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Research article

Uniqueness of meromorphic functions sharing small functions in the k-punctured complex plane

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Abstract: The main purpose of this article is concerned with the uniqueness of meromorphic functions in the k-punctured complex plane Ω sharing five small functions with finite weights. We proved that for any two admissible meromorphic functions f and g in Ω, if $\widetilde{E}_{\Omega}(\alpha_j, l; f) = \widetilde{E}_{\Omega}(\alpha_j, l; g)$ and an integer $l \geq 22$, then $f \equiv g$, where α_j $(j = 1, 2, \ldots, 5)$ are five distinct small functions with respect to f and g. Our results are extension and improvement of previous theorems given by Ge and Wu, Cao and Yi.

Keywords: small function; k-punctured; admissible meromorphic function; weighted shared
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1. Introduction

As is known to all, Nevanlinna value distribution is a powerful tool in studying the properties of meromorphic functions in the fields of complex analysis. In 1926, R. Nevanlinna gave the definition of characterise function $T(r, f)$ of meromorphic function, and established the famous first and second main theorem, lemma on the logarithmic derivatives etc. of Nevalinna theory, (see Hayman [5], Yang [20] and Yi and Yang [21]). Moreover, the two main theorems occupy a central place in the value distribution theory of meromorphic functions. By using these results, R. Nevanlinna in 1926 proved the following well-known five-values theorem and gave a problem about small function.

Theorem 1.1. *(see [21]). If f and g are two non-constant meromorphic functions that share five distinct values a_1, a_2, a_3, a_4, a_5 IM in $X = \mathbb{C}$, then $f(z) \equiv g(z)$.*

Question 1.1. *(see [21]). Does Theorem 1.1 still hold if the five distinct values $a_1, a_2, a_3,$ a_4, a_5 are* *replaced by five distinct small functions a_j $(j = 1, 2, \ldots, 5)$?*

After this theorem and question, many mathematicians had paid considerate attention to the uniqueness of meromorphic functions with shared values in the whole complex plane (see [21]). There were a series of beautiful uniqueness theorems about value-shared and small functions shared (see [10, 11, 17, 18, 22, 23]). Especially, Yi [22] gave a positive answer to Question 1.1 and extend the five value theorem to the case of sharing five distinct small functions.

Theorem 1.2. *(see [22] (The five small functions theorem)) Let f and g be two non-constant meromorphic functions in complex plane $\mathbb{C}$, and $a_j(j = 1, 2, 3, 4, 5)$ be five distinct small functions with respect to f and g. If f and g share $a_j(j = 1, 2, 3, 4, 5)$ IM in $\mathbb{C}$, then $f \equiv g$.*

In the past several decades, it is an increasing interest to investigate the uniqueness of meromorphic functions on a subset of complex plane $\mathbb{C}$, such as: the unit disc, the angular domain and the whole complex plane, and a lot of important theorems were obtained (see [21]). In 1999, Fang [3] discussed the uniqueness of admissible functions sharing some sets in the unit disc. Around 2003, Zheng in [24, 25] studied the uniqueness problem and obtained five-values theorem and four-values theorem in some angular domain of $\mathbb{C}$. In 2015, Liu and Mao [12] further extended Theorem 1.2 to an angular domain and obtained the following theorem.

Theorem 1.3. *(see [12, Theorem 1.1]). Let f and g be two nonconstant meromorphic functions in an angular domain $\mathbb{X}(\alpha,\beta) := \{z : \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leq 2\pi$ such that*

$$\lim_{r\to+\infty} \frac{\mathfrak{T}_{\alpha,\beta}(r, f)}{\log r} = +\infty,$$

and let $a_j(j = 1, \ldots, 5)$ be five distinct small functions with respect to f and g in $\mathbb{X}(\alpha,\beta)$. If f and g share $a_j(j = 1, \ldots, 5)$ IM in $\mathbb{X}(\alpha,\beta)$, then $f \equiv g$.

As we all know, the whole complex plane, unit disc and angular domain can all be regarded as a simply connected region. Of late, with the establishment of Nevanlinna theory for meromorphic functions on annuli given by Khrystiyanyn and Kondratyuk [6, 7] or [8] in 2005 or [9] in 2004, it is very interesting to consider the uniqueness of meromorphic function on doubly connected regions (see [1, 2, 13, 14]). In 2009 and 2011, Cao [1, 2] investigated the uniqueness of meromorphic functions on annuli sharing some values and some sets, and obtained an analog of Nevanlinna's famous five-values theorem.

Theorem 1.4. *(see [1, Thereom 3.2] or [2, Corollary 3.3]).* Let f and g be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let a_j $(j = 1, 2, 3, 4, 5)$ be five distinct complex numbers in $\overline{\mathbb{C}}$. If f and g share a_j *IM* for $j = 1, 2, 3, 4, 5$, then $f(z) \equiv g(z)$.

In 2015, Wu and Ge [14] extended Theorem 1.4 when five values are replaced by five small functions and obtained

Theorem 1.5. *(see [14, Theorem 1.1]). Let f and g be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_j(j = 1, 2, \ldots, 5)$ be five distinct small functions with respect to f and g on the annulus $\mathbb{A}$. If f and g share $a_j(j = 1, 2, \ldots, 5)$ IM, then $f(z) \equiv g(z)$.*

Very recently, the authors [16] investigated the uniqueness of meromorphic function in a special multiply connested region-k-punctured complex plane, and obtained an analog of Nevanlinna's famous five-values theorem for meromorphic functions f and g in a k-punctured complex plane, and gave a question as follows.

Theorem 1.6. *(see [16, Theorem 3.2]). Let f and g be two admissible meromorphic functions in Ω, if f, g share five distinct values a_1, a_2, a_3, a_4, a_5 IM in Ω, then $f(z) \equiv g(z)$.*

Question 1.2. *(see [16, Remark 3.3]). Does the conclusion of Theorem 1.6 still holds if $a_j (j = 1, \ldots, 5)$ are replaced by small functions $a_j(z)(j = 1, \ldots, 5)$.*

2. Results

Motivated by Question 1.2 and Theorems 1.2–1.6, the main purpose of this article is to investigate the uniqueness of meromorphic functions concerning small functions, and we obtain an analog of Nevanlinna's famous five-values theorem for meromorphic functions in k-punctured complex plane. To state our main results, let us recall some basic notations about function-shared in a k-punctured plane as follows.

Let f be a non-constant meromorphic function in a k-punctured plane Ω, we denote $S(f)$ a set of meromorphic function $a(z)$ in a k-punctured plane Ω satisfying $T_0(r, a) = S(r, f)$, and such a meromorphic function $a(z)$ in a k-punctured plane Ω is called as a small function with respect to f.

Let f be a non-constant meromorphic function in a k-punctured plane Ω, a small function $\alpha(z) \in S(f) \cup \{\infty\}$ and a positive integer l (or $+\infty$). We use $E_\Omega(\alpha, l; f)$ to denote the set of zeros of $f(z) - \alpha$ in Ω of multiplicity less than l (counting the multiplicities), and $\widetilde{E}_\Omega(\alpha, l; f)$ to denote the set of zeros of $f(z)-\alpha(z)$ in Ω of multiplicity less than l (ignoring the multiplicities), especially, $E_\Omega(\alpha, l; f) = E_\Omega(\alpha, f)$ and $\widetilde{E}_\Omega(\alpha, l; f) = \widetilde{E}_\Omega(\alpha, f)$ as $l = \infty$.

For two non-constant meromorphic functions f and g in Ω and $\alpha \in (S(f) \cap S(g)) \cup \{\infty\}$, if $E_\Omega(\alpha, f) = E_\Omega(\alpha, g)$, then we say that f and g share α CM in Ω, if $\widetilde{E}_\Omega(\alpha, f) = \widetilde{E}_\Omega(\alpha, g)$, then we say that f and g share α IM in Ω.

The following inequality concerning five small functions plays a key role in proving our main theorem.

