Research article

Ruled surfaces with constant Disteli-axis

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Abstract: This work develops the kinematic-geometry for ruled surfaces by using the analogy with spherical kinematics. This provides the ability to compute set of curvature functions which define the local shape of ruled surfaces. Hence, the well known equation of the Plücker's conoid has been obtained and its kinematic-geometry are examined in details. Finally, a characterization for a line trajectory to be a constant Disteli-axis is derived, and investigated.

Keywords: Blaschke frame; E. study map; line complex; Disteli-axis
Mathematics Subject Classification: 53A04, 53A05, 53A17

1. Introduction

Line geometry possesses a close relation to spatial kinematics and has therefore found applications in mechanism design and robot kinematics. In spatial kinematics, it is interesting to study the intrinsic properties of the line trajectory from the concepts of ruled surface in differential geometry [1–4]. As it is known, dual numbers are used to investigate the motion of a line trajectory. Hence, the E. Study's map leads to the conclusion that the family of all directed lines in Euclidean three-space $\mathbb{E}^3$ is one-to- one correspondent with family of points of the dual unit sphere in the dual three- space $\mathbb{D}^3$ (see [1–3, 6, 9–12]).

This work offers an approach for constructing ruled surfaces with constant Disteli-axis by using E. Study map. Meanwhile, we determine kinematic- geometry of the Plücker conoid and its characterization. Finally, the well-known Dupin's indicatrix in surfaces theory has been defined for the kinematic geometry of the Plücker's conoid. Moreover, some characterizations equations for special ruled surfaces undrgoing one-parameter scrow motion are obtained and investigated. In addition, we have given some necessary and sufficient conditions to have constant dual angles with respect to constant Disteli-axis and we have discussed some special cases which lead to some special ruled surfaces such as the constant helicoids, One-sheeted circular hyperboloids, Archimedecs

helicoids , circular cones and right helicoids.

2. Basic concepts

Here, we give a brief introduction of some basic definitions and formulae that are associated to the topic (See for instance Refs. [6,10–12]). Recall that any directed line L in the Euclidean 3-space $\mathbb{E}^3$ has a representation in terms of a point $\mathbf{p}$ on L and a directed vector $\mathbf{x}$ of L such that $\|\mathbf{x}\| = 1$. Therefore,the moment vector about the origin 0 in $\mathbb{E}^3$ can be given as

$$\mathbf{x}^* = \mathbf{p} \times \mathbf{x}. \tag{1}$$

$\mathbf{p}$ can be replaced by any point

$$\mathbf{q} = \mathbf{p} + \mu\mathbf{x}, \ \mu \in \mathbb{R} \tag{2}$$

on L, then $\mathbf{x}^*$ does not depend on the choice of $\mathbf{p}$ on line. Clearly, $\mathbf{x}$ and $\mathbf{x}^*$ are dependent of each other. Moreover, we have

$$< \mathbf{x}^*, \mathbf{x} >=0, \quad < \mathbf{x}, \mathbf{x} >=1. \tag{3}$$

The components x_i^*, $x_i (i = 1, 2, 3)$ of $\mathbf{x}^*$ and $\mathbf{x}$ are referred to as the normalized Plűcker coordinates of L. Therefore, L can be determined by the two vectors $\mathbf{x}$ and $\mathbf{x}^*$.

The dual numbers set is

$$\mathbb{D} = \{X = x + \varepsilon x^* \mid x, \ x^* \in \mathbb{R}, \ \varepsilon \neq 0, \ \varepsilon^2 = 0\}, \tag{4}$$

which is a commutative ring. A dual number $X = a + \varepsilon x^*$, is referred to as proper if $x \neq 0$. A dual angle formed by two skew lines in $\mathbb{E}^3$ gives an example of dual number in which it is given by $\Theta = \vartheta + \varepsilon\vartheta^*$; where ϑ and ϑ^* are the projected angle and the minimal distance between the lines, respectively.

Now, the set

$$\mathbb{D}^3 = \{\mathbf{X} = \mathbf{x} + \varepsilon\mathbf{x}^*; \ (\mathbf{x}, \mathbf{x}^*) \in \mathbb{E}^3 \times \mathbb{E}^3 \text{ and } \varepsilon \neq 0, \ \varepsilon^2 = 0\}, \tag{5}$$

with the inner product

$$< \mathbf{X}, \mathbf{Y} >=< \mathbf{x}, \mathbf{y} > + \varepsilon(< \mathbf{y}, \mathbf{x}^* > + < \mathbf{y}^*, \mathbf{x} >), \tag{6}$$

constructs the dual 3-space $\mathbb{D}^3$. Hence, a point $\mathbf{X} = (X_1, X_2, X_3)^t$ has dual coordinates $X_i = (x_i + \varepsilon x_i^*) \in \mathbb{D}$. The norm is given as

$$< \mathbf{X}, \mathbf{X} >^{\frac{1}{2}} := \|\mathbf{X}\| = \|\mathbf{x}\| \, (1 + \varepsilon \frac{< \mathbf{x}, \mathbf{x}^* >}{\|\mathbf{x}\|^2}). \tag{7}$$

The set

$$\mathbb{K} = \{\mathbf{X} \in \mathbb{D}^3 \mid \|\mathbf{X}\|^2 = X_1^2 + X_2^2 + X_3^2 = 1\}. \tag{8}$$

gives the dual unit sphere in $\mathbb{D}^3$. All points $\mathbf{X}$ in $\mathbb{K}$ satisfy the two formulae

$$x_1^2 + x_2^2 + x_3^2 = 1, \quad x_1 x_1^* + x_2 x_2^* + x_3 x_3^* = 0. \tag{9}$$

Therefore, E. Study's map ([6, 10–12]) states that: There is a one to one map between the set of all oriented lines in $\mathbb{E}^3$ and the set of points of dual unit sphere in $\mathbb{D}^3$.

Definition 1. A family of oriented lines $L = (\mathbf{a}, \mathbf{a}^*) \in \mathbb{E}^3$ satisfying

$$C : \ < \mathbf{a}^*, \mathbf{x} > + < \mathbf{x}^*, \mathbf{a} >=0, \tag{10}$$

where $X = (\mathbf{x}, \mathbf{x}^*) \in \mathbb{E}^3$, is referred to as a line complex when $< \mathbf{x}, \mathbf{x}^* > \neq 0$ and is a singular (special) line complex when

$$< \mathbf{x}^*, \mathbf{x} >=0, \quad < \mathbf{x}, \mathbf{x} >=1. \tag{11}$$

Geometrically, a singular line complex is a family of all lines $L = (\mathbf{a}, \mathbf{a}^*)$ intersecting the line $\mathbf{X} = (\mathbf{x}, \mathbf{x}^*)$ [6, 10–12]. From E. Study's map, line complex can be defined using four independent parameters, so we can produce a finite number of lines, line congruence, given by intersecting any two line complexes. The intersection of two independent line congruences forms a differentiable set of lines in $\mathbb{E}^3$ defined as a ruled surface.

