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Research article

Positive periodic solution of first-order neutral differential equation with infinite distributed delay and applications

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Abstract: In this paper, we consider first-order neutral differential equation with infinite distributed delay, where nonlinear term may satisfy sub-linearity, semi-linearity and super-linearity conditions. By virtue of a fixed point theorem of Leray-Schauder type, we prove the existence of positive periodic solutions. As applications, we prove that Hematopoiesis model, Nicholson's blowflies model and the model of blood cell production have positive periodic solutions.

Keywords: neutral operator with infinite distributed delay; positive periodic solutions; Hematopoiesis model; Nicholson's blowflies model
Mathematics Subject Classification: 34C25

1. Introduction

This paper is devoted to investigate the existence of positive ω-periodic solutions of the following first-order neutral differential equation with infinite distributed delay

$$\left(u(t) - c\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma\right)' + a(t)u(t) = b(t)\int_{-\infty}^{0} P(\sigma)f(t, u(t+\sigma))d\sigma, \tag{1.1}$$

where c is a constant with $|c| \neq 1$, $P(t) \in C((-\infty, 0], [0, +\infty))$ with $\int_{-\infty}^{0} P(\sigma)d\sigma = 1$, $a(t) \in C(\mathbb{R}, \mathbb{R})$, $b(t) \in C(\mathbb{R}, (0, +\infty))$ and the nonlinear term $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ are ω-periodic functions with respect to t where $\int_0^{\omega} a(t)dt > 0$.

Equation (1.1) includes many mathematical ecological models and population models. For example, the Hematopoiesis model [1–3]

$$\left(u(t) - c\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma\right)' + a(t)u(t) = b(t)\int_{-\infty}^{0} P(\sigma)e^{-\beta(t)u(t)}d\sigma, \tag{1.2}$$

where $\beta(t) \in C(\mathbb{R}, \mathbb{R})$ is a continuous ω-periodic function, the Nicholson's blowflies model [4–6]

$$\left(u(t) - c\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma\right)' + a(t)u(t) = b(t)\int_{-\infty}^{0} P(\sigma)u(t)e^{-\beta(t)u(t)}d\sigma, \tag{1.3}$$

and the model of blood cell production [7–9]

$$\left(u(t) - c\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma\right)' + a(t)u(t) = b(t)\int_{-\infty}^{0} P(\sigma)\frac{u(t)}{1+u^n(t)}d\sigma. \tag{1.4}$$

It is well known that neutral equations play a significant role in the applied science. Many scholars have studied the above equations from different perspectives [5, 7, 10–17]. For Eq (1.1), using Krasnoselskii's fixed-point theorem in cones, Li and Wang [7] proved the existence of positive ω-periodic solutions, where $c \in [0, 1)$ and $a(t) \in C(\mathbb{R}, (0, +\infty))$. Luo et al. [5] obtained sufficient conditions for the existence of positive ω-periodic solutions for $|c| < 1$ and $a(t) \in C(\mathbb{R}, (0, +\infty))$. However, the above results are only related to the case of sublinearity, and can not applicable to models (1.2)-(1.4). A naturally question is that whether or not there is any positive ω-periodic solution for (1.1) with semi-linearity and super-linearity? Especially for models (1.2)-(1.4)?

In this paper, we provide some sufficient conditions for the existence of positive ω-periodic solutions of Eq (1.1) where the nonlinear term f may satisfy sub-linearity, semi-linearity and super-linearity conditions. The main tool is the fixed point theorem of Leray-Schauder type. As applications, we prove that models (1.2)-(1.4) exist positive ω-periodic solutions.

Compared with [5, 7], we have following five differences. Firstly, we give the property of neutral operator $(Au)(t) := u(t) - c\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma$ for the first time. Secondly, we enlarge the range of the parameter c, i.e., $|c| < 1$ and $|c| > 1$. Thirdly, we weaken conditions of the nonlinear term f, i.e., f satisfies sub-linearity, semi-linearity and super-linearity conditions. Fourthly, we prove that the models (1.2)-(1.4) have at least one positive ω-periodic solution for the first time. Fifthly, in addition to $a(t)$ is a positive function, we study that $a(t)$ may change sign.

2. Preliminaries

Define

$$C_\omega := \{u \in C(\mathbb{R}, \mathbb{R}) : u(t+\omega) = u(t),\ \text{for } t \in \mathbb{R}\},$$

with $\|u\| := \max_{t\in\mathbb{R}} |u(t)|$. Obviously, $(C_\omega, \|\cdot\|)$ is a Banach space.

Firstly, we recall a fixed point theorem of Leray-Schauder type, which will be used in our proof.

Lemma 2.1. [18, Theorem 5] Let $B(0, r_1)$ (respectively, $B[0, r_1]$) be the open ball (respectively, the closed ball) in a Banach space $X = (X, \|\cdot\|)$ with center 0 and radius r_1. Suppose $\mathcal{A}, \mathcal{B} : X \to X$ are two operators satisfying the following conditions:

(a) $\mathcal{A}$ is a contraction;
(b) $\mathcal{B}$ is continuous and completely continuous.

