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Research article

Integral transforms of an extended generalized multi-index Bessel function

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Abstract: In this paper, we discuss the extended generalized multi-index Bessel function by using the extended beta type function. Then we investigate its several properties including integral representation, derivatives, beta transform, Laplace transform, Mellin transforms, and some relations of extension of extended generalized multi-index Bessel function (E^1GMBF) with the Laguerre polynomial and Whittaker functions. Further, we also discuss the composition of the generalized fractional integral operator having Appell function as a kernel with the extension of extended generalized multi-index Bessel function and establish these results in terms of Wright functions.

Keywords: extended multi-index Bessel function; fractional integrals and derivatives; Appell function; extended beta transform
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1. Introduction

The Bessel function [1–8] has great importance in the field of mathematics, physics and engineering due to its applications. Researchers and mathematicians developed a new class of Bessel functions in the sense of multi-index functions, which motivate the future research work in the field of special functions and fractional calculus. The theory of multi-index multivariate Bessel function discussed by Dattoli *et al.* [9] in 1997.

Generalized multi-index Mittag-Leffler function was defined by Choi *et al.* in [10]. Kamarujjama *et al.* [11] introduced and studied the extended multi-index Bessel function. Suthar *et al.* [12] discussed a large number of results for the generalized multi-index Bessel function. Recently, many authors worked on generalized multi-index Bessel functions [13–15]. We describe extension of extended generalized multi-index Bessel function (E^1GMBF) which is generalized version of generalized multi-index Bessel function.

Definition 1.1. *[11] Kamarujjama et al. introduced and studied the extended generalized multi-index Bessel function, defined as:*

$$\mathbb{J}^{(\alpha_j)_m,\gamma,c}_{(\beta_j)_m,k,b,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-cz)^n}{(\delta)_n\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})},\qquad m\in\mathbb{N}. \tag{1.1}$$

where $\alpha_j,\beta_j,b,\delta,\gamma,c\in\mathbb{C}$ $(j=1,2\cdots m)$ *be such that* $\sum_{j=1}^{m}\Re(\alpha_j)>max\{0;\Re(k)-1\}$; $k>0$, $\Re(\beta_j)>-1,\Re(\gamma)>0$, $\Re(\delta)>0$.

Definition 1.2. *[16] Generalized fractional integral operator is defined for* $\alpha,\acute{\alpha},\beta,\grave{\beta},\lambda\in\mathbb{C}$, *and* $x>0$ *as follows:*

$$I^{\alpha,\acute{\alpha},\beta,\acute{\beta},\lambda}_{0^+}f(t)=\frac{x^{-\alpha}}{\Gamma(\lambda)}\int_0^x(x-t)^{\lambda-1}t^{-\acute{\alpha}}F_3(\alpha,\acute{\alpha},\beta,\acute{\beta};\lambda;1-\frac{t}{x};1-\frac{x}{t})f(t)dt, \tag{1.2}$$

and

$$I^{\alpha,\acute{\alpha},\beta,\acute{\beta},\lambda}_{-}f(t)=\frac{x^{-\acute{\alpha}}}{\Gamma(\lambda)}\int_x^\infty(t-x)^{\lambda-1}t^{-\alpha}F_3(\alpha,\acute{\alpha},\beta,\acute{\beta};\lambda;1-\frac{x}{t};1-\frac{t}{x})f(t)dt. \tag{1.3}$$

where F_3 *is the Appell function.*

Definition 1.3. *[17] Appell function* F_3 *also called the (Horn function) and defined for* $\alpha,\acute{\alpha},\beta,\grave{\beta},\lambda\in\mathbb{C}$, *as follows:*

$$F_3(\alpha,\acute{\alpha},\beta,\acute{\beta};\lambda;x;y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\acute{\alpha})_n(\beta)_m(\acute{\beta})_n}{(\lambda)_{m+n}m!n!}x^my^n,\qquad max\{|x|,|y|\}<1 \tag{1.4}$$

Definition 1.4. *[18, 19] The integral representation of gamma function is defined for* $\Re(s)>0$, *as follows:*

$$\Gamma(s)=\int_0^\infty u^{s-1}e^{-u}du. \tag{1.5}$$

Definition 1.5. *[18, 19] Classical beta function is defined for* $\Re(x)>0$ *and* $\Re(y)>0$, *as follows:*

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{1.6}$$

$$=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.7}$$

Definition 1.6. *[20, 21] Extended beta function is defined for* $\Re(x) > 0$, $\Re(y) > 0$, $\Re(p) > 0$ *as follows:*

$$B_p(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(\frac{-p}{t(1-t)}\right)dt, \tag{1.8}$$

if $p = 0$*, then extended beta function* $B_p(x, y)$ *reduces into the classical beta function.*

Definition 1.7. *[22] Generalized Wright type hypergeometric function is defined as follows:*

$${}_r\psi_s(z) = {}_r\psi_s\left[\begin{array}{c}(y_j, h_j)_{1,r}\\(x_i, q_i)_{1,s}\end{array}\middle| z\right] \equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r}\Gamma(y_j + h_j n)}{\prod_{i=1}^{s}\Gamma(x_i + q_i n)} \frac{z^n}{n!}. \tag{1.9}$$

where $z \in \mathbb{C}$, $y_j, x_i \in \mathbb{C}$ *and* $h_j, q_i \in \Re$ $(j = 1, 2 \cdots r; i = 1, 2 \cdots s)$.

Definition 1.8. *[23] Laplace transform is defined* $\Re(s) > 0$*, as follows:*

$$Ł[f(t)] = f(s) = \int_0^{\infty} e^{-st} f(t)dt. \tag{1.10}$$

Definition 1.9. *[24] Euler transform of a function* $f(z)$ *is defined as follows:*

$$B\{f(z); a, b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)dz \qquad (\Re(a) > 0, \Re(b) > 0). \tag{1.11}$$

Definition 1.10. *[24] Mellin transform of the function* $f(z)$ *is defined as follows:*

$$\mathfrak{M}\{f(z); s\} = \int_0^{\infty} z^{s-1} f(z)dz = f^*(s), \qquad \Re(s) > 0, \tag{1.12}$$

then inverse Mellin transform

$$f(z) = \mathfrak{M}^{-1}[f^*(s); z] = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} f^* z^{-s} ds, \qquad \lambda > 0. \tag{1.13}$$

Definition 1.11. *The Pochhammer symbol defined as*

$$(\delta)_n = \begin{cases} 1, & n = 0 \\ \delta(\delta+1)(\delta+2)\cdots(\delta+n-1), & n = 1, 2\cdots \end{cases} \tag{1.14}$$

or

$$(\delta)_n = \frac{\Gamma(\delta+n)}{\Gamma(\delta)} \tag{1.15}$$

$$(\delta)_{kn} = \frac{\Gamma(\delta+kn)}{\Gamma(\delta)}, \tag{1.16}$$

where $\delta \in \mathbb{C}$ *and* $n, k \in \mathbb{N}$.

Definition 1.12. *The E^1GMBF $\mathbb{J}^{(\alpha_j)_m,\gamma,c}_{(\beta_j)_m,k,b,\delta}(z)$ is defined in the following way:*

$$\mathbb{J}^{c,b,\delta}_{(\gamma,d);k}[(\alpha_j,\beta_j)_m;(z;p)] = \mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p)$$
$$= \sum_{n=0}^{\infty} \frac{B_p(\gamma+kn,d-\gamma)}{B(\gamma,d-\gamma)} \frac{c^n(d)_{kn}(-z)^n}{(\delta)_n \prod_{j=1}^{m} \Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})}. \tag{1.17}$$

where $\alpha_j,\beta_j,b,d,\delta,\gamma,c \in \mathbb{C}$ $(j=1,2\cdots m)$, $p \geq 0$ be such that $\sum_{j=1}^{m}\Re(\alpha_j) > max\{0;\Re(k)-1\}$; $k>0$, $\Re(\beta_j)>-1$, $\Re(d)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$.

