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Research article

On graded 2-absorbing I_e-prime submodules of graded modules over graded commutative rings

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Abstract: Let G be an abelian group with identity e. Let R be a G-graded commutative ring with identity and M a graded R-module. In this paper, we introduce the concept of graded 2-absorbing I_e-prime submodule as a generalization of a graded 2-absorbing prime submodule for $I = \oplus_{g \in G} I_g$ a fixed graded ideal of R. We give a number of results concerning these classes of graded submodules and their homogeneous components. A proper graded submodule N of M is said to be a graded 2-absorbing I_e-prime submodule of M if whenever $r_h, s_\lambda \in h(R)$ and $m_\alpha \in h(M)$ with $r_h s_\lambda m_\alpha \in N \backslash I_e N$, implies either $r_h s_\lambda \in (N :_R M)$ or $r_h m_\alpha \in N$ or $s_\lambda m_\alpha \in N$.

Keywords: graded 2-absorbing I_e-prime submodules; graded I_e-prime submodules; graded 2-absorbing submodules; graded prime submodules; graded 2-absorbing I_e-prime ideals
Mathematics Subject Classification: 13A02, 16W50

1. Introduction and preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary.

Badawi in [15] introduced the concept of 2-absorbing ideals of commutative rings. The notion of 2-absorbing ideals was extended to 2-absorbing submodules in [17] and [24]. Recently, Farshadifar in [18] introduced and studied the concept of 2-absorbing I-prime submodules.

Refai and Al-Zoubi in [25] introduced the concept of graded primary ideal. The concept of graded 2-absorbing ideal was introduced and studied by Al-Zoubi, Abu-Dawwas and Ceken in [4]. The concept of graded prime submodule was introduced and studied by many authors, see for example [1, 2, 10–12, 14, 23]. The concept of graded 2-absorbing submodule, generalizations of graded prime submodule, was introduced by Al-Zoubi and Abu-Dawwas in [3] and studied in [7, 8]. Then many generalizations of graded 2-absorbing submodules were studied such as graded 2-absorbing primary (see [16]), graded weakly 2-absorbing primary (see [6]) and graded classical 2-absorbing (see [5]). Recently, Alghueiri and Al-Zoubi in [13] introduced the concept of graded I_e-prime submodule over a commutative ring as a new generalization of graded prime submodule. Here, we introduce the concept of graded 2-absorbing I_e-prime submodule as a new generalization of a graded 2-absorbing prime submodule on the one hand and a generalization of a graded I_e-prime submodule on other hand.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [19–22] for these basic properties and more information on graded rings and modules.

Let G be an abelian multiplicative group with identity element e. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_h of R indexed by the elements $h \in G$ such that $R = \oplus_{h\in G} R_h$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The non-zero elements of R_h are said to be homogeneous of degree h and all the homogeneous elements are denoted by $h(R)$, i.e., $h(R) = \cup_{h\in G} R_h$. If $a \in R$, then a can be written uniquely as $\sum_{h\in G} a_h$, where a_g is called a homogeneous component of a in R_h. Moreover, R_e is a subring of R and $1 \in R_e$ (see [22]). Let $R = \oplus_{h\in G} R_h$ be a G-graded ring. An ideal J of R is said to be a graded ideal if $J = \sum_{h\in G}(J \cap R_h) := \sum_{h\in G} J_h$ (see [22]).

Let $R = \oplus_{h\in G} R_h$ be a G-graded ring. A left R-module M is said to be a graded R-module (or G-graded R-module) if there exists a family of additive subgroups $\{M_h\}_{h\in G}$ of M such that $M = \oplus_{h\in G} M_h$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Also if an element of M belongs to $\cup_{h\in G} M_h = h(M)$, then it is called a homogeneous. Note that M_h is an R_e-module for every $h \in G$. Let $R = \oplus_{h\in G} R_h$ be a G-graded ring. A submodule N of M is said to be a graded submodule of M if $N = \oplus_{h\in G}(N \cap M_h) := \oplus_{h\in G} N_h$. In this case, N_h is called the h-component of N. Moreover, M/N becomes a G-graded R-module with h-component $(M/N)_h := (M_h + N)/N$ for $h \in G$ (see [22]).