Then either

(i) $\exists\, u \in B[0, r_1]$ with $u = \mathcal{A}u + \mathcal{B}u$; or

(ii) $\exists\, u \in \partial B[0, r_1]$ and $\lambda \in (0, 1)$ with $u = \lambda\mathcal{A}(\frac{u}{\lambda}) + \lambda\mathcal{B}u$.

Lemma 2.2. [5, Remark 2.2] The following first-order linear differential equation

$$v'(t) + a(t)v(t) = h(t),$$

has an ω-periodic solution

$$v(t) = \int_0^{\omega} G(t, s)h(s)ds,$$

where

$$G(t, s) = \begin{cases} \dfrac{e^{-\int_s^t a(t)dt}}{1 - e^{-\int_0^{\omega} a(t)dt}}, & 0 \le s \le t \le \omega, \\ \dfrac{e^{-\int_s^{\omega+t-s} a(t)dt}}{1 - e^{-\int_0^{\omega} a(t)dt}}, & 0 \le t < s \le \omega, \end{cases}$$

and $h \in C_\omega^+ := \{h \in C(\mathbb{R}, (0, \infty)) : \ h(t + \omega) = h(t), \text{ for } t \in \mathbb{R}\}$. Moreover, since $\int_0^{\omega} a(t)dt > 0$, it is clear that $G(t, s)$ is positive for all $(t, s) \in [0, \omega] \times [0, \omega]$.

Next, we give the property of operator A.

Lemma 2.3. If $|c| < 1$, then the operator A has a continuous inverse A^{-1} on C_ω satisfying

$$\left|\left(A^{-1} f\right)(t)\right| \le \frac{\|f\|}{1 - |c|}, \quad \forall\, f \in C_\omega.$$

Proof. First, define an operator $S : C_\omega \to C_\omega$ by

$$(S u)(t) = c \int_{-\infty}^{0} P(\sigma)u(t + \sigma)d\sigma.$$

Then, we arrive that

$$(S f)(t) = c \int_{-\infty}^{0} P(\sigma_1)f(t + \sigma_1)d\sigma_1,$$

$$(S^2 f)(t) = c^2 \int_{-\infty}^{0} \int_{-\infty}^{0} P(\sigma_2)P(\sigma_1)f(t + \sigma_1 + \sigma_2)d\sigma_1 d\sigma_2.$$

Therefore, we get

$$\begin{aligned} (S^j f)(t) =& c^j \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} P(\sigma_j) \cdots P(\sigma_1)f(t + \sigma_1 + \cdots + \sigma_j)d\sigma_1 \cdots d\sigma_j \\ =& c^j \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \prod_{i=1}^{j} P(\sigma_i)f\left(t + \sum_{i=1}^{j} \sigma_i\right) d\sigma_1 \cdots d\sigma_j. \end{aligned}$$

Since $A = I - S$, where I is an identity operator, and

$$\|S\| \le |c| \left|\int_{-\infty}^{0} P(\sigma)d\sigma\right| \le |c| < 1,$$

we obtain that A has a continuous inverse $A^{-1}: C_\omega \to C_\omega$ by

$$A^{-1} = (I - S)^{-1} = I + \sum_{j=1}^{\infty} S^j = \sum_{j=0}^{\infty} S^j.$$

Thus we have

$$(A^{-1}f(t)) = \sum_{j=0}^{\infty}[S^j f](t) = f(t) + \sum_{j=1}^{\infty} c^j \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \prod_{i=1}^{j} P(\sigma_i) f\left(t + \sum_{i=1}^{j} \sigma_i\right) d\sigma_1 \cdots d\sigma_j.$$

Therefore, we obtain

$$\begin{aligned}
|(A^{-1}f)(t)| &= \left|\sum_{j=0}^{\infty}[S^j f](t)\right| \\
&= \left|\sum_{j=0}^{\infty} c^j \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \prod_{i=1}^{j} P(\sigma_i) f\left(t + \sum_{i=1}^{j} \sigma_i\right) d\sigma_1 \cdots d\sigma_j\right| \\
&\le \left|\sum_{j=0}^{\infty} c^j \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \prod_{i=1}^{j} P(\sigma_i) d\sigma_1 \cdots d\sigma_j\right| \|f\| \\
&\le \sum_{j=0}^{\infty} |c|^j \left|\int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \prod_{i=1}^{j} P(\sigma_i) d\sigma_1 \cdots d\sigma_j\right| \|f\| \\
&\le \frac{\|f\|}{1 - |c|}.
\end{aligned}$$

□

3. Equation (1.1) with small constant c

In this section, we consider the existence of a positive ω-periodic solution of Eq (1.1). Moreover, we suppose the absolute value of constant c is smaller than 1, to be precise, $c \in (-\frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}, \frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}})$, where $\bar{a} := \frac{1}{\omega}\int_0^\omega a(t)dt$. We divide the discussion into the following two cases $c \in \left(0, \frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}\right)$ and $c \in \left(-\frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}, 0\right]$.

