, 2, \cdots, n.$$

As it is supposed that one of the conditions (3.11)-(3.13) holds, then $\dot{s}(t) \ge 0$, which implies that $k_i < 0$, $i = 1, 2, \cdots, n$. Substituting (6.4) into the boundary condition (6.3) gives

$$p_i + q_i \exp(k_i \lambda_i^*) = \gamma_i, \tag{6.5}$$

$$p_{i+1} + q_{i+1} \exp(k_{i+1}\lambda_i^*) = \gamma_i, \tag{6.6}$$

and

$$q_i k_i \exp(k_i \lambda_i^*) = q_{i+1} k_{i+1} \exp(k_{i+1}\lambda_i^*) \tag{6.7}$$

for $i = 1, 2, \cdots, n-1$. Also, substituting (6.4) into the boundary condition (6.2) gives

$$p_1 = 0, \ p_n + q_n = 1. \tag{6.8}$$

It is easy to see that coefficient system (6.5)-(6.8) can be equivalently rewritten as

$$\log q_i + k_i \lambda_i^* = \log(\gamma_i - p_i), \tag{6.9}$$

$$\log q_{i+1} + k_{i+1}\lambda_i^* = \log(\gamma_i - p_{i+1}), \tag{6.10}$$

$$\log q_i + \log k_i + k_i\lambda_i^* = \log q_{i+1} + \log k_{i+1} + k_{i+1}\lambda_i^* \tag{6.11}$$

for $i = 1, 2, \cdots, n-1$, and

$$\log(\gamma_1 - p_1) = \log \gamma_1, \ \log q_n = \log(1 - p_n). \tag{6.12}$$

For $i = 1, 2, \cdots, n-2$, from the equations

$$p_{i+1} + q_{i+1}\exp(k_{i+1}\lambda_i^*) = \gamma_i, \ p_{i+1} + q_{i+1}\exp(k_{i+1}\lambda_{i+1}^*) = \gamma_{i+1},$$

we can derive their relationship as

$$\log(\gamma_i - p_{i+1}) = \log(\gamma_{i+1} - p_{i+1}) + k_{i+1}\lambda_i^* - k_{i+1}\lambda_{i+1}^*.$$

Thus, denote by $x_i = \log q_i$ for $i = 1, 2, \cdots, n$, $y_i = \log(\gamma_i - p_i)$ for $i = 1, 2, \cdots, n-1$ and $y_n = \log(1-p_n)$, $z_i = \log(\gamma_i - p_{i+1})$, $i = 1, 2, \cdots, n-1$. Then (6.9)-(6.12) can be rewritten as

$$x_i - y_i + k_i\lambda_i^* = 0, \ i = 1, 2, \cdots, n-1, \tag{6.13}$$

$$x_{i+1} - z_i + k_{i+1}\lambda_i^* = 0, \ i = 1, 2, \cdots, n-1, \tag{6.14}$$

$$x_i - x_{i+1} + (k_i - k_{i+1})\lambda_i^* + \log k_i - \log k_{i+1} = 0, \ i = 1, 2, \cdots, n-1, \tag{6.15}$$

$$z_i - y_{i+1} - k_{i+1}\lambda_i^* + k_{i+1}\lambda_{i+1}^* = 0, \ i = 1, 2, \cdots, n-1, \tag{6.16}$$

$$\lambda_n^* = 0, \ y_1 = \log \gamma_1, \ x_n = y_n, \tag{6.17}$$

where a virtual parameter $\lambda_n^*$ is added. (6.13)-(6.17) is a linearized system from (6.5)-(6.8) and can be solved according to fundamental linear algebra theory. Thus, we obtain the explicit solution of the static problem (6.1)-(6.3).

### *6.2. Convergence*

The coefficients in the model of [35] are time-homogeneous and moreover, the solution is decreasing in time. However, both of these are not the cases in our model. We cannot take advantage of the decreasing property of solution to obtain the convergence. In this paper, we are motivated by the idea of [12] and obtain the convergence by two steps. The first step is to show that the solution of problem (3.6)-(3.10) converges to the solution of some auxiliary problem defined below, while in the second step, we show that the solution of auxiliary problem converges to the solution of static problem (6.1)-(6.3). Now define an auxiliary free boundary problem as follows

$$\frac{\partial \bar{u}}{\partial t} - \frac{\bar{\bar{\sigma}}^2}{2}\frac{\partial^2 \bar{u}}{\partial x^2} - \Big(\frac{\bar{\bar{\sigma}}^2}{2} + \theta - \frac{\sigma_r^2}{2a^2}\Big)\frac{\partial \bar{u}}{\partial x} = 0, \ 0 < x < \infty, \ t > 0, \tag{6.18}$$

with initial condition $\bar{u}_0$ and boundary condition $\bar{u}(t, 0) = 1$,

$$\bar{u}(t, \bar{\lambda}_i(t)-) = \bar{u}(t, \bar{\lambda}_i(t)+) = \gamma_i, \ \frac{\partial \bar{u}}{\partial x}(t, \bar{\lambda}_i(t)-) = \frac{\partial \bar{u}}{\partial x}(t, \bar{\lambda}_i(t)+), \ i = 1, 2, \cdots, n-1. \tag{6.19}$$

All the results derived above involving the existence, uniqueness and some properties of solution presented in Lemmas 4.1-4.6 hold for solution of problem (6.18)-(6.19). We have to notice that as the presence of default boundary, although we follow the idea of [12], the technical proofs are different, especially in the step, i.e., the convergence from the original solution to the auxiliary solution.

