AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 7500–7509.
DOI: 10.3934/math.2020480
Received: 31 July 2020
Accepted: 18 September 2020
Published: 23 September 2020

Research article

A certain two-term exponential sum and its fourth power means

Zhang Jin[1] **and Zhang Wenpeng**[2,*]

[1] School of Information Engineering, Xi'an University, Xi'an, Shaanxi, 710065, P.R.China
[2] School of Mathematics, Northwest University, Xi'an, Shaanxi, 710127, P. R. China

* **Correspondence:** Email: wpzhang@nwu.edu.cn.

Abstract: The main purpose of this article is using the properties of the Legendre's symbol and the classical Gauss sums to study the calculating problem of the fourth power mean of a certain two-term exponential sums, and give an interesting calculating formula for it.

Keywords: the two-term exponential sums; the fourth power mean; elementary method; calculating formula
Mathematics Subject Classification: 11L03, 11L05

1. Introduction

Let $q \geq 3$ be a fixed integer. For any integers $k \geq 2$ and m with $(m, q) = 1$, we define the two-term exponential sums $G(m, k; q)$ as

$$G(m,k;q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k + a}{q}\right),$$

where as usual, $e(y) = e^{2\pi i y}$ and $i^2 = -1$.

This sum play a very important role in the study of analytic number theory, many number theory problems are closely related to it. Such is the case with the famous Waring problem. Therefore, it is necessary to study the various properties of $G(m, k; q)$, in order to promote the development of research work in related fields. The work in this area mainly includes two aspects, one is the upper bound estimate of the exponential sum, the other is the mean value of the exponential sum. In fact, a great deal of work has been done in this field and a series of meaningful research results have been obtained, we do not want to enumerate here, interested readers can refer to references [5–15].

For example, H. Zhang and W. P. Zhang [5] proved that for any odd prime p, one has the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right)\right|^4 = \begin{cases} 2p^3 - p^2 & \text{if } 3 \nmid p-1, \\ 2p^3 - 7p^2 & \text{if } 3 | p-1, \end{cases}$$

where n represents any integer with $(n, p) = 1$.

W. P. Zhang and D. Han [6] obtained the identity

$$\sum_{a=1}^{p-1}\left|\sum_{n=0}^{p-1} e\left(\frac{n^3+an}{p}\right)\right|^6 = 5p^4 - 8p^3 - p^2,$$

where p denotes an odd prime with $3 \nmid (p-1)$.

X. Y. Liu and W. P. Zhang [8] proved that for any prime p with $3 \nmid (p-1)$, one has the identity

$$\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a) e\left(\frac{ma^3+a}{p}\right)\right|^6 = p(p-1)\left(6p^3 - 28p^2 + 39p + 5\right),$$

where $\sum\limits_{\chi \bmod p}$ denotes the summation over all Dirichlet characters modulo p.

In this paper, we will use the properties of Legendre's symbol and the classical Gauss sums to prove the following main result:

Theorem. Let $p > 3$ be a prime, then we have the identity

$$\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4 = \begin{cases} p^2(\delta-3), & \text{if } p \equiv 1 \bmod 6; \\ p^2(\delta+3), & \text{if } p \equiv -1 \bmod 6, \end{cases}$$

where as usual, $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol modulo p, $d \cdot \overline{d} \equiv 1 \bmod p$, $\delta = \sum\limits_{d=1}^{p-1}\left(\frac{d-1+\overline{d}}{p}\right)$ is an integer which satisfies the estimate $|\delta| \leq 2\sqrt{p}$.

From this theorem and H. Zhang and W. P. Zhang [5] we may immediately deduce the following three corollaries:

Corollary 1. Let $p > 3$ be a prime, then we have the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{m^2\cdot a^3+a}{p}\right)\right|^4 = \begin{cases} 2p^3 + p^2(\delta-10), & \text{if } p \equiv 1 \bmod 6; \\ 2p^3 + p^2(\delta+2), & \text{if } p \equiv -1 \bmod 6. \end{cases}$$

Corollary 2. Let $p > 3$ be a prime, n be any quadratic non-residue modulo p. Then we have the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{n\cdot m^2\cdot a^3+a}{p}\right)\right|^4 = 2p^3 - p^2(\delta+4).$$

Corollary 3. Let $p > 3$ be a prime. Then for any integer n with $(n, p) = 1$, we have the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{n\cdot m^2\cdot a^3+a}{p}\right)\right|^4 = 2p^3 + O\left(p^{\frac{5}{2}}\right).$$

Some notes: From our theorem and A. Weil's work [15] we can also deduce the following nontrivial estimate:

$$\left|\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4\right| = O\left(p^{\frac{5}{2}}\right).$$

Obviously, this estimate saves $p^{\frac{1}{2}}$ compared to the trivial estimate p^3, and may be used in some analytic number theory problems.