3.1. *Equation (1.1) in the case that* $c \in \left(0, \frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}\right)$

Theorem 3.1. Suppose $c \in \left(0, \frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}\right)$ holds. Furthermore, assume that there exists a constant $r > 0$ such that

(H_1) There exist continuous, non-negative functions $q(u)$ and $k(t)$ such that

$$0 \le f(t, u) \le k(t)q(u), \text{ for all } (t, u) \in [0, \omega] \times [0, r],$$

where $q(u)$ is non-decreasing in $[0, r]$.

(H_2) The following inequality holds

$$K^* < \frac{r[1 - e^{-\bar{a}\omega} + c(e^{-\bar{a}\omega} - 1 - \|a\|\omega)]}{(1 - e^{-\bar{a}\omega})\|b\|q(r)},$$

where $K(t) := \int_0^\omega G(t, s)k(s)ds$, and $K^* := \max_{t\in[0,\omega]} K(t)$.
Then Eq (1.1) has at least one positive ω-periodic solution with $u \in [0, r]$.

Proof. Consider Eq (1.1)

$$\left(u(t) - c\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma\right)' + a(t)u(t) = b(t)\int_{-\infty}^{0} P(\sigma)f(t, u(t+\sigma))d\sigma,$$

and a family of the equations

$$\left(u(t) - c\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma\right)' + a(t)u(t) = \lambda b(t)\int_{-\infty}^{0} P(\sigma)f(t, u(t+\sigma))d\sigma, \ \ \lambda \in (0, 1). \tag{3.1}$$

Let $v(t) = (Au)(t)$. From Lemma 2.3, we have $u(t) = (A^{-1}v)(t)$. Then Eq (1.1) and (3.1) can be written in the following forms

$$v'(t) + a(t)v(t) - a(t)H(v(t)) = b(t)\int_{-\infty}^{0} P(\sigma)f(t, u(t+\sigma))d\sigma,$$

and

$$v'(t) + a(t)v(t) - a(t)H(v(t)) = \lambda b(t)\int_{-\infty}^{0} P(\sigma)f(t, u(t+\sigma))d\sigma, \quad \lambda \in (0, 1), \tag{3.2}$$

where

$$H(v(t)) = -c\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma = -c\int_{-\infty}^{0} P(\sigma)(A^{-1}v)(t+\sigma)d\sigma.$$

Let

$$h(t) = b(t)\int_{-\infty}^{0} P(\sigma)f(t, u(t+\sigma))d\sigma,$$

then $h(t) \in C_\omega^+$ and Eq (3.2) can be written as the following linear differential equation

$$v'(t) + a(t)v(t) - a(t)H(v(t)) = \lambda h(t). \tag{3.3}$$

Define operators $\mathcal{T}, \mathcal{N} : C_\omega \to C_\omega$ by

$$(\mathcal{T}h)(t) = \int_0^\omega G(t, s)h(s)ds, \quad (\mathcal{N}v)(t) = a(t)H(v(t)), \tag{3.4}$$

where $G(t, s)$ is defined in Lemma 2.2. Therefore, $v(t)$ satisfied by

$$v(t) = \lambda(\mathcal{T}h)(t) + (\mathcal{T}\mathcal{N}v)(t) \tag{3.5}$$

is the positive ω-periodic solution of Eq (3.1). Moreover, Eq (3.5) is equivalent to

$$(I - \mathcal{T}\mathcal{N})v(t) = \lambda(\mathcal{T}h)(t).$$

Since $c \in \left(0, \frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega - e^{-\bar{a}\omega}}\right)$, using Lemma 2.2, we obtain
Case 1: If $0 \le s \le t \le \omega$,

$$\begin{aligned}\|\mathcal{T}\mathcal{N}\| \le&\|\mathcal{T}\|\|\mathcal{N}\| \\ \le&\int_0^\omega \frac{e^{-\int_s^t a(t)dt}}{1-e^{-\int_0^\omega a(t)dt}}dt\frac{\|a\|c}{1-c} \\ \le&\int_0^\omega \frac{1}{1-e^{-\int_0^\omega a(t)dt}}dt\frac{\|a\|c}{1-c} \\ \le&\frac{\omega}{1-e^{-\bar{a}\omega}}\frac{\|a\|c}{1-c} \\ <&1.\end{aligned} \tag{3.6}$$

Case 2: If $0 \le t \le s \le \omega$, similarly, we obtain the same result that $\|\mathcal{T}\mathcal{N}\| < 1$.
Hence, $I - \mathcal{T}\mathcal{N}$ is an invertible linear operator and

$$v(t) = \lambda(I - \mathcal{T}\mathcal{N})^{-1}(\mathcal{T}h)(t).$$

Now we define $\mathcal{P} : C_\omega \to C_\omega$ by

$$(\mathcal{P}h)(t) = (I - \mathcal{T}\mathcal{N})^{-1}(\mathcal{T}h)(t).$$

Since $\|\mathcal{T}\mathcal{N}\| < 1$, applying Neumann expansion of $\mathcal{P}$, we have

$$\begin{aligned}\mathcal{P} =&(I - \mathcal{T}\mathcal{N})^{-1}\mathcal{T} \\ =&(I + \mathcal{T}\mathcal{N} + (\mathcal{T}\mathcal{N})^2 + (\mathcal{T}\mathcal{N})^3 + \cdots)\mathcal{T} \\ =&\mathcal{T} + \mathcal{T}\mathcal{N}\mathcal{T} + (\mathcal{T}\mathcal{N})^2\mathcal{T} + (\mathcal{T}\mathcal{N})^3\mathcal{T} + \cdots \\ =&(I + (\mathcal{T}\mathcal{N})^2 + (\mathcal{T}\mathcal{N})^4 + \cdots)(I + \mathcal{T}\mathcal{N})\mathcal{T}.\end{aligned}$$