2. Results

Definition 2.1. Let R be a G-graded ring, M a graded R-module, $I = \oplus_{g\in G} I_g$ a graded ideal of R, $N = \oplus_{g\in G} N_g$ a graded submodule of M and $g \in G$.

(i) We say that N_g is a g-2-absorbing I_e-prime submodule of the R_e-module M_g, if $N_g \neq M_g$; and whenever $r_e, s_e \in R_e$ and $m_g \in M_g$ with $r_e s_e m_g \in N_g \backslash I_e N_g$, implies either $r_e s_e \in (N_g :_{R_e} M_g)$ or $r_e m_g \in N_g$ or $s_e m_g \in N_g$.

(ii) We say that N is a graded 2-absorbing I_e-prime submodule of M, if $N \neq M$; and whenever $r_h, s_\lambda \in h(R)$ and $m_\alpha \in h(M)$ with $r_h s_\lambda m_\alpha \in N \backslash I_e N$, implies either $r_h s_\lambda \in (N :_R M)$ or $r_h m_\alpha \in N$ or $s_\lambda m_\alpha \in N$.

Proposition 2.2. *Let R be a G-graded ring, M a graded R-module, $I = \oplus_{g\in G} I_g$ a graded ideal of R and $N = \oplus_{g\in G} N_g$ a graded submodule of M. If N is a graded 2-absorbing I_e-prime submodule of M, then for any $g \in G$ with $N_g \neq M_g$, N_g is a g-2-absorbing I_e-prime submodule of the R_e-module M_g.*

Proof. Let $r_e, s_e \in R_e$ and $m_g \in M_g$ such that $r_e s_e m_g \in N_g \backslash I_e N_g$, so $r_e s_e m_g \in N \backslash I_e N$ and then either $r_e s_e \in (N :_R M)$ or $r_e m_g \in N$ or $s_e m_g \in N$ as N is a graded 2-absorbing I_e-prime submodule of M. Since $M_g \subseteq M$ and $N_g = N \cap M_g$, we conclude that either $r_e s_e \in (N_g :_{R_e} M_g)$ or $r_e m_g \in N_g$ or $s_e m_g \in N_g$. Therefore, N_g is a g-2-absorbing I_e-prime submodule of M_g. □

Recall from [3] that a proper graded submodule N of a graded R-module M is said to be a graded weakly 2-absorbing submodule of M if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $0 \neq r_g s_h m_\lambda \in N$, then either $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$ or $r_g s_h \in (N :_R M)$.

Remark 2.3. Let R be a G-graded ring, M a graded R-module and $I = \oplus_{g\in G} I_g$ a graded ideal of R. If $I = (0)$, then the notion of graded 2-absorbing I_e-prime submodule is exactly the notion of graded weakly 2-absorbing submodule.

Recall from [3] that a proper graded submodule N of a graded R-module M is said to be a graded 2-absorbing submodule of M if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $r_g s_h m_\lambda \in N$, then either $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$ or $r_g s_h \in (N :_R M)$.

It is easy to see that every graded 2-absorbing submodule is a graded 2-absorbing I_e-prime submodule. The following example shows that the converse is not true in general.

Example 2.4. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$ be a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}_{12}$ be a graded R-module with $M_0 = \mathbb{Z}_{12}$ and $M_1 = \{\bar{0}\}$. Now, consider the graded submodule $N = (\bar{0})$ of M, then N is not a graded 2-absorbing submodule of M since $\bar{2} \cdot \bar{2} \cdot \bar{3} \in N$ and neither $\bar{2} \cdot \bar{3} \in N$ nor $\bar{2} \cdot \bar{2} \in (N :_{\mathbb{Z}} \mathbb{Z}_{12})$. However, for any graded ideal $I = \oplus_{g\in G} I_g$ of R, N is a graded 2-absorbing I_e-prime submodule of M.