In addition, the distribution of exponential sums is very irregular, so it should be meaningful to study their power means and obtain an accurate calculation formula.

It is interesting to ask whether a formula analogous to our theorem can be obtained for the general $2h$-th power means

$$\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^{2h}, \text{ where } h \geq 3.$$

This is an open problem. We hope that interested readers will join us in considering this problem.

2. Several lemmas

In this section, we will give some basic lemmas. Of course, the proofs of these lemmas need some knowledge of elementary and analytic number theory. They can be found in references [1–4], and we do not repeat them. At the beginning, we are going to introduce the definition of the classical Gauss sums $\tau(\chi)$ as follows: For any integer $q > 1$, let χ denotes any Dirichlet character modulo q, then the classical Gauss sums $\tau(\chi)$ is defined as

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right),$$

where $e(y) = e^{2\pi i y}$. Based on this definition and its properties:

$$\sum_{a=1}^{q} \chi(a) e\left(\frac{ma}{q}\right) = \overline{\chi}(m) \cdot \tau(\chi),$$

providing $(m, q) = 1$ or χ is a primitive character modulo q, we have the following:

Lemma 1. If $p > 3$ be an odd prime, then we have the identity

$$\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2 = -\left(\frac{-3}{p}\right) \cdot p,$$

where $\left(\frac{*}{p}\right) = \chi_2$ denotes the Legendre symbol mod p.

Proof. Note that $\chi_2^3 = \chi_2$ and $\tau^2(\chi_2) = \chi_2(-1) \cdot p$, from the definition and properties of the Gauss sums and reduced residue system modulo p we have

$$\sum_{m=1}^{p-1} \chi_2(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2$$

$$
\begin{aligned}
&= \sum_{m=1}^{p-1}\chi_2(m)\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\left(1+\sum_{b=1}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right)\\
&= \sum_{m=1}^{p-1}\chi_2(m)\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\\
&\quad+\sum_{m=1}^{p-1}\chi_2(m)\sum_{a=0}^{p-1}\sum_{b=1}^{p-1} e\left(\frac{ma^3+mb^3+a+b}{p}\right)\\
&= \tau(\chi_2)\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{a}{p}\right)+\tau(\chi_2)\sum_{a=0}^{p-1}\sum_{b=1}^{p-1}\chi_2\left(a^3+b^3\right)e\left(\frac{a+b}{p}\right)\\
&= \tau(\chi_2)\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{a}{p}\right)+\tau(\chi_2)\sum_{a=0}^{p-1}\chi_2\left(a^3+1\right)\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{b(a+1)}{p}\right)\\
&= \tau^2(\chi_2)+\tau^2(\chi_2)\cdot\sum_{a=0}^{p-1}\chi_2\left(a^3+1\right)\chi_2(a+1)\\
&= \chi_2(-1)\cdot p+\chi_2(-1)\cdot p\cdot\sum_{a=0}^{p-2}\chi_2\left(a^2-a+1\right)\\
&= \chi_2(-1)\cdot p+\chi_2(-1)\cdot p\cdot\sum_{a=0}^{p-2}\left(\frac{(2a-1)^2+3}{p}\right)\\
&= \chi_2(-1)\cdot p-\left(\frac{-3}{p}\right)\cdot p+\chi_2(-1)\cdot p\cdot\sum_{b=0}^{p-1}\left(\frac{b^2+3}{p}\right). \qquad (2.1)
\end{aligned}
$$

Note that for any odd prime p, we have the identity

$$
\begin{aligned}
&\sum_{a=0}^{p-1}\left(\frac{a^2+n}{p}\right)=\left(\frac{n}{p}\right)+\sum_{a=1}^{p-1}\left(1+\left(\frac{a}{p}\right)\right)\left(\frac{a+n}{p}\right)\\
&= \sum_{a=0}^{p-1}\left(\frac{a+n}{p}\right)+\sum_{a=1}^{p-1}\left(\frac{a(a+n)}{p}\right)=\sum_{a=1}^{p-1}\left(\frac{1+n\cdot\overline{a}}{p}\right)\\
&= \sum_{a=1}^{p-1}\left(\frac{1+na}{p}\right)=\begin{cases} p-1 & \text{if } p\mid n,\\ -1 & \text{if } p\nmid n.\end{cases} \qquad (2.2)
\end{aligned}
$$

From (2.1) and (2.2) we have

$$
\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2=-\left(\frac{-3}{p}\right)\cdot p.
$$

This proves Lemma 1.