From inequality (3.6), we obtain that

$$\begin{aligned}(\mathcal{P}h)(t) =&(I - \mathcal{T}\mathcal{N})^{-1}(\mathcal{T}h)(t) \\ \le&\frac{\|\mathcal{T}h\|}{I - \|\mathcal{T}\mathcal{N}\|} \\ \le&\frac{(1-e^{-\bar{a}\omega})(1-c)}{1-e^{-\bar{a}\omega} + c(e^{-\bar{a}\omega} - 1 - \|a\|\omega)}\|\mathcal{T}h\| \\ :=&C\|\mathcal{T}h\|,\end{aligned} \tag{3.7}$$

for all $h(t) \in C_\omega^+$. Define operators $\mathcal{A}$, $\mathcal{B} : C_\omega \to C_\omega$ by

$$\begin{aligned}(\mathcal{A}u)(t) :&= c\int_{-\infty}^0 P(\sigma)u(t+\sigma)d\sigma, \\ (\mathcal{B}u)(t) :&= \mathcal{P}\left(b(t)\int_{-\infty}^0 P(\sigma)f(t, u(t+\sigma))d\sigma\right) = (\mathcal{P}h)(t).\end{aligned}$$

According to the above analysis, the existence of a positive ω-periodic solution of Eq (3.1) is just a fixed point of the following operator equation

$$u = \lambda\mathcal{A}\left(\frac{u}{\lambda}\right) + \lambda\mathcal{B}u \tag{3.8}$$

in C_ω. Similarly, the existence of a positive ω-periodic solution of Eq (1.1) is just a fixed point of the following operator equation

$$u = \mathcal{A}u + \mathcal{B}u \tag{3.9}$$

in C_ω.

Next, we use a fixed point theorem of Leray-Schauder type, see Lemma 2.1, to prove the existence of fixed point of Eq (3.9). Define

$$B[0,r] := \{u \in C_\omega : \ 0 \le u \le r, \ \text{for } t \in \mathbb{R}\},$$

where r is defined in Theorem 3.1. Obviously, $B[0,r]$ is a bounded closed convex set in C_ω. Then, we obtain at

$$\begin{aligned}
(\mathcal{A}u)(t+\omega) =& c\int_{-\infty}^{0} P(\sigma)u(t+\omega+\sigma)d\sigma \\
=& c\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma = (\mathcal{A}u)(t), \\
(\mathcal{B}u)(t+\omega) =& \mathcal{P}\left(b(t+\omega)\int_{-\infty}^{0} P(\sigma)f(t+\omega, u(t+\omega+\sigma))d\sigma\right) \\
=& \mathcal{P}\left(b(t)\int_{-\infty}^{0} P(\sigma)f(t, u(t+\sigma))d\sigma\right) = (\mathcal{B}u)(t),
\end{aligned}$$

for any $u \in B[0,r]$, and $t \in \mathbb{R}$. Obviously, $(\mathcal{A}u)(t)$ and $(\mathcal{B}u)(t)$ are ω-periodic. Moreover, we obtain

$$\begin{aligned}
|(\mathcal{A}u_1)(t) - (\mathcal{A}u_2)(t)| =& \left|c\int_{-\infty}^{0} P(\sigma)u_1(t+\sigma)d\sigma - c\int_{-\infty}^{0} P(\sigma)u_2(t+\sigma)d\sigma\right| \\
\le& c\int_{-\infty}^{0} |P(\sigma)||u_1(t+\sigma) - u_2(t+\sigma)|d\sigma \\
\le& c\int_{-\infty}^{0} P(\sigma)||u_1 - u_2||d\sigma \\
\le& c||u_1 - u_2||\int_{-\infty}^{0} P(\sigma)d\sigma \\
\le& c||u_1 - u_2||,
\end{aligned} \tag{3.10}$$

for any u_1, $u_2 \in B[0,r]$. Since $c \in \left(0, \frac{1-e^{-\bar{a}\omega}}{1+||a||\omega - e^{-\bar{a}\omega}}\right)$, we know that $\mathcal{A}$ is contractive. Moreover, it is easy to obtain that $\mathcal{B}$ is completely continuous (for details, please see [10, Theorem 3.1]).