Let R be a G-graded ring, M a graded R-module and $I = \oplus_{g\in G} I_g$ a graded ideal of R. Recall from [13] that a proper graded submodule N of M is said to be a graded I_e-prime submodule of M if whenever $r_h \in h(R)$ and $m_\lambda \in h(M)$ with $r_h m_\lambda \in N - I_e N$, implies either $m_\lambda \in N$ or $r_h \in (N :_R M)$.

It is easy to see that every graded I_e-prime submodule is a graded 2-absorbing I_e-prime submodule. The following example shows that the converse is not true in general.

Example 2.5. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$ be a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$. Now, consider the graded ideal $I = 2\mathbb{Z}$ of R and the graded submodule $N = 4\mathbb{Z}$ of M. Then N is not a graded I_e-prime submodule of M since $2 \cdot 2 \in 4\mathbb{Z} \backslash 8\mathbb{Z}$ and neither $2 \in 4\mathbb{Z}$ nor $2 \in (4\mathbb{Z} :_{\mathbb{Z}} \mathbb{Z})$. However, easy computations show that N is a graded 2-absorbing submodule of M and then a graded 2-absorbing I_e-prime.

Let R be a G-graded ring, M a graded R-module, $N = \oplus_{g\in G} N_g$ a graded submodule of M and $g \in G$. Recall from [3] that N_g is said to be a *g-2-absorbing submodule* of the R_e-module M_g if $N_g \neq M_g$; and whenever $r, s \in R_e$ and $m \in M_g$ with $rsm \in N_g$, then either $rs \in (N_g :_{R_e} M_g)$ or $rm \in N_g$ or $sm \in N_g$.

Theorem 2.6. *Let R be a G-graded ring, M a graded R-module, $I = \oplus_{g\in G} I_g$ a graded ideal of R and $N = \oplus_{g\in G} N_g$ a graded 2-absorbing I_e-prime submodule of M. Then for any $g \in G$ with $N_g \neq M_g$, either N_g is g-2-absorbing submodule of the R_e-module M_g or $(N_g :_{R_e} M_g)^2 N_g \subseteq I_e N_g$.*