Remark 1.1. *The E^1GMBF can also be write as*

$$\mathbb{J}^{c,b,\delta}_{(\gamma,d);k}[(\alpha_j,\beta_j)_m;(z;p)] = \mathbb{J}^{b,\delta}_{(\gamma,d);k}[(\alpha_j,\beta_j)_m;(cz;p)]. \tag{1.18}$$

2. Particular special cases

In this section, we establish some particular special cases of E^1GMBF as below

- if we set $p=0$, then E^1GMBF reduce into extended multi-index Bessel function

$$\mathbb{J}^{c,b,\delta}_{\gamma;k}[(\alpha_j,\beta_j)_m;(z)] = \mathbb{J}^{(\alpha_j)_m,\gamma,c}_{(\beta_j)_m,k,b,\delta}(z) = \sum_{n=0}^{\infty} \frac{(c)^n(\gamma)_{kn}(-z)^n}{(\delta)_n \prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})}. \tag{2.1}$$

- when $p=0$, $c=b=\delta=1$, then

$$\mathbb{J}^{1,1,1}_{\gamma;k}[(\alpha_j,\beta_j)_m;(z)] = J^{(\alpha_j)_m,\gamma}_{(\beta_j)_m,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}(-z)^n}{n!\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+1)}. \tag{2.2}$$

- if we put $p=0$, $c=b=\delta=m=1$, then E^1GMBF reduce to the generalized Bessel-Maitland function as,

$$\mathbb{J}^{1,1,1}_{\gamma;k}[(\alpha,\beta);(z)] = \mathrm{J}^{\alpha,\gamma}_{\beta,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}(-z)^n}{n!\Gamma(\alpha n+\beta+1)}. \tag{2.3}$$

- when $p=0$, $k=0$, $\delta=c=b=1$, then E^1GMBF reduce to the Bessel-Maitland function as given below

$$\mathbb{J}^{1,1,1}_{\gamma}[(\alpha,\beta);(z)] = \mathrm{J}^{\alpha}_{\beta}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(\alpha n+\beta+1)}. \tag{2.4}$$

- if we put $p=0$, $c=\delta=1$, $z=-z$ and set $\beta_j=\beta_j-1$, then E^1GMBF reduce to the multi-index Mittag Leffler function as given below

$$\mathbb{J}^{1,b,1}_{\gamma;k}[(\alpha_j,\beta_j)_m;(-z)] = E_{\gamma,k}[(\alpha_j,\beta)_j)^{m}_{j=1}] = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}}{\prod_{j=1}^{m}(\alpha_j n+\beta_j)} \frac{z^n}{n!}. \tag{2.5}$$

- if we set $p=k=0$, $b=c=m=1$, $\alpha_1=\delta=1$, $\beta_1=\nu$ and replace $z=\frac{z^2}{4}$ then E^1GMBF reduce into Bessel function of fist kind

$$\mathbb{J}^{1,1,1}_{\gamma;0}[(1,\nu)_m;(\frac{z^2}{4})] = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(n+\nu+1)}. \tag{2.6}$$

3. Results of E¹GMBF

In this section, we investigate the E¹GMBF, and studied some important observations. Moreover, we develop integral and differential of E¹GMBF in the form of theorems.

Theorem 3.1. *The E¹GMBF can be able to represent with* $\alpha_j, \beta_j, b, \delta, \gamma, c \in \mathbb{C}$ $(j = 1, 2 \cdots m)$ *be such that* $\sum_{j=1}^{m} \Re(\alpha_j) > max\{0; \Re(k) - 1\}$*;* $k > 0$*,* $\Re(\beta_j) > -1, \Re(\gamma) > 0$*,* $\Re(\delta) > 0$ *then following relation holds*

$$\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(z;p) = \frac{1}{B(\gamma, d-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{d-\gamma-1} e^{\frac{-p}{t(1-t)}} \mathbb{J}_{(\beta_j)_m,k,b,\delta}^{(\alpha_j)_m,\gamma,c}(t^k z)dt. \tag{3.1}$$

Proof. Using the definition of Eq (1.8) in (1.17), we obtain

$$\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(z;p) = \sum_{n=0}^{\infty} \left\{ \int_0^1 t^{\gamma+kn-1}(1-t)^{d-\gamma-1} e^{\frac{-p}{t(1-t)}} \right\} \times \frac{c^n (d)_{kn} (-z)^n}{B(\gamma, d-\gamma)(\delta)_n \prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j + \frac{1+b}{2})} dt. \tag{3.2}$$

Changing the order of summation and integration, and after simplification of Eq (3.2), we get

$$\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(z;p) = \frac{1}{B(\gamma, d-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{d-\gamma-1} e^{\frac{-p}{t(1-t)}} \sum_{n=0}^{\infty} \frac{c^n (d)_{kn} (-t^k z)^n}{(\delta)_n \prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j + \frac{1+b}{2})} dt. \tag{3.3}$$

Using Eq (1.1) in Eq (3.3), we obtain the desired result in theorem 3.1. □

Corollary 3.1. *Let* $\alpha_j, \beta_j, b, \delta, \gamma, c \in \mathbb{C}$ $(j = 1, 2 \cdots m)$ *be such that* $\sum_{j=1}^{m} \Re(\alpha_j) > max\{0; \Re(k) - 1\}$*;* $k > 0$*,* $\Re(\beta_j) > -1$*,* $\Re(\gamma) > 0$*,* $\Re(\delta) > 0$*. Taking* $t = \frac{r}{1+r}$ *in theorem 3.1, then following relation holds*

$$\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(z;p) = \frac{1}{B(\gamma, d-\gamma)} \int_0^{\infty} \frac{r^{\gamma-1}}{(1+r)^d} e^{\frac{-p(1+r)^2}{r}} \mathbb{J}_{(\beta_j)_m,k,b,\delta}^{(\alpha_j)_m,\gamma,c}\left(\frac{r^k z}{(1+r)^k}\right)dr. \tag{3.4}$$

Corollary 3.2. *Let* $\alpha_j, \beta_j, b, \delta, \gamma, c \in \mathbb{C}$ $(j = 1, 2 \cdots m)$ *be such that* $\sum_{j=1}^{m} \Re(\alpha_j) > max\{0; \Re(k) - 1\}$*;* $k > 0$*,* $\Re(\beta_j) > -1, \Re(\gamma) > 0$*,* $\Re(\delta) > 0$ *and consider* $t = \cos^2\theta$ *in theorem 3.1, then following relation holds*

$$\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(z;p) = \frac{2}{B(\gamma, d-\gamma)} \int_0^{\frac{\pi}{2}} (\cos\theta)^{2\gamma-1}(\sin\theta)^{2d-2\gamma-1} \exp\left(\frac{-p}{\sin^2\theta\cos^2\theta}\right) \times \mathbb{J}_{(\beta_j)_m,k,b,\delta}^{(\alpha_j)_m,\gamma,c}(z\cos^{2k}\theta)d\theta. \tag{3.5}$$

Theorem 3.2. *Let $\alpha, \beta, b, \delta, \gamma, c \in \mathbb{C}$ be such that $\Re(\alpha) > max\{0; \Re(k) - 1\}$; $k > 0$, $\Re(\beta) > -1, \Re(\gamma) > 0$, $\Re(\delta) > 0$, then the following recurrence relation holds in the definition* (1.17) *for* $j = 1$ *as*

$$\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,\alpha,\beta}(z;p) = (\beta + \frac{b+1}{2})\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,\alpha,\beta+1}(z;p) + \alpha z \frac{d}{dz}\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha,\beta+1)}(z;p). \tag{3.6}$$