On the other hand, we claim that any fixed point u of Eq (3.8) for any $\lambda \in (0,1)$ must satisfy $||u|| \ne r$. Through the reverse proving, we assume that the above claim does not holds. Then, there exists a fixed point u of Eq (3.8) for some $\lambda \in (0,1)$ such that $\|u\| = r$. From Eq (3.7), conditions (H_1) and (H_2), we obtain

$$\begin{aligned}
u(t) =&\lambda(\mathcal{B}u)(t) + \lambda\left(\mathcal{A}\left(\frac{u}{\lambda}\right)\right)(t)\\
=&\lambda\mathcal{P}\left(b(t)\int_{-\infty}^{0} P(\sigma)f(t,u(t+\sigma))d\sigma\right) + c\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma\\
\le& C\left\|\int_{0}^{\omega} G(t,s)b(s)\int_{-\infty}^{0} P(\sigma)f(s,u(s+\sigma)d\sigma ds\right\| + c\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma\\
\le& C\max_{t\in[0,\omega]}\int_{0}^{\omega} G(t,s)b(s)\int_{-\infty}^{0} P(\sigma)f(s,u(s+\sigma)d\sigma ds + c\int_{-\infty}^{0} rP(\sigma)d\sigma\\
\le& C\max_{t\in[0,\omega]}\int_{0}^{\omega} G(t,s)b(s)\int_{-\infty}^{0} P(\sigma)k(s)q(u)d\sigma ds + cr\int_{-\infty}^{0} P(\sigma)d\sigma\\
\le& C\max_{t\in[0,\omega]}\int_{0}^{\omega} G(t,s)b(s)k(s)q(r)ds + cr\\
\le& CK^*\|b\|q(r) + cr\\
<&r.
\end{aligned}$$

Thus, $r = \|u\| < r$, this is a contradiction. Using Lemma 2.1, we obtain that $u = \mathcal{A}u + \mathcal{B}u$ has a fixed point u in $B[0,r]$. Therefore, Equation (1.1) has at least one positive ω-periodic solution u with $u \in [0,r]$. □

In the following, applying Theorem 3.1, we consider the existence of positive ω-periodic solutions of the Hematopoiesis model (1.2), the Nicholson's blowflies model (1.3) and the model of blood cell production (1.4).

Corollary 3.1. Assume $c \in \left(0, \frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}\right)$ holds, then model (1.2) has at least one positive ω-periodic solution.

Proof. We apply Theorem 3.1 in which we set. Let us set

$$k(t) = 1, \quad q(u) = e^{\|\beta\|r}.$$

Then condition (H_1) is satisfied and the existence condition (H_2) becomes

$$\frac{r}{e^{\|\beta\|r}} > \frac{\|b\|\omega}{1 - e^{-\bar{a}\omega} + c(e^{-\bar{a}\omega} - 1 - \|a\|\omega)}. \tag{3.11}$$

We can choose r appropriately large such that (3.11) holds. □

Corollary 3.2. Assume $c \in \left(0, \frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}\right)$ holds. Furthermore, the following inequality holds:

$$\frac{1 - e^{-\bar{a}\omega} + c(e^{-\bar{a}\omega} - 1 - \|a\|\omega)}{\|b\|\omega} > 1. \tag{3.12}$$

Then, model (1.3) has at least one positive ω-periodic solution.

Proof. We apply Theorem 3.1 in which we set. Let us set

$$k(t) = 1, \quad q(u) = ue^{\|\beta\| r}.$$

Then condition (H_1) is satisfied and the existence condition (H_2) becomes

$$r < \frac{\ln \frac{1-e^{-\bar{a}\omega}+c(e^{-\bar{a}\omega}-1-\|a\|\omega)}{\|b\|\omega}}{\|\beta\|}. \tag{3.13}$$

From (3.12), we know $\ln \frac{1-e^{-\bar{a}\omega}+c(e^{-\bar{a}\omega}-1-\|a\|\omega)}{\|b\|\omega} > 0$, we can take r appropriately small such that (3.13) holds. □

Corollary 3.3. Assume $c \in \left(0, \frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}\right)$ and (3.12) hold. Then, model (1.4) has at least one positive ω-periodic solution.

Proof. We apply Theorem 3.1 in which we set. Let us set

$$k(t) = 1, \quad q(u) = u.$$

Then condition (H_1) is satisfied and the existence condition (H_2) becomes

$$\frac{1 - e^{-\bar{a}\omega} + c(e^{-\bar{a}\omega} - 1 - \|a\|\omega)}{\|b\|\omega} > 1. \tag{3.14}$$

□

3.2. *Equation* (1.1) *in the case that* $c \in \left(-\frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}, 0\right]$

Theorem 3.2. Suppose $c \in \left(-\frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}, 0\right]$ and (H_1) hold. Furthermore, assume that the following condition is satisfied:

(H_3) There exists a constant $r > 0$ such that

$$K^* < \frac{r[1 - e^{-\bar{a}\omega} + |c|(e^{-\bar{a}\omega} - 1 - \|a\|\omega)]}{(1 - e^{-\bar{a}\omega})(1 - |c|)\|b\|q(r)}.$$

Then Eq (1.1) has at least one positive ω-periodic solution with $u \in [0, r]$.