Theorem 2.1. *Let f and g be two transcendental or admissible meromorphic functions in Ω, $\alpha_j(z) \in S(f) \cap S(g)$ $(j = 1, 2, \ldots, 5)$ be five distinct small functions. If l is a positive integer ≥ 4, f and g satisfy*

$$\widetilde{E}_\Omega(\alpha_j, l; f) = \widetilde{E}_\Omega(\alpha_j, l; g), \quad (j = 1, 2, \ldots, 5). \tag{2.1}$$

Then we have $S(r) := S(r, f) = S(r, g)$,

$$(2 - \frac{3}{l}) [T_0(r, f) + T_0(r, g)] \leq \sum_{j=1}^{5} [\widetilde{N}_0(r, \alpha_j; f| \leq l) + \widetilde{N}_0(r, \alpha_j; g| \leq l)] + S(r), \tag{2.2}$$

and

$$(1 - \frac{3}{2l}) [T_0(r, f) + T_0(r, g)] \leq \sum_{j=1}^{5} \widetilde{N}_0(r, \alpha_j; f| \leq l) + S(r), \tag{2.3}$$

where $\widetilde{n}_0(r,\alpha_j; f| \leq l)$ *is the counting function of distinct poles of* $\frac{1}{f-\alpha_j}$ *in* $\overline{\Omega}_r$ *with the multiplicities less than* l*,*

$$\widetilde{N}_0(r,\alpha_j; f| \leq l) = \int_{r_0}^{r} \frac{\widetilde{n}_0(r,\alpha_j; f| \leq l)}{t} dt, \quad r \geq r_0.$$

By applying the above result, we obtain the main theorem below, which gives a positive answer to Question 1.2.

Theorem 2.2. *Let* f *and* g *be two transcendental or admissible meromorphic functions in* Ω*,* $\alpha_j \in S(f) \cap S(g)$*,* $j = 1, 2, \ldots, 5$ *be five distinct small functions and* l *a positive integer. If* $\widetilde{E}_\Omega(\alpha_j, l; f) = \widetilde{E}_\Omega(\alpha_j, l; g)$ *for* $j = 1, 2, \ldots, 5$*, and* $l \geq 22$*, then* $f(z) \equiv g(z)$*.*

Remark 2.1. *By comparing with Theorem 2.2 and Theorem 1.6, we can see that five constants have been replaced by five small functions, and IM shared has been also relaxed to partial weighted shared (only under the condition that the distinct zeros of* $f - \alpha_j$ *and* $g - \alpha_j$ *in* Ω *of multiplicity less than* l *are same for* $j = 1, 2, \ldots, 5$*). Hence, Theorem 2.2 is an extension and improvement of Theorem 1.6.*

Remark 2.2. *In view of Theorem 2.2, two naturel questions arise:*

Question 2.1. *What condition on* $l_j, (j = 1, 2, , \ldots, 5)$ *can guarantee that the conclusion of Theorem 2.2 still holds if* $\widetilde{E}_\Omega(\alpha_j, l; f) = \widetilde{E}_\Omega(\alpha_j, l; g)$ *for* $j = 1, 2, \ldots, 5$ *are replaced by* $\widetilde{E}_\Omega(\alpha_j, l_j; f) = \widetilde{E}_\Omega(\alpha_j, l; g)$ *for* $j = 1, 2, \ldots, 5$*, and* l_j *are five positive integers?*

Question 2.2. *Does the conclusion of Theorem 2.2 still hold, if the k-punctured complex plane* Ω *is turned to a more general domain*

$$\Omega' = G_0 \setminus \bigcup_{j=1}^{m} K_j,$$

where G_0 *is a bounded simply connected domain, and* $\{K_j\}, K_j \subset G_0$*,* $j = 1, 2, \ldots, m$*, are connected compacts not degenerating to a point and such that* $G_j = \mathbb{C} \backslash K_j$ *is a domain,* $j = 1, 2, \ldots, m$*?*

Moreover, similar to the argument as in the proof of Theorem 2.2, we can easily get the following result, which is an improvement of Theorem 1.4.

Corollary 2.1. *Let* f *and* g *be two transcendental or admissible meromorphic functions on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$*, where* $1 < R_0 \leq +\infty$*. Let* $a_j (j = 1, 2, \ldots, 5)$ *be five distinct small functions with respect to* f *and* g *on the annulus* $\mathbb{A}$*. If* $\widetilde{E}_{\mathbb{A}}(a_j, l; f) = \widetilde{E}_{\mathbb{A}}(a_j, l; g)$ *for* $j = 1, 2, \ldots, 5$*, and* $l \geq 22$*, then* $f(z) \equiv g(z)$*.*

When $l = +\infty$, one can get the following conclusion.

Theorem 2.3. *Let* f *and* g *be two transcendental or admissible meromorphic functions in* Ω*,* $\alpha_j \in S(f) \cap S(g), (j = 1, 2, \ldots, 5)$ *be five distinct small functions. If* $\widetilde{E}_\Omega(\alpha_j, f) = \widetilde{E}_\Omega(\alpha_j, g)$*, for* $(j = 1, 2, \ldots, 5)$*, then* $f(z) \equiv g(z)$*.*

Remark 2.3. *Theorem 2.3 can be called as 5 IM theorem for meromorphic functions in k-punctured complex plane.*

3. Nevanlinna theory in k-punctured complex planes

Let k be a positive integer, for k distinct points $c_j \in \mathbb{C}$, $j \in \{1, 2, \dots, k\}$, we say that $\Omega = \mathbb{C} \setminus \bigcup_{j=1}^{k} \{c_j\}$ is a k-punctured complex plane. The main purpose of this article is to discuss meromorphic functions in those k-punctured planes for which $k \geq 2$.

Let $d = \frac{1}{2} \min\{|c_s - c_j| : j \neq s\}$ and $r_0 = \frac{1}{d} + \max\{|c_j| : j \in \{1, 2, \dots, k\}\}$, thus it follows $\frac{1}{r_0} < d$ and

$$\overline{D}_{1/r_0}(c_j) \bigcap \overline{D}_{1/r_0}(c_s) = \emptyset, \quad for \;\; j \neq s,$$

and

$$\overline{D}_{1/r_0}(c_j) \subset D_{r_0}(0), \quad for \quad j \in \{1, 2, \dots, k\},$$

where $D_\delta(c) = \{z : |z - c| < \delta\}$ and $\overline{D}_\delta(c) = \{z : |z - c| \leq \delta\}$. Now, we define

$$\Omega_r = D_r(0) \setminus \bigcup_{j=1}^{k} \overline{D}_{1/r}(c_j), \quad for \;\; any \quad r \geq r_0.$$

Thus, it yields that $\Omega_r \supset \Omega_{r_0}$ for $r_0 < r \leq +\infty$. Moreover, it is easy to see that Ω_r is $k + 1$ connected region.

In 2007, Hanyak and Kondratyuk [4] gave some extension of the Nevanlinna value distribution theory for meromorphic functions in a k-punctured complex plane and proved a series of theorems which is an analog of the result on the whole plane $\mathbb{C}$.

Let f be a meromorphic function in a k-punctured plane Ω, we denote $n_0(r, f)$ to be the counting function of its poles in $\overline{\Omega}_r$, $r_0 \leq r < +\infty$ and

$$N_0(r, f) = \int_{r_0}^{r} \frac{n_0(t, f)}{t} dt,$$

and we also define

$$\begin{aligned} m_0(r, f) =& \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta + \frac{1}{2\pi} \sum_{j=1}^{m} \int_0^{2\pi} \log^+ \left| f(c_j + \frac{1}{r} e^{i\theta}) \right| d\theta - \\ & - \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(r_0 e^{i\theta}) \right| d\theta - \frac{1}{2\pi} \sum_{j=1}^{m} \int_0^{2\pi} \log^+ \left| f(c_j + \frac{1}{r_0} e^{i\theta}) \right| d\theta, \end{aligned}$$

where $\log^+ x = \max\{\log x, 0\}$ and $r_0 \leq r < +\infty$, then

$$T_0(r, f) = m_0(r, f) + N_0(r, f)$$

is called as the Nevanlinna characteristic of f. Besides, we use $S(r, f)$ to denote any quantity satisfying $S(r, f) = o(T_0(r, f))$ for all r outside a possible exceptional set of finite linear measure.

Theorem 3.1. *(see [4, Theorem 3]). Let f, f_1, f_2 be meromorphic functions in a k-punctured plane Ω. Then*

(i) the function $T_0(r, f)$ is non-negative, continuous, non-decreasing and convex with respect to $\log r$ *on* $[r_0, +\infty)$, $T_0(r_0, f) = 0$*;*

(ii) if f identically equals a constant, then $T_0(r, f)$ vanishes identically;

(iii) if f is not identically equal to zero, then $T_0(r, f) = T_0(r, 1/f), r_0 \leq r < +\infty$;

(iv) $T_0(r, f_1 f_2) \leq T_0(r, f_1) + T_0(r, f_2) + O(1)$ and $T_0(r, f_1 + f_2) \leq T_0(r, f_1) + T_0(r, f_2) + O(1)$, for $r_0 \leq r < +\infty$;

(v) $T_0(r, \frac{1}{f-a}) = T_0(r, f) + O(1)$, for any fixed $a \in \mathbb{C}$.