Ruled surfaces (such as cones and cylinders) contain generating lines in which the tangent plane touches the surface along the generator. Such generating lines are referred to as torsal lines:
• Ruled surfaces formed by only torsal generating line are known as developable surfaces;
• Ruled surfaces, in which most of its generators are non-torsal, are known as skew ruled surfaces;
• Ruled surfaces consisting of tangents of a spatial curve, , cones, and cylinders are developable surfaces.

One-parameter dual spherical motions

Let $\mathbb{K}_m$ and $\mathbb{K}_f$ be two dual unit spheres with the common center $\mathbf{O}$ and two orthonormal dual coordinate frames $\{\mathbf{O}; \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and $\{\mathbf{O}; \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3\}$, respectively. Let us assume that the frame $\{\mathbf{O}; \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3\}$ is fixed, and the set $\{\mathbf{O}; \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ consists of real functions of parameter t (say the time). Then under assumption, $\mathbb{K}_m$ moves about $\mathbb{K}_f$ and such motion is denoted by $\mathbb{K}_m/\mathbb{K}_f$ and referred to as a one-parameter dual spherical motion. If the line spaces $\mathbb{H}_m$ and $\mathbb{H}_f$ correspond to $\mathbb{K}_m$ and $\mathbb{K}_f$, respectively, then the one-parameter spatial motion $\mathbb{H}_m/\mathbb{H}_f$ corresponds to $\mathbb{K}_m/\mathbb{K}_f$. Therefore $\mathbb{H}_m$ moves about the fixed space $\mathbb{H}_f$. Let $A = (A_{ij})$ be a dual matrix. By putting $< \mathbf{F}_i, \mathbf{E}_j >= A_{ij}$, the E. Study map can be written as [2, 3, 5]:

$$\mathbb{K}_m/\mathbb{K}_f : \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix}. \tag{12}$$

The dual matrix $A := (A_{ij}) = a + \varepsilon a^* = (a_{ij}) + \varepsilon(a_{ij}^*)$ satisfies that $A(u)A(u)^t = \mathbb{I}_3$, where $\mathbb{I}_3$ is the 3×3 identity matrix. Hence, $\det A(t) = 1$ and $A(t)$ is an orthogonal dual matrix, . The Lie algebra $\mathbb{L}(\mathbb{O}_{\mathbb{D}^{3\times3}})$ of the group of positive orthogonal dual matrices A of rank 3×3 is the algebra of skew-symmetric dual matrices of rank 3×3

$$\Omega(t) = A'A^t = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix},$$

where $A' = \dfrac{dA(t)}{dt}$. Let us consider the following convention

$$\Omega(t) = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = \boldsymbol{\Omega}. \tag{13}$$

Consequently, we can consider the elements in $\mathbb{L}(\mathbb{O}_{\mathbb{D}^{3\times3}})$ as vectors of the usual 3-dimensional dual vector space or skew-symmeteric matrices. If dash refers to the derivation about t, then the derivative equation of the motion $\mathbb{K}_m/\mathbb{K}_f$ is:

$$\begin{pmatrix} \mathbf{E}'_1 \\ \mathbf{E}'_2 \\ \mathbf{E}'_3 \end{pmatrix} = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix} = \boldsymbol{\Omega}\times\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix}. \tag{14}$$

where $\boldsymbol{\Omega}(t)= \omega+\varepsilon\omega^* = (\Omega_1, \Omega_2, \Omega_3)$ is known as the instantaneous dual rotation vector of $\mathbb{K}_m/\mathbb{K}_f$. ω^* and ω, respectively, are real vectors corresponding to the instantaneous translational differential velocity vector and the instantaneous rotational differential velocity vector of the motion $\mathbb{H}_m/\mathbb{H}_f$ [2–5].

3. Ruled surfaces with constant Disteli-axis

During the motion $\mathbb{H}_m/\mathbb{H}_f$, every fixed line $\mathbf{X}$ of the moving space $\mathbb{H}_m$, constructs a ruled surface in the fixed space $\mathbb{H}_f$ denoted by (X). In kinematics, such ruled surface is called line trajectory. To study the geometrical aspects of a line trajectory, we choose a moving frame, called the Blaschke frame, associated with the point on $\mathbb{K}_m$ [2–5] as:

$$\mathbf{X} = \mathbf{X}(t), \ \mathbf{T}(t) := \mathbf{t}(t)+\varepsilon\mathbf{t}^*(t) = \frac{\mathbf{X}'}{\|\mathbf{X}'\|}, \ \mathbf{G}(t) := \mathbf{g}(t)+\varepsilon\mathbf{g}^*(t) = \mathbf{X} \times \mathbf{T}. \tag{15}$$

$\mathbf{X}$, $\mathbf{T}$, and $\mathbf{G}$ are three dual unit vectors corresponding to three concurrent mutually perpendicular lines $\mathbb{E}^3$ and they intersect at a point $\mathbf{C}$ on $\mathbf{X}$ called central point. $\mathbf{G}(t)$ is the limit position of the shared perpendicular to $\mathbf{X}(t)$ and $\mathbf{X}(t + dt)$, and is referred to as the central tangent of (X) at $\mathbf{C}$. All central points form the striction curve. The vector $\mathbf{T}$ is known as the central normal of $\mathbf{X}$ at $\mathbf{C}$. By construction, the Blaschke motion $\mathbb{K}_m/\mathbb{K}_f$ is given by

$$\mathbb{K}_m/\mathbb{K}_f : \begin{pmatrix} \mathbf{X}' \\ \mathbf{T}' \\ \mathbf{G}' \end{pmatrix} = \begin{pmatrix} 0 & P & 0 \\ -P & 0 & Q \\ 0 & -Q & 0 \end{pmatrix}\begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix} = \boldsymbol{\Omega}\times\begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix}, \tag{16}$$

where

$$P = p + \varepsilon p^* = \left\|\mathbf{X}'\right\|, \ Q := q + \varepsilon q^* = \det(\mathbf{X}, \mathbf{X}', \mathbf{X}'')\left\|\mathbf{X}'\right\|^{-2}, \tag{17}$$

are called the Blaschke invariants of (X), and $\boldsymbol{\Omega}(t) =Q\mathbf{X}+P\mathbf{G}$. The tangent of the striction curve $\mathbf{C}$ may be written as

$$\mathbf{C}' = q^*\mathbf{x} + p^*\mathbf{g}. \tag{18}$$

Furthermore, the distribution parameters of the surfaces (X), (T), and (G), respectively, are:

$$\mu(t) = \frac{p^*}{p}, \delta(t) = \frac{pp^* + qq^*}{q^2 + p^2}, \text{ and } \Gamma(t) = \frac{q^*}{q}. \tag{19}$$