Proof. Consider the definition of (1.17) for $j = 1$, and the right side of the Eq (3.6), we get

$$\begin{aligned}
&(\beta + \frac{b+1}{2})\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha,\beta+1)}(z;p) + \alpha z \frac{d}{dz}\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha,\beta+1)}(z;p) \\
&\quad =(\beta + \frac{b+1}{2})\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn, d-\gamma)}{B(\gamma, d-\gamma)}\frac{c^n(d)_{kn}(-z)^n}{(\delta)_n\Gamma(\alpha n + \beta + 1 + \frac{1+b}{2})} \\
&\quad +\alpha z\frac{d}{dz}\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn, d-\gamma)}{B(\gamma, d-\gamma)}\frac{c^n(d)_{kn}(-z)^n}{(\delta)_n\Gamma(\alpha n + \beta + 1 + \frac{1+b}{2})} \\
&\quad =\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn, d-\gamma)}{B(\gamma, d-\gamma)}\frac{c^n(d)_{kn}}{(\delta)_n} \\
&\quad \times\Big[\frac{(\beta + \frac{b+1}{2})(-z)^n}{\Gamma(\alpha n + \beta + 1 + \frac{1+b}{2})} + \frac{\alpha z\frac{d}{dz}(-z)^n}{\Gamma(\alpha n + \beta + 1 + \frac{1+b}{2})}\Big] \\
&\quad =\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn, d-\gamma)}{B(\gamma, d-\gamma)}\frac{c^n(d)_{kn}(-z)^n(\alpha n + \beta + \frac{1+b}{2})}{(\delta)_n\Gamma(\alpha n + \beta + 1 + \frac{1+b}{2})} \\
&\quad =\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn, d-\gamma)}{B(\gamma, d-\gamma)}\frac{c^n(d)_{kn}(-z)^n}{(\delta)_n\Gamma(\alpha n + \beta + \frac{1+b}{2})} \\
&\quad =\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha,\beta)}(z;p)
\end{aligned} \tag{3.7}$$

□

Theorem 3.3. *For the E^1GMBF we have the following higher derivative formula for* $\delta = 1$, *is given below*

$$\frac{d^n}{dz^n}\mathbb{J}^{k,c,(\gamma,d);k}_{1,b,(\alpha_j,\beta_j)_m}(z;p) = (-c)^n(d)_k(d+k)_k\cdots(d+(n-1)k)_k\mathbb{J}^{k,c,(\gamma+kn,d+kn);k}_{1,b,(\alpha_j,\beta_j+\alpha_j n)_m}(z;p). \tag{3.8}$$

where $\alpha_j, \beta_j, b, \gamma, c \in \mathbb{C}$ $(j = 1, 2\cdots m)$ be such that $\sum_{j=1}^{m}\Re(\alpha_j) > max\{0; \Re(k) - 1\}$; $k > 0$, $\Re(\beta_j) > -1, \Re(\gamma) > 0$.

Proof. Differentiation with respect to z in Eq (1.17), we get

$$\begin{aligned}
\frac{d}{dz}\mathbb{J}^{k,c,(\gamma,d);k}_{1,b,(\alpha_j,\beta_j)_m}(z;p) &= \sum_{n=1}^{\infty}\frac{B_p(\gamma+kn, d-\gamma)}{B(\gamma, d-\gamma)}\frac{c^n(d)_{kn}(-1)^n n z^{n-1}}{n!\prod_{j=1}^{m}\Gamma(\alpha_j n + \beta_j + \frac{1+b}{2})} \\
&= \sum_{n=1}^{\infty}\frac{B_p(\gamma+k(n-1)+k, d-\gamma)}{B(\gamma, d-\gamma)}\frac{(-c)^n(d)_{k(n-1)+k} n z^{n-1}}{n(n-1)!\prod_{j=1}^{m}\Gamma(\alpha_j n + \beta_j + \frac{1+b}{2})}
\end{aligned} \tag{3.9}$$

we can write the pochhammer symbols as

$$\begin{aligned}(d)_{k(n-1)+k} &= \frac{\Gamma(d+k(n-1)+k)}{\Gamma(d)}\\ &= \frac{\Gamma(d+k(n-1)+k)}{\Gamma(d+k)}\frac{\Gamma(d+k)}{\Gamma(d)}\\ &= (d+k)_{(n-1)k}(d)_k. \end{aligned}\tag{3.10}$$

Now, using the Eq (3.10) in Eq (3.9), we have

$$\begin{aligned}&\frac{d}{dz}\mathbb{J}^{k,c,(\gamma,d);k}_{1,b,(\alpha_j,\beta_j)_m}(z;p)\\ &= (-c)(d)_k\sum_{n=1}^{\infty}\frac{B_p(\gamma+k+k(n-1),d-\gamma)(-c)^{n-1}(d+k)_{k(n-1)}z^{n-1}}{B(\gamma,d-\gamma)(n-1)!\prod_{j=1}^{m}\Gamma(\alpha_j(n-1)+\alpha_j+\beta_j+\frac{1+b}{2})}\\ &= (-c)(d)_k\mathbb{J}^{k,c,(\gamma+k,d+k);k}_{1,b,(\alpha_j,\beta_j+\alpha_j)_m}(z;p).\end{aligned}\tag{3.11}$$

Again differentiation with respect to z in Eq (3.9), we have

$$\frac{d^2}{dz^2}\mathbb{J}^{k,c,(\gamma,d);k}_{1,b,(\alpha_j,\beta_j)_m}(z;p) = (-c)^2(d)_k(d+k)_k\mathbb{J}^{k,c,(\gamma+2k,d+2k);k}_{1,b,(\alpha_j,\beta_j+2\alpha_j)_m}(z;p),$$

continue this technique up to n times, we obtain the desired result which state in the theorem 3.3. □

Theorem 3.4. *Let $\alpha_j,\beta_j,d,\gamma,c,\lambda\in\mathbb{C}$ $(j=1,2\cdots m)$, $p\geq 0$ be such that $\sum_{j=1}^{m}\Re(\alpha_j)>max\{0;\Re(k)-1\}$; $k>0$, $\Re(\beta_j)>0$, $\Re(d)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ then the following relation holds as:*

$$\frac{d^n}{dz^n}\{z^{\beta_1\cdots\beta_m-1}\mathbb{J}^{k,c,(\gamma,d);k}_{1,-1,(\alpha_j,\beta_j)_m}(\lambda z^{\alpha_1\cdots\alpha_m};p)\} = \frac{\mathbb{J}^{k,c,(\gamma,d);k}_{1,-1,(\alpha_j,\beta_j-n)_m}(\lambda z^{\alpha_1\cdots\alpha_m};p)}{z^{n-\beta_1\cdots-\beta_m+1}}.\tag{3.12}$$

Proof. Replacing z by $\lambda z^{\alpha_j\cdots\alpha_j}$, $b=-1$ and $\delta=1$ in Eq (1.17), take its product $z^{\beta_1\cdots\beta_j}$, and after taking differentiation with respect to z up to n times, we obtain our required result. □

4. Integral transforms of E[1]GMBF

In this section, we establish some integral transforms (Euler, Mellin and Laplace transform) of E^1GMBF in the form of theorems, and also discuss its sub cases.

Theorem 4.1. *Euler transform of E^1GMBF holds for $\alpha_j,\beta_j,b,\delta,\gamma,c\in\mathbb{C}$ $(j=1,2\cdots m)$ be such that $\sum_{j=1}^{m}\Re(\alpha_j)>max\{0;\Re(k)-1\}$; $k>0$, $\Re(\beta_j)>-1,\Re(\gamma)>0$, $\Re(\delta)>0$.*

$$B\{\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,-1,(\alpha_j,\beta_j)_m}(\lambda z^{\alpha_j};p);\beta_1\cdots\beta_m,1\} = \mathbb{J}^{k,c,(\gamma,d);k}_{\delta,-1,(\alpha_j,\beta_j+1)_m}(\lambda;p).\tag{4.1}$$

Proof. Apply the definition of Euler transform (1.9) in Eq (1.17), we get

$$B\{\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,-1,(\alpha_j,\beta_j)_m}(\lambda z^{\alpha_j};p);\beta_1\cdots\beta_m,1\}$$
$$=\int_0^1 z^{\beta_1\cdots\beta_m-1}(1-z)^{1-1}\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn,d-\gamma)}{B(\gamma,d-\gamma)}\times\frac{c^n(d)_{kn}(-1)^n(\lambda z^{\alpha_j})^n}{(\delta)_n\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j)}dz. \tag{4.2}$$

Interchanging the order of summations and integration in Eq (4.2), we get

$$B\{\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,-1,(\alpha_j,\beta_j)_m}(\lambda z^{\alpha_j};p);\beta_1\cdots\beta_m,1\}$$
$$=\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn,d-\gamma)}{B(\gamma,d-\gamma)}\frac{c^n(d)_{kn}(-\lambda)^n}{(\delta)_n\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j)}\times\int_0^1 z^{\beta_1\cdots\beta_m+\alpha_j n-1}(1-z)^{1-1}dz. \tag{4.3}$$