Remark 3.1. *Theorem 3.1 (i)–(iv) show the elementary properties of meromorphic function $f(z)$ in the k-punctured plane Ω, Theorem 3.1 (v) can be said the Jensen-Nevanlinna formula for meromorphic function $f(z)$ in the k-punctured plane Ω, which is another expression of Jensen's formula and exhibits the relations between characteristic functions of f and $\frac{1}{f}$ in Ω. Theorem 3.1 (v) is also called as the first fundamental theorem of the value distribution theory in the k-punctured plane.*

Definition 3.1. *Let f be a nonconstant meromorphic function in k-punctured plane Ω. The function f is called admissible in k-punctured plane Ω provided that*

$$\limsup_{r\to+\infty} \frac{T_0(r, f)}{\log r} = +\infty, \quad r_0 \leq r < +\infty.$$

Remark 3.2. *From Theorem 5 in [4], we have that a meromorphic function f in k-punctured plane is rational if f satisfies*

$$\limsup_{r\to+\infty} \frac{T_0(r, f)}{\log r} < +\infty, \quad r_0 \leq r < +\infty.$$

By using Lemma 6 in [4], we can get the following lemma easily.

Theorem 3.2. *Let f be a nonconstant meromorphic function in a k-punctured plane Ω, and p a positive integer, then*

$$m_0(r, \frac{f^{(p)}}{f}) = O(\log T_0(r, f)) + O(\log^+ r) := S(r, f), \quad r \to +\infty,$$

outside a set of finite linear measure.

Remark 3.3. *Obviously, if f is admissible in a k-punctured plane Ω, then*

$$m_0(r, \frac{f^{(p)}}{f}) = S(r, f) = o(T_0(r, f)),$$

outside a set of finite linear measure.

Remark 3.4. *This is an important result which is often used in this paper.*

4. The proof of Theorem 2.1

To prove Theorem 2.1, we require the following lemma.

Lemma 4.1. *(see [15, Lemma 4.2]). Let $f_1(z)$ and $f_2(z)$ be two admissible meromorphic functions in a k-punctured plane Ω, $\alpha_j(z)(\not\equiv 0, 1) \in S(f_1) \cap S(f_2)$ $j = 1, 2, \ldots, 5$ be five distinct meromorphic functions in a k-punctured plane Ω, then*

$$2T_0(r, f_s) < \sum_{j=1}^{5} \widetilde{N}_0(r, \frac{1}{f_s - \alpha_j}) + S(r, f_1) + S(r, f_2), \quad s = 1, 2. \tag{4.1}$$

The proof of Theorem 2.1: In view of the assumptions of Theorem 2.1, it follows

$$\begin{aligned}
2T_0(r,f) &\leq \sum_{j=1}^{5} \widetilde{N}_0(r, \frac{1}{f-\alpha_j}) + S(r,f) \\
&\leq \frac{l}{1+l} \sum_{j=1}^{5} \widetilde{N}_0(r, \alpha_j; f| \leq l) + \frac{5}{l+1} T_0(r,f) + S(r,f) \\
&\leq \frac{l}{1+l} N_0(r, \frac{1}{f-g}) + \frac{5}{1+l} T_0(r,f) + S(r,f) \\
&\leq \frac{l+5}{l+1} T_0(r,f) + \frac{l}{l+1} T_0(r,g) + S(r,f),
\end{aligned}$$

that is,

$$T_0(r,f) \leq \frac{l}{l-3} T_0(r,g) + S(r,f). \tag{4.2}$$

Similarly, we have

$$T_0(r,g) \leq \frac{l}{l-3} T_0(r,f) + S(r,g). \tag{4.3}$$

From (4.2) and (4.3), we can deduce immediately that $S(r) = S(r,f) = S(r,g)$.

By applying Lemma 4.1 for $f(z)$ and $\alpha_j(z)(j = 1, 2, \ldots, 5)$ again, we have

$$\begin{aligned}
&2T_0(r,f) + l \sum_{j=1}^{5} \widetilde{N}_0(r, \alpha_j; f| \geq l+1) \\
\leq &\sum_{j=1}^{5} \widetilde{N}_0(r, \alpha_j; f) + l \sum_{j=1}^{5} \widetilde{N}_0(r, \alpha_j; f| \geq l+1) + S(r) \leq 5T_0(r,f) + S(r),
\end{aligned}$$

where $\widetilde{n}_0(r, \alpha_j; f| \geq l+1)$ is the counting function of distinct poles of $\frac{1}{f-\alpha_j}$ in $\overline{\Omega}_r$ with the multiplicities great than $l+1$,

$$\widetilde{N}_0(r, \alpha_j; f| \geq l+1) = \int_{r_0}^{r} \frac{\widetilde{n}_0(r, \alpha_j; f| \geq l+1)}{t} dt, \quad r \geq r_0.$$

Thus, it follows

$$\sum_{j=1}^{5} \widetilde{N}_0(r, \alpha_j; f| \geq l+1) \leq \frac{3}{l} T_0(r,f) + S(r). \tag{4.4}$$

Similarly, we have

$$\sum_{j=1}^{5} \widetilde{N}_0(r, \alpha_j; g| \geq l+1) \leq \frac{3}{l} T_0(r,g) + S(r). \tag{4.5}$$

By Lemma 4.1 and (4.4), it yields

$$2T_0(r,f) \leq \sum_{j=1}^{5} \widetilde{N}_0(r, \alpha_j; f| \leq l) + \sum_{j=1}^{5} \widetilde{N}_0(r, \alpha_j; f| \geq l+1) + S(r)$$

$$\leq \sum_{j=1}^{5} \widetilde{N}_0(r, \alpha_j; f| \leq l) + \frac{3}{l} T_0(r, f) + S(r),$$

that is,

$$(2 - \frac{3}{l}) T_0(r, f) \leq \sum_{j=1}^{5} \widetilde{N}_0(r, \alpha_j; f| \leq l) + S(r).$$

Similarly, we have

$$(2 - \frac{3}{l}) T_0(r, g) \leq \sum_{j=1}^{5} \widetilde{N}_0(r, \alpha_j; g| \leq l) + S(r).$$

Thus, from the above two inequalities, it follows

$$(2 - \frac{3}{l})[T_0(r, f) + T_0(r, g)] \leq \sum_{j=1}^{5} (\widetilde{N}_0(r, \alpha_j; f| \leq l) + \widetilde{N}_0(r, \alpha_j; g| \leq l)) + S(r). \tag{4.6}$$

And since $\widetilde{E}_\Omega(\alpha_j, l; f) = \widetilde{E}_\Omega(\alpha_j, l; g), (j = 1, 2, \ldots, 5)$, that is, $\widetilde{N}_0(r, \alpha_j; f| \leq l) + \widetilde{N}_0(r, \alpha_j; g| \leq l)$ for $j = 1, 2, \ldots, 5$, then we can prove (2.3) easily by combining (4.6).

Thus, we complete the proof of Theorem 2.1.

5. The proof of Theorem 2.2

To prove Theorem 2.2, some lemmas below will be required.

Lemma 5.1. *(see [16]). Let f be a nonconstant meromorphic function in k-punctured plane Ω, and let*

$$R(f) = \sum_{i=0}^{n} a_i f^i / \sum_{j=0}^{m} b_j f^j$$

be an irreducible rational function in f with coefficients $\{a_i\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = max\{n, m\}$.

Lemma 5.2. *Let f be a transcendental or admissible meromorphic functions in Ω, $a(z), b(z) \in S(f)$ be two distinct small functions with respect to f. Set*

$$L(f, a, b) := \begin{vmatrix} f & f' & 1 \\ a & a' & 1 \\ b & b' & 1 \end{vmatrix}.$$

Then we have $L(f, a, b) \not\equiv 0$ and for $i = 0, 1$,

$$m_0\left(r, \frac{L(f, a, b) f^i}{(f - a)(f - b)}\right) = S(r, f).$$

Proof. Utilizing the determinant nature, we obtain that

$$L(f,a,b) = \begin{vmatrix} f & f' & 1 \\ f-a & f'-a' & 0 \\ f-b & f'-b' & 0 \end{vmatrix} = (f-a)(f'-b') - (f-b)(f'-a'),$$

which further yields

$$\frac{L(f,a,b)}{(f-a)(f-b)} = \frac{f'-b'}{f-b} - \frac{f'-a'}{f-a}. \tag{5.1}$$