Proof. Let $g \in G$ with $N_g \neq M_g$. Then N_g is a g-2-absorbing I_e-prime submodule of the R_e-module M_g by Proposition 2.2. Suppose that $(N_g :_{R_e} M_g)^2 N_g \nsubseteq I_e N_g$. Now, let $r_e, s_e \in R_e$ and $m_g \in M_g$ such that $r_e s_e m_g \in N_g$. If $r_e s_e m_g \notin I_e N_g$, then either $r_e s_e \in (N_g :_{R_e} M_g)$ or $r_e m_g \in N_g$ or $s_e m_g \in N_g$ as N_g is a g-2-absorbing I_e-prime submodule of the R_e-module M_g. So now we can assume that $r_e s_e m_g \in I_e N_g$. First, suppose that $r_e s_e N_g \nsubseteq I_e N_g$, so there exists $n_g \in N_g$ such that $r_e s_e n_g \notin I_e N_g$ and it follows that $r_e s_e (m_g + n_g) \in N_g \backslash I_e N_g$. Then we get either $r_e s_e \in (N_g :_{R_e} M_g)$ or $r_e(m_g + n_g) \in N_g$ or $s_e(m_g + n_g) \in N_g$ as N_g is a g-2-absorbing I_e-prime submodule of M_g. Hence, either $r_e s_e \in (N_g :_{R_e} M_g)$ or $r_e m_g \in N_g$ or $s_e m_g \in N_g$. Now, we may assume that $r_e s_e N_g \subseteq I_e N_g$. If $r_e (N_g :_{R_e} M_g) m_g \nsubseteq I_e N_g$, then there exists $t_e \in (N_g :_{R_e} M_g)$ such that $r_e t_e m_g \notin I_e N_g$. This yields that $r_e(s_e + t_e) m_g \in N_g \backslash I_e N_g$ and then we have either $r_e(s_e + t_e) \in (N_g :_{R_e} M_g)$ or $r_e m_g \in N_g$ or $(s_e + t_e) m_g \in N_g$ as N_g is a g-2-absorbing I_e-prime submodule of the R_e-module M_g. Thus, either $r_e s_e \in (N_g :_{R_e} M_g)$ or $r_e m_g \in N_g$ or $s_e m_g \in N_g$. We get the same result if $s_e(N_g :_{R_e} M_g)m_g \nsubseteq I_eN_g$, so assume that $r_e(N_g :_{R_e} M_g)m_g \subseteq I_eN_g$ and $s_e(N_g :_{R_e} M_g)m_g \subseteq I_eN_g$. Now, since $(N_g :_{R_e} M_g)^2N_g \nsubseteq I_eN_g$, there exist $r'_e, s'_e \in (N_g :_{R_e} M_g)$ and $n'_g \in N_g$ with $r'_e s'_e n'_g \notin I_eN_g$. If $r_e s'_e n'_g \notin I_eN_g$, then $r_e(s_e + s'_e)(m_g + n'_g) \in N_g \backslash I_eN_g$ implies that either $r_e(s_e + s'_e) \in (N_g :_{R_e} M_g)$ or $r_e(m_g + n'_g) \in N_g$ or $(s_e + s'_e)(m_g + n'_g) \in N_g$. Hence, either $r_e s_e \in (N_g :_{R_e} M_g)$ or $r_e m_g \in N_g$ or $s_e m_g \in N_g$. Now, assume that $r_e s'_e n'_g \in I_eN_g$. Similarly, assume that $r'_e s'_e m_g \in I_eN_g$ and $r'_e s_e n'_g \in I_eN_g$. Then from $(r_e + r'_e)(s_e + s'_e)(m_g + n'_g) \in N_g \backslash I_eN_g$, we get $(r_e + r'_e)(s_e + s'_e) \in (N_g :_{R_e} M_g)$ or $(r_e + r'_e)(m_g + n'_g) \in N_g$ or $(s_e + s'_e)(m_g + n'_g) \in N_g$ and it follows that either $r_e s_e \in (N_g :_{R_e} M_g)$ or $r_e m_g \in N_g$ or $s_e m_g \in N_g$. Therefore, N_g is a g-2-absorbing submodule of the R_e-module M_g. □

Theorem 2.7. *Let R be a G-graded ring, M a graded R-module, $I = \oplus_{g\in G} I_g$ a graded ideal of R, N a graded 2-absorbing I_e-prime submodule of M and $K = \oplus_{\lambda\in G} K_\lambda$ a graded submodule of M. If $r_g, s_h \in h(R)$ and $\lambda \in G$ with $r_g s_h K_\lambda \subseteq N$ and $2r_g s_h K_\lambda \nsubseteq I_eN$, then either $r_g s_h \in (N :_R M)$ or $r_g K_\lambda \subseteq N$ or $s_h K_\lambda \subseteq N$.*

Proof. Suppose that $r_g s_h \notin (N :_R M)$. Now, let $k_{\lambda_1} \in K_\lambda$. If $r_g s_h k_{\lambda_1} \notin I_eN$, then either $r_g k_{\lambda_1} \in N$ or $s_h k_{\lambda_1} \in N$ as N is a graded 2-absorbing I_e-prime submodule of M and $r_g s_h \notin (N :_R M)$, which yields that $k_{\lambda_1} \in (N :_M r_g) \cup (N :_M s_h)$. Now, we can assume that $r_g s_h k_{\lambda_1} \in I_eN$. Since $2r_g s_h K_\lambda \nsubseteq I_eN$, there exists $k_{\lambda_2} \in K_\lambda$ such that $2r_g s_h k_{\lambda_2} \notin I_eN$ and then $r_g s_h k_{\lambda_2} \in N \backslash I_eN$. Hence, we get either $r_g k_{\lambda_2} \in N$ or $s_h k_{\lambda_2} \in N$ as N is a graded 2-absorbing I_e-prime and $r_g s_h \notin (N :_R M)$. Also, $r_g s_h(k_{\lambda_1} + k_{\lambda_2}) \in N \backslash I_eN$ implies either $r_g(k_{\lambda_1} + k_{\lambda_2}) \in N$ or $s_h(k_{\lambda_1} + k_{\lambda_2}) \in N$. Hence, we consider three cases.