Using the Eq (1.6) and Eq (1.7) in Eq (4.3), then we obtain

$$B\{\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,-1,(\alpha_j,\beta_j)_m}(\lambda z^{\alpha_j};p);\beta_1\cdots\beta_m,1\}$$
$$=\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn,d-\gamma)c^n(d)_{kn}(-\lambda)^n}{B(\gamma,d-\gamma)(\delta)_n\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j+1)}$$
$$=\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,-1,(\alpha_j,\beta_j+1)_m}(\lambda;p). \tag{4.4}$$

□

Theorem 4.2. *The Mellin transform of* E^1*GMBF is given by for* $\alpha_j,\beta_j,b,\delta,\gamma,c\in\mathbb{C}$ $(j=1,2\cdots m)$ *be such that* $\sum_{j=1}^m\Re(\alpha_j)>max\{0;\Re(k)-1\}$*;* $k>0$*,* $\Re(\beta_j)>-1,\Re(\gamma)>0$*,* $\Re(\delta)>0$*. Then the following relation holds*

$$\mathfrak{M}\{\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p);s\}$$
$$=\frac{\Gamma(s)\Gamma(\delta)\Gamma(d)\Gamma(d-\gamma+s)}{[\Gamma(\gamma)]^2\Gamma(d-\gamma)}{}_3\psi_{m+2}\left[\begin{matrix}(\gamma,k)(\gamma+s,k)(1,1)\\(\delta,1)(d+2s,k)(\beta_j+\frac{1+b}{2},\alpha_j)|_{j=1}^m\end{matrix}\middle|-cz\right].$$

Proof. By applying the definition of the Mellin transform to the E^1GMBF, we have

$$\mathfrak{M}\{\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p);s\}=\int_0^\infty p^{s-1}\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p)dp. \tag{4.5}$$

Using theorem 3.1 in right side of Eq (4.5), we get

$$\mathfrak{M}\{\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p);s\}$$
$$= \frac{1}{B(\gamma,d-\gamma)}\int_0^\infty p^{s-1}\left\{\int_0^1 t^{\gamma-1}(1-t)^{d-\gamma-1} e^{\frac{-p}{t(1-t)}}\mathbb{J}^{(\alpha_j)_m,\gamma,c}_{(\beta_j)_m,k,b,\delta}(t^k z)dt\right\}dp. \tag{4.6}$$

Interchanging the order of integration in Eq (4.6), then we have

$$\mathfrak{M}\{\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p);s\}$$
$$= \frac{1}{B(\gamma,d-\gamma)}\int_0^1 t^{\gamma-1}(1-t)^{d-\gamma-1}\mathbb{J}^{(\alpha_j)_m,\gamma,c}_{(\beta_j)_m,k,b,\delta}(t^k z)\left\{\int_0^\infty p^{s-1}e^{\frac{-p}{t(1-t)}}dp\right\}dt. \tag{4.7}$$

Now, putting $\frac{p}{t(1-t)} = u$ in Eq (4.7), and applying the mathematical formula of Eq (1.5), we get

$$\mathfrak{M}\{\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p);s\} = \frac{\Gamma(s)}{B(\gamma,d-\gamma)}\int_0^1 t^{\gamma+s-1}(1-t)^{d-\gamma+s-1}\mathbb{J}^{(\alpha_j)_m,\gamma,c}_{(\beta_j)_m,k,b,\delta}(t^k z)dt. \tag{4.8}$$

Using Eq (1.1), and interchanging the order of integration and summation in Eq (4.8), we obtain

$$\mathfrak{M}\{\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p);s\}$$
$$= \frac{\Gamma(s)}{B(\gamma,d-\gamma)}\sum_{n=0}^{\infty}\frac{(c)^n(\gamma)_{kn}(-z)^n}{(\delta)_n\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})}\int_0^1 t^{\gamma+s+kn-1}(1-t)^{d-\gamma+s-1}dt. \tag{4.9}$$

Using Eq (1.6) and Eq (1.7) in Eq (4.9), we get

$$\mathfrak{M}\{\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p);s\}$$
$$= \frac{\Gamma(s)}{B(\gamma,d-\gamma)}\sum_{n=0}^{\infty}\frac{(c)^n(\gamma)_{kn}(-z)^n}{(\delta)_n\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})}\frac{\Gamma(\gamma+s+kn)\Gamma(d-\gamma+s)}{\Gamma(2s+kn+d)}. \tag{4.10}$$

After simplification in Eq (4.10), we get

$$\mathfrak{M}\{\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p);s\}$$
$$=\frac{\Gamma(s)\Gamma(\delta)\Gamma(d)\Gamma(d-\gamma+s)}{[\Gamma(\gamma)]^2\Gamma(d-\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+kn)(-cz)^n}{\Gamma(\delta+n)\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})}\frac{\Gamma(\gamma+s+kn)}{\Gamma(2s+kn+d)}$$
$$=\frac{\Gamma(s)\Gamma(\delta)\Gamma(d)\Gamma(d-\gamma+s)}{[\Gamma(\gamma)]^2\Gamma(d-\gamma)}{}_3\psi_{m+2}\left[\begin{array}{c}(\gamma,k)(\gamma+s,k)(1,1)\\(\delta,1)(d+2s,k)(\beta_j+\frac{1+b}{2},\alpha_j)|_{j=1}^m\end{array}\middle|-cz\right].$$

□

Corollary 4.1. *Let $\alpha_j, \beta_j, b, \delta, \gamma, c \in \mathbb{C}$ $(j = 1, 2 \cdots m)$ be such that $\sum_{j=1}^{m} \Re(\alpha_j) > max\{0; \Re(k) - 1\}$; $k > 0$, $\Re(\beta_j) > -1, \Re(\gamma) > 0$, $\Re(\delta) > 0$. Taking $s = 1$ in theorem 4.2, then the following relation holds*

$$\int_0^\infty \mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p)dp = \frac{(d-\gamma)\Gamma(\delta)}{\Gamma(\gamma)} {}_3\psi_{m+2}\left[\begin{array}{c} (\gamma,k)(\gamma+1,k)(1,1) \\ (\delta,1)(d+2,k)(\beta_j+\frac{1+b}{2},\alpha_j)|_{j=1}^m \end{array}\Bigg| -cz\right]. \tag{4.11}$$

Corollary 4.2. *Let $\alpha_j, \beta_j, b, \delta, \gamma, c \in \mathbb{C}$ $(j = 1, 2 \cdots m)$ be such that $\sum_{j=1}^{m} \Re(\alpha_j) > max\{0; \Re(k) - 1\}$; $k > 0$, $\Re(\beta_j) > -1, \Re(\gamma) > 0$, $\Re(\delta) > 0$. Applying the inverse Mellin transform on left and right side of Eq* (1.17)*, we gain the important complex integral representation as follows:*

$$\begin{aligned}\mathfrak{M}^{-1}\{\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p);s\} =& \frac{1}{2\pi i\Gamma(\gamma)\Gamma(d-\gamma)} \int_{\lambda-i\infty}^{\lambda+i\infty} \Gamma(s)\Gamma(\delta)\Gamma(d-\gamma+s) \\ &\times {}_3\psi_{m+2}\left[\begin{array}{c} (\gamma,k)(\gamma+s,k)(1,1) \\ (\delta,1)(d+2s,k)(\beta_j+\frac{1+b}{2},\alpha_j)|_{j=1}^m \end{array}\Bigg| -cz\right] p^{-s}ds. \end{aligned} \tag{4.12}$$

Theorem 4.3. *The Laplace transform of E[1]GMBF is given as for $\alpha_j, \beta_j, b, \delta, \gamma, c \in \mathbb{C}$ $(j = 1, 2 \cdots m)$ be such that $\sum_{j=1}^{m} \Re(\alpha_j) > max\{0; \Re(k) - 1\}$; $k > 0$, $\Re(\beta_j) > -1, \Re(\gamma) > 0$, $\Re(\delta) > 0$.*

$$\text{Ł}(\mathbb{J}^{k,c,(\gamma,d);k}_{1,b,(\alpha_j,\beta_j)_m}(z;p)) = \frac{1}{s}\mathbb{J}^{k,c,(\gamma,d);k}_{1,b,(\alpha_j,\beta_j)_m}(\frac{1}{s};p). \tag{4.13}$$