6.2.1. Convergence from $u$ to $\bar{u}$

Since $\hat{\sigma}_i(t) \to \bar{\hat{\sigma}}_i$, $i = 1, 2, \cdots, n$, and $\dot{s}(t) \to \theta - \frac{\sigma_r^2}{2a^2}$ as $t \to \infty$, then for any $\epsilon > 0$, there exists a $T > 0$ such that for $t \geq T$,

$$\frac{1}{\bar{\hat{\sigma}}_i^2} - \epsilon \leq \frac{1}{\hat{\sigma}_i^2(t)} \leq \frac{1}{\bar{\hat{\sigma}}_i^2} + \epsilon, \ \theta - \frac{\sigma_r^2}{2a^2} - \epsilon \leq \dot{s}(t) \leq \theta - \frac{\sigma_r^2}{2a^2} + \epsilon. \tag{6.20}$$

Let $\bar{u}_0(x) = u(T, x)$ and denote by $u_T(t, x) = u(t + T, x)$ for $t \geq 0$. We have $u_T(t, \lambda_i(t)) = \bar{u}(t, \bar{\lambda}_i(t)) = \gamma_i$, $i = 1, 2, \cdots, n-1$, and

$$u_T(t, \lambda_i^T(t)) - \bar{u}(t, \lambda_i^T(t)) = \bar{u}(t, \bar{\lambda}_i(t)) - \bar{u}(t, \lambda_i^T(t)), \ i = 1, 2, \cdots, n-1,$$

where $\lambda_i^T(t) = \lambda_i(t + T)$, $i = 1, 2, \cdots, n-1$. Similarly to the proof of Theorem 5.2, by the implicit function theorem, there exists a $\rho > 0$, such that when $0 < t < \rho$,

$$|\bar{\lambda}_i(t) - \lambda_i^T(t)| \leq C \max_{0 \leq x \leq \infty} |u_T(t, x) - \bar{u}(t, x)|, \ i = 1, 2, \cdots, n-1, \tag{6.21}$$

where $C$ is a positive constant, whose value may change line on line but makes no difference. Let $w = u_T - \bar{u}$. Then $w$ satisfies

$$\frac{1}{\hat{\sigma}_T^2} \frac{\partial w}{\partial t} = \frac{1}{2} \frac{\partial^2 w}{\partial x^2} + \Big(\frac{1}{2} + \frac{\dot{s}_T(t)}{\hat{\sigma}_T^2}\Big) \frac{\partial w}{\partial x} + h_1 + h_2, \tag{6.22}$$

where $\dot{s}_T(t) = \dot{s}(t + T)$,

$$h_1 = \Big(\frac{1}{\bar{\hat{\sigma}}^2} - \frac{1}{\hat{\sigma}_T^2}\Big)\Big(\frac{\partial \bar{u}}{\partial t} - \Big(\theta - \frac{\sigma_r^2}{2a^2}\Big)\frac{\partial \bar{u}}{\partial x}\Big),$$

and

$$h_2 = \Big(\dot{s}_T(t) - \theta + \frac{\sigma_r^2}{2a^2}\Big)\frac{\partial \bar{u}}{\partial x}.$$

As $u$, $\bar{u}$ and their derivatives decay exponentially fast to 0 as $x \to \infty$, multiplying (6.22) by $w$ on both sides and integrating $x$ from 0 to $\infty$ gives

$$\int_0^\infty \frac{w}{\hat{\sigma}_T^2} \frac{\partial w}{\partial t} dx = \int_0^\infty \frac{w}{2} \frac{\partial^2 w}{\partial x^2} dx + \int_0^\infty \Big(\frac{1}{2} + \frac{\dot{s}_T(t)}{\hat{\sigma}_T^2}\Big) \frac{\partial w}{\partial x} w dx + \int_0^\infty h_1 w dx + \int_0^\infty h_2 w dx. \tag{6.23}$$