Suppose that $L(f,a,b) \equiv 0$, that is,

$$\frac{f'-b'}{f-b} \equiv \frac{f'-a'}{f-a},$$

by a simply integral, we get $f = \frac{1}{1-\eta}(b - \eta a)$, where η is a constant, which is a contradiction with $a(z), b(z) \in S(f)$. So, we have $L(f,a,b) \not\equiv 0$. If $i = 0$, by applying Theorem 3.2 for (5.1), it follows

$$m_0\left(r, \frac{L(f,a,b)}{(f-a)(f-b)}\right) = S(r,f).$$

If $i = 1$, we have

$$\begin{aligned} \frac{L(f,a,b)f}{(f-a)(f-b)} &= \frac{f(f-a)(f'-b') - f(f-b)(f'-a')}{(f-a)(f-b)} \\ &= \frac{(f-b+b)(f-a)(f'-b') - (f-a+a)(f-b)(f'-a')}{(f-a)(f-b)} \\ &= b\frac{f'-b'}{f-b} - a\frac{f'-a'}{f-a} + (a'-b'). \end{aligned} \tag{5.2}$$

By combining $a(z), b(z) \in S(f)$ and applying Theorem 3.2 for (5.2), we get

$$m_0\left(r, \frac{L(f,a,b)f}{(f-a)(f-b)}\right) = S(r,f).$$

Therefore, this completes the proof of Lemma 5.2. □

Let $h_1(z)$ and $h_2(z)$ be two non-constant meromorphic functions in Ω and $\alpha(z)$ (or ∞) be the common small function of $h_1(z)$ and $h_2(z)$, we use $\widetilde{N}_0(r, h_1(z) = \alpha(z) = h_2(z))$ ($\widetilde{N}_0^E(r, h_1(z) = \alpha(z) = h_2(z))$) to denote the counting function of those common zeros of $h_1(z) - \alpha(z)$ and $h_2(z) - \alpha(z)$ in Ω, regardless of multiplicity (with the same multiplicity), and each zeros counted only once. Moreover, if $\widetilde{N}_0(r, \frac{1}{h_j(z)-\alpha(z)}) - \widetilde{N}_0^E(r, h_1(z) = \alpha(z) = h_2(z)) = S(r, h_j)$, $j = 1, 2$, then we can say that $h_1(z)$ and $h_2(z)$ share $\alpha(z)$ CM^*; if $\widetilde{N}_0(r, \frac{1}{h_j(z)-\alpha(z)}) - \widetilde{N}_0(r, h_1(z) = \alpha(z) = h_2(z)) = S(r, h_j)$, $j = 1, 2$, then we can say that $h_1(z)$ and $h_2(z)$ share $\alpha(z)$ IM^*.

Lemma 5.3. *Let f and g be two transcendental or admissible meromorphic functions in Ω, $b_j(z) \in S(f) \cap S(g)$ $(j = 1, 2, 3)$. If f, g share $b_j(z) (j = 1, 2, 3)$ and ∞ IM^*. Set*

$$\mathcal{H}_1 := \frac{L(f,b_1,b_2)(f-g)L(g,b_2,b_3)}{(f-b_1)(f-b_2)(g-b_2)(g-b_3)} - \frac{L(g,b_1,b_2)(f-g)L(f,b_2,b_3)}{(g-b_1)(g-b_2)(f-b_2)(f-b_3)}.$$

Then $T_0(r, \mathcal{H}_1) = S(r,f) + S(r,g)$.

Proof. By applying Lemma 5.2 for $\mathcal{H}_1$, we can obtain

$$m_0(r, \mathcal{H}_1) = S(r, f) + S(r, g). \tag{5.3}$$

Now, we will estimate $N_0(r, \mathcal{H}_1)$. We know that the poles of $\mathcal{H}_1$ in Ω can come from the zeros of $f - b_j(z), g - b_j(z)$ in Ω for $j = 1, 2, 3$, the poles of f, g in Ω. Because $\mathcal{H}_1$ can be represented as

$$\begin{aligned}\mathcal{H}_1 \equiv &(f-g)\left\{\frac{L(f, b_1, b_2)}{b_2 - b_1}\left(\frac{1}{f - b_2} - \frac{1}{f - b_1}\right)\frac{L(g, b_2, b_3)}{b_3 - b_2}\left(\frac{1}{g - b_3} - \frac{1}{g - b_2}\right)\right.\\ &\left.\frac{L(g, b_1, b_2)}{b_2 - b_1}\left(\frac{1}{g - b_2} - \frac{1}{g - b_1}\right)\frac{L(f, b_2, b_3)}{b_3 - b_2}\left(\frac{1}{f - b_3} - \frac{1}{f - b_2}\right)\right\}\\ \equiv &(f-g)\left\{\left(\frac{f' - b_2'}{f - b_2} - \frac{f' - b_1'}{f - b_1}\right)\left(\frac{g' - b_3'}{g - b_3} - \frac{g' - b_2'}{g - b_2}\right)\right.\\ &\left.-\left(\frac{g' - b_2'}{g - b_2} - \frac{g' - b_1'}{g - b_1}\right)\left(\frac{f' - b_3'}{f - b_3} - \frac{f' - b_2'}{f - b_2}\right)\right\},\end{aligned} \tag{5.4}$$

and since f, g share $b_j(z)$ IM^* in Ω and $b_j(z) \in S(f) \cap S(g)$, from (5.4), it is easy to see that the poles of $\mathcal{H}_1$ in Ω which come from the zeros of $f - b_1$ and $f - b_3$ in Ω are only $S(r, f) + S(r, g)$.

Further, if z_1 is a pole of f in Ω with multiplicity p, a pole of g in Ω with multiplicity q, and $b_j(z_1)(b_j(z_1) - 1) \neq 0, \infty$ for $j = 1, 2, 3$. W.l.g., assume $p \geq q$, then for $z \to z_1$, it follows

$$\begin{aligned}\frac{L(f, b_1, b_2)(f - g)L(g, b_2, b_3)}{(f - b_1)(f - b_2)(g - b_2)(g - b_3)} &\sim (b_2 - b_1)(b_3 - b_2)(1 - \frac{g}{f})\frac{f'g'}{fg^2},\\ \frac{L(g, b_1, b_2)(f - g)L(f, b_2, b_3)}{(g - b_1)(g - b_2)(f - b_2)(f - b_3)} &\sim (b_2 - b_1)(b_3 - b_2)(1 - \frac{g}{f})\frac{f'g'}{fg^2}.\end{aligned}$$

Thus, $\mathcal{H}_1$ is analytic at z_1.

In addition, we can rewrite $\mathcal{H}_1$ as the following form

$$\begin{aligned}\mathcal{H}_1 \equiv &\frac{(g - f)}{(f - b_1)(g - b_3)}\left\{\left[(b_1 - b_2)\frac{f' - b_2'}{f - b_2} - (b_1' - b_2')\right]\left[(b_3 - b_2)\frac{g' - b_2'}{g - b_2} - (b_3' - b_2')\right]\right.\\ &\left.-\frac{f(g-1)}{g(f-1)}\left[(b_1 - b_2)\frac{g' - b_2'}{g - b_2} - (b_1' - b_2')\right]\left[(b_3 - b_2)\frac{f' - b_2'}{f - b_2} - (b_3' - b_2')\right]\right\}.\end{aligned}$$

Thus, if z_2 is a zero of $f - b_2$ in Ω and $b_j(z_1)(b_j(z_1) - 1) \neq 0, \infty$ $j = 1, 2, 3$, since f, g share $b_j(z)(j = 1, 2, 3)$ IM^*, from the above expression, we know that $\mathcal{H}_1$ is analytic at z_2.

Hence, we have $N_0(r, \mathcal{H}_1) = S(r, f) + S(r, g)$, and by combining (5.3), it follows $T_0(r, \mathcal{H}_1) = S(r, f) + S(r, g)$.