Proof. We follow the same notations and use a similar method as in the proof of Theorem 3.1. For Eq (3.8)

$$u = \lambda\mathcal{A}\left(\frac{u}{\lambda}\right) + \lambda\mathcal{B}u,$$

we claim that any fixed point u of Eq (3.8) for any $\lambda \in (0, 1)$ must satisfy $\|u\| \neq r$. Through the reverse proving, we assume that the above claim does not holds. Then, there exists a fixed point u of Eq (3.8) for some $\lambda \in (0, 1)$ such that $\|u\| = r$. From Eq (3.7), conditions (H_1) and (H_3), we get

$$\begin{aligned} u(t) =& \lambda(\mathcal{B}u)(t) + \lambda\left(\mathcal{A}\left(\frac{u}{\lambda}\right)\right)(t) \\ =& \lambda\mathcal{P}\left(b(t)\int_{-\infty}^{0} P(\sigma)f(t, u(t+\sigma))d\sigma\right) + \lambda c\int_{-\infty}^{0} P(\sigma)\frac{1}{\lambda}u(t+\sigma)d\sigma \end{aligned}$$

$$
\begin{aligned}
&\leq \widetilde{C} \max_{t\in[0,\omega]} \int_0^{\omega} G(t,s)b(s) \int_{-\infty}^{0} P(\sigma) f(s, u(s+\sigma)) d\sigma ds \\
&\leq \widetilde{C} \max_{t\in[0,\omega]} \int_0^{\omega} G(t,s)b(s) \int_{-\infty}^{0} P(\sigma) k(s) q(u) d\sigma ds \\
&\leq \widetilde{C} \int_0^{\omega} G(t,s)b(s)k(s)q(r) ds \\
&\leq \widetilde{C} K^{*} \|b\| q(r) \\
&< r.
\end{aligned}
$$

where

$$
\widetilde{C} := \frac{(1-e^{-\bar{a}\omega})(1-|c|)}{1-e^{-\bar{a}\omega}+|c|(e^{-\bar{a}\omega}-1-\|a\|\omega)}.
$$

Thus, $r = \|u\| < r$, this is a contradiction. Therefore, using Lemma 2.1, we obtain that $u = \mathcal{A}u + \mathcal{B}u$ has a fixed point u in $B[0, r]$. Hence, Equation (1.1) has at least one positive ω-periodic solution u with $u \in [0, r]$. □

By Theorem 3.2 and Corollary 3.1, Corollary 3.2, Corollary 3.3, we get the following conclusions.

Corollary 3.4. Assume $c \in \left(-\frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}, 0\right]$ holds, then model (1.2) has at least one positive ω-periodic solution.

Corollary 3.5. Assume $c \in \left(-\frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}, 0\right]$ holds. Furthermore, The following inequality holds:

$$
\frac{1-e^{-\bar{a}\omega}+|c|(e^{-\bar{a}\omega}-1-\|a\|\omega)}{\|b\|\omega(1-|c|)} > 1. \tag{3.15}
$$

Then, model (1.3) has at least one positive ω-periodic solution.

Corollary 3.6. Assume $c \in \left(-\frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}, 0\right]$ and (3.15) hold. Then, model (1.4) has at least one positive ω-periodic solution.

Remark 3.1. If $|c| > 1$, from (3.10), we do not obtain that $\mathcal{A}$ is contractive. Therefore, the above method does not apply to the case that $|c| > 1$. Next, we use another way to get over this problem.

4. Equation (1.1) with large constant c

In this section, we consider the existence of a positive ω-periodic solution of Eq (1.1). Moreover, we suppose the absolute value of constant c is larger than 1. We divide the discussion into the following two cases $c \in (1, +\infty)$ and $c \in (-\infty, -1)$.

4.1. *Equation* (1.1) *in the case that* $c \in (1, +\infty)$

Consider Eq (1.1), it can be transformed into

$$
\begin{aligned}
&-c\left(\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma - \frac{1}{c}u(t)\right)' - ca(t)\left(\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma - \frac{1}{c}u(t)\right) \\
&= b(t)\int_{-\infty}^{0} P(\sigma) f(t, u(t+\sigma)) d\sigma - ca(t)\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma.
\end{aligned} \tag{4.1}
$$

Define

$$F(t,u) = a(t)\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma - \frac{b(t)}{c}\int_{-\infty}^{0} P(\sigma)f(t,u(t+\sigma))d\sigma,$$

then Eq (1.1) can be written as

$$\left(\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma - \frac{1}{c}u(t)\right)' + a(t)\left(\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma - \frac{1}{c}u(t)\right) = F(t,u). \tag{4.2}$$

Theorem 4.1. Suppose $c \in (1,+\infty)$ holds. Furthermore, assume that there exists a constant $r > 0$ such that

(H_4) There exist continuous, non-negative functions $q(u)$ and $k(t)$ such that

$$0 \le F(t,u) \le k(t)q(u), \text{ for all } (t,u) \in [0,\omega] \times [0,r],$$

where $h(u)$ is non-decreasing in $[0,r]$.

(H_5) The following condition holds

$$K^* < \frac{(c-1)r}{cq(r)},$$

where K^* is defined by Theorem (3.1).
Then Eq (1.1) has at least one positive ω-periodic solution u with $u(t) \in [0,r]$.