By Lemmas 4.2 and 4.3, we know that $\frac{\partial \bar{u}}{\partial t}$ and $\frac{\partial \bar{u}}{\partial x}$ are uniformly bounded as the initial data is set to be $\bar{u}_0(x) = u(T, x)$ for a sufficiently large $T$. It follows that

$$\begin{aligned}
\int_0^\infty h_1 w dx &\leq \max_{0 \leq x \leq \infty} |w(t, x)| \sum_{i=1}^{n-1} \int_{\lambda_i^T \wedge \bar{\lambda}_i}^{\lambda_i^T \vee \bar{\lambda}_i} \Big|\frac{1}{\bar{\hat{\sigma}}^2} - \frac{1}{\hat{\sigma}_T^2}\Big| dx + h_3 \\
&\leq \max_{0 \leq x \leq \infty} |w(t, x)| \sum_{i=1}^{n-1} |\lambda_i^T(t) - \bar{\lambda}_i(t)| + h_3 \text{ (by (6.21))} \\
&\leq \max_{0 \leq x \leq \infty} |w(t, x)|^2 + h_3,
\end{aligned}$$

where

$$h_3 = \left(\int_0^{\lambda_{n-1}^T \wedge \bar{\lambda}_{n-1}} + \sum_{i=1}^{n-2} \int_{\lambda_{i+1}^T \vee \bar{\lambda}_{i+1}}^{\lambda_i^T \wedge \bar{\lambda}_i} + \int_{\lambda_1^T \vee \bar{\lambda}_1}^{\infty}\right)\left|\frac{1}{\bar{\hat{\sigma}}^2} - \frac{1}{\hat{\sigma}_T^2}\right| w dx \le \epsilon \int_0^\infty w dx \le C\epsilon,$$

hold by (6.20) and the exponential decay of $w$. Using (5.4), there holds

$$\int_0^\infty h_1 w dx \le \frac{C}{\epsilon}\int_0^\infty w^2 dx + \epsilon \int_0^\infty \left(\frac{\partial w}{\partial x}\right)^2 dx + \epsilon C. \tag{6.24}$$

On the other hand, there holds

$$\int_0^\infty h_2 w dx \le \epsilon \int_0^\infty \left|\frac{\partial \bar{u}}{\partial x}\right| w dx \le C\epsilon \int_0^\infty w dx \le C\epsilon. \tag{6.25}$$

We now proceed to estimate the left side of (6.23) by

$$\begin{aligned}\int_0^{+\infty} \frac{w}{\hat{\sigma}_T^2}\frac{\partial w}{\partial t} dx =& \int_0^{\lambda_{n-1}^T} \frac{w}{(\hat{\sigma}_T)_n^2}\frac{\partial w}{\partial t} dx + \sum_{i=1}^{n-2}\int_{\lambda_{i+1}^T}^{\lambda_i^T} \frac{w}{(\hat{\sigma}_T)_{i+1}^2}\frac{\partial w}{\partial t} dx + \int_{\lambda_1^T}^\infty \frac{w}{(\hat{\sigma}_T)_1^2}\frac{\partial w}{\partial t} dx \\ =& g_1'(t) + g_2(t) + \sum_{i=1}^{n-1}\left(\frac{1}{(\hat{\sigma}_T)_i^2} - \frac{1}{(\hat{\sigma}_T)_{i+1}^2}\right)\frac{1}{2}\frac{d\lambda_i^T}{dt} w^2(t, \lambda_i^T(t)) \\ \ge& g_1'(t) + \sum_{i=1}^{n-1}\left(\frac{1}{(\hat{\sigma}_T)_i^2} - \frac{1}{(\hat{\sigma}_T)_{i+1}^2}\right)\frac{1}{2}\frac{d\lambda_i^T}{dt} w^2(t, \lambda_i^T(t)),\end{aligned} \tag{6.26}$$

where

$$g_1(t) = \int_0^{\lambda_{n-1}^T} \frac{w^2}{2(\hat{\sigma}_T)_n^2} dx + \sum_{i=1}^{n-2}\int_{\lambda_{i+1}^T}^{\lambda_i^T} \frac{w^2}{2(\hat{\sigma}_T)_{i+1}^2} dx + \int_{\lambda_1^T}^\infty \frac{w^2}{2(\hat{\sigma}_T)_1^2} dx,$$

and

$$g_2(t) = \int_0^{\lambda_{n-1}^T} \frac{w^2}{2(\hat{\sigma}_T)_n^4}\frac{\partial(\hat{\sigma}_T)_n^2}{\partial t} dx + \sum_{i=1}^{n-2}\int_{\lambda_{i+1}^T}^{\lambda_i^T} \frac{w^2}{2(\hat{\sigma}_T)_{i+1}^4}\frac{\partial(\hat{\sigma}_T)_{i+1}^2}{\partial t} dx + \int_{\lambda_1^T}^\infty \frac{w^2}{2(\hat{\sigma}_T)_1^4}\frac{\partial(\hat{\sigma}_T)_1^2}{\partial t} dx.$$

The remaining terms in (6.23) can be estimated by

$$\int_0^\infty \frac{w}{2}\frac{\partial^2 w}{\partial x^2} dx = -\frac{1}{2}\int_0^\infty \left(\frac{\partial w}{\partial x}\right)^2 dx \tag{6.27}$$

and

$$\int_0^\infty \left(\frac{\dot{s}_T}{\hat{\sigma}_T^2} + \frac{1}{2}\right) w \frac{\partial w}{\partial x} dx \le \epsilon \int_0^\infty \left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{C}{\epsilon}\int_0^\infty w^2 dx. \tag{6.28}$$