Therefore, this completes the proof of Lemma 5.3. □

Lemma 5.4. *Let f and g be two transcendental or admissible meromorphic functions in Ω, $b_j(z) \in S(f) \cap S(g)$ $(j = 1, 2, 3)$. Let $\mathcal{H}_1$ be stated as in Lemma 5.2 and*

$$\begin{aligned}\mathcal{H}_2 :&= \frac{L(f, b_2, b_1)(f - g)L(g, b_1, b_3)}{(f - b_2)(f - b_1)(g - b_1)(g - b_3)} - \frac{L(g, b_2, b_1)(f - g)L(f, b_1, b_3)}{(g - b_2)(g - b_1)(f - b_1)(f - b_3)},\\ \mathcal{H}_3 :&= \frac{L(f, b_1, b_3)(f - g)L(g, b_3, b_2)}{(f - b_1)(f - b_3)(g - b_3)(g - b_2)} - \frac{L(g, b_1, b_3)(f - g)L(f, b_3, b_2)}{(g - b_1)(g - b_3)(f - b_3)(f - b_2)}.\end{aligned}$$

Then

(i) $\mathcal{H}_1 \equiv -\mathcal{H}_2 \equiv -\mathcal{H}_3$*;*

(ii) $\mathcal{H}_1 \equiv 0 \Longleftrightarrow \widetilde{\mathcal{H}}_1 \equiv 0$*, where*

$$\widetilde{\mathcal{H}}_1 := \frac{L(F,\widetilde{b}_1,\widetilde{b}_2)(F-G)L(G,\widetilde{b}_2,\widetilde{b}_3)}{(F-\widetilde{b}_1)(F-\widetilde{b}_2)(G-\widetilde{b}_2)(G-\widetilde{b}_3)} - \frac{L(G,\widetilde{b}_1,\widetilde{b}_2)(F-G)L(F,\widetilde{b}_2,\widetilde{b}_3)}{(G-\widetilde{b}_1)(G-\widetilde{b}_2)(F-\widetilde{b}_2)(F-\widetilde{b}_3)},$$

and

$$F = \frac{1}{f-b_1} + b_1, \;\; G = \frac{1}{g-b_1} + b_1,$$
$$\widetilde{b}_1 = b_1, \;\; \widetilde{b}_2 = \frac{1}{b_2-b_1} + b_1, \;\; \widetilde{b}_3 = \frac{1}{b_3-b_1} + b_1.$$

Proof. (i) From (5.4), we have

$$\begin{aligned}\mathcal{H}_1 :=&(f-g)\left\{\frac{f'-b_2'}{f-b_2}\frac{g'-b_3'}{g-b_3} - \frac{g'-b_2'}{g-b_2}\frac{f'-b_3'}{f-b_3} - \frac{f'-b_1'}{f-b_1}\frac{g'-b_3'}{g-b_3}\right.\\ &\left.+\frac{g'-b_1'}{g-b_1}\frac{f'-b_3'}{f-b_3} + \frac{g'-b_2'}{g-b_2}\frac{f'-b_1'}{f-b_1} - \frac{g'-b_1'}{g-b_1}\frac{f'-b_2'}{f-b_2}\right\},\end{aligned} \tag{5.5}$$

by interchanging between b_1 and b_2 in (5.5), we can get $\mathcal{H}_1 \equiv -\mathcal{H}_2$.

Similarly, by interchanging between b_2 and b_3 in (5.5), we can get $\mathcal{H}_1 \equiv -\mathcal{H}_3$. Thus, (i) is proved.

(ii) Without loss of generality, assume that $b_1 = 0, b_2 = 1, b_3 = b$, and $b \neq (0,1,\infty)$. From (i), it follows

$$-\mathcal{H}_1 \equiv \mathcal{H}_2 = \frac{L(f,1,0)(f-g)L(g,0,b)}{(f-1)fg(g-b)} - \frac{L(g,1,0)(f-g)L(f,0,b)}{(g-1)gf(f-b)}.$$

Thus, let $F = \frac{1}{f}, G = \frac{1}{g}, \widetilde{b}_1 = 0, \widetilde{b}_2 = 1, \widetilde{b}_3 = \frac{1}{b}$. Similar to the definition of $\mathcal{H}_2$, we have

$$\begin{aligned}-\widetilde{\mathcal{H}}_1 \equiv \widetilde{\mathcal{H}}_2 =&\frac{L(F,1,0)(F-G)L(G,0,\frac{1}{b})}{(F-1)FG(G-\frac{1}{b})} - \frac{L(G,1,0)(F-G)L(F,0,\frac{1}{b})}{G(G-1)F(F-\frac{1}{b})}\\ =&\frac{L(f^{-1},1,0)(f^{-1}-g^{-1})L(g^{-1},0,b^{-1})}{(f^{-1}-1)f^{-1}g^{-1}(g^{-1}-b^{-1})}\\ &-\frac{L(g^{-1},1,0)(f^{-1}-g^{-1})L(f^{-1},0,b^{-1})}{(g^{-1}-1)f^{-1}g^{-1}(g^{-1}-b^{-1})}\\ =&\frac{\frac{f'}{f^2}\frac{g-f}{fg}\begin{vmatrix}\frac{1}{g} & \frac{g'}{g^2}\\ \frac{1}{b} & \frac{b'}{b^2}\end{vmatrix}}{\frac{1-f}{f}\frac{1}{f}\frac{1}{g}\frac{b-g}{bg}} - \frac{\frac{g'}{g^2}\frac{g-f}{fg}\begin{vmatrix}\frac{1}{f} & \frac{f'}{f^2}\\ \frac{1}{b} & \frac{b'}{b^2}\end{vmatrix}}{\frac{1-g}{g}\frac{1}{g}\frac{1}{f}\frac{b-f}{bf}}\\ =&\frac{1}{b}\left\{\frac{-f'(g-f)\begin{vmatrix}g & g'\\ b & b'\end{vmatrix}}{(f-1)fg(g-b)} - \frac{-g'(f-g)\begin{vmatrix}f & f'\\ b & b'\end{vmatrix}}{(g-1)gf(f-b)}\right\}\\ =&\frac{1}{b}\mathcal{H}_2 = -\frac{1}{b}\mathcal{H}_1,\end{aligned}$$

which implies $\mathcal{H}_1 \equiv 0 \Longleftrightarrow \widetilde{\mathcal{H}}_1 \equiv 0$ by $b(z) \not\equiv 0$.

Therefore, this completes the proof of Lemma 5.4. □

5.1. The proof of Theorem 2.2

We will adopt the idea of Yi and Li [23], Yao [19]. W.l.o.g, we can consider a quasi-Möbius transformation

$$\beta_j = \frac{\alpha_j - \alpha_4}{\alpha_j - \alpha_5}\frac{\alpha_3 - \alpha_5}{\alpha_3 - \alpha_4}, \quad j = 1, 2, \ldots, 5,$$

that is, $\beta_1 = \alpha_1(z), \beta_2 = \alpha_2(z), \beta_3 = 1, \beta_4 = 0$ and $\beta_5(z) = \infty$. Set

$$\mathscr{H}_1 := \frac{L(f,0,1)(f-g)L(g,1,\beta_2)}{f(f-1)(g-1)(g-\beta_2)} - \frac{L(g,0,1)(f-g)L(f,1,\beta_2)}{g(g-1)(f-1)(f-\beta_2)}.$$

If $\mathscr{H}_1 \not\equiv 0$, from Lemma 5.3, we have

$$m_0(r, \mathscr{H}_1) = S(r),$$

and by Lemma 5.4, it follows

$$\begin{aligned}
N_0(r, \mathscr{H}_1) \leq & \sum_{j=2}^{5} \widetilde{N}_0(r, \alpha_j; f| \geq l+1) + \sum_{j=2}^{5} \widetilde{N}_0(r, \alpha_j; g| \geq l+1) + S(r) \\
\leq & \frac{4}{l+1}(T_0(r,f) + T_0(r,g)) - \frac{2}{l+1}\sum_{j=1}^{5} \widetilde{N}_0(r, \alpha_j; f| \leq l) + \frac{2}{l+1}\widetilde{N}_0(r, \alpha_1; f| \leq l) + S(r) \\
\leq & \frac{4}{l+1}(T_0(r,f) + T_0(r,g)) - \frac{2}{l+1}(1 - \frac{3}{2l})(T_0(r,f) + T_0(r,g)) \\
& + \frac{2}{l+1}\widetilde{N}_0(r, \alpha_1; f| \leq l) + S(r) \\
\leq & \frac{2l+3}{l(l+1)}(T_0(r,f) + T_0(r,g)) + \frac{2}{l+1}\widetilde{N}_0(r, \alpha_1; f| \leq l) + S(r),
\end{aligned}$$

and because

$$\widetilde{N}_0(r, \alpha_1; f| \leq l) \leq N_0(r, \frac{1}{\mathscr{H}_1}) \leq N_0(r, \mathscr{H}_1) + S(r),$$

thus it follows

$$\widetilde{N}_0(r, \alpha_1; f| \leq l) \leq \frac{2l+3}{l(l-1)}(T_0(r,f) + T_0(r,g)) + S(r).$$

Noting that $\widetilde{E}_\Omega(\alpha_j, l; f) = \widetilde{E}_\Omega(\alpha_j, l; g)$, that is, $\widetilde{N}_0(r, \alpha_j; f| \leq l) = \widetilde{N}_0(r, \alpha_j; g| \leq l)$ for $j = 1, 2, \ldots, 5$, then we have

$$\widetilde{N}_0(r, \alpha_1; f| \leq l) + \widetilde{N}_0(r, \alpha_1; g| \leq l) \leq \frac{2(2l+3)}{l(l-1)}(T_0(r,f) + T_0(r,g)) + S(r).$$