Proof. Using the definition of Laplace transform (1.8) in Eq (1.17), we have

$$\begin{aligned}\text{Ł}(\mathbb{J}^{k,c,(\gamma,d);k}_{1,b,(\alpha_j,\beta_j)_m}(z;p)) &= \int_0^\infty e^{-st}\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn,d-\gamma)}{B(\gamma,d-\gamma)}\frac{c^n(d)_{kn}(-t)^n}{n!\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})}dt \\ &= \sum_{n=0}^{\infty}\frac{B_p(\gamma+kn,d-\gamma)}{B(\gamma,d-\gamma)}\frac{c^n(d)_{kn}(-1)^n}{n!\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})}\int_0^\infty e^{-st}t^n dt \\ &= \sum_{n=0}^{\infty}\frac{B_p(\gamma+kn,d-\gamma)}{B(\gamma,d-\gamma)}\frac{c^n(d)_{kn}(-1)^n}{n!\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})}\frac{n!}{s^{n+1}} \\ &= \frac{1}{s}\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn,d-\gamma)}{B(\gamma,d-\gamma)}\frac{c^n(d)_{kn}(-s)^n}{\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})} \\ &= \frac{1}{s}\mathbb{J}^{k,c,(\gamma,d);k}_{1,b,(\alpha_j,\beta_j)_m}(\frac{1}{s};p). \end{aligned} \tag{4.14}$$

□

5. Relation of the E[1]GMBF with the Laguerre polynomial and Whittaker function

In this section, the authors represent the E[1]GMBF in terms of Laguerre polynomial, and Whittaker function in the form of theorems.

Theorem 5.1. *Let $\alpha_j, \beta_j, b, d, \delta, \gamma, c \in \mathbb{C}$ $(j = 1, 2 \cdots m)$, $p \geq 0$ be such that $\sum_{j=1}^{m} \Re(\alpha_j) > max\{0; \Re(k) - 1\}$; $k > 0$, $\Re(\beta_j) > 0$, $\Re(d) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, then the E[1]GMBF holds*

$$e^{2p}\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p)=\frac{\Gamma(\delta)\sum_{a,b=0}^{\infty}L_b(p)L_a(p)\Gamma(b+d-\gamma+1)}{\Gamma(\gamma)B(\gamma,d-\gamma)}$$
$$\times{}_3\psi_{m+2}\left[\begin{matrix}(\gamma,k)(a+\gamma+1,k)(1,1)\\(\delta,1)(a+b+d+2,k)(\beta_j+\frac{1+b}{2},\alpha_j)|_{j=1}^{m}\end{matrix}\middle|-cz\right].\tag{5.1}$$

Proof. We being recalling the valuable identity [25] which is

$$e^{\frac{-p}{t(1-t)}}=e^{-2p}\sum_{a,b=0}^{\infty}L_b(p)L_a(p)t^{a+1}(1-t)^{b+1},\quad(0<t<1).\tag{5.2}$$

Applying Eq (5.2) in theorem 3.1, we get

$$\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p)$$
$$=\frac{1}{B(\gamma,d-\gamma)}\int_0^1 t^{\gamma-1}(1-t)^{d-\gamma-1}e^{-2p}\sum_{a,b=0}^{\infty}L_b(p)L_a(p)t^{a+1}(1-t)^{b+1}\mathbb{J}^{(\alpha_j)_m,\gamma,c}_{(\beta_j)_m,k,b,\delta}(t^kz)dt$$
$$=\frac{1}{B(\gamma,d-\gamma)}\int_0^1 t^{\gamma-1}(1-t)^{d-\gamma-1}e^{-2p}\sum_{a,b=0}^{\infty}L_b(p)L_a(p)t^{a+1}(1-t)^{b+1}$$
$$\times\sum_{n=0}^{\infty}\frac{(c)^n(\gamma)_{kn}(-t^kz)^n}{(\delta)_n\prod_{j=1}^{m}\Gamma(\alpha_jn+\beta_j+\frac{1+b}{2})}dt.\tag{5.3}$$

Interchanging the order of integration and summations in Eq (5.3), we obtain

$$\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p)$$
$$=\frac{e^{-2p}}{B(\gamma,d-\gamma)}\sum_{a,b,n=0}^{\infty}\frac{L_b(p)L_a(p)(\gamma)_{kn}(-cz)^n}{(\delta)_n\prod_{j=1}^{m}\Gamma(\alpha_jn+\beta_j+\frac{1+b}{2})}\int_0^1 t^{a+kn+\gamma}(1-t)^{b+d-\gamma}dt.\tag{5.4}$$

Using Eq (1.6) and Eq (1.7) in Eq (5.4), then we have

$$e^{2p}\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p)$$
$$=\frac{1}{B(\gamma,d-\gamma)}\sum_{a,b,n=0}^{\infty}\frac{L_b(p)L_a(p)(\gamma)_{kn}(-cz)^n}{(\delta)_n\prod_{j=1}^{m}\Gamma(\alpha_jn+\beta_j+\frac{1+b}{2})}\frac{\Gamma(a+kn+\gamma+1)\Gamma(b+d-\gamma+1)}{\Gamma(a+b+d+kn+2)}$$
$$=\frac{\Gamma(\delta)\sum_{a,b=0}^{\infty}L_b(p)L_a(p)\Gamma(b+d-\gamma+1)}{\Gamma(\gamma)B(\gamma,d-\gamma)}$$
$$\times{}_3\psi_{m+2}\left[\begin{matrix}(\gamma,k)(a+\gamma+1,k)(1,1)\\(\delta,1)(a+b+d+2,k)(\beta_j+\frac{1+b}{2},\alpha_j)|_{j=1}^{m}\end{matrix}\middle|-cz\right].\tag{5.5}$$

□

Theorem 5.2. *For the E^1GMBF with $\alpha_j, \beta_j, b, \delta, \gamma, c \in \mathbb{C}$ $(j = 1, 2 \cdots m)$ be such that $\sum_{j=1}^{m} \Re(\alpha_j) > max\{0; \Re(k) - 1\}$; $k > 0$, $\Re(\beta_j) > -1, \Re(\gamma) > 0$, $\Re(\delta) > 0$, we have*

$$e^{\frac{3p}{2}} \mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p) = \frac{e^{-p}}{B(\gamma, d-\gamma)} \sum_{a,n=0}^{\infty} \frac{L_a(p)(\delta)_{kn}(-cz)^n}{(\delta)_n \prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j + \frac{b+1}{2})} \times \Gamma(d-\gamma+1) p^{\frac{\gamma+kn}{2}} \mathrm{W}_{\frac{-1+\gamma-2d-kn}{2}, \frac{\gamma+kn}{2}}. \tag{5.6}$$

Proof. Allowing for the following equality $e^{\frac{-p}{t(1-t)}} = e^{(\frac{-p}{1-t})} e^{(\frac{-p}{t})}$ and via generating function related to the Laguerre polynomial [25], we obtain

$$e^{\frac{-p}{t(1-t)}} = e^{-p} e^{\frac{-p}{t}} (1-t) \sum_{a=0}^{\infty} L_a(p) t^n. \tag{5.7}$$

Using Eq (5.7) in Eq (1.17), we have

$$\begin{aligned} &\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p) \\ &= \frac{1}{B(\gamma, d-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{d-\gamma-1} e^{-p} e^{\frac{-p}{t}} (1-t) \sum_{a=0}^{\infty} L_a(p) t^n \mathbb{J}^{(\alpha_j)_m,\gamma,c}_{(\beta_j)_m,k,b,\delta}(t^k z) dt \\ &= \frac{e^{-p}}{B(\gamma, d-\gamma)} \sum_{a,n=0}^{\infty} \frac{L_a(p)(\delta)_{kn}(-cz)^n}{(\delta)_n \prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j + \frac{b+1}{2})} \int_0^1 t^{\gamma+kn-1}(1-t)^{d-\gamma} e^{\frac{-p}{t}} dt. \end{aligned} \tag{5.8}$$