Now,

$$\mathbf{B}(t) = \mathbf{b}(t) + \varepsilon \mathbf{b}^*(t) = \frac{\mathbf{\Omega}}{\|\mathbf{\Omega}\|} = \frac{Q\mathbf{X} + P\mathbf{G}}{\sqrt{Q^2 + P^2}}, \tag{20}$$

is a dual unit vector such that $\Omega := \omega + \varepsilon \omega^* = \|\mathbf{\Omega}\|$ is referred to as dual angular speed of the Blaschke frame. Clearly, $\mathbf{B}$ is the Disteli-axis (striction axis or evolute) of (X). For each generating line of a cylindrical surface, the Disteli-axis and the curvature axis of the perpendicular sections are identified. From Eq (20), the Disteli-axis is perpendicular to the central normal $\mathbf{T}$, and is parallel to the tangent plane of (X). Let $\Psi = \psi + \varepsilon \psi^*$ be a dual angle between $\mathbf{B}$ and $\mathbf{X}$, then we have that:

$$\mathbf{B}(t) = \cos \Psi \mathbf{X} + \sin \Psi \mathbf{G}, \tag{21}$$

where

$$\cot \Psi = \frac{Q}{P}. \tag{22}$$

Note that Ψ is the dual spherical radius of curvature. The trigonometric function in Eq (22) of Ψ can be expanded as:

$$\cot \Psi = \cot \psi - \varepsilon \psi^*(1 + \coth^2 \psi) = \frac{q + \varepsilon q^*}{p + \varepsilon p^*}. \tag{23}$$

One of the invariants of the dual curve $\mathbf{X}(t)$ on $\mathbb{K}_f$ is

$$\Sigma(t) := \gamma + \varepsilon (\Gamma - \mu\gamma) = \frac{Q}{P}, \tag{24}$$

called the dual geodesic curvature. Here $\gamma(t) = \frac{q}{p}$ is the geodesic curvature of the spherical image curve $t \in I \mapsto \mathbf{x}(t)$ of (X). Hence, from the dual and real parts of Eqs (23) and (24), respectively, we get

$$\psi^*(t) = \frac{1}{2} (\mu - \Gamma) \sin 2\psi, \tag{25}$$

and

$$\gamma(t) = \cot \psi = \frac{q}{p}. \tag{26}$$

Here, ψ^* is the normal distance from $\mathbf{X}$ to the Distelli-axis $\mathbf{B}$ along the central tangent of (X). The functions $\Gamma(t)$, $\mu(t)$ and $\gamma(t)$ are referred to as construction parameters of (X) or the curvature functions.

3.1. Kinematic-geometry and Plücker's Conoid

We now are interesting to research kinematics and the geometrical properties of the line trajectory (X). So, we are going to do a detailed study of the Blaschke invariants $P(t)$, and $Q(t)$. To carry out this, from Eq (16), we can write the following equations:

$$\mathbf{X}'(t) = (\|\mathbf{\Omega}\| \mathbf{B}) \times \mathbf{X}, \quad \mathbf{T}'(t) = (\|\mathbf{\Omega}\| \mathbf{B}) \times \mathbf{T}, \quad \mathbf{G}'(t) = (\|\mathbf{\Omega}\| \mathbf{B}) \times \mathbf{G}. \tag{27}$$

Hence, at any instant, it is seen that: $\|\mathbf{\Omega}\| = \Omega = \omega + \varepsilon \omega^*$ is the dual angular speed of the motion $\mathbb{K}_m / \mathbb{K}_f$ about $\mathbf{B}$. The parts

$$\omega^*(t) = \frac{pp^* + qq^*}{\sqrt{q^2 + p^2}}, \text{ and } \omega(t) = \sqrt{p^2 + q^2}, \tag{28}$$

corresponding to translational angular speed and the rotational angular speed of the motion $\mathbb{H}_m/\mathbb{H}_f$ along **B**, respectively.

Hence, we have the following corollary:

Corollary 1. For the one-parameter Blaschke motion, at any instant t, the pitch of the motion $h(t)$ can be given by

$$h(t) := \frac{<\omega, \omega^*>}{\|\omega\|^2} = \mu \cos^2 \psi + \Gamma \sin^2 \psi. \tag{29}$$

Now, assume that the dual vector $\mathbf{\Omega} = \omega + \varepsilon\omega^*$ is given, then the list of the following physical properties of the Blaschke motion can be determined:
(i) The Disteli-axis **B** can be represented by Eq (20).
(ii) The dual angular speed of its dual angular velocity can be expressed as $\|\mathbf{\Omega}\| = \omega(1 + \varepsilon h)$.
(iii) If **y** is a point on the Disteli-axis **B**, then we get:

$$\mathbf{y}(t, v) = \mathbf{b} \times \mathbf{b}^* + v\mathbf{b}, \ v \in \mathbb{R}. \tag{30}$$

This parametrization defines a non-developable ruled surface (B). If $h(u) = 0$, i.e., the Blaschke motion is pure rotation, then

$$\mathbf{B}(t) = \mathbf{b}(t) + \varepsilon\mathbf{b}^*(t) = \frac{1}{\|\omega\|}(\omega + \varepsilon\omega^*), \tag{31}$$

whereas if $h(t) = 0$, and $\|\omega\|^2 = 1$, then $\mathbf{\Omega}$ is an directed line. However, if $\mathbf{\Omega} = 0 + \varepsilon\omega^*$, i.e., of the Blaschke motion is pure translational, we put $\omega^* = \|\mathbf{\Omega}^*\|$; $\omega^*\mathbf{b} = \omega^*$ for arbitrary $\mathbf{b}^*$ such that $\omega^* \neq 0$, otherwise **b** can be arbitrarily chosen, too.