Lemma 2. Let $p>3$ be an odd prime, then we have the identity

$$
\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2\left(\sum_{b=0}^{p-1} e\left(\frac{-mb^3-b}{p}\right)\right)
$$

$$= \left(\frac{-1}{p}\right) \cdot p^2 + \left(\frac{-6}{p}\right) \cdot p^2 \cdot \sum_{\substack{a=0 \\ a^3+2\equiv 0 \bmod p}}^{p-1} \left(\frac{a+2}{p}\right).$$

Proof. From Lemma 1 and its proving methods we have

$$\begin{aligned}
& \sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2\left(\sum_{b=0}^{p-1} e\left(\frac{-mb^3-b}{p}\right)\right) \\
= & \sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2\left(1+\sum_{b=1}^{p-1} e\left(\frac{-mb^3-b}{p}\right)\right) \\
= & -\left(\frac{-3}{p}\right)\cdot p+\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2\left(\sum_{b=1}^{p-1} e\left(\frac{-mb^3-b}{p}\right)\right) \\
= & -\chi_2(-3)\cdot p+\chi_2(-1)\cdot p\cdot\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\left(\frac{a^3+b^3-1}{p}\right)\left(\frac{a+b-1}{p}\right). \qquad (2.3)
\end{aligned}$$

Let $c = b-1$, if b pass through a complete residue system $\bmod p$, then c also passes a complete residue system $\bmod p$, from (2.2), (2.3) and the properties of the complete residue system $\bmod\ p$ we have

$$\begin{aligned}
& \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\left(\frac{a^3+b^3-1}{p}\right)\left(\frac{a+b-1}{p}\right) \\
= & \sum_{a=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^3+(c+1)^3-1}{p}\right)\left(\frac{a+c}{p}\right) \\
= & \sum_{a=0}^{p-1}\sum_{c=1}^{p-1}\left(\frac{a^3c^3+c^3+3c^2+3c}{p}\right)\left(\frac{ac+c}{p}\right)+\sum_{a=1}^{p-1}\left(\frac{a^4}{p}\right) \\
= & (p-1)+\sum_{a=0}^{p-1}\left(\frac{a+1}{p}\right)\sum_{c=1}^{p-1}\left(\frac{a^3+1+3\bar{c}+3\bar{c}^2}{p}\right) \\
= & (p-1)-\sum_{a=0}^{p-1}\left(\frac{\left(a^3+1\right)(a+1)}{p}\right)+\sum_{a=0}^{p-1}\left(\frac{a+1}{p}\right)\sum_{c=0}^{p-1}\left(\frac{3c^2+3c+a^3+1}{p}\right) \\
= & (p-1)-\sum_{a=0}^{p-2}\left(\frac{a^2-a+1}{p}\right)+\sum_{a=0}^{p-1}\left(\frac{a+1}{p}\right)\sum_{c=0}^{p-1}\left(\frac{3(2c+1)^2+4a^3+1}{p}\right) \\
= & p-1+\left(\frac{3}{p}\right)-\sum_{a=0}^{p-1}\left(\frac{(2a-1)^2+3}{p}\right)+\sum_{a=0}^{p-1}\left(\frac{a+1}{p}\right)\sum_{c=0}^{p-1}\left(\frac{3c^2+4a^3+1}{p}\right) \\
= & p+\left(\frac{3}{p}\right)+\left(\frac{3}{p}\right)p\sum_{\substack{a=0 \\ 4a^3+1\equiv 0 \bmod p}}^{p-1}\left(\frac{a+1}{p}\right)-\left(\frac{3}{p}\right)\sum_{a=0}^{p-1}\left(\frac{a+1}{p}\right)
\end{aligned}$$

$$= p + \left(\frac{3}{p}\right) + \left(\frac{3}{p}\right) \cdot p \cdot \sum_{\substack{a=0 \\ \overline{2} \cdot a^3+1 \equiv 0 \bmod p}}^{p-1} \left(\frac{\overline{2}a+1}{p}\right)$$

$$= p + \left(\frac{3}{p}\right) + \left(\frac{3}{p}\right) \cdot p \cdot \sum_{\substack{a=0 \\ a^3+2 \equiv 0 \bmod p}}^{p-1} \left(\frac{a+2}{p}\right)\left(\frac{2}{p}\right)$$

$$= p + \left(\frac{3}{p}\right) + \left(\frac{6}{p}\right) \cdot p \cdot \sum_{\substack{a=0 \\ a^3+2 \equiv 0 \bmod p}}^{p-1} \left(\frac{a+2}{p}\right). \tag{2.4}$$