Proof. Let us set

$$\tilde{v}(t) = \int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma - \frac{1}{c}u(t),$$

then Eq (4.2) can be written as the following form

$$\tilde{v}'(t) + a(t)\tilde{v}(t) = F(t,u).$$

Next we study the following equation

$$\tilde{v}'(t) + a(t)\tilde{v}(t) = \lambda F(t,u), \quad \lambda \in (0,1).$$

Then we obtain

$$\begin{aligned} u(t) =& u(t)\int_{-\infty}^{0} P(\sigma)d\sigma = \int_{-\infty}^{0} P(\sigma)u(t)d\sigma = \tilde{v}(t-\sigma) + \frac{1}{c}u(t-\sigma) \\ =& \lambda\int_{0}^{\omega} G(t-\sigma,s)F(s-\sigma,u(s))ds + \lambda\frac{1}{\lambda}\frac{1}{c}u(t-\sigma). \end{aligned} \tag{4.3}$$

Define operators $\tilde{\mathscr{A}}, \tilde{\mathscr{B}} : C_\omega \to C_\omega$ by

$$(\tilde{\mathscr{A}}u)(t) = \frac{1}{c}u(t-\sigma), \quad (\tilde{\mathscr{B}}u)(t) = \int_{0}^{\omega} G(t-\sigma,s)F(s-\sigma,u(s))ds.$$

According to the above analysis, the existence of a positive ω-periodic of Eq (4.3) is equivalent to the existence of solution for the operator equation

$$u = \lambda\tilde{\mathscr{A}}\left(\frac{u}{\lambda}\right) + \lambda\tilde{\mathscr{B}}u \tag{4.4}$$

in C_ω. Similarly, the existence of a positive ω-periodic of Eq (1.1) is equivalent to the existence of solution for the operator equation

$$u = \tilde{\mathcal{A}}u + \tilde{\mathcal{B}}u \tag{4.5}$$

in C_ω.

Next, we use a fixed point theorem of Leray-Schauder type, see Lemma 2.1, to prove the existence of fixed point of Eq (4.5). First, we have

$$\begin{aligned}(\tilde{\mathcal{A}}u)(t+\omega) &= \frac{1}{c}u(t+\omega-\sigma) = \frac{1}{c}u(t-\sigma) = (\tilde{A}u)(t),\\ (\tilde{\mathcal{B}}u)(t+\omega) &= \int_0^\omega G(t+\omega-\sigma, s)F(s-\sigma, u(s))ds\\ &= \int_0^\omega G(t-\sigma, s)F(s-\sigma, u(s))ds = (\tilde{B}u)(t),\end{aligned}$$

for any $u \in B[0, r]$, and $t \in \mathbb{R}$. Obviously, $(\tilde{\mathcal{A}}u)(t)$ and $(\tilde{\mathcal{B}}u)(t)$ are ω-periodic. Moreover, we get

$$\begin{aligned}|(\tilde{\mathcal{A}}u_1)(t) - (\tilde{\mathcal{A}}u_2)(t)| =& \left|\frac{1}{c}u_1(t-\sigma) - \frac{1}{c}u_2(t-\sigma)\right|\\ =& \left|\frac{1}{c}\right| |u_1(t-\sigma) - u_2(t-\sigma)|\\ \le& \frac{1}{c}\|u_1 - u_2\|\end{aligned}$$

for any $u \in B[0, r]$, and $t \in \mathbb{R}$. Thus, we know that $\tilde{\mathcal{A}}$ is contractive since $c \in (1, +\infty)$. By using the same notations and a similar method as in the proof of Theorem 3.1, we can get that $\tilde{B}$ is completely continuous.

Next, we claim that any fixed point u of Eq (4.4) for any $\lambda \in (0, 1)$ must satisfy $\|u\| \neq r$. Through the reverse proving, we assume that the above claim does not holds. Then, there exists a fixed point u of Eq (4.4) for some $\lambda \in (0, 1)$ such that $\|u\| = r$. From conditions (H_4) and (H_5), we have

$$\begin{aligned}u(t) =& \lambda\int_0^\omega G(t-\sigma, s)F(s-\sigma, u(s))ds + \lambda\frac{1}{\lambda c}u(t-\sigma)\\ \le& \lambda\int_0^\omega G(t-\sigma, s)k(s)q(u)ds + \frac{1}{c}u(t-\sigma)\\ \le& K^* q(r) + \frac{r}{c}\\ <& r.\end{aligned}$$

Thus, $r = \|u\| < r$, this is a contradiction. Using Lemma 2.1, we see that $u = \tilde{\mathcal{A}}u + \tilde{\mathcal{B}}u$ has a fixed point u in $B[0, r]$. Therefore, Equation (1.1) has at least one positive ω-periodic solution $u(t)$ with $u(t) \in [0, r]$. □

Corollary 4.1. Assume $c \in (1, +\infty)$ holds. Furthermore, The following inequality holds:

$$1 - e^{-\bar{a}\omega} - \omega\|a\| > 0 \quad \text{and} \quad c > \frac{1 - e^{-\bar{a}\omega}}{1 - e^{-\bar{a}\omega} - \omega\|a\|}.$$

Then, models (1.2), (1.3) and (1.4) respectively have at least one positive ω-periodic solution.