Similarly, let

$$\mathscr{H}_2 := \frac{L(f,1,0)(f-g)L(g,0,\beta_1)}{f(f-1)g(g-\beta_1)} - \frac{L(g,1,0)(f-g)L(f,0,\beta_1)}{g(g-1)f(f-\beta_1)},$$

$$\mathscr{H}_3 := \frac{L(f,0,\beta_2)(f-g)L(g,\beta_2,\beta_1)}{f(f-\beta_2)(g-\beta_2)(g-\beta_1)} - \frac{L(g,0,\beta_2)(f-g)L(f,\beta_2,\beta_1)}{g(g-\beta_2)(f-\beta_2)(f-\beta_1)},$$
$$\mathscr{H}_4 := \frac{L(f,1,\beta_1)(f-g)L(g,\beta_1,\beta_2)}{(f-1)(f-\beta_1)(g-\beta_1)(g-\beta_2)} - \frac{L(g,1,\beta_1)(f-g)L(f,\beta_1,\beta_2)}{(g-1)(g-\beta_1)(f-\beta_1)(f-\beta_2)},$$

then if $\mathscr{H}_j \not\equiv 0$ for $j = 2,3,4$, by using the same argument as in the above, we have

$$\widetilde{N}_0(r,\alpha_j; f| \le l) + \widetilde{N}_0(r,\alpha_j; g| \le l) \le \frac{2(2l+3)}{l(l-1)}(T_0(r,f) + T_0(r,g)) + S(r), \tag{5.6}$$

for $j = 2,3,4$. Next, we will prove that

$$\widetilde{N}_0(r,\alpha_5; f| \le l) + \widetilde{N}_0(r,\alpha_5; g| \le l) \le \frac{2(2l+3)}{l(l-1)}(T_0(r,f) + T_0(r,g)) + S(r). \tag{5.7}$$

Let

$$F(z) = \frac{1}{f(z)},\quad G(z) = \frac{1}{g(z)},\quad \alpha_1^*(z) = \frac{1}{\alpha_1(z)} = \frac{1}{\beta_1(z)},$$
$$\alpha_2^*(z) = \frac{1}{\alpha_2(z)} = \frac{1}{\beta_2(z)},\quad \alpha_3^*(z) = \frac{1}{\alpha_3(z)} = 1,\quad \alpha_4^*(z) = \infty,\quad \alpha_5^*(z) = 0.$$

Thus, $\alpha_j^*(z)(j = 1,2,\ldots,5)$ are small functions of $F(z)$ and $G(z)$, and $\widetilde{E}_\Omega(\alpha_j^*, l; F) = \widetilde{E}_\Omega(\alpha_j^*, l; G)$ $(j = 1,2,\ldots,5)$. Further, by applying Theorem 3.1 and Lemma 5.1 for $F(z), G(z)$, we have

$$T_0(r,F) = T_0(r,f),\quad T_0(r,G) = T_0(r,g),$$
$$S(r,F) = S(r,f) = S(r),\quad S(r,G) = S(r,g) = S(r).$$

Set

$$\mathscr{H}_5 := \frac{L(F,1,\beta_1^{-1})(F-G)L(G,\beta_2^{-1},\beta_1^{-1})}{(F-1)(F-\beta_1^{-1})(G-\beta_1^{-1})(G-\beta_2^{-1})} - \frac{L(G,1,\beta_1^{-1})(F-G)L(F,\beta_2^{-1},\beta_1^{-1})}{(G-1)(G-\beta_1^{-1})(F-\beta_1^{-1})(F-\beta_2^{-1})},$$

if $\mathscr{H}_5 \not\equiv 0$, from Lemma 5.3, we have

$$m_0(r,\mathscr{H}_5) = S(r),$$

and by Lemma 5.4, it follows

$$\begin{aligned}
N_0(r,\mathscr{H}_5) &\le \sum_{j=1}^{4} \widetilde{N}_0(r,\alpha_j^*; F| \ge l+1) + \sum_{j=1}^{4} \widetilde{N}_0(r,\alpha_j^*; G| \ge l+1) + S(r)\\
&\le \sum_{j=1}^{4} \widetilde{N}_0(r,\alpha_j; f| \ge l+1) + \sum_{j=1}^{4} \widetilde{N}_0(r,\alpha_j; f| \ge l+1) + S(r)\\
&\le \frac{4}{l+1}(T_0(r,f) + T_0(r,g)) - \frac{2}{l+1}\sum_{j=1}^{5} \widetilde{N}_0(r,\alpha_j; f| \le l)\\
&\quad + \frac{2}{l+1}\widetilde{N}_0(r,\alpha_1; f| \le l) + S(r)
\end{aligned}$$

$$\leq\frac{4}{l+1}(T_0(r,f)+T_0(r,g))-\frac{2}{l+1}(1-\frac{3}{2l})(T_0(r,f)+T_0(r,g))$$
$$+\frac{2}{l+1}\widetilde{N}_0(r,\alpha_5;f|\leq l)+S(r)$$
$$\leq\frac{2l+3}{l(l+1)}(T_0(r,f)+T_0(r,g))+\frac{2}{l+1}\widetilde{N}_0(r,\alpha_5;f|\leq l)+S(r),$$

and since

$$\widetilde{N}_0(r,\alpha_5;f|\leq l)\leq N_0(r,\frac{1}{\mathscr{H}_5})\leq N_0(r,\mathscr{H}_5)+S(r),$$

thus it follows

$$\widetilde{N}_0(r,\alpha_5;f|\leq l)\leq\frac{2l+3}{l(l-1)}(T_0(r,f)+T_0(r,g))+S(r). \tag{5.8}$$

Noting that $\widetilde{E}_\Omega(\alpha_j,l;f)=\widetilde{E}_\Omega(\alpha_j,l;g)$, that is, $\widetilde{N}_0(r,\alpha_j;f|\leq l)=\widetilde{N}_0(r,\alpha_j;g|\leq l)$ for $j=1,2,\dots,5$, then from (5.8) we prove (5.7).

By applying Theorem 2.1, we can see that there are at least two of the five $\widetilde{N}_0(r,\alpha_j;f|\leq l)+\widetilde{N}_0(r,\alpha_j;g|\leq l)$ $(j=1,2,\dots,5)$, w.l.g. assume $j=2,3$, such that

$$\widetilde{N}_0(r,\alpha_2;f|\leq l)+\widetilde{N}_0(r,\alpha_2;g|\leq l)\geq\left(\frac{l-3}{4l}+o(1)\right)(T_0(r,f)+T_0(r,g)), \tag{5.9}$$

and

$$\widetilde{N}_0(r,\alpha_3;f|\leq l)+\widetilde{N}_0(r,\alpha_3;g|\leq l)\geq\left(\frac{l-3}{4l}+o(1)\right)(T_0(r,f)+T_0(r,g)), \tag{5.10}$$

for $r\geq r_0, r\in I\subseteq[r_0,+\infty)$ and $mesI=+\infty$. Hence, if $\mathscr{H}_2\not\equiv 0$, then it follows from (5.6) and (5.9) that

$$\frac{l-3}{4l}\leq\frac{2(2l+3)}{l(l-1)},$$

which implies a contradiction with $l\geq 22$. Thus, $\mathscr{H}_2\equiv 0$. Similarly, from (5.6) and (5.10) and $l\geq 22$, we have $\mathscr{H}_3\equiv 0$.