Combining (2.3) and (2.4) we have the identity

$$\sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2 \left(\sum_{b=0}^{p-1} e\left(\frac{-mb^3-b}{p}\right)\right)$$
$$= \left(\frac{-1}{p}\right) \cdot p^2 + \left(\frac{-6}{p}\right) \cdot p^2 \cdot \sum_{\substack{a=0 \\ a^3+2 \equiv 0 \bmod p}}^{p-1} \left(\frac{a+2}{p}\right).$$

This proves Lemma 2.

Lemma 3. If $p > 3$ is an odd prime, then we have

$$\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^3+b^3-c^3-1}{p}\right)\left(\frac{a+b-c-1}{p}\right)$$
$$= \left(\frac{-1}{p}\right) \cdot p \cdot \sum_{d=1}^{p-1} \left(\frac{d-1+\overline{d}}{p}\right) - p - 3\left(\frac{3}{p}\right) \cdot p - \left(\frac{6}{p}\right) \cdot p \cdot \sum_{\substack{d=1 \\ d^3+2 \equiv 0 \bmod p}}^{p-1} \left(\frac{d+2}{p}\right).$$

Proof. Let $d = a - 1$, $e = b - c$. Then from the properties of the complete residue system modulo p we have

$$\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^3+b^3-c^3-1}{p}\right)\left(\frac{a+b-c-1}{p}\right)$$
$$= \sum_{d=0}^{p-1} \sum_{e=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{(d+1)^3+(e+c)^3-c^3-1}{p}\right)\left(\frac{d+e}{p}\right)$$
$$= \sum_{d=0}^{p-1} \sum_{e=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{d^3+3d^2+3d+e^3+3e^2c+3ec^2}{p}\right)\left(\frac{d+e}{p}\right)$$
$$= \sum_{d=0}^{p-1} \sum_{e=1}^{p-1} \sum_{c=0}^{p-1} \left(\frac{d^3+3d^2e+3de^2+1+3ec+3e^2c^2}{p}\right)\left(\frac{d+1}{p}\right)$$
$$+p \sum_{d=1}^{p-1} \left(\frac{d^2+3d+3}{p}\right). \tag{2.5}$$

If c pass through a complete residue system $\bmod\ p$, then $2ce+1$ also passes a complete residue system $\bmod\ p$, so from (2.2) and (2.5) we have