The Eqs in (26) and (29), respectively, are resemblances to the Mannhiem and Hamilton formulae of surfaces theory in Euclidean 3-space. Now we intend to give geometrical interpretations of these formulae. Next, the parametric representation of ψ^* in Eq (26) is the well known Plücker's conoid, or cylindroid. The Plücker conoid is a differentiable regular ruled surface [6, 10, 11]. The parametric representation can also be given by using point coordinates. To do that, let **T** and y−axis of a fixed Euclidean space frame $(oxyz)$ be coincident and the position of the dual unit vector **B** be determined by angle ψ and distance ψ^* along the positive direction of y−axis. The dual unit vectors **X** and **G** can be chosen in terms of x and z-axes, respectively. This shows that the dual unit vectors **X** and **G** together with **T** form the coordinate system of the Plücker's conoid, as depicted in Figure 1. Let **y** be a point on the surface, then we have the point coordinates

$$M : \mathbf{y}(t, v) = (0, \psi^*, 0) + v(\cos \psi, 0, \sin \psi), \ v \in \mathbb{R}, \tag{32}$$

Using such representation, the generator **B** is clearly visible passing through the y−axis. Thus,

$$\psi^* := y = \frac{1}{2}(\mu - \Gamma)\sin 2\psi, \ x = v\cos\psi, \text{ and } z = v\sin\psi, \tag{33}$$

where ψ^* gives the point of the intersection of the principal axes **X** and **G** which locates at a half of the conoid height. By direct calculations,

$$\left(y^2 + z^2\right)y + (\Gamma - \mu)zy = 0, \tag{34}$$

which is the algebraic equation for Plücker's conoid. The Plücker's conoid described by Eq (34) depends only on the difference of its two integral invariants of the first order; $\Gamma - \mu = 1$, $0 \le \psi \le 2\pi$, $-4 \le \upsilon \le 4$ (Figure 2). Furthermore, one can get a second-order equation in $\frac{y}{z}$ in which its solutions are given by

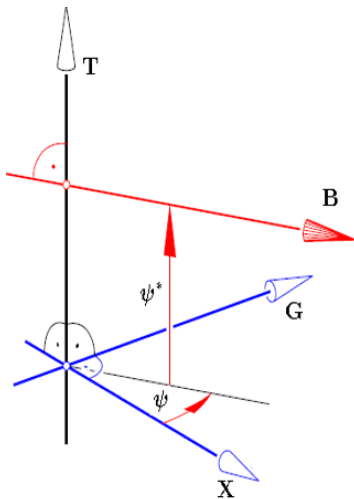$$\frac{y}{z} = \frac{1}{2y}\left[\mu - \Gamma \pm \sqrt{(\Gamma - \mu)^2 - 4y^2}\right]. \tag{35}$$

Figure 1. $\mathbf{B} = cos\Psi\mathbf{X} + sin\Psi\mathbf{G}$.
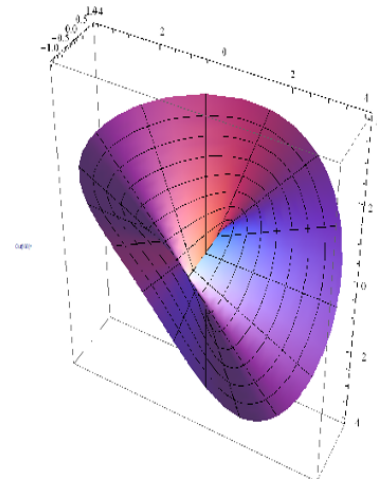
Figure 2. Plücker's conoid.

By equating the discriminant of Eq (35) to zero, we get the limits of the Plücker's conoid. Hence, the two extreme positions are given by,

$$y = \pm (\Gamma - \mu)/2. \tag{36}$$

Equation (36) gives the positions of the two torsal planes, each of which contains one torsal line L.

On the other hand, $h(u)$ in Eq (29) is periodic function and to have at most two extreme values, the curvature functions μ and Γ. Therefore, the directed lines $\mathbf{X}$ and $\mathbf{G}$ are principal axes of the Plücker's conoid. However, the geometric aspects of the Plücker's conoid are discussed as follows:

(i)- If $h(t) \ne 0$, then we have two real generating lines moving through the point $(0, y, 0)$ if $y < (\Gamma - \mu)/2$; and for the two limit points $y = \pm (\Gamma - \mu)/2$, the generators and the principal axes $\mathbf{X}$ and $\mathbf{G}$ are coincident.

(ii)- If $h(t) = 0$, then the two torsal lines L_1, L_2 are represented by

$$L_1, \ L_2 : \ \frac{x}{z} = \cot\psi = \pm \sqrt{-\frac{\mu}{\Gamma}}, \ y = \pm (\Gamma - \mu)/2, \tag{37}$$

which shows that the two torsal lines L_1, and L_2 are perpendicular each other. So, if μ and Γ are equal, then the Plücker's conoid becomes pencil of lines through the origin "**o**" in the torsal plane $y = 0$. In this case L_1 and L_2 are the principal axes of an elliptic line congruence. However, if μ and Γ have opposite signs, then L_1 and L_2 are real and to be coincident with the principal axes of a hyperbolic line congruence. If either μ or Γ is zero then the lines L_1 and L_2 both coincide with y−axis; for $\mu \ne 0$, $\Gamma = 0$ or for $\Gamma \ne 0$, $\mu = 0$ they coincide with z−axis.

Furthermore, to convert from polar coordinates to Cartesian, we use the substitution

$$y = \frac{\cos \psi}{\sqrt{h}}, \ z = \frac{\sin \psi}{\sqrt{h}},$$

in Hamilton's formula, one obtain the equation

$$D : |\mu| y^2 + |\Gamma| z^2 = 1,$$

of a conic section. This conic section is the Dupin's indicatrix of the Plücker's conoid. We now examine three cases in detail:

(1) If μ, and Γ are both positive, the Dupin's indicatrix is an ellipse has the principal semi-axes are $\frac{1}{\sqrt{\mu}}$, and $\frac{1}{\sqrt{\Gamma}}$. The lines through the center intersects the ellipse in the points:

$$y = \pm \frac{\cos \psi}{\sqrt{h}}, \ z = \pm \frac{\sin \psi}{\sqrt{h}}.$$

The distance intercepted by the ellipse on the line $\frac{z}{y} = \tan \psi$ is:

$$\sqrt{y^2 + z^2} = \frac{1}{\sqrt{h}}.$$

(2) If μ, and Γ have opposite signs the Dupin's indicatrix is set of conjugate hyperbolas

$$D : \mu x^2 \mp \Gamma y^2 = \pm 1.$$

The two asymptotic directions of the hyperbolas represent the torsal lines at which $h = 0$;

(3) If either μ or Γ is zero the Dupin's indicatrix is a set of parallels lines corresponding to one of the forms:

$$z^2 = \left| \frac{1}{\Gamma} \right| \text{ with } \mu = 0, \text{ or } y^2 = \left| \frac{1}{\mu} \right| \text{ with } \Gamma = 0.$$