Case 1: $r_g k_{\lambda_2} \in N$ and $s_h k_{\lambda_2} \in N$. Then $r_g(k_{\lambda_1} + k_{\lambda_2}) \in N$ or $s_h(k_{\lambda_1} + k_{\lambda_2}) \in N$ implies either $r_g k_{\lambda_1} \in N$ or $s_h k_{\lambda_1} \in N$.

Case 2: $r_g k_{\lambda_2} \in N$ and $s_h k_{\lambda_2} \notin N$. Assume that $r_g k_{\lambda_1} \notin N$. Then $r_g(k_{\lambda_1} + k_{\lambda_2}) \notin N$ and so $s_h(k_{\lambda_1} + k_{\lambda_2}) \in N$. Thus, $r_g(k_{\lambda_1} + 2k_{\lambda_2}) \notin N$ and $s_h(k_{\lambda_1} + 2k_{\lambda_2}) \notin N$. Now, we get $r_g s_h(k_{\lambda_1} + 2k_{\lambda_2}) \in I_eN$ as N is a graded 2-absorbing I_e-prime submodule of M and $r_g s_h \notin (N :_R M)$, and so $2r_g s_h k_{\lambda_2} \in I_eN$, a contradiction. Thus, $r_g k_{\lambda_1} \in N$.

Case 3: $r_g k_{\lambda_2} \notin N$ and $s_h k_{\lambda_2} \in N$. Then the proof is similar to that of Case 2. Therefore, $K_\lambda \subseteq (N :_M r_g) \cup (N :_M s_h)$ and then either $r_g K_\lambda \subseteq N$ or $s_h K_\lambda \subseteq N$. □

Theorem 2.8. *Let R be a G-graded ring, M a graded R-module, $I = \oplus_{g\in G} I_g$ a graded ideal of R and N a graded 2-absorbing I_e-prime submodule of M. Let $J = \oplus_{h\in G} J_h$ be a graded ideal of R and $K = \oplus_{\lambda\in G} K_\lambda$ a graded submodule of M. If $r_g \in h(R)$ and $h, \lambda \in G$ with $r_g J_h K_\lambda \subseteq N$ and $4r_g J_h K_\lambda \nsubseteq I_eN$, then either $r_g J_h \subseteq (N :_R M)$ or $r_g K_\lambda \subseteq N$ or $J_h K_\lambda \subseteq N$.*