$$
\begin{aligned}
&\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^3+b^3-c^3-1}{p}\right)\left(\frac{a+b-c-1}{p}\right)\\
=\ &\sum_{d=0}^{p-1}\sum_{e=1}^{p-1}\sum_{c=0}^{p-1}\left(\frac{3(2ce+1)^2+1+4d^3+12d^2e+12de^2}{p}\right)\left(\frac{d+1}{p}\right)\\
&\quad +p\sum_{d=0}^{p-1}\left(\frac{(2d+3)^2+3}{p}\right)-\left(\frac{3}{p}\right)p\\
=\ &\sum_{d=0}^{p-1}\sum_{e=1}^{p-1}\sum_{c=0}^{p-1}\left(\frac{3c^2+1+4d^3+12d^2e+12de^2}{p}\right)\left(\frac{d+1}{p}\right)\\
&\quad +p\sum_{d=0}^{p-1}\left(\frac{d^2+3}{p}\right)-\left(\frac{3}{p}\right)p\\
=\ &\left(\frac{3}{p}\right)p\sum_{\substack{d=0\\ 12de^2+12d^2e+4d^3+1\equiv 0 \bmod p}}^{p-1}\sum_{e=1}^{p-1}\left(\frac{d+1}{p}\right)\\
&\quad -\left(\frac{3}{p}\right)\sum_{d=0}^{p-1}\sum_{e=1}^{p-1}\left(\frac{d+1}{p}\right)-p-\left(\frac{3}{p}\right)p\\
=\ &\left(\frac{3}{p}\right)p\sum_{\substack{d=1\\ 3d^2(2e+d)^2+d^4+d\equiv 0 \bmod p}}^{p-1}\sum_{e=1}^{p-1}\left(\frac{d+1}{p}\right)-p-\left(\frac{3}{p}\right)p\\
=\ &\left(\frac{3}{p}\right)p\sum_{\substack{d=1\\ 3e^2+d^4+d\equiv 0 \bmod p}}^{p-1}\sum_{e=0}^{p-1}\left(\frac{d+1}{p}\right)-\left(\frac{3}{p}\right)p\sum_{\substack{d=1\\ 8d^3+2\equiv 0 \bmod p}}^{p-1}\left(\frac{d+1}{p}\right)-p-\left(\frac{3}{p}\right)p\\
=\ &\left(\frac{3}{p}\right)p\sum_{d=1}^{p-1}\left(1+\left(\frac{-3(d^4+d)}{p}\right)\right)\left(\frac{d+1}{p}\right)-p-\left(\frac{3}{p}\right)p\\
&-\left(\frac{3}{p}\right)p\sum_{\substack{d=1\\ (2d)^3+2\equiv 0 \bmod p}}^{p-1}\left(\frac{2d+2}{p}\right)\left(\frac{2}{p}\right)\\
=\ &p\sum_{d=1}^{p-1}\left(\frac{-d^4-d}{p}\right)\left(\frac{d+1}{p}\right)-p-2\left(\frac{3}{p}\right)p-\left(\frac{6}{p}\right)p\sum_{\substack{d=1\\ d^3+2\equiv 0 \bmod p}}^{p-1}\left(\frac{d+2}{p}\right)\\
=\ &\left(\frac{-1}{p}\right)p\sum_{d=1}^{p-2}\left(\frac{d}{p}\right)\left(\frac{d^2-d+1}{p}\right)-p-2\left(\frac{3}{p}\right)p-\left(\frac{6}{p}\right)p\sum_{\substack{d=1\\ d^3+2\equiv 0 \bmod p}}^{p-1}\left(\frac{d+2}{p}\right)
\end{aligned}
$$

$$= \left(\frac{-1}{p}\right) p \sum_{d=1}^{p-1} \left(\frac{d-1+\overline{d}}{p}\right) - p - 3\left(\frac{3}{p}\right) p - \left(\frac{6}{p}\right) p \sum_{\substack{d=1 \\ d^3+2\equiv 0 \bmod p}}^{p-1} \left(\frac{d+2}{p}\right).$$

This proves Lemma 3.

3. Proof of the theorem

Applying the basic lemmas of the previous section, we can easily complete the proof of our theorem. Our proof idea is to decompose the proof process of the theorem into the form of lemmas, and then apply Lemma 2 and Lemma 3 directly. In fact for any prime $p > 3$, note that $\tau^2(\chi_2) = \chi_2(-1) \cdot p$ and the identity

$$\begin{aligned}
&\sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2 \left(\sum_{b=0}^{p-1} e\left(\frac{-mb^3-b}{p}\right)\right) \left(\sum_{c=1}^{p-1} e\left(\frac{-mc^3-c}{p}\right)\right) \\
= \;& \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=1}^{p-1}\sum_{m=1}^{p-1} \left(\frac{m}{p}\right) e\left(\frac{m\left(a^3+b^3-c^3-d^3\right)+a+b-c-d}{p}\right) \\
= \;& \tau(\chi_2) \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=1}^{p-1} \left(\frac{a^3+b^3-c^3-d^3}{p}\right) e\left(\frac{a+b-c-d}{p}\right) \\
= \;& \tau(\chi_2) \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{a^3+b^3-c^3-1}{p}\right) \sum_{d=1}^{p-1} \left(\frac{d}{p}\right) e\left(\frac{d(a+b-c-1)}{p}\right) \\
= \;& \tau^2(\chi_2) \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{a^3+b^3-c^3-1}{p}\right) \left(\frac{a+b-c-1}{p}\right) \\
= \;& \left(\frac{-1}{p}\right) \cdot p \cdot \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{a^3+b^3-c^3-1}{p}\right) \left(\frac{a+b-c-1}{p}\right), \qquad (3.1)
\end{aligned}$$