Serret–Frenet motion

If $p^* = 0$ in Eq (18), then (X) is a tangential developable ruled surface, that is, $\mathbf{C}' = \mathbf{x}$. In this case, the Blaschke frame $\{\mathbf{x}, \mathbf{t}, \mathbf{g}\}$ coincides with the classic Serret–Frenet frame and then the striction curve $\mathbf{C}$ becomes the edge of regression of (X). Hence, p and q are the curvature κ, and the torsion τ of $\mathbf{C}$, respectively. Moreover, $q^* = 1$, and $\Gamma = 1/\tau$. Also Γ is the radius of torsion of $\mathbf{C}$. By similar argument, we can also have the following equations:

$$h(u) = \frac{\tau}{\tau^2 + \kappa^2} = \frac{1}{\tau} \sin^2 \psi, \quad \psi^* = \frac{1}{2\tau} \sin 2\psi, \quad \text{with } \tan \psi = \frac{\tau}{\kappa}. \tag{38}$$

The corresponding Plücker conoid is

$$M : \zeta(t, v) = (0, \psi^*, 0) + v(\cos \psi, 0, \sin \psi), \quad \upsilon \in \mathbb{R}. \tag{39}$$

This yields

$$\left(z^2 + y^2 \right) y + \Gamma z y = 0. \tag{40}$$

3.2. Constant Disteli-axis ruled surfaces

In the following, when we say (X) is a constant Disteli-axis, we mean that all the generators of the ruled surface (X) have a constant dual angle from its Disteli-axis. The dual arc length $\widehat{ds} = ds + \varepsilon ds^*$ of $\mathbf{X}(t)$ is

$$\widehat{s}(t) = \int_0^u P dt = \int_0^u p(1 + \varepsilon\mu)dt. \tag{41}$$

Now, we use the dual arc length parameter $\widehat{s}$ instead of t. If the prime means to differentiate with respect to $\widehat{s}$, then from Eq (16), we get

$$\begin{pmatrix} \mathbf{X}' \\ \mathbf{T}' \\ \mathbf{G}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \Sigma \\ 0 & -\Sigma & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix} = \Omega\times \begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix}, \tag{42}$$

where $\Omega = \Sigma\mathbf{X} + \mathbf{G}$. Thus, we may write the following relationships:

$$\widehat{\kappa} := \kappa + \varepsilon\kappa^* = \sqrt{1 + \Sigma^2} = \sin\Psi = \frac{1}{\widetilde{\rho}}, \widehat{\tau} := \tau + \varepsilon\tau^* = \pm\frac{\Sigma'}{\sqrt{1 + \Sigma^2}} = \pm\Psi'. \tag{43}$$

Here, $\widehat{\kappa}$ and $\widehat{\tau}$ refer to the dual curvature function and the dual torsion function of $\mathbf{X} = \mathbf{X}(\widehat{s})$, respectively. The terms found in Eq (43) are analogous to their counterparts in 3-dimensional Euclidean spherical geometry.

Definition 2. For one-parameter Blaschke motion, at the instant $\widehat{s} \in \mathbb{D}$, a directed line $\mathbf{Y}$ in fixed space is said to be $\mathbf{B}_k$-Disteli-axis of (X)

$$\left. \begin{array}{l} < \mathbf{Y}, \mathbf{X}^i(\widehat{s}) >= 0 \ , \ \ 1 \leq i \leq k \\ < \mathbf{Y}, \mathbf{X}^{k+1}(\widehat{s}) > \neq 0. \end{array} \right\}$$

Here $\mathbf{X}^i$ refers to the i-th differentiation of $\mathbf{X}$.

Based on above, consider

$$\widetilde{\rho} = \cos^{-1}(< \mathbf{Y}, \mathbf{X} >), \tag{44}$$

which a dual angle such that $\mathbf{Y}$, and $\widetilde{\rho}$ remain constant up to the second order at $\widehat{s} = \widehat{s}_0$, hence

$$\widetilde{\rho}' \mid \widehat{s} = \widehat{s}_0 = 0, \ \ \mathbf{X}' \mid \widehat{s} = \widehat{s}_0 = \mathbf{0}, \tag{45}$$

and

$$\widetilde{\rho}'' \mid \widehat{s} = \widehat{s}_0 = 0, \ \mathbf{X}'' \mid \widehat{s} = \widehat{s}_0 = \mathbf{0}.$$

Therefore,

$$< \mathbf{X}', \mathbf{Y} > = 0, \tag{46}$$

and

$$< \mathbf{X}'', \mathbf{Y} > = 0.$$

Then, $\widetilde{\rho}$ is invariant in the second approximation if and only if $\mathbf{Y}$ is the Disteli-axis $\mathbf{B}$ of (X), i.e.,

$$\widetilde{\rho}' = \widetilde{\rho}'' = 0 \Leftrightarrow \mathbf{Y} = \frac{\mathbf{X}' \times \mathbf{X}''}{\|\mathbf{X}' \times \mathbf{X}''\|} = \pm \mathbf{B}. \tag{47}$$

Now, we shall act as we have done in the last. Using the definition of the Disteli-axis, we get the dual frame;

$$\mathbf{U}_1 = \mathbf{B}(\widehat{s}), \ \mathbf{U}_2(\widehat{s}) = \frac{\mathbf{B}'}{\|\mathbf{B}'\|}, \ \mathbf{U}_3(\widehat{s}) = \mathbf{B} \times \mathbf{U}_2, \tag{48}$$

as the Blaschke frame along $\mathbf{B}(t)$. Thus, the calculations give that:

$$\begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \end{pmatrix} = \begin{pmatrix} \cos\Psi & 0 & \sin\Psi \\ -\sin\Psi & 0 & \cos\Psi \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix}. \tag{49}$$

The variations of this frame are analogous to Eqs. (16) and is given by:

$$\begin{pmatrix} \mathbf{U}_1' \\ \mathbf{U}_2' \\ \mathbf{U}_3' \end{pmatrix} = \begin{pmatrix} 0 & \Psi' & 0 \\ -\Psi' & 0 & \widehat{\kappa} \\ 0 & -\widehat{\kappa} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \end{pmatrix} = \widetilde{\mathbf{\Omega}} \times \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \end{pmatrix}, \tag{50}$$

where $\widetilde{\mathbf{\Omega}}(\widehat{s}) = \widehat{\kappa}\mathbf{U}_1 + \Psi'\mathbf{U}_3$ is the Darboux vector. From Eqs (42) and (50), we obtain that:

$$\mathbf{\Omega} - \widetilde{\mathbf{\Omega}} = -\Psi'\mathbf{T}. \tag{51}$$

From Eqs (43) and (51), we obtain that: If $T(t) = \tau + \varepsilon\tau^* = 0(\Sigma' = 0)$, i.e. ψ and ψ^* are constants, then the line $\mathbf{X}$ moves about it with constant pitch h and the Disteli-axis is fixed to the second order. Therefore, the surface (X) is locally traced during a one-parameter screw motion with respect to the constant Disteli-axis $\mathbf{B}$, by the line $\mathbf{X}$ positioned at a constant distance ψ^* and constant angle ψ relative to $\mathbf{B}$. Consequently, we have:

Theorem 1. Any non-developable ruled surface (X) is a constant Disteli-axis if and only if (a) γ =constant, and (b) $\Gamma - \mu\gamma$ =constant.