Combining the inequalities (6.24)-(6.28), we have

$$\begin{aligned} g_1'(t) \le& \epsilon \int_0^\infty \left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{C}{\epsilon}\int_0^\infty w^2 dx + \sum_{i=1}^{n-1}\left(\frac{1}{(\hat{\sigma}_T)_{i+1}^2} - \frac{1}{(\hat{\sigma}_T)_i^2}\right)\frac{\partial \lambda_i^T}{\partial t}\frac{w^2(t, \lambda_i^T(t))}{2} + C\epsilon \\ \le& \epsilon \int_0^\infty \left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{C}{\epsilon}\int_0^\infty w^2 dx + C \max_{0\le x\le\infty} |w(t,x)|^2 + C\epsilon \\ \le& C\epsilon \int_0^\infty \left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{C}{\epsilon}\int_0^\infty w^2 dx + C\epsilon. \end{aligned}$$

It follows that

$$C_0\int_0^\infty w^2dx \le g_1(t) \le C\epsilon\int_0^t\int_0^\infty\left(\frac{\partial w}{\partial x}\right)^2dxds + \frac{C}{\epsilon}\int_0^t\int_0^\infty w^2dxds + C\epsilon t.$$

Then taking a sufficiently small $\epsilon$, which means a sufficiently large $T$, by applying the Gronwall's inequality, we conclude that $w = 0$, namely that $u_T(t,x) = \bar{u}(t,x)$ for $0 \le t \le \rho$. The result can be extended to any interval of $t$.

6.2.2. Convergence from $\bar{u}$ to $\psi$

The convergence from the solution of auxiliary problem (6.18)-(6.19) to the solution of problem (6.1)-(6.3) is proved by a Lyapunov argument, which is similar to the procedure in [12, 21, 32], but with some necessary modifications to fit the model in this paper. For instance, we have extend the solution with domain $[0,\infty)$ to the whole real line. The first step is to present the formal construction of a Lyapunov function, ignoring the integrability of any arisen integral. The second step is to verify the integrability of those integrals arisen in the formal construction. The third step is to complete the proof of convergence.

Denote by $\bar{U}$ the extension of $\bar{u}$, who is the solution of auxiliary problem (6.18)-(6.19), from $x \in [0,\infty)$ to the real line $\mathbb{R}$, namely that

$$\bar{U}(t,x) = 1 \text{ for } x < 0,\ \bar{U}(t,x) = \bar{u}(t,x) \text{ for } x \ge 0.$$

Following [12], let $V(x,u,q)$ be a undetermined function and set

$$E[\bar{U}](t) = \int_{-\infty}^\infty V(x,\bar{U}(t,x),\bar{U}_x(t,x))dx.$$

Formally, assuming the integrability, we also have

$$\begin{aligned}
\frac{d}{dt}E[\bar{U}] &= \int_{-\infty}^\infty (V_u\bar{U}_t + V_q\bar{U}_{xt})dx \\
&= \int_{-\infty}^\infty \bar{U}_t(V_u - V_{qx} - V_{qu}\bar{U}_x - V_{qq}\bar{U}_{xx})dx \\
&= \int_{-\infty}^\infty \bar{U}_t\left(V_u - V_{qx} - V_{qu}\bar{U}_x - V_{qq}\left(\frac{2}{\bar{\tilde{\sigma}}^2}\bar{U}_t - \frac{2}{\bar{\tilde{\sigma}}^2}\left(\frac{\bar{\tilde{\sigma}}^2}{2} + \delta\right)\bar{U}_x\right)\right)dx \\
&= -\int_{-\infty}^\infty \frac{2}{\bar{\tilde{\sigma}}^2}V_{qq}\bar{U}_t^2dx + \int_{-\infty}^\infty \bar{U}_t\left(V_u - V_{qx} - V_{qu}\bar{U}_x + V_{qq}\left(1 + \frac{2\delta}{\bar{\tilde{\sigma}}^2}\right)\bar{U}_x\right)dx \\
&= -\int_{-\infty}^\infty \frac{2}{\bar{\tilde{\sigma}}^2}V_{qq}\bar{U}_t^2dx,
\end{aligned}$$

where $\delta = \theta - \frac{\sigma_r^2}{2a^2}$, provided taking $V$ satisfying

$$V_u - V_{qx} - qV_{qu} + qV_{qq}\left(1 + \frac{2\delta}{\bar{\tilde{\sigma}}^2(u)}\right) = 0. \tag{6.29}$$

Denote by $\rho = V_{qq}$. Suppose that $V(x,u,0) = V_q(x,u,0) = 0$ as in [12]. Then we have

$$\int_0^q (q-m)\rho(x,u,m)dm = \int_0^q (q-m)dV_q(x,u,m) = \int_0^q V_q(x,u,m)dm = V(x,u,q).$$