from (3.1), Lemma 2 and Lemma 3 we have

$$\begin{aligned}
&\sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4 \\
= \;& \sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2 \left(\sum_{b=0}^{p-1} e\left(\frac{-mb^3-b}{p}\right)\right) \left(1 + \sum_{b=1}^{p-1} e\left(\frac{-mb^3-b}{p}\right)\right) \\
= \;& \sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2 \left(\sum_{b=0}^{p-1} e\left(\frac{-mb^3-b}{p}\right)\right) \\
&+ \sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2 \left(\sum_{b=0}^{p-1} e\left(\frac{-mb^3-b}{p}\right)\right) \left(\sum_{b=1}^{p-1} e\left(\frac{-mb^3-b}{p}\right)\right) \\
= \;& \sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2 \left(\sum_{b=0}^{p-1} e\left(\frac{-mb^3-b}{p}\right)\right)
\end{aligned}$$

$$+p\left(\frac{-1}{p}\right)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^3+b^3-c^3-1}{p}\right)\left(\frac{a+b-c-1}{p}\right)$$

$$= \left(\frac{-1}{p}\right)p^2+\left(\frac{-6}{p}\right)p^2\sum_{\substack{a=0\\ a^3+2\equiv 0 \bmod p}}^{p-1}\left(\frac{a+2}{p}\right)+p^2\sum_{d=1}^{p-1}\left(\frac{d-1+\overline{d}}{p}\right)$$

$$-\left(\frac{-1}{p}\right)p^2-3\left(\frac{-3}{p}\right)p^2-\left(\frac{-6}{p}\right)p^2\sum_{\substack{d=1\\ d^3+2\equiv 0 \bmod p}}^{p-1}\left(\frac{d+2}{p}\right)$$

$$= p^2\cdot\sum_{d=1}^{p-1}\left(\frac{d-1+\overline{d}}{p}\right)-3\left(\frac{-3}{p}\right)\cdot p^2. \tag{3.2}$$

Note that $\left(\frac{-3}{p}\right)=1$, if $p\equiv 1 \bmod 6$; And $\left(\frac{-3}{p}\right)=-1$, if $p\equiv -1 \bmod 6$. From (3.2) we may immediately deduce our theorem.

4. Conclusions

In this article, we obtained an identity for the fourth power mean of a certain two-term exponential sum with the weight of the Legendre's symbol modulo p. This result is the further promotion and extension of [5,6], and it is new contribution in the related fields.

Acknowledgments

The authors would like to express their heartfelt thanks to the reviewers for their valuable comments and suggestions. In particular, the calculation errors are pointed out, which makes this paper further improved and perfected.

This work is supported by the N. S .F. (11771351) of P. R. China and Sci-Tech Program of Xi'an (Grant No. 2020KJWL08) of P. R. China.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.
2. W. P. Zhang, H. L. Li, *Elementary Number Theory*, Shaanxi Normal University Press, Xi'an, 2013.
3. W. Narkiewicz, *Classical Problems in Number Theory*, Polish Scientifc Publishers, WARSZAWA, 1986.
4. K. Ireland, M. Rosen, *A classical introduction to modern number theory*, Springer-Verlag, New York, 1982.

5. H. Zhang, W. P. Zhang, *The fourth power mean of two-term exponential sums and its application*, Mathematical Reports, **19** (2017), 75–81.
6. W. P. Zhang, D. Han, *On the sixth power mean of the two-term exponential sums*, J. Number Theory, **136** (2014), 403–413.
7. D. Han, *A Hybrid mean value involving two-term exponential sums and polynomial character sums*, Czechoslovak Mathematical J., **64** (2014), 53–62.
8. X. Y. Liu, W. P. Zhang, *On the high-power mean of the generalized Gauss sums and Kloosterman sums*, Mathematics, **7** (2019), 907.
9. X. X. Li, J. Y. Hu, *The hybrid power mean quartic Gauss sums and Kloosterman sums*, Open Math., **15** (2017), 151–156.
10. L. Chen, Z. Y. Chen, *Some new hybrid power mean formulae of trigonometric sums*, Advances Differences Equation, (2020) 2020: 220.
11. L. Chen, J. Y. Hu, *A linear Recurrence Formula Involving Cubic Gauss Sums and Kloosterman Sums*, Acta Mathematica Sinica (Chinese Series), **61** (2018), 67–72.
12. S. Chowla, J. Cowles, M. Cowles, *On the number of zeros of diagonal cubic forms*, J. Number Theory, **9** (1977), 502–506.
13. W. P. Zhang, J. Y. Hu, *The number of solutions of the diagonal cubic congruence equation* mod p, Mathematical Reports, **20** (2018), 70–76.
14. A. Weil, *Basic number theory*, Springer-Verlag, New York, 1974
15. A. Weil, *On some exponential sums*, Proc. Nat. Acad. Sci. U.S.A., **34** (1948), 203–210.