Proof. Suppose that $r_g J_h \nsubseteq (N :_R M)$ and $r_g K_\lambda \nsubseteq N$. Now, since $r_g J_h \nsubseteq (N :_R M)$, there exists $j_{h_1} \in J_h$ such that $r_g j_{h_1} \notin (N :_R M)$. Also, since $4r_g J_h K_\lambda \nsubseteq I_eN$, there exists $j_{h_2} \in J_h$ such that $4r_g j_{h_2} K_\lambda \nsubseteq I_eN$ and then $2r_g j_{h_2} K_\lambda \nsubseteq I_eN$. Now, let $j_h \in J_h$, if $2r_g j_h K_\lambda \nsubseteq I_eN$, then by Theorem 2.7, we get $j_h \in ((N :_R M) :_R r_g) \cup (N :_R K_\lambda)$ as N is a graded 2-absorbing I_e-prime submodule of M. So we can assume that $2r_g j_h K_\lambda \subseteq I_eN$. If $4r_g j_{h_1} K_\lambda \nsubseteq I_eN$, then $2r_g j_{h_1} K_\lambda \nsubseteq I_eN$. Thus $j_{h_1} K_\lambda \subseteq N$ by Theorem 2.7 as N is a graded 2-absorbing I_e-prime submodule of M**.** So, $2r_g(j_h + j_{h_1})K_\lambda \nsubseteq I_eN$ implies that $j_h + j_{h_1} \in ((N :_R M) :_R r_g) \cup (N :_R K_\lambda)$. Assume that $j_h + j_{h_1} \in ((N :_R M) :_R r_g) \backslash (N :_R K_\lambda)$ then consider $2r_g(j_h + j_{h_1} + j_{h_1})K_\lambda = 2r_g j_h K_\lambda + 4r_g j_{h_1} K_\lambda \nsubseteq I_eN$, which yields that $j_h + j_{h_1} + j_{h_1} \in ((N :_R M) :_R r_g) \cup (N :_R K_\lambda)$. But $j_{h_1} K_\lambda \subseteq N$ and $(j_h + j_{h_1})K_\lambda \nsubseteq N$ implies that $(j_h + j_{h_1} + j_{h_1})K_\lambda \nsubseteq N$, also $r_g j_{h_1} \notin (N :_R M)$ and $r_g(j_h + j_{h_1}) \in (N :_R M)$ implies that $r_g(j_h + j_{h_1} + j_{h_1}) \notin (N :_R M)$, a contradiction. Hence, $j_h + j_{h_1} \in (N :_R K_\lambda)$. Thus $j_h K_\lambda \subseteq N$ since $j_{h_1} K_\lambda \subseteq N$. Similarly, if $r_g j_{h_2} \notin (N :_R M)$, then we get the result in the same manner. So now we can assume that $r_g j_{h_2} \in (N :_R M)$ and $4r_g j_{h_1} K_\lambda \subseteq I_e N$. Thus, $4r_g(j_{h_1} + j_{h_2})K_\lambda \nsubseteq I_e N$, then $2r_g(j_{h_1} + j_{h_2})K_\lambda \nsubseteq I_e N$. It follows that $(j_{h_1} + j_{h_2})K_\lambda \subseteq N$ by Theorem 2.7 as N is a graded 2-absorbing I_e-prime submodule of M and $r_g(j_{h_1} + j_{h_2}) \notin (N :_R M)$. So, $2r_g(j_h + (j_{h_1} + j_{h_2}))K_\lambda \nsubseteq I_e N$ implies that $j_h + (j_{h_1} + j_{h_2}) \in ((N :_R M) :_R r_g) \cup (N :_R K_\lambda)$. Assume that $j_h + (j_{h_1} + j_{h_2}) \in ((N :_R M) :_R r_g) \backslash (N :_R K_\lambda)$ then consider $2r_g(j_h + 2(j_{h_1} + j_{h_2}))K_\lambda = 2r_g j_h K_\lambda + 4r_g(j_{h_1} + j_{h_2})K_\lambda \nsubseteq I_e N$, which yields that $j_h + 2(j_{h_1} + j_{h_2}) \in ((N :_R M) :_R r_g) \cup (N :_R K_\lambda)$. But $(j_{h_1} + j_{h_2})K_\lambda \subseteq N$ and $(j_h + (j_{h_1} + j_{h_2}))K_\lambda \nsubseteq N$ implies that $(j_h + 2(j_{h_1} + j_{h_2}))K_\lambda \nsubseteq N$, also $r_g(j_{h_1} + j_{h_2}) \notin (N :_R M)$ and $r_g(j_h + (j_{h_1} + j_{h_2})) \in (N :_R M)$ implies that $r_g(j_h + 2(j_{h_1} + j_{h_2})) \notin (N :_R M)$, a contradiction. Hence, $j_h + (j_{h_1} + j_{h_2}) \in (N :_R K_\lambda)$. Thus $j_h K_\lambda \subseteq N$ since $(j_{h_1} + j_{h_2})K_\lambda \subseteq N$. Therefore, $J_h \subseteq ((N :_R M) :_R r_g) \cup (N :_R K_\lambda)$ and then $r_g J_h \subseteq (N :_R M)$ or $J_h K_\lambda \subseteq N$, but $r_g J_h \nsubseteq (N :_R M)$, so $J_h K_\lambda \subseteq N$. □