Now, integral representation of Whittaker function is defined [26] as follows

$$\int_0^1 t^{\mu-1}(1-t)^{\nu-1} e^{\frac{-p}{t}} dt = \Gamma(\nu) p^{\frac{\mu-1}{2}} e^{\frac{-p}{2}} \mathrm{W}_{\frac{1-\mu-2\nu}{2}, \frac{\mu}{2}}(p). \tag{5.9}$$

Using Eq (5.9) in Eq (5.8), then we have

$$\mathbb{J}^{k,c,(\gamma,d);k}_{\delta,b,(\alpha_j,\beta_j)_m}(z;p) = \frac{e^{-p}}{B(\gamma, d-\gamma)} \sum_{a,n=0}^{\infty} \frac{L_a(p)(\delta)_{kn}(-cz)^n}{(\delta)_n \prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j + \frac{b+1}{2})} \times \Gamma(d-\gamma+1) p^{\frac{\gamma+kn}{2}} e^{\frac{-p}{2}} \mathrm{W}_{\frac{-1+\gamma-2d-kn}{2}, \frac{\gamma+kn}{2}}. \tag{5.10}$$

□

Theorem 5.3. *Let $\alpha_j, \beta_j, b, d, \delta, \gamma, \sigma, \eta, c \in \mathbb{C}$ $(j = 1, 2 \cdots m)$, $p \geq 0$ be such that $\sum_{j=1}^{m} \Re(\alpha_j) > max\{0; \Re(k) - 1\}$; $k > 0$, $\Re(\beta_j) > 0$, $\Re(d) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$ the E^1GMBF holds*

$$(I_{0^+}^{\alpha,\acute{\alpha},\beta,\acute{\beta},\lambda}\mathbb{J}_{(\gamma,d);k}^{c,b,\delta}[(\alpha_j,\beta_j)_m;(t^{-\sigma+\eta};p)])(x)$$
$$=\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn,d-\gamma)(-c)^n}{x^{\sigma n-\eta n+\alpha-\lambda+\acute{\alpha}}}\Gamma\left[\begin{array}{l}(d+kn)(\delta)(\lambda-\acute{\alpha}-\sigma n+\eta n-\alpha-\beta+1)\\(\gamma)(d-\gamma)(\delta+n)(\lambda-\acute{\alpha}-\sigma n+\eta n-\beta+1)\end{array}\right.$$
$$\times\left.\begin{array}{l}(1-\acute{\alpha}-\sigma n+\eta n+\acute{\beta})(1-\sigma n+\eta n)\\(1-\sigma n+\eta n+\acute{\beta})(\lambda-\acute{\alpha}-\sigma n+\eta n-\alpha+1)(\alpha_j n+\beta_j+\frac{1+b}{2})|_{j=1}^{m}\end{array}\right].$$

Proof. Consider the composition of generalized fractional integral operator having Appell function as its kernel with the E^1GMBF,

$$(I_{0^+}^{\alpha,\acute{\alpha},\beta,\acute{\beta},\lambda}\mathbb{J}_{(\gamma,d);k}^{c,b,\delta}[(\alpha_j,\beta_j)_m;(t^{-\sigma+\eta};p)])(x)$$
$$=\frac{x^{-\alpha}}{\Gamma(\lambda)}\int_0^x(x-t)^{\lambda-1}t^{-\acute{\alpha}}F_3(\alpha,\acute{\alpha},\beta,\acute{\beta};\lambda;1-\frac{t}{x};1-\frac{x}{t})\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn,d-\gamma)}{B(\gamma,d-\gamma)}$$
$$\times\frac{c^n(d)_{kn}(-1)^n t^{-\sigma n+\eta n}}{(\delta)_n\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})}dt$$
$$=\frac{x^{-\alpha+\lambda-1}}{\Gamma(\lambda)}\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(1;p)|_{n=0}^{\infty}\int_0^x(1-\frac{t}{x})^{\lambda-1}t^{-\acute{\alpha}-\sigma n+\eta n}\sum_{m,s=0}^{\infty}\frac{(\alpha)_m(\acute{\alpha})_s(\beta)_m(\acute{\beta})_s}{\lambda_{m+s}m!s!}$$
$$\times(1-\frac{t}{x})^m(1-\frac{x}{t})^s dt$$
$$=\frac{x^{-\alpha+\lambda-1}}{\Gamma(\lambda)}\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(1;p)|_{n=0}^{\infty}\sum_{m,s=0}^{\infty}\frac{(\alpha)_m(\acute{\alpha})_s(\beta)_m(\acute{\beta})_s}{\lambda_{m+s}m!s!}\int_0^x(1-\frac{t}{x})^{m+\lambda-1}$$
$$\times(1-\frac{x}{t})^s t^{-\acute{\alpha}-\sigma n+\eta n}dt. \tag{5.11}$$

Putting these values $\frac{t}{x}=\tau\Rightarrow d\tau=xdt$, $t=x\Rightarrow\tau=1$ and $t=0\Rightarrow\tau=0$ in Eq (5.11), then we have

$$(I_{0^+}^{\alpha,\acute{\alpha},\beta,\acute{\beta},\lambda}\mathbb{J}_{(\gamma,d);k}^{c,b,\delta}[(\alpha_j,\beta_j)_m;(t^{-\sigma+\eta};p)])(x)$$
$$=\frac{x^{-\alpha+\lambda-1}}{\Gamma(\lambda)}\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(1;p)|_{n=0}^{\infty}\sum_{m,s=0}^{\infty}\frac{(\alpha)_m(\acute{\alpha})_s(\beta)_m(\acute{\beta})_s}{\lambda_{m+s}m!s!}\int_0^1(1-\tau)^{m+\lambda-1}$$
$$\times(1-\frac{1}{\tau})^s(x\tau)^{-\acute{\alpha}-\sigma n+\eta n}xd\tau$$
$$=\frac{x^{-\alpha-\acute{\alpha}+\lambda}}{\Gamma(\lambda)}\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(\frac{x^{\eta}}{x^{\sigma}};p)|_{n=0}^{\infty}\sum_{m,s=0}^{\infty}\frac{(\alpha)_m(\acute{\alpha})_s(\beta)_m(\acute{\beta})_s(-1)^s}{\lambda_{m+s}m!s!}\int_0^1(1-\tau)^{s+m+\lambda-1}$$
$$\times\tau^{-\acute{\alpha}-\sigma n+\eta n-s}d\tau. \tag{5.12}$$

Using Eqs (1.6) and (1.7) in Eq (5.12), we obtain

$$
\begin{aligned}
&(I_{0^+}^{\alpha,\acute{\alpha},\beta,\acute{\beta},\lambda}\mathbb{J}_{(\gamma,d);k}^{c,b,\delta}[(\alpha_j,\beta_j)_m;(t^{-\sigma+\eta};p)])(x)-x^{-\alpha+\lambda-\acute{\alpha}}\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(\frac{x^{\eta}}{x^{\sigma}};p)|_{n=0}^{\infty}\\
&=\sum_{m,s=0}^{\infty}\frac{(\alpha)_m(\acute{\alpha})_s(\beta)_m(\acute{\beta})_s(-1)^s}{\lambda_{m+s}m!s!}\frac{\Gamma(s+m+\lambda)\Gamma(-\acute{\alpha}-\sigma n+\eta n-s+1)}{\Gamma(\lambda)\Gamma(m+\lambda-\acute{\alpha}-\sigma n+\eta n+1)}\\
&=\frac{\Gamma(-\acute{\alpha}-\sigma n+\eta n+1)}{\Gamma(\lambda-\acute{\alpha}-\sigma n+\eta n+1)}\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\lambda-\acute{\alpha}-\sigma n+\eta n+1)_m m!}\sum_{s=0}^{\infty}\frac{(\acute{\alpha})_s(\acute{\beta})_s}{(\acute{\alpha}+\sigma n-\eta n)_s s!}\\
&=\frac{\Gamma(-\acute{\alpha}-\sigma n+\eta n+1)\Gamma(\lambda-\acute{\alpha}-\sigma n+\eta n-\alpha-\beta+1)\Gamma(\acute{\alpha}+\sigma n-\eta n)\Gamma(\sigma n-\eta n-\acute{\beta})}{\Gamma(\lambda-\acute{\alpha}-\sigma n+\eta n-\alpha+1)\Gamma(\lambda-\acute{\alpha}-\sigma n+\eta n-\beta+1)\Gamma(\sigma n-\eta n)\Gamma(\acute{\alpha}+\sigma n-\eta n-\acute{\beta})}\\
&=\frac{\Gamma(\lambda-\acute{\alpha}-\sigma n+\eta n-\alpha-\beta+1)\Gamma(1-\acute{\alpha}-\sigma n+\eta n+\acute{\beta})\Gamma(1-\sigma n+\eta n)}{\Gamma(\lambda-\acute{\alpha}-\sigma n+\eta n-\alpha+1)\Gamma(\lambda-\acute{\alpha}-\sigma n+\eta n-\beta+1)\Gamma(1-\sigma n+\eta n+\acute{\beta})}.
\end{aligned}
\tag{5.13}
$$