It follows that

$$V_u(x,u,q) = \int_0^q (q-m)\rho_u(x,u,m)dm,$$

$$V_q(x,u,q) = \int_0^q \rho(x,u,m)dm,$$

$$V_{qx}(x,u,q) = \int_0^q \rho_x(x,u,m)dm,$$

$$V_{qu}(x,u,q) = \int_0^q \rho_u(x,u,m)dm,$$

and

$$qV_{qq} = q\rho = \int_0^q \frac{d}{dm}(\rho(x,u,m)m)dm = \int_0^q (\rho(x,u,m) + \rho_q(x,u,m))dm.$$

Then (6.29) can be written as

$$\begin{aligned}&\int_0^q (q\rho_u(x,u,m) - m\rho_u(x,u,m) - \rho_x(x,u,m) - q\rho_u(x,u,m))dm\\&+ \int_0^q \Big(1 + \frac{2\delta}{\bar{\hat{\sigma}}^2(u)}\Big)(\rho(x,u,m) + \rho_q(x,u,m)m)dm = 0.\end{aligned}$$

To ensure (6.29) holds, it should be

$$m\rho_u(x,u,m) + \rho_x(x,u,m) - \Big(1 + \frac{2\delta}{\bar{\hat{\sigma}}^2(u)}\Big)(\rho(x,u,m) + \rho_q(x,u,m)m) = 0. \tag{6.30}$$

Formally, denote by $v$ the solution of the following equation

$$-v_{xx} - \Big(1 + \frac{2\delta}{\bar{\hat{\sigma}}^2(v)}\Big)v_x = 0, \tag{6.31}$$

with boundary conditions

$$v(x_0) = u_0, \; v_x(x_0) = q_0.$$

Then by (6.30) and (6.31), there holds

$$\frac{d}{dx}\rho = \rho_x + \rho_u v_x + \rho_q v_{xx} = \rho_x + \rho_u v_x - \rho_q\Big(1 + \frac{2\delta}{\bar{\hat{\sigma}}^2(v)}\Big)v_x = \Big(1 + \frac{2\delta}{\bar{\hat{\sigma}}^2(v)}\Big)\rho,$$

which can be solved as

$$\rho(x_0,u_0,q_0) = \exp\Big(\int_0^{x_0}\Big(1 + \frac{2\delta}{\bar{\hat{\sigma}}^2(v(z))}\Big)dz\Big).$$

Replacing $x_0$ by $x$, $u_0$ by $u$ and $q_0$ by $q$, we have

$$\rho(x,u,q) = \exp\Big(\int_0^{x}\Big(1 + \frac{2\delta}{\bar{\hat{\sigma}}^2(v(z))}\Big)dz\Big). \tag{6.32}$$

Integrating the Lyapunov function and assuming $E(t) \geq 0$, we have

$$\int_{t_0}^t \int_{-\infty}^{\infty} \frac{2}{\bar{\hat{\sigma}}^2}\rho(x,\bar{U}(s,x),\bar{U}_x(s,x))\bar{U}_s^2(s,x)dxds = E(t_0) - E(t) \leq E(t_0).$$

Following the formal construction, we proceed to the second step, namely that verify the integrability of those integrals arisen in the formal construction of the Lyapunov function. As indicated by Liang et al. [21], there are two problems, where the first one is that $\rho$ grows exponentially as $x \to \pm\infty$, while the second one is that the coefficient in (6.31) is discontinuous and the theory of ODE cannot be applied directly. These two problems will be overcame through the approximated solution of problem (4.1)-(4.3) with all the uniform estimates. We begin this step by defining

$$E_R[\bar{U}_\epsilon](t) = \int_{-R}^{R} V_\epsilon\Big(x, \bar{U}_\epsilon(t,x), \frac{\partial \bar{U}_\epsilon}{\partial x}(t,x)\Big)dx,$$

for $R > 0$, where $\bar{U}_\epsilon$ is the approximation of $\bar{U}$ with the coefficient of the approximated problem given as

$$\bar{\bar{\sigma}}_\epsilon = \bar{\bar{\sigma}}_1 + \sum_{i=1}^{n-1}(\bar{\bar{\sigma}}_{i+1} - \bar{\bar{\sigma}}_i)H_\epsilon(\bar{U}_\epsilon - \gamma_i),$$

and $V_\epsilon$ satisfies

$$V_\epsilon(x,u,q) = \int_0^q (q-m)\rho_\epsilon(x,u,m)dm, \ \ V_\epsilon(x,u,0) = \frac{\partial V_\epsilon}{\partial q}(x,u,0) = 0,$$

and $\rho_\epsilon$ is defined by (6.32) with $\bar{\bar{\sigma}}^2$ replaced by $\bar{\bar{\sigma}}_\epsilon^2$. Thus, (6.31) can be solved on the real line $x \in \mathbb{R}$ and $V_\epsilon$ is well defined. Meanwhile, it also satisfies

$$\frac{\partial V_\epsilon}{\partial u} - \frac{\partial^2 V_\epsilon}{\partial q \partial x} - q\frac{\partial^2 V_\epsilon}{\partial q \partial u} + q\frac{\partial^2 V_\epsilon}{\partial q^2}\Big(1 + \frac{2\delta}{\bar{\bar{\sigma}}_\epsilon^2(u)}\Big) = 0.$$