Proof. We apply Theorem 4.1 in which we set. Let us set

$$k(t) = \|a\|r, \quad q(u) = 1.$$

Then condition (H_4) is satisfied and the existence condition (H_5) becomes

$$c > \frac{1 - e^{-\bar{a}\omega}}{1 - e^{-\bar{a}\omega} - \omega\|a\|},$$

since $1 - e^{-\bar{a}\omega} - \omega\|a\| > 0$. Then, models (1.2), (1.3) and (1.4) respectively have at least one positive ω-periodic solution. □

Remark 4.1. If $c \in (-\frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}}, \frac{1-e^{-\bar{a}\omega}}{1+\|a\|\omega-e^{-\bar{a}\omega}})$, the method of proving the positive ω-periodic solutions of models (1.2), (1.3) and (1.4) in Corollaries 3.1-3.6 is more general than the above method (Corollary 4.1).

4.2. *Equation* (1.1) *in the case that* $c \in (-\infty, -1)$

Theorem 4.2. Suppose $c \in (-\infty, -1)$ and (H_4) hold. Furthermore, the following condition is satisfied:
(H_6) There exists a constant $r > 0$ such that

$$K^* < \frac{r}{q(r)}.$$

Then Eq (1.1) has at least one positive ω-periodic solution u with $u(t) \in [0, r]$.

Proof. We follow the same notations and use the same method in the proof of Theorem 4.1. We claim that any fixed point u of Eq (4.4) for any $\lambda \in (0, 1)$ must satisfy $\|u\| \neq r$. Through the reverse proving, we assume that the above claim does not holds. Then, there exists a fixed point u of Eq (4.4) for some $\lambda \in (0, 1)$ such that $\|u\| = r$. From conditions (H_4) and (H_6), we get

$$\begin{aligned}
u(t) =& \lambda \int_0^\omega G(t-\sigma, s)F(s-\sigma, u(s))ds + \lambda \frac{1}{\lambda}\frac{u(t-\sigma)}{c} \\
\leq& \lambda \int_0^\omega G(t-\sigma, s)k(s)q(u)ds - \left|\frac{1}{c}\right| u(t-\sigma) \\
\leq& K^* q(r) \\
<& r.
\end{aligned}$$

Thus, $r = \|u\| < r$, this is a contradiction. Using Lemma 2.1, we see that $u = \tilde{\mathcal{A}}u + \tilde{\mathcal{B}}u$ has a fixed point u in $B[0, r]$. Therefore, Equation (1.1) has at least one positive ω-periodic solution $u(t)$ with $u(t) \in [0, r]$. □

Remark 4.2. If $c \in (-\infty, -1)$, from the definition of $F(t, x)$ and models (1.2), (1.3) and (1.4), we can not find appropriate $k(t)$ and $h(u)$ such that conditions (H_4) and (H_6) are satisfied. Therefore, the above method does not apply to models (1.2), (1.3) and (1.4).

Finally, we present an example to illustrate our results.

Example 4.1. Consider the following neutral equation

$$\left(u(t) - \frac{1}{20}\int_{-\infty}^{0} P(\sigma)u(t+\frac{\pi}{5})d\sigma\right)' + (\cos 8t + 2)u(t) = (\sin 8t + 2)\int_{-\infty}^{0} P(\sigma)(\cos 8t + 2)8u^2(t+\frac{\pi}{5})d\sigma. \tag{4.6}$$

Comparing Eq (4.6) to Eq (1.1), we have $\omega = \frac{\pi}{4}, \sigma = \frac{\pi}{5}$, $\bar{a} = 2$, $c = \frac{1}{20} < \frac{1-\frac{1}{e^{\frac{\pi}{2}}}}{1+\frac{3\pi}{4}-\frac{1}{e^{\frac{\pi}{2}}}} \approx 0.9340$, $a(t) = \cos 8t + 2$, $b(t) = \sin 8t + 2$, $f(t,u) = (\cos 8t + 2)8u^2(t+\frac{\pi}{5})$. Let $k(t) = \cos 8t + 2$, $q(u) = 8u^2(t+\frac{\pi}{5})$, we get condition (H_1) is satisfied. Let $r = \frac{1}{40}$, we can verify that condition (H_2) is satisfied. Applying Theorem 3.1, Equation (4.6) has at least one $\frac{\pi}{4}$-periodic solution with $u \in [0, \frac{1}{40}]$.

5. Conclusion

By virtue of a fixed point theorem of Leray-Schauder type, we prove the existence of positive periodic solutions of the following first-order neutral differential equation with infinite distributed delay

$$\left(u(t) - c\int_{-\infty}^{0} P(\sigma)u(t+\sigma)d\sigma\right)' + a(t)u(t) = b(t)\int_{-\infty}^{0} P(\sigma)f(t,u(t+\sigma))d\sigma,$$

and we prove that Hematopoiesis model, Nicholson's blowflies model and the model of blood cell production have positive periodic solutions.

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Conflict of interest

The authors declare that they have no competing interests in this paper.

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