we have the required result

$$
\begin{aligned}
&(I_{0^+}^{\alpha,\acute{\alpha},\beta,\acute{\beta},\lambda}\mathbb{J}_{(\gamma,d);k}^{c,b,\delta}[(\alpha_j,\beta_j)_m;(t^{-\sigma+\eta};p)])(x)\\
&=\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn,d-\gamma)(-c)^n}{x^{\sigma n-\eta n+\alpha-\lambda+\acute{\alpha}}}\Gamma\Bigg[\begin{array}{l}(d+kn)(\delta)(\lambda-\acute{\alpha}-\sigma n+\eta n-\alpha-\beta+1)\\(\gamma)(d-\gamma)(\delta+n)(\lambda-\acute{\alpha}-\sigma n+\eta n-\beta+1)\end{array}\\
&\times\begin{array}{l}(1-\acute{\alpha}-\sigma n+\eta n+\acute{\beta})(1-\sigma n+\eta n)\\(1-\sigma n+\eta n+\acute{\beta})(\lambda-\acute{\alpha}-\sigma n+\eta n-\alpha+1)(\alpha_j n+\beta_j+\frac{1+b}{2})|_{j=1}^{m}\end{array}\Bigg].
\end{aligned}
$$

□

Theorem 5.4. *Let* $\alpha_j,\beta_j,b,d,\delta,\gamma,\sigma,\eta,c\in\mathbb{C}$ $(j=1,2\cdots m)$*,* $p\geq 0$ *be such that* $\sum_{j=1}^{m}\Re(\alpha_j)>max\{0;\Re(k)-1\}$*;* $k>0$*,* $\Re(\beta_j)>0$*,* $\Re(d)>0$*,* $\Re(\gamma)>0$*,* $\Re(\delta)>0$*, then the* E^1*GMBF holds true:*

$$
\begin{aligned}
&(I_{-}^{\alpha,\acute{\alpha},\beta,\acute{\beta},\lambda}\mathbb{J}_{(\gamma,d);k}^{c,b,\delta}[(\alpha_j,\beta_j)_m;(t^{\frac{\sigma-d}{\eta+b}};p)])(x)\\
&=\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn,d-\gamma)(-c)^n}{x^{\sigma n-\eta n+\alpha-\lambda+\acute{\alpha}}}\Gamma\Bigg[\begin{array}{l}(d+kn)(\delta)(\frac{(d-\sigma)n}{(\eta+b)n}-\beta)(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\lambda+\acute{\beta})\\(\gamma)(d-\gamma)(\delta+n)(\frac{(d-\sigma)n}{(\eta+b)n})(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\beta)\end{array}\\
&\times\begin{array}{l}(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\lambda+\acute{\alpha})\\(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\lambda+\acute{\alpha}+\acute{\beta})(\alpha_j n+\beta_j+\frac{1+b}{2})|_{j=1}^{m}\end{array}\Bigg].
\end{aligned}
$$

Proof. Consider the composition of right side generalized fractional integral operator with the E^1GMBF

$$\begin{aligned}
&(I_{-}^{\alpha,\acute{\alpha},\beta,\acute{\beta},\lambda}\mathbb{J}_{(\gamma,d);k}^{c,b,\delta}[(\alpha_j,\beta_j)_m;(t^{\frac{\sigma-d}{\eta+b}};p)])(x)\\
&=\frac{x^{-\acute{\alpha}}}{\Gamma(\lambda)}\int_x^{\infty}(t-x)^{\lambda-1}t^{-\alpha}F_3(\alpha,\acute{\alpha},\beta,\acute{\beta};\lambda;1-\frac{x}{t};1-\frac{t}{x})\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn,d-\gamma)}{B(\gamma,d-\gamma)}\\
&\times\frac{c^n(d)_{kn}(-1)^n t^{\frac{\sigma n-dn}{\eta n+bn}}}{(\delta)_n\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})}dt\\
&=\frac{x^{-\acute{\alpha}}}{\Gamma(\lambda)}\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(1;p)|_{n=0}^{\infty}\int_x^{\infty}(1-\frac{x}{t})^{\lambda-1}t^{\frac{(\sigma-d)n}{(\eta+b)n}-\alpha+\lambda-1}\sum_{m,s=0}^{\infty}\frac{(\alpha)_m(\acute{\alpha})_s(\beta)_m(\acute{\beta})_s}{\lambda_{m+s}m!s!}\\
&\times(1-\frac{x}{t})^m(1-\frac{t}{x})^s dt\\
&=\frac{x^{-\acute{\alpha}}}{\Gamma(\lambda)}\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(1;p)|_{n=0}^{\infty}\sum_{m,s=0}^{\infty}\frac{(\alpha)_m(\acute{\alpha})_s(\beta)_m(\acute{\beta})_s}{\lambda_{m+s}m!s!}\int_x^{\infty}(1-\frac{x}{t})^{\lambda+m-1}(1-\frac{t}{x})^s\\
&\times t^{\frac{(\sigma-d)n}{(\eta+b)n}-\alpha+\lambda-1}dt.
\end{aligned} \tag{5.14}$$

Putting these values $\frac{x}{t}=u\Rightarrow\frac{-x}{u^2}du=dt$, $t=x\Rightarrow u=1$ and $t=\infty\Rightarrow u=0$ in Eq (5.14), then we have

$$\begin{aligned}
&(I_{-}^{\alpha,\acute{\alpha},\beta,\acute{\beta},\lambda}\mathbb{J}_{(\gamma,d);k}^{c,b,\delta}[(\alpha_j,\beta_j)_m;(t^{\frac{\sigma-d}{\eta+b}};p)])(x)-\frac{x^{-\acute{\alpha}}}{\Gamma(\lambda)}\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(1;p)|_{n=0}^{\infty}\\
&=\sum_{m,s=0}^{\infty}\frac{(\alpha)_m(\acute{\alpha})_s(\beta)_m(\acute{\beta})_s}{\lambda_{m+s}m!s!}\int_1^0(1-u)^{\lambda+m-1}(1-\frac{1}{u})^s(\frac{x}{u})^{\frac{(\sigma-d)n}{(\eta+b)n}-\alpha+\lambda-1}(\frac{-x}{u^2})du\\
&=\sum_{m,s=0}^{\infty}\frac{(\alpha)_m(\acute{\alpha})_s(\beta)_m(\acute{\beta})_s(-1)^s}{\lambda_{m+s}m!s!}x^{\frac{(\sigma-d)n}{(\eta+b)n}-\alpha+\lambda}\int_0^1(1-u)^{\lambda+m+s-1}u^{\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\lambda-s-1}du.
\end{aligned} \tag{5.15}$$