Then we have

$$\frac{d}{dt}E_R[\bar{U}_\epsilon] = \int_{-R}^{R}\Big(\frac{\partial V_\epsilon}{\partial u}\frac{\partial \bar{U}_\epsilon}{\partial t} + \frac{\partial V_\epsilon}{\partial q}\frac{\partial^2 \bar{U}_\epsilon}{\partial x \partial t}\Big)dx = \frac{\partial V_\epsilon}{\partial q}\frac{\partial \bar{U}_\epsilon}{\partial t}\Big|_{-R}^{R} - \int_{-R}^{R}\frac{2\rho_\epsilon}{\bar{\bar{\sigma}}_\epsilon^2}\Big(\frac{\partial \bar{U}_\epsilon}{\partial t}\Big)^2 dx.$$

**Lemma 6.1.** *Let $\bar{u}_\epsilon$ be the approximated solution of problem (6.18)-(6.19). Then for any $K_1 > 0$, there exist constants $C_0$, $K_2 > 0$ such that for $x > C_2$, there holds*

$$\left|\frac{\partial \bar{u}_\epsilon}{\partial x}\right| + \left|\frac{\partial \bar{u}_\epsilon}{\partial t}\right| \le C_0 e^{K_2 t - K_1 x},$$

*where $C_2$ is the constant given in Lemma 4.5.*

*Proof.* We know that for $x > C_2$

$$\mathscr{L}_{\sigma_1}^\epsilon[\bar{u}_\epsilon] = \mathscr{L}_{\sigma_1}^\epsilon\Big[\frac{\partial \bar{u}_\epsilon}{\partial t}\Big] = \mathscr{L}_{\sigma_1}^\epsilon\Big[\frac{\partial \bar{u}_\epsilon}{\partial x}\Big],$$

where the operator

$$\mathscr{L}_{\sigma_1}^\epsilon[\ \cdot\ ] = \frac{\partial}{\partial t} - \frac{\bar{\bar{\sigma}}_1^2}{2}\frac{\partial^2}{\partial x^2} - \Big(\frac{\bar{\bar{\sigma}}_1^2}{2} + \delta\Big)\frac{\partial}{\partial x}.$$

By Lemmas 4.2 and 4.3, it has been shown that

$$\sup_{0<t<\infty}\left(\left|\frac{\partial \bar{u}_\epsilon}{\partial t}(t,C_2)\right| + \left|\frac{\partial \bar{u}_\epsilon}{\partial x}(t,C_2)\right|\right) \le C,$$

for some $C > 0$. On the other hand, there holds

$$\frac{\partial \bar{u}_\epsilon}{\partial t}(0,x) = -\delta e^{-(x+\log K)},\ \frac{\partial \bar{u}_\epsilon}{\partial x}(0,x) = -e^{-(x+\log K)}$$

for $x > C_2$. Then for any $K_1 > 0$, there exist constants $C_0, K_2 > 0$ such that

$$\mathscr{L}^\epsilon_{\sigma_1}[C_0 e^{K_2 t - K_1 x}] \geq 0,\ x > C_2,\ t > 0,$$

and $C_0 \geq C$, $C_0 e^{-K_1 x} \geq (\delta \vee 1) e^{-(x+\log K)}$, which implies that

$$-C_0 e^{K_2 t - K_1 x} \leq \frac{\partial \bar{u}_\epsilon}{\partial x} \leq C_0 e^{K_2 t - K_1 x},\ -C_0 e^{K_2 t - K_1 x} \leq \frac{\partial \bar{u}_\epsilon}{\partial t} \leq C_0 e^{K_2 t - K_1 x}$$

for $x > C_2$. □

From the formula of $\rho_\epsilon$

$$\rho_\epsilon(x,u,q) = \exp\Big(\int_0^x \Big(1 + \frac{2\delta}{\bar{\bar{\sigma}}^2_\epsilon(v(z))}\Big)dz\Big),$$

we have

$$\exp\Big(\Big(1 + \frac{2\delta}{\bar{\bar{\sigma}}^2_n}\Big)x\Big) \leq \rho_\epsilon(x,u,q) \leq \exp\Big(\Big(1 + \frac{2\delta}{\bar{\bar{\sigma}}^2_1}\Big)x\Big), \tag{6.33}$$

which also clearly implies that

$$\exp\Big(\Big(1 + \frac{2\delta}{\bar{\bar{\sigma}}^2_n}\Big)x\Big)q \leq \frac{\partial V_\epsilon}{\partial q}(x,u,q) \leq \exp\Big(\Big(1 + \frac{2\delta}{\bar{\bar{\sigma}}^2_1}\Big)x\Big)q$$

and

$$\exp\Big(\Big(1 + \frac{2\delta}{\bar{\bar{\sigma}}^2_n}\Big)x\Big)q^2 \leq V_\epsilon(x,u,q) \leq \exp\Big(\Big(1 + \frac{2\delta}{\bar{\bar{\sigma}}^2_1}\Big)x\Big)q^2$$