Theorem 2.9. *Let R be a G-graded ring, M a graded R-module, $I = \oplus_{g\in G} I_g$ be a graded ideal of R and N a proper graded submodule of M. Then the following statements are equivalent:*

(i) *N is a graded 2-absorbing I_e-prime submodule of M.*

(ii) *$N/I_e N$ is a graded weakly 2-absorbing submodule of $M/I_e N$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that N is a graded 2-absorbing I_e-prime submodule of M. Now, let $r_g, s_h \in h(R)$ and $(m_\lambda + I_e N) \in h(M/I_e N)$ with $0_{M/I_e N} \neq (r_g s_h m_\lambda + I_e N) \in N/I_e N$, this yields that $r_g s_h m_\lambda \in N \backslash I_e N$. Hence, either $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$ or $r_g s_h M \subseteq N$ as N is a graded 2-absorbing I_e-prime submodule of M. Then either $(r_g m_\lambda + I_e N) \in N/I_e N$ or $(s_h m_\lambda + I_e N) \in N/I_e N$ or $r_g s_h (M/I_e N) \subseteq N/I_e N$. Therefore, $N/I_e N$ is a graded weakly 2-absorbing submodule of $M/I_e N$.

$(i) \Rightarrow (ii)$ Suppose that $N/I_e N$ is a graded weakly 2-absorbing submodule of $M/I_e N$. Let $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ such that $r_g s_h m_\lambda \in N \backslash I_e N$. This follows that $0_{M/I_e N} \neq (r_g s_h m_\lambda + I_e N) = r_g s_h (m_\lambda + I_e N) \in N/I_e N$. Thus, either $r_g s_h \in (N/I_e N :_R M/I_e N)$ or $(r_g m_\lambda + I_e N) \in N/I_e N$ or $(s_h m_\lambda + I_e N) \in N/I_e N$ and then either $r_g s_h \in (N :_R M)$ or $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$. Therefore, N is a graded 2-absorbing I_e-prime submodule of M. □

Recall from [9] that a graded zero-divisor on a graded R-module M is an element $r_g \in h(R)$ for which there exists $m_h \in h(M)$ such that $m_h \neq 0$ but $r_g m_h = 0$. The set of all graded zero-divisors on M is denoted by $G\text{-}Zdv_R(M)$.

The following result studies the behavior of graded 2-absorbing I_e-prime submodules under localization.

Theorem 2.10. *Let R be a G-graded ring, M a graded R-module, $S \subseteq h(R)$ be a multiplicatively closed subset of R and $I = \oplus_{g\in G} I_g$ a graded ideal of R.*

(i) *If N is a graded 2-absorbing I_e-prime submodule of M with $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a graded 2-absorbing I_e-prime submodule of $S^{-1}M$.*

(ii) *If $S^{-1}N$ is a graded 2-absorbing I_e-prime submodule of $S^{-1}M$ with $S \cap G\text{-}Zdv_R(M/N) = \emptyset$, then N is a graded 2-absorbing I_e-prime submodule of M.*

Proof. (i) Since $(N :_R M) \cap S = \emptyset$, $S^{-1}N$ is a proper graded submodule of $S^{-1}M$. Let $\frac{r_g}{s_1}, \frac{s_h}{s_2} \in h(S^{-1}R)$ and $\frac{m_\lambda}{s_3} \in h(S^{-1}M)$ such that $\frac{r_g}{s_1}\frac{s_h}{s_2}\frac{m_\lambda}{s_3} \in S^{-1}N \backslash I_e S^{-1}N$. Then there exists $t \in S$ such that $t r_g s_h m_\lambda \in N \backslash I_e N$

which yields that either $tr_g m_\lambda \in N$ or $ts_h m_\lambda \in N$ or $r_g s_h \in (N :_R M)$ as N is a graded 2-absorbing I_e-prime submodule of M. Hence, either $\frac{r_g m_\lambda}{s_1 s_3} = \frac{tr_g m_\lambda}{ts_1 s_3} \in S^{-1}N$ or $\frac{s_h m_\lambda}{s_2 s_3} = \frac{ts_h m_\lambda}{ts_2 s_3} \in S^{-1}N$ or $\frac{r_g s_h}{s_1 s_2} \in S^{-1}(N :_R M) = (S^{-1}N :_{S^{-1}R} S^{-1}M)$. Therefore, $S^{-1}N$ is a graded 2-absorbing I_e-prime submodule of $S^{-1}M$.