We will now construct ruled surfaces with the constant Disteli-axis. From Eq (42), we have the following ordinary dual differential equation

$$\mathbf{X}''' + \widetilde{\kappa}^2\mathbf{X}' = \mathbf{0}. \tag{52}$$

Case (I). If $\Sigma \neq 0$, then without loss of generality, we may assume $\mathbf{X}'(0) = (0, 1, 0)$. Under such initial condition, a dual unit vector $\mathbf{X}'$ is given as

$$\mathbf{X}'(\widehat{s}) = A_1 \sin(\widetilde{\kappa s})\mathbf{F}_1 + (\cos(\widetilde{\kappa s}) + A_2 \sin(\widetilde{\kappa s}))\mathbf{F}_2 + A_3 \sin(\widetilde{\kappa s})\mathbf{F}_3,$$

where A_1, A_2, and A_3 are some dual constants satisfying $A_1^2 + A_3^2 = 1$, and $A_2 = 0$. From this, we can obtain

$$\mathbf{X}(\widehat{s}) = (-\widetilde{\rho}A_1 \cos(\widetilde{\kappa s}) + D_1)\mathbf{F}_1 + \widetilde{\rho} \sin(\widetilde{\kappa s})\mathbf{F}_2 + (-\widetilde{\rho}A_3 \cos(\widetilde{\kappa s}) + D_3)\mathbf{F}_3,$$

where D_1, D_3, are some dual constants satisfying $A_1D_1 + A_3D_3 = 0$, and $D_1^2 + D_3^2 = 1$. If we adopt the dual coordinates transformation such that

$$\begin{pmatrix} \overline{X_1} \\ \overline{X_2} \\ \overline{X_3} \end{pmatrix} = \begin{pmatrix} A_1 & 0 & A_3 \\ 0 & 1 & 0 \\ -A_3 & 0 & A_1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix},$$

with respect to the new coordinates $\overline{X_i}$, the dual vector $\mathbf{X}(\widehat{s})$ becomes

$$\mathbf{X}(\widehat{s}) = -\widetilde{\rho}\cos(\widetilde{\kappa s})\,\mathbf{F}_1 + \widetilde{\rho}\sin(\widetilde{\kappa s})\,\mathbf{F}_2 + D\mathbf{F}_3, \tag{53}$$

for a dual constant $D = A_1D_3 - A_3D_3$, with $D = \pm\cos\Psi$. It is noted that $\mathbf{X}(\widehat{s})$ does not depend on the choice of the upper or lower sign of $\pm$. Hence, throughout the study we choose upper sign, that is,

$$\mathbf{X}(\Phi) = -\cos\Phi\sin\Psi\mathbf{F}_1 + \sin\Phi\sin\Psi\mathbf{F}_2 + \cos\Psi\mathbf{F}_3. \tag{54}$$

where $\Phi = \varphi + \varepsilon\varphi^* = \widetilde{\kappa s}$. This means that the lines $\mathbf{B}$ and $\mathbf{F}_3$ are coincident, and

$$\psi = c_1(\text{real } const.), \ \psi^* = c_2(\text{real } const.). \tag{55}$$

Since φ, and φ^* are two-independent parameters, the surface (X) is a line congruence $\mathbb{E}^3$.

Now, the Plücker coordinates can be used to determine the equation of this line congruence. Computing the dual and real parts of Eq (54), we get

$$\mathbf{x}^*(\varphi, \varphi^*) = \begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} -\psi^*\cos\psi\cos\varphi + \varphi^*\sin\psi\sin\varphi \\ \psi^*\cos\psi\sin\varphi + \varphi^*\sin\psi\cos\varphi \\ -\psi^*\sin\psi \end{pmatrix}. \tag{56}$$

and

$$\mathbf{x}(\varphi) = (-\cos\varphi\sin\psi, \sin\varphi\sin\psi, \cos\psi), \tag{57}$$

Let $\mathbf{r}(r_1, r_2, r_3)$ be a point on the line $\mathbf{X}$. Since $\mathbf{r} \times \mathbf{x} = \mathbf{x}^*$, we get a system of linear equations in r_1, r_2, and r_3:

$$\left. \begin{array}{r} -r_3\sin\varphi\sin\psi + r_2\cos\psi = x_1^*, \\ -r_3\cos\varphi\sin\psi - r_1\cos\psi = x_2^*, \\ r_1\sin\varphi\sin\psi + r_2\cos\varphi\sin\psi = x_3^*. \end{array} \right\}$$

The matrix of coefficients of unknowns r_1, r_2, and r_3 is the skew 3×3 matrix

$$\begin{pmatrix} 0 & \cos\psi & -\sin\psi\sin\varphi \\ -\cos\psi & 0 & -\sin\psi\cos\varphi \\ \sin\psi\sin\varphi & \sin\psi\cos\varphi & 0 \end{pmatrix},$$

and therefore its rank is 2 with $\psi \neq 0$, and $\varphi \neq 0$. Moreover, the rank of the augmented matrix

$$\begin{pmatrix} 0 & \cos\psi & -\sin\psi\sin\varphi & x_1^* \\ -\cos\psi & 0 & -\sin\psi\cos\varphi & x_2^* \\ \sin\psi\sin\varphi & \sin\psi\cos\varphi & 0 & x_3^* \end{pmatrix},$$

equals 2. Therefore, the system has infinitely many solutions represented by

$$
\begin{aligned}
-r_1 &= (r_3 + \varphi^*) \cos\varphi \tan\psi + \psi^* \sin\varphi, \\
r_2 &= (r_3 + \varphi^*) \sin\varphi \tan\psi - \psi^* \cos\varphi, \\
r_1 \sin\psi \sin\varphi &+ r_2 \sin\psi \cos\varphi = x_3^*.
\end{aligned}
\tag{58}
$$

Since r_3 is taken at random, then we may take $r_3 + \varphi^* = 0$. Then, Eq (58) becomes

$$
\left.
\begin{aligned}
r_1(t) &= -\psi^* \sin\varphi, \\
r_2(t) &= -\psi^* \cos\varphi, \\
r_3(t) &= -\varphi^*.
\end{aligned}
\right\}
\tag{59}
$$