On the basis of $\mathscr{H}_2\equiv 0$ and $\mathscr{H}_3\equiv 0$, we have

$$\frac{L(f,1,0)L(g,0,\beta_1)}{(f-1)(g-\beta_1)}\equiv\frac{L(g,1,0)L(f,0,\beta_1)}{(g-1)(f-\beta_1)}, \tag{5.11}$$

$$\frac{L(f,1,\beta_2)L(g,\beta_2,\beta_1)}{f(g-\beta_1)}\equiv\frac{L(g,0,\beta_2)L(f,\beta_2,\beta_1)}{f(f-\beta_1)}. \tag{5.12}$$

The five cases will hereinafter be taken into consideration. Let

$$J(z):=\{z\in\Omega|\beta_2(z)=0,1,\infty,\ or\ \beta_1(z)=0,1,\infty,\ or\ \beta_2(z)-\beta_1(z)=0\}.$$

Case 1. Assume that $\beta_2'(z)\not\equiv 0$. If z_1 is a zero of $g-\beta_1$ in Ω, and not a zero of $f-\beta_1$ in Ω, and $z_1\notin J(z),\beta_2'(z_1)\neq 0$, then from (5.11) and (5.12) we have z_1 is not a pole of $f-\beta_1$. In fact, if z_1 is a zero of $g-\beta_1$ and a pole of $f-\beta_1$ in Ω, and we can rewrite the right sides of (5.11) as $\vartheta_1:=-\frac{g'}{g-1}\frac{f'\beta_1-f\beta_1'}{f-\beta_1}$, then z_1 is a pole of ϑ_1 with multiplicity 1, but the left sides of (5.11) is $\vartheta_2 := -\frac{f'}{f-1}\frac{g'\beta_1-g\beta_1'}{g-\beta_1}$, and z_1 is a pole of ϑ_2 with multiplicity 2, a contradiction. Similarly, we can get a contradiction from (5.12) when z_1 is a zero of $g-\beta_1$ and a pole of $f-\beta_1$ in Ω. Thus, the right sides of (5.11) and (5.12) are analytic at z_1. Thus we get $f'(z_1)\neq 0$, $f(z_1)\neq 1$ and $L(f,1,0)_{z=z_1}=0$, $L(f,0,\beta_2)_{z=z_1}=0$. Hence it follows $f'(z_1)\neq 0$ and $f(z_1)\beta_2'(z_1)-f'(z_1)\beta_2(z_1)=0$, that is, $f(z_1)\beta_2'(z_1)-0$, a contradiction with $\beta_2'(z_1)\neq 0$ and $f(z_1)\neq 0$. Thus, we have $f(z_1)-\beta_2(z_1)=0$.

Hence we get

$$g-\beta_1=0\Longrightarrow f-\beta_1=0,\quad (r\notin E), \tag{5.13}$$

where $E\subseteq[r_0,+\infty)$, and $mesE<+\infty$. Similarly, we get

$$f-\beta_1=0\Longrightarrow g-\beta_1=0,\quad (r\notin E), \tag{5.14}$$

which implies that f,g share β_1 IM^* in Ω.

(ii) Assume that $\beta_2'(z)\equiv 0$, that is, $\beta_2(z)$ is a constant, set $\beta_2(z)\equiv\gamma$. Thus, by Lemma 5.4, (5.12) is equivalent to

$$\frac{L(f,\beta_2,0)L(g,0,\beta_1)}{(f-\beta_2)(g-\beta_1)}\equiv\frac{L(g,\beta_2,0)L(f,0,\beta_1)}{(g-\beta_2)(f-\beta_1)}. \tag{5.15}$$

Due to $\beta_2\equiv\gamma$, we have $L(f,\beta_2,0)\equiv\gamma L(f,1,0)$, $L(g,\beta_2,0)\equiv\gamma L(g,1,0)$. Substituting them into (5.15), and combining (5.11), we can get

$$\frac{f-\beta_2}{f-1}\equiv\frac{g-\beta_2}{g-1},$$

which implies $f\equiv g$.

Case 2. By Lemma 5.4, (5.11) is equivalent to the following equation

$$\frac{L(f,0,1)L(g,1,\beta_1)}{f(g-\beta_1)}\equiv\frac{L(g,0,1)L(f,1,\beta_1)}{g(f-\beta_1)}. \tag{5.16}$$

(i) $(\beta_1(z)-1)\beta_2'(z)-(\beta_2(z)-1)\beta_1'(z)\not\equiv 0$. If z_2 is a zero of f in Ω, but not a zero of g in Ω, and $z_2\notin J(z)$, $(\beta_1(z_2)-1)\beta_2'(z_2)-(\beta_2(z_2)-1)\beta_1'(z_2)\neq 0$, then z_2 is not a pole of g in Ω. In fact, if z_2 is a zero of f and a pole of g in Ω, and we can rewrite the right sides of (5.16) as $\vartheta_3 := \frac{g'}{g}\frac{\beta_1 f'-\beta_1' f+\beta_1'-f'}{f-\beta_1}$, then z_2 is a pole of ϑ_3 with multiplicity 1, but the left sides of (5.16) is $\vartheta_4 := \frac{f'}{f}\frac{\beta_1 g'-\beta_1' g+\beta_1'-g'}{g-\beta_1}$, and by a simply calculation, z_2 is a pole of ϑ_4 with multiplicity 2, a contradiction. Similarly, we can get a contradiction from (5.12) when z_2 is a zero of f and a pole of g in Ω. Thus, the right sides of (5.12) and (5.16) are analytic at z_2. Thus we get $f(z_2)-\beta_1(z_2)\neq 0$ and $L(g,1,\beta_1)_{z=z_2}=0$, $L(g,\beta_2,\beta_1)_{z=z_2}=0$. Hence it follows

$$\begin{vmatrix} g(z_2)-\beta_1(z_2) & g'(z_2)-\beta_1'(z_2)\\ \beta_1(z_2)-1 & \beta_1'(z_2)\end{vmatrix}=0,\quad \begin{vmatrix} g(z_2)-\beta_1(z_2) & g'(z_2)-\beta_1'(z_2)\\ \beta_1(z_2)-\beta_2(z_2) & \beta_1'(z_2)-\beta_2'(z_2)\end{vmatrix}=0,$$

that is,

$$\begin{vmatrix} \beta_1(z_2)-1 & \beta_1'(z_2)\\ \beta_1(z_2)-\beta_2(z_2) & \beta_1'(z_2)-\beta_2'(z_2)\end{vmatrix}=0,$$

as $g(z_2)-\beta_1(z_2)\neq 0$. It means that $(\beta_1(z_2)-1)\beta_2'(z_2)-(\beta_2(z_2)-1)\beta_1'(z_2)=0$, a contradiction. Hence z_2 is a zero of $g(z)$ in Ω, which implies

$$f=0\Longrightarrow g=0,\quad (r\notin E),$$

where $E \subseteq [r_0, +\infty)$, and $mesE < +\infty$. Similarly, we get

$$g = 0 \Longrightarrow f = 0, \quad (r \notin E).$$

Therefore, it means that f, g share 0 IM^* in Ω.

By Lemma 5.3 and from (5.11) and (5.12), it follows

$$\frac{L(f, 0, 1)L(g, 1, \beta_2)}{f(g - \beta_2)} \equiv \frac{L(g, 0, 1)L(f, 1, \beta_2)}{g(f - \beta_2)}. \tag{5.17}$$

(ii) $(\beta_1(z) - 1)\beta_2'(z) - (\beta_2(z) - 1)\beta_1'(z) \equiv 0$. It follows that $\beta_2 - 1 = \gamma(\beta_1 - 1)$ and $\beta_2' = \gamma\beta_1'$, where $\gamma(\neq 0)$ a constant. Then, we have

$$L(g, 1, \beta_2) \equiv \begin{vmatrix} g - 1 & g' \\ \gamma(\beta_1 - 1) & \gamma\beta_1' \end{vmatrix} \equiv \gamma L(g, 1, \beta_1),$$
$$L(f, 1, \beta_2) \equiv \begin{vmatrix} f - 1 & f' \\ \gamma(\beta_1 - 1) & \gamma\beta_1' \end{vmatrix} \equiv \gamma L(f, 1, \beta_1).$$

Substituting the above equations into (5.17), and combining (5.16), we get

$$\frac{f - \beta_2}{f - \beta_1} \equiv \frac{g - \beta_2}{g - \beta_1},$$

which implies $f \equiv g$.

Case 3. From (5.11) and (5.15), we have

$$\frac{L(f, \beta_2, 0)L(g, 0, 1)}{(f - \beta_2)(g - 1)} \equiv \frac{L(g, \beta_2, 0)L(f, 0, 1)}{(g - \beta_2)(f - 1)}. \tag{5.18}$$

(i) $\beta_2(z)\beta_1'(z) - \beta_2'(z)\beta_1(z) \not\equiv 0$. If z_3 is a zero of $g - 1$ in Ω, but not a zero of $f - 1$ in Ω, and $z_3 \notin J(z)$, $\beta_2(z_3)\beta_1'(z_3) - \beta_2'(z_3)\beta_1(z_3) \neq 0$, then z_3 is not a pole of $f - 1$ in Ω. In fact, if z_3 is a zero of $g - 1$ and a pole of $f - 1$ in Ω, and we can rewrite the right sides of (5.11) as $\vartheta_1 := -\frac{g'}{g-1}\frac{f'\beta_1 - f\beta_1'}{f-\beta_1}$, then z_3 is a pole of ϑ_1 with multiplicity 2, but the left sides of (5.11) is $\vartheta_2 := -\frac{f'}{f-1}\frac{g'\beta_1 - g\beta_1'}{g-\beta_1}$, and by a simply calculation, z_3 is a pole of ϑ_2 with multiplicity 1, a contradiction. Similarly, we can get a contradiction from (5.18) when z_3 is a zero of $g - 1$ and a pole of $f - 1$ in Ω. Thus, the left side of (5.11) and the right side of (5.18) are analytic at z_3. Hence we get $f(z_3) - \beta_1(z_3) \neq 0$, $f(z_3) - \beta_2(z_3) \neq 0$, but $L(f, \beta_2, 0)_{z=z_3} = 0$ and $L(f, 0, \beta_1)_{z=z_3} = 0$. Then

$$\begin{vmatrix} f(z_3) & f'(z_3) \\ \beta_2(z_3) & \beta_2'(z_3) \end{vmatrix} = 0, \quad \begin{vmatrix} f(z_3) & f'(z_3) \\ \beta_1(z_3) & \beta_1'(z_3) \end{vmatrix} = 0.$$