(ii) Let $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ such that $r_g s_h m_\lambda \in N \backslash I_e N$. Then $\frac{r_g}{1}\frac{s_h}{1}\frac{m_\lambda}{1} \in S^{-1}N \backslash I_e S^{-1}N$. Since $S^{-1}N$ is a graded 2-absorbing I_e-prime submodule of $S^{-1}M$, either $\frac{r_g}{1}\frac{m_\lambda}{1} \in S^{-1}N$ or $\frac{s_h}{1}\frac{m_\lambda}{1} \in S^{-1}N$ or $\frac{r_g}{1}\frac{s_h}{1} \in (S^{-1}N :_{S^{-1}R} S^{-1}M)$. If $\frac{r_g m_\lambda}{1} \in S^{-1}N$, then there exists $t_1 \in S$ such that $t_1 r_g m_\lambda \in N$. This yields that $r_g m_\lambda \in N$ since $S \cap G\text{-}Zdv_R(M/N) = \emptyset$. Similarly, if $\frac{s_h m_\lambda}{1} \in S^{-1}N$, then there exists $t_2 \in S$ such that $t_2 s_h m_\lambda \in N$. This yields that $s_h m_\lambda \in N$ since $S \cap G\text{-}Zdv_R(M/N) = \emptyset$. Now, if $\frac{r_g s_h}{1} \in (S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}(N :_R M)$, then there exists $t_3 \in S$ such that $t_3 r_g s_h M \subseteq N$ and hence $r_g s_h \in (N :_R M)$ since $S \cap G\text{-}Zdv_R(M/N) = \emptyset$. Therefore, N is a graded 2-absorbing I_e-prime submodule of M. □

Proposition 2.11. *Let R be a G-graded ring, M_1 and M_2 be two graded R-modules, $I = \oplus_{g\in G} I_g$ a graded ideal of R and N_1 and N_2 be two graded submodules of M_1 and M_2, respectively. Then:*

(i) *If N_1 is a graded 2-absorbing I_e-prime submodule of M_1, then $N_1 \times M_2$ is a graded 2-absorbing I_e-prime submodule of $M_1 \times M_2$.*

(ii) *If N_2 is a graded 2-absorbing I_e-prime submodule of M_2, then $M_1 \times N_2$ is a graded 2-absorbing I_e-prime submodule of $M_1 \times M_2$.*

Proof. (i) Suppose that N_1 is a graded I_e-prime submodule of M_1. Now, let $r_g, s_h \in h(R)$ and $(m_{\lambda 1}, m_{\lambda 2}) \in h(M_1 \times M_2)$ such that $r_g s_h (m_{\lambda 1}, m_{\lambda 2}) = (r_g s_h m_{\lambda 1}, r_g s_h m_{\lambda 2}) \in (N_1 \times M_2) \backslash I_e(N_1 \times M_2) = (N_1 \backslash I_e N_1) \times (M_2 \backslash I_e M_2)$, which follows that $r_g s_h m_{\lambda 1} \in N_1 \backslash I_e N_1$. Hence, either $r_g m_{\lambda 1} \in N_1$ or $s_h m_{\lambda 1} \in N_1$ or $r_g s_h M_1 \subseteq N_1$ and then either $r_g(m_{\lambda 1}, m_{\lambda 2}) \in N_1 \times M_2$ or $s_h(m_{\lambda 1}, m_{\lambda 2}) \in N_1 \times M_2$ or $r_g s_h (M_1 \times M_2) \subseteq N_1 \times M_2$. Therefore, $N_1 \times M_2$ is a graded 2-absorbing I_e-prime submodule of $M_1 \times M_2$.

(ii) The proof is similar to that in part (i). □

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Conflict of interest

The authors declare that they have no any competing interests

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