Let **y** be a point on such line congruence. According to Eq (2), we get

$$
\mathbf{y}(\varphi, \varphi^*, v) = \mathbf{r}(\varphi, \varphi^*) + v\mathbf{x}(\varphi, \varphi^*), \; v \in \mathbb{R}.
\tag{60}
$$

By means of Eqs (57) and (59) we obtain:

$$
\mathbf{y}(\varphi, \varphi^*, v) = \begin{pmatrix} -\psi^* \sin\varphi - v\cos\varphi \sin\psi \\ -\psi^* \cos\varphi + v\sin\varphi \sin\psi \\ -\varphi^* + v\cos\psi \end{pmatrix},
\tag{61}
$$

which contains the family of ruled surfaces $\mathbf{y}(\varphi_0, \varphi^*, v)$, $\mathbf{y}(\varphi, \varphi_0^*, v)$, and $\mathbf{y}(\varphi(t), \varphi^*(t), v)$. Here φ_0^*, φ_0, and t are real constants. Let (x, y, z) be coordinates of **y**, then Eq (61) yield

$$
\left.
\begin{aligned}
x &= -\psi^* \sin\varphi - v\cos\varphi \sin\psi, \\
y &= -\psi^* \cos\varphi + v\sin\varphi \sin\psi, \\
z &= -\varphi^* + v\cos\psi,
\end{aligned}
\right\}
\tag{62}
$$

By eliminating the parameter φ, we get

$$
(X) : \frac{x^2}{\psi^{*2}} + \frac{y^2}{\psi^{*2}} - \frac{Z^2}{k^2} = 1,
\tag{63}
$$

where $k = \psi^* \cot\psi$ and $Z = z + \varphi^*$. Then (X) is a one-parameter set of one-sheeted hyperboloids. The intersection of every hyperboloid and its corresponding plane $z + \varphi^* = 0$ is the circular cylinder $y^2 + x^2 = \psi^{*2}$. Therefore the envelope of (X) is the cylinder $y^2 + x^2 = \psi^{*2}$. Notice that if $\varphi^* = 0$, then

$$
(X) : \frac{x^2}{\psi^{*2}} + \frac{y^2}{\psi^{*2}} - \frac{z^2}{(\psi^* \coth\psi)^2} = 1.
\tag{64}
$$

Constant parameter ruled surfaces

A correlation between the two parameters φ and φ^* such as $f(\varphi, \varphi^*) = 0$ reduces the line congruence Eq (54) (resp. (61)) to a one-parameter set of lines (a ruled surface). So, if we set $\varphi^* = h\varphi$ in which h and φ are the pitch and the parameter of the motion $\mathbb{H}_m/\mathbb{H}_f$, resp., then Eq (54) (resp. (61)) gives a ruled surface in $\mathbb{H}_f-$space. Hence we get the following relation:

$$
\begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix} = \begin{pmatrix} -\sin\Psi \cos\Phi & \sin\Psi \sin\Phi & \cos\Psi \\ \sin\Phi & \cos\Phi & 0 \\ -\cos\Psi \cos\Phi & \cos\Psi \sin\Phi & -\sin\Psi \end{pmatrix} \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_3 \end{pmatrix}.
$$

Hence, the Blaschke equations are given as the following matrix:

$$\frac{d}{d\varphi} \begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix} = \begin{pmatrix} 0 & P & 0 \\ -P & 0 & Q \\ 0 & -Q & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix},$$

where

$$P = (1 + \varepsilon h) \sin \Psi, \ Q = (1 + \varepsilon h) \cos \Psi, \text{ and } \cot \Psi = \frac{Q}{P}. \tag{65}$$

Thus we reach, by computing the real and dual parts of Eq (65), to the result

$$\mu = h + \psi^* \cot \psi, \ \Gamma = h - \psi^* \tan \psi, \ \gamma = \tan \psi, \tag{66}$$

where μ, Γ, and γ are constants. Hence, as a direct consequence of Definition 2, (X) is a constant parameter ruled surface. We can also show that $< \frac{d\mathbf{r}}{d\varphi}, \frac{d\mathbf{x}}{d\varphi} > = 0$; so the base curve $\mathbf{r}(\varphi)$ of (X) is its striction curve. Also, the curvature and the torsion of $\mathbf{r}(\varphi)$ can be given, respectively, by

$$\kappa(\varphi) = \frac{\psi^*}{\psi^{*2} + h^2}, \text{ and } \tau(\varphi) = \frac{h}{\psi^{*2} + h^2},$$

which leads to $\mathbf{r}(\varphi)$ is a circular helix. Therefore, the constant parameter ruled surface can be characterized as follows:
(1) The constant helicoid: If we take $h = 1$, $\psi = \frac{\pi}{4}$, $\psi^* = 1$, $\varphi \in [-\pi, \pi]$, and $v \in [-4, 4]$, the surface is shown in Figure 3. (2) The one-sheeted circular hyperboloid: Figure 4 shows the surface with $\psi = \frac{\pi}{4}$, $\psi^* = 1$, $\varphi \in [-\pi, \pi]$, $v \in [-4, 4]$, and $h = 0$.(3) Archimedes helicoid: Figure 5 shows the surface with $\psi = \frac{\pi}{4}$, $h = 1$, $\varphi \in [-\pi, \pi]$, $v \in [-4, 4]$, and $\psi^* = 0$. (4) A circular cone: Figure 6 shows the surface with $\psi = \frac{\pi}{4}$, $\varphi \in [-\pi, \pi]$, $v \in [-4, 4]$, and $\psi^* = h = 0$.
Case (II). If $\Sigma = 0$, then we have:

$$\mathbf{X}''' + \mathbf{X}' = \mathbf{0}. \tag{67}$$

Also, from Eq (26), we can see that

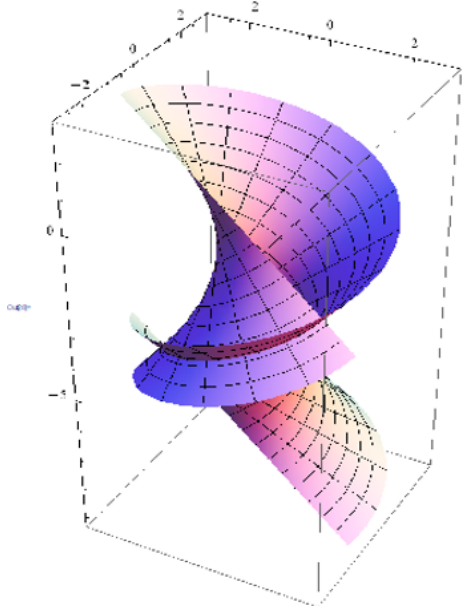$$\Sigma = 0 \Leftrightarrow \cot \Psi = 0 \Leftrightarrow \psi = 0, \text{ and } \psi^* = 0. \tag{68}$$
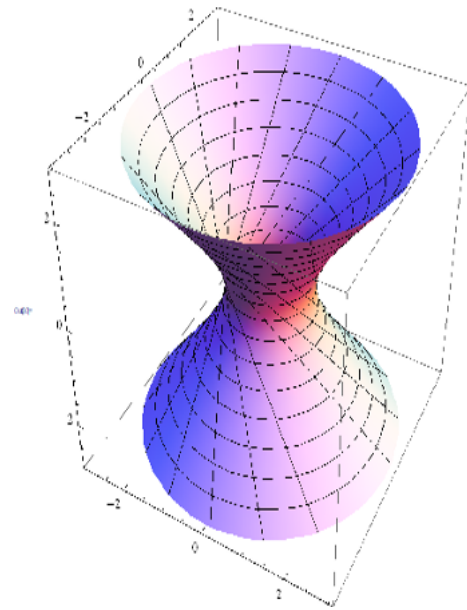
Figure 3. The constant helicoid.

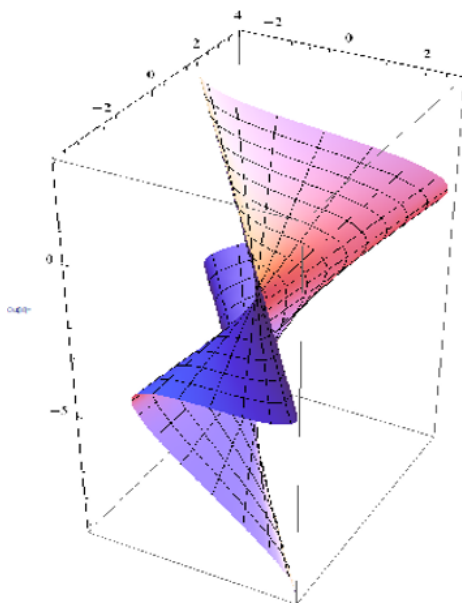**Figure 4.** One-sheeted circular hyperboloid.
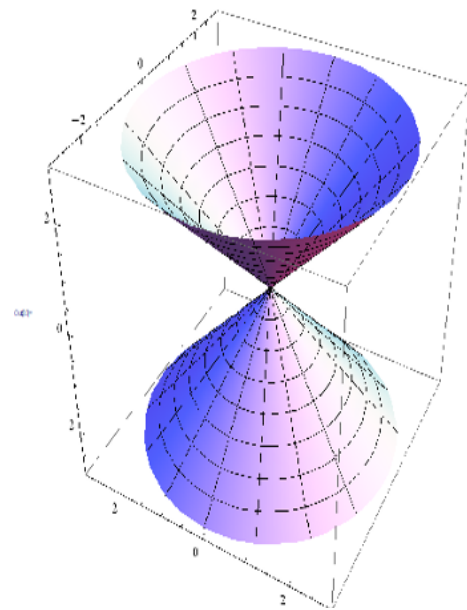

Figure 5. Archimedecs helicoid.

Figure 6. A circular cone.

In such instant, the lines $\mathbf{X}$, $\mathbf{T}$ and $\mathbf{B}$ construct the Blaschke frame and meet at the striction point of the (X). Moreover, the oriented lines $\mathbf{B}$ and $\mathbf{G}$ coincide, That is, the line $\mathbf{X}$ moves with respect to $\mathbf{B}$ with constant pitch. Hence, we have a great circle on $\mathbb{K}_f$ is described by

$$\mathbb{S} = \left\{ \mathbf{X}(S) \in \mathbb{K}_f \mid\, <\mathbf{X}, \mathbf{B}>= 0, \text{ with } \|\mathbf{B}\|^2 = 1 \right\}.$$

Therefore, we conclude that all lines of (X) intersect perpendicularly the Disteli-axis $\mathbf{B}$. Quite similarly

as we did, we obtain

$$\mathbf{X}(\widehat{s}) = (-\cos\widehat{s}, \sin\widehat{s}, 0).$$

The central normal can be found as $\mathbf{T}(\widehat{s}) = (\sin\widehat{s}, \cos\widehat{s}, 0)$. Thus $\mathbf{G}$ has the form $\mathbf{G} = (0, 0, -1)$. Furthermore, an analogous arguments show that: $\mu = h$, and $\gamma = \Gamma = 0$. Also, it can be shown that the ruled surface (X) is

$$x = v\cos s, \ y = hs, \ \text{and} \ z = v\sin s. \tag{69}$$

Or, in Cartesian coordinates,

$$y = h\cot^{-1}\left(\frac{x}{z}\right), \tag{70}$$

which represents a one-parameter family of right helicoids. If we take $h = 1$, $0 \le s \le 2\pi$, and $-3 \le v \le 3$, then a member of such family can be formed, see Figure 7.
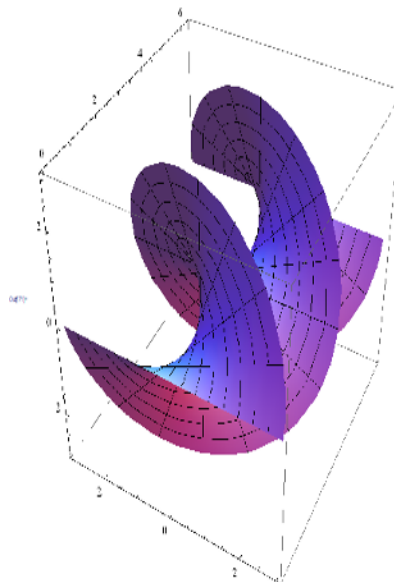

Figure 7. A right helicoid.

Corollary 2. Let $\mathbf{X} = \mathbf{X}(\widehat{s})$ be a dual spherical curve. Then, the curve $\mathbf{X}$ is a great circle if its dual geodesic curvature function $\Sigma = \Sigma(\widehat{s})$ vanishes. Moreover, the ruled surface is classified as a right helicoid and its striction curve is geodesic.

4. Conclusions

This paper presents a new approach for constructing ruled surfaces having constant Disteli-axis by means of the E. Study map. In terms of this, for the motion of the Blaschke frame on a regular dual spherical curve, some well-known formulae of surfaces theory into line space and their geometrical explanations are investigated. Finally, the geometric invariants of a line trajectory undrgoing one-parameter scrow motion are examined. The study of spatial kinematics in Euclidean 3-space $\mathbb{E}^3$ via the E. Study map may used to solve some problems and conclude new applications.

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Conflict of interest

Author has no conflict of interest.

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