Since f, g share 0 IM^*, and $g(z_3) = 1$, then it follows $f(z_3) \neq 0$. So, we have

$$\begin{vmatrix} \beta_2(z_3) & \beta_2'(z_3) \\ \beta_1(z_3) & \beta_1'(z_3) \end{vmatrix} = 0,$$

which implies $\beta_2(z)\beta_1'(z) - \beta_2'(z)\beta_1(z) = 0$, a contradiction. Hence, it follows that z_3 is a zero of $f(z) - 1$ in Ω, which implies

$$g - 1 = 0 \Longrightarrow f - 1 = 0, \quad (r \notin E),$$

where $E \subseteq [r_0, +\infty)$, and $mesE < +\infty$. Similarly, we get

$$f - 1 = 0 \Longrightarrow f - 1 = 0, \quad (r \notin E).$$

Therefore, it means that f, g share 1 IM^* in Ω.

(ii) $\beta_2(z)\beta_1'(z) - \beta_2'(z)\beta_1(z) \equiv 0$. Then it follows $\beta_2 = \gamma\beta_1$ and $\beta_2' = \gamma\beta_1'$, where $\gamma(\neq 0)$ a constant. Thus, we have

$$L(g, \beta_2, 0) \equiv \begin{vmatrix} g & g' \\ \gamma\beta_1 & \gamma\beta_1' \end{vmatrix} \equiv -\gamma L(g, 0, \beta_1),$$

and

$$L(f, \beta_2, 0) \equiv \begin{vmatrix} f & f' \\ \gamma\beta_1 & \gamma\beta_1' \end{vmatrix} \equiv -\gamma L(f, 0, \beta_1).$$

By substituting the above two equivalents into (5.18) and combining (5.11), we have

$$\frac{f - \beta_2}{f - \beta_1} \equiv \frac{g - \beta_2}{g - \beta_1},$$

which implies $f \equiv g$.

Case 4. $\beta_1'(z) \not\equiv 0$. If z_4 is a zero of $f - \beta_2$ in Ω, but not a zero of $g - \beta_2$ in Ω, and $z_4 \notin J(z)$, $\beta_1'(z_4) \neq 0$, then we have that z_4 is not a pole of $g - \beta_2$ in Ω. In fact, if z_4 is a zero of $f - \beta_2$ and a pole of $g - \beta_2$ in Ω, and we can rewrite the right sides of (5.15) as $\vartheta_5 := \frac{\beta_2 g' - \beta_2' g}{g - \beta_2} \frac{f'\beta_1 - f\beta_1'}{f - \beta_1}$, then z_4 is a pole of ϑ_5 with multiplicity 1, but the left sides of (5.15) is $\vartheta_6 := \frac{\beta_1 g' - \beta_1' g}{g - \beta_1} \frac{f'\beta_2 - f\beta_2'}{f - \beta_2}$, and by a simply calculation, z_4 is a pole of ϑ_6 with multiplicity 2, a contradiction. Similarly, z_4 is a pole of the right side of (5.18) $\vartheta_7 := \frac{f'}{f-1} \frac{\beta_2 g' - \beta_2' g}{g - \beta_2}$ with multiplicity 1, and z_4 is a pole of the left of (5.18) $\vartheta_8 := \frac{g'}{g-1} \frac{\beta_2 f' - \beta_2' f}{f - \beta_2}$ with multiplicity 2. Hence, this is a contradiction. Thus, the right side of (5.15) and (5.18) are analytic at z_4. Hence we get $f(z_4) - \beta_1(z_4) \neq 0$, $f(z_4) - 1 \neq 0$, but $L(g, 0, 1)_{z=z_4} = 0$ and $L(g, 0, \beta_1)_{z=z_4} = 0$. Then $g'(z_4) = 0$, and $g(z_4)\beta_1'(z_4) - g'(z_4)\beta_1(z_4) = 0$, that is, $g(z_4)\beta_1'(z_4) = 0$. Since $\beta_1'(z_4) \neq 0$, then it follows $g(z_4) = 0$, which is a contradiction with f, g share 0 IM^* in Ω. Thus, it means that z_4 is a zero of $g - \beta_2$ in Ω, which implies

$$f - \beta_2 = 0 \Longrightarrow g - \beta_2 = 0, \quad (r \notin E),$$

where $E \subseteq [r_0, +\infty)$, and $mesE < +\infty$. Similarly, we get

$$g - \beta_2 = 0 \Longrightarrow f - \beta_2 = 0, \quad (r \notin E).$$

Therefore, it means that f, g share β_2 IM^* in Ω.

(ii) $\beta_1'(z) \equiv 0$. That is $\beta_1(z) \equiv \delta$, where $\delta(\neq 0)$ a constant. So, it yields $L(f, 0, \beta_1) \equiv \delta L(f, 0, 1)$, and $L(g, 0, \beta_1) \equiv \delta L(g, 0, 1)$. Substituting these into (5.15), and combining (5.18), we can deduce

$$\frac{f - \beta_1}{f - 1} \equiv \frac{g - \beta_1}{g - 1},$$

which implies $f \equiv g$.

Case 5. Let

$$\widetilde{\mathcal{H}}_2 = \frac{L(F, 1, 0)(F - G)L(G, 0, \beta_1^{-1})}{(F - 1)FG(G - \beta_1^{-1})} - \frac{L(G, 1, 0)(F - G)L(F, 0, \beta_1^{-1})}{G(G - 1)F(F - \beta_1^{-1})},$$

$$\widetilde{\mathcal{H}_3} = \frac{L(F,0,\beta_2^{-1})(F-G)L(G,\beta_2^{-1},\beta_1^{-1})}{(F-\beta_2^{-1})F(G-\beta_2^{-1})(G-\beta_1^{-1})} - \frac{L(G,0,\beta_2^{-1})(F-G)L(F,\beta_2^{-1},\beta_1^{-1})}{G(G-\beta_2^{-1})(F-\beta_2^{-1})(F-\beta_1^{-1})}.$$

Then from Lemma 5.4, we have

$$\mathscr{H}_2 \equiv 0 \Longleftrightarrow \widetilde{\mathscr{H}_2} \equiv 0,$$

and

$$\mathscr{H}_3 \equiv 0 \Longleftrightarrow \widetilde{\mathscr{H}_3} \equiv 0.$$

Further, it follows from $\widetilde{\mathscr{H}_2} \equiv 0$ and $\widetilde{\mathscr{H}_3} \equiv 0$ that

$$\frac{L(F,1,0)L(G,0,\beta_1^{-1})}{(F-1)(G-\beta_1^{-1})} \equiv \frac{L(G,1,0)L(F,0,\beta_1^{-1})}{(G-1)(F-\beta_1^{-1})},$$

and

$$\frac{L(F,0,\beta_2^{-1})L(G,\beta_2^{-1},\beta_1^{-1})}{F(G-\beta_1^{-1})} \equiv \frac{L(G,0,\beta_2^{-1})L(F,\beta_2^{-1},\beta_1^{-1})}{G(F-\beta_1^{-1})}.$$

By using the same argument as in Case 2 of Theorem 2.2, we obtain either F, G share 0 IM^* in Ω or $F \equiv G$. If F, G share 0 IM^* in Ω, it means that f, g share ∞ IM^* in Ω. If $F \equiv G$, by a simply calculation, we have $f \equiv g$.

From Cases 1–5, we get that either f, g share $0, 1, \infty, \beta_1(z), \beta_2(z)$ IM^* in Ω, or $f \equiv g$. If f, g share $0, 1, \infty, \beta_1(z), \beta_2(z)$ IM^* in Ω, by using the same method as in [22], it follows $f \equiv g$. Hence, we obtain $f \equiv g$.

Thus, this completes the proof of Theorem 2.2.

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Conflict of interest

The authors declare that they have no competing interests.

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