for $x > 0$. Thus by Lemma 6.1, we can choose $K_1 > 1 + \frac{2\delta}{\bar{\bar{\sigma}}^2_1}$ such that

$$\lim_{R\to\infty} \frac{\partial V_\epsilon}{\partial q}(t,R)\frac{\partial \bar{U}_\epsilon}{\partial t}(t,R) = 0,$$

and

$$\lim_{R\to\infty}\int_{-R}^R \frac{2\rho_\epsilon}{\bar{\bar{\sigma}}^2_\epsilon}\Big(\frac{\partial \bar{U}_\epsilon}{\partial t}\Big)^2 dx = \int_{-\infty}^\infty \frac{2\rho_\epsilon}{\bar{\bar{\sigma}}^2_\epsilon}\Big(\frac{\partial \bar{U}_\epsilon}{\partial t}\Big)^2 dx.$$

Then there holds

$$\int_{t_0}^t \int_{-\infty}^\infty \frac{2\rho_\epsilon}{\bar{\bar{\sigma}}^2_\epsilon}\Big(\frac{\partial \bar{U}_\epsilon}{\partial s}\Big)^2 dxds \leq E_\infty[\bar{U}_\epsilon](t_0) \leq C,$$

where the constant $C$ is independent of $\epsilon$, which implies that

$$\int_{t_0}^\infty \int_0^\infty \bar{u}_t^2(s,x)dxds \leq C, \tag{6.34}$$

according to (6.28) and the setting of $\bar{U}$.

Now we arrive at the third step, completing the proof of convergence from $\bar{u}$ to $\psi$. Denote by $\bar{u}_m(t,x) = \bar{u}(t+m,x)$ and consider $\bar{u}_m$ as a sequence of functions on $[0,1]\times(0,\infty)$. Since $\bar{u}_m$ is a bounded sequence in $W^{1,2}_\infty([0,1]\times(0,\infty))$, we derive by the embedding theorem that there exists a subsequence $m_j$ of $m$ and a function $\tilde{\psi}$ such that as $m_j\to\infty$, there holds

$$\bar{u}_{m_j} \to \tilde{\psi} \text{ in } C^{\frac{(1+\alpha)}{2},1+\alpha}([0,1]\times(0,R)),\ 0<\alpha<1, \tag{6.35}$$

for any $R>1$. Furthermore, by taking a further subsequence if necessary, there holds that

$$\frac{\partial \bar{u}_{m_j}}{\partial t} \xrightarrow{w^*} \tilde{\psi}_t,\ \frac{\partial^2 \bar{u}_{m_j}}{\partial x^2} \xrightarrow{w^*} \tilde{\psi}_{xx} \text{ in } L^\infty([0,1]\times(0,\infty)),$$

and thus

$$\|\tilde{\psi}_t\|_{L^\infty}\le \liminf_{m\to\infty}\|\frac{\partial \bar{u}_{m_j}}{\partial t}\|_{L^\infty}\le C,\ \ \|\tilde{\psi}_{xx}\|_{L^\infty}\le \liminf_{m\to\infty}\|\frac{\partial^2 \bar{u}_{m_j}}{\partial x^2}\|_{L^\infty}\le C,$$

for some constant $C>0$. As (6.34) suggests that

$$\int_0^1\int_0^\infty \Big(\frac{\partial \bar{u}_m}{\partial t}\Big)^2 dxds = \int_m^{m+1}\int_0^\infty \bar{u}_t^2 dxds \to 0 \text{ as } m=m_j\to\infty.$$

we have

$$\int_0^1\int_0^\infty \tilde{\psi}_t^2 dxdt = 0,$$

which implies that $\tilde{\psi}_t\equiv 0$ and $\tilde{\psi}$ is independent of $t$ and only depends on $x$. Now we proceed to prove that $\tilde{\psi}$ satisfies (6.1). Take a test function $f\in C_c^\infty(0,\infty)$, then there holds

$$\int_0^\infty \frac{\partial \bar{u}_m}{\partial t} f dx = \int_0^\infty \frac{1}{2}\bar{\bar{\sigma}}^2(\bar{u}_m)\Big(\frac{\partial^2 \bar{u}_m}{\partial x^2}+\frac{\partial \bar{u}_m}{\partial x}\Big) f dx + \int_0^\infty \delta\frac{\partial \bar{u}_m}{\partial x} f dx. \tag{6.36}$$