Using Eqs (1.6) and (1.7) in Eq (5.15), we have

$$\begin{aligned}
&(I_{-}^{\alpha,\acute{\alpha},\beta,\acute{\beta},\lambda}\mathbb{J}_{(\gamma,d);k}^{c,b,\delta}[(\alpha_j,\beta_j)_m;(t^{\frac{\sigma-d}{\eta+b}};p)])(x)-x^{-\acute{\alpha}+\lambda-\alpha}\mathbb{J}_{\delta,b,(\alpha_j,\beta_j)_m}^{k,c,(\gamma,d);k}(x^{\frac{\sigma-d}{\eta+b}};p)|_{n=0}^{\infty}\\
&=\sum_{m,s=0}^{\infty}\frac{(\alpha)_m(\acute{\alpha})_s(\beta)_m(\acute{\beta})_s(-1)^s}{\lambda_{m+s}m!s!}\frac{\Gamma(\lambda+m+s)\Gamma(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\lambda-s)}{\Gamma(\lambda)\Gamma(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha+m)}\\
&=\frac{\Gamma(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\lambda)}{\Gamma(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha)_m m!}\sum_{s=o}^{\infty}\frac{(\acute{\alpha})_s(\acute{\beta})_s}{(1-\frac{(d-\sigma)n}{(\eta+b)n}-\alpha+\lambda)_s s!}\\
&=\frac{\Gamma(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\lambda)\Gamma(\frac{(d-\sigma)n}{(\eta+b)n}-\beta)}{\Gamma(\frac{(d-\sigma)n}{(\eta+b)n})\Gamma(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\beta)}\frac{\Gamma(1-\frac{(d-\sigma)n}{(\eta+b)n}-\alpha+\lambda)\Gamma(1-\frac{(d-\sigma)n}{(\eta+b)n}-\alpha+\lambda-\acute{\alpha}-\acute{\beta})}{\Gamma(1-\frac{(d-\sigma)n}{(\eta+b)n}-\alpha+\lambda-\acute{\alpha})\Gamma(1-\frac{(d-\sigma)n}{(\eta+b)n}-\alpha+\lambda-\acute{\beta})}\\
&=\frac{\Gamma(\frac{(d-\sigma)n}{(\eta+b)n}-\beta)\Gamma(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\lambda+\acute{\beta})\Gamma(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\lambda+\acute{\alpha})}{\Gamma(\frac{(d-\sigma)n}{(\eta+b)n})\Gamma(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\beta)\Gamma(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\lambda+\acute{\alpha}+\acute{\beta})}.
\end{aligned}$$

We have a desired result

$$
\begin{aligned}
&(I_{-}^{\alpha,\acute{\alpha},\beta,\acute{\beta},\lambda}\mathbb{J}_{(\gamma,d);k}^{c,b,\delta}[(\alpha_j,\beta_j)_m;(t^{\frac{\sigma-d}{\eta+b}};p)])(x)\\
&=\sum_{n=0}^{\infty}\frac{B_p(\gamma+kn,d-\gamma)(-c)^n}{x^{\sigma n-\eta n+\alpha-\lambda+\acute{\alpha}}}\Gamma\left[\begin{array}{l}(d+kn)(\delta)(\frac{(d-\sigma)n}{(\eta+b)n}-\beta)(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\lambda+\acute{\beta})\\ (\gamma)(d-\gamma)(\delta+n)(\frac{(d-\sigma)n}{(\eta+b)n})(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\beta)\end{array}\right.\\
&\times\left.\begin{array}{l}(\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\lambda+\acute{\alpha})\\ (\frac{(d-\sigma)n}{(\eta+b)n}+\alpha-\lambda+\acute{\alpha}+\acute{\beta})(\alpha_j n+\beta_j+\frac{1+b}{2})|_{j=1}^{m}\end{array}\right].
\end{aligned}
$$

□

6. Conclusions

In this research, we described extension of extended generalized multi-index Bessel function (E^1GMBF) and developed some results with the Laguerre polynomial and Whittaker function, integral representation, derivatives and solved integral transforms (beta transform, Laplace transform, Mellin transforms). Moreover, we discussed the composition of the generalized fractional integral operator having Appell function as a kernel with the E^1GMBF and obtained results in terms of Wright functions.

Conflict of interest

The authors declare that they have no competing interests.

References

1. G. Dattoli, S. Lorenzutta, G. Maino, et al. *Generalized Bessel functions and exact solutions of partial differential equations*, Rend. Mat., **7** (1992), 12.
2. V. S. Kiryakova, *Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus*, J. Comput. Appl. Math., **118** (2000), 241–259.
3. M. Kamarujjama, *Computation of new class of integrals involving generalized Galue type Struve function*, J. Comput. Appl. Math., **351** (2019), 228–236.
4. O. Khan, M. Kamarujjama, N. U. Khan, *Certain integral transforms involving the product of Galue type struve function and Jacobi polynomial*, Palestine J. Math., **8** (2019), 191–199.
5. M. Kamarujjama, N. U. Khan, O. Khan, *The generalized pk-Mittag-Leffler function and solution of fractional kinetic equations*, J. Anal., **27** (2019), 1029–1046.
6. N. U. Khan, M. Ghayasuddin, W. A. Khan, et al. *Certain unified integral involving Generalized Bessel-Maitland function*, South East Asian J. Math. Math. Sci., **11** (2015), 27–36.
7. N. U. Khan, M. Ghayasuddin, *Study of the unified double integral associated with generalized Bessel-Maitland function*, Pure Appl. Math. Lett., **2016** (2016), 15–19.

8. S. D. Purohit, P. Agarwal, et al. *Certain new integral formulas involving the generalized Bessel functions*, Bull. Korean Math. Soc., **51** (2014), 995–1003.
9. G. Dattoli, S. Lorenzutta, G. Maino, et al. *Theory of multiindex multivariable Bessel functions and Hermite polynomials*, Le Mat., **52** (1997), 179–197.
10. J. Choi, P. Agarwal, *A note on fractional integral operator associated with multiindex Mittag-Leffler functions*, Filomat, **30** (2016), 1931–1939.
11. M. Kamarujjama, N. U. Khan, O. Khan, *Estimation of certain integrals with extended multi-index Bessel function*, Malaya J. Mat. (MJM), **7** (2019), 206–212.
12. D. L. Suthar, S. D. Purohit, R. K. Parmar, *Generalized fractional calculus of the multi-index Bessel function*, Math. Nat. Sci., **1** (2017), 26–32.
13. K. S. Nisar, S. D. Purohit, D. L. Suthar, et al. *Fractional calculus and certain integrals of generalized multiindex Bessel function*, arXiv preprint arXiv:1706.08039, 2017.
14. D. L. Suthar, T. Tsagye, *Riemann-Liouville fractional integrals and differential formula involving Multi-index Bessel-function*, Math. Sci. Lett., **6** (2017), 233–237.
15. D. L. Suthar, D. Kumar, H. Habenom, *Solutions of fractional Kinetic equation associated with the generalized multiindex Bessel function via Laplace transform*, Differ. Equ. Dyn. Syst., (2019), 1–14.
16. O. I. Marichev, *Volterra equation of Mellin convolution type with a Horn function in the kernel*, Izv. Akad. Nauk BSSR. Ser. Fiz.-Mat. Nauk, **1** (1974), 128–129.
17. F. W. Olver, D. W. Lozier, R. F. Boisvert, et al. *NIST handbook of mathematical functions hardback and CD-ROM*, Cambridge university press, 2010.
18. S. Mubeen, R. S. Ali, *Fractional operators with generalized Mittag-Leffler k-function*, Adv. Differ. Equ., **2019** (2019), 520.
19. A. Petojevic, *A note about the Pochhammer symbols*, Math. Moravica, **12** (2008), 37–42.
20. M. A. Chaudhry, A. Qadir, H. M. Srivastava, et al. *Extended hypergeometric and confluent hypergeometric functions*, Appl. Math. Comput., **159** (2004), 589–602.
21. M. A. Chaudhry, S. M. Zubair, et al. *On a class of incomplete gamma functions with applications*, CRC press, 2001.
22. H. M. Srivastava, P. W. Karlsson, *Multiple gaussian hypergeometric series*, Halsted Press (Ellis Horwood Limited, Chichester), 1985.
23. D. V. Widder, *Laplace transform (PMS-6)*, Princeton university press, 2015.
24. I. N. Sneddon, *The use of integral transform*, Tata McGraw Hill, New Delhi, 1979.
25. M. A. Özarslan, B. Yilmaz, *The extended Mittag-Leffler function and its properties*, J. Inequal. Appl., **2014** (2014), 85.
26. M. A. Özarslan, *Some remarks on extended hypergeometric, extended confluent hypergeometric and extended Appell's functions*, J. Comput. Anal. Appl., **14** (2012).