Clearly, the second term on the right side converges to the corresponding integral of $\tilde{\psi}$ as $m=m_j\to\infty$. With regard to the first term on the right side of (6.36), there holds

$$\begin{aligned}&\int_0^\infty \frac{1}{2}\bar{\bar{\sigma}}^2(\bar{u}_m)\Big(\frac{\partial^2 \bar{u}_m}{\partial x^2}+\frac{\partial \bar{u}_m}{\partial x}\Big) f dx\\ =&\int_0^\infty \frac{1}{2}\bar{\bar{\sigma}}^2(\tilde{\psi})\Big(\frac{\partial^2 \bar{u}_m}{\partial x^2}+\frac{\partial \bar{u}_m}{\partial x}\Big) f dx + \int_0^\infty \frac{1}{2}(\bar{\bar{\sigma}}^2(\bar{u}_m)-\bar{\bar{\sigma}}^2(\tilde{\psi}))\Big(\frac{\partial^2 \bar{u}_m}{\partial x^2}+\frac{\partial \bar{u}_m}{\partial x}\Big) f dx.\end{aligned} \tag{6.37}$$

By the weak-star convergence, the first term on the right side of (6.32) converges to the corresponding integral of $\tilde{\psi}$. The second term on the right side of (6.37) is bounded by $C\int_0^\infty |\bar{\bar{\sigma}}^2(\bar{u}_m)-\bar{\bar{\sigma}}^2(\tilde{\psi})| f dx$, which converges to 0 by the dominated convergence theorem. By the convergence (6.35), we have

$$\int_0^1\int_0^\infty \frac{\partial \bar{u}_m}{\partial t} f dxdt \to 0 \text{ as } m\to\infty.$$

Thus integrating (6.36) with respect to $t$ over $[0,1]$ and letting $m\to\infty$, there holds

$$\int_0^\infty \frac{1}{2}\bar{\bar{\sigma}}^2(\tilde{\psi})(\tilde{\psi}_{xx}+\tilde{\psi}_x) f dx + \int_0^\infty \delta\tilde{\psi}_x f dx = 0.$$

It follows that $\tilde{\psi}$ satisfies (6.1). The convergence (6.35) also suggests that $\tilde{\psi}$ satisfies (6.2). Now suppose that

$$\liminf_{m_j\to\infty}\inf_{0\le t\le 1}\bar{\lambda}_i(t+m_j)=\bar{\lambda}_i^{min}\le\bar{\lambda}_i^{max}=\limsup_{m_j\to\infty}\sup_{0\le t\le 1}\bar{\lambda}_i(t+m_j),\ i=1,2,\cdots,n-1. \tag{6.38}$$

We choose $t_{i,j}^{min}, t_{i,j}^{max}\in[0,1]$ such that

$$\inf_{0\le t\le 1}\bar{\lambda}_i(t+m_j)=\bar{\lambda}_i(t_{i,j}^{min}+m_j),\quad \sup_{0\le t\le 1}\bar{\lambda}_i(t+m_j)=\bar{\lambda}_i(t_{i,j}^{max}+m_j).$$

Taking the subsequences along which the liminf and limsup in (6.38) are attained, together with the boundary conditions

$$\bar{u}_m(t,\bar{\lambda}_i(t+n))=\gamma_i,\ i=1,2,\cdots,n-1,$$

and (6.35), it is deduced that

$$\tilde{\psi}(\bar{\lambda}_i^{min})=\tilde{\psi}(\bar{\lambda}_i^{max})=\gamma_i,\ i=1,2,\cdots,n-1.$$

However, by the uniqueness of solution to static problem (6.1)-(6.3), there should hold

$$\bar{\lambda}_i^{min}=\bar{\lambda}_i^{max}=\lambda_i^*,\ i=1,2,\cdots,n-1,$$

and $\psi\equiv\tilde{\psi}$. In addition, the uniqueness implies that all subsequences limit should be uniform and thus the full sequence must converge as $m\to\infty$.

## 7. Conclusion

In this paper, we study a free boundary problem for pricing a defaultable corporate bond with multiple credit rating migration risk and stochastic interest rate. By using PDE techniques, the existence, uniqueness some regularities of solution are obtained to support the rationality of the model to pricing a defaultable corporate bond. In [35], it is shown that the solution and rating boundaries of the free boundary problem pricing a defaultable corporate bond with constant interest rate are all decreasing with respect to time, which do not hold any more in our model with stochastic interest rate, as the coefficients of model are all time heterogeneous. Furthermore, we present the asymptotic behavior of solution to this pricing model. Asymptotic behaviors of solution to free boundary problems pricing corporate bonds with credit rating migration risk have been analyzed in [12, 21, 32]. The asymptotic solution of the model with only one migration boundary can be solved explicitly [21], while in the works [12, 32], where the models are subject to multiple migration boundaries, it is not the case. However, interestingly, in this paper, although our model is also subject to multiple migration boundaries, it is proved that the asymptotic solution can be solved explicitly. We conclude that if the maturity $T$ is sufficiently large, we can valuate the defaultable corporate bond with multiple credit rating migration risk and stochastic interest rate by an explicit pricing formula as follows:

$$\phi(t)=S(t)\psi(\log S(t)-\log K),$$

where $S(t)$ is the corporate value, $K$ is default threshold and $\psi$ is the steady status given in Section 6, whose explicit form can be obtained by solving the linear algebraic equation set.

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## Conflict of interest

All authors declare no conflict of interest.

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