



Research article

# Properties of positive solutions for a fractional boundary value problem involving fractional derivative with respect to another function

Yitao Yang\* and Dehong Ji

College of Science, Tianjin University of Technology, Tianjin 300384, China

\* **Correspondence:** Email: yitaoyangqf@163.com.

**Abstract:** In this article, we develop the existence and uniqueness of positive solutions to a class of fractional boundary value problems involving fractional order derivative with respect to another function for any given parameter. The analysis is based upon the fixed point theorems of concave operators in partial ordering Banach spaces. For the sake of discussing the existence of solutions for the problem, we first construct Green function and study its properties. Furthermore, some properties of positive solutions to the boundary value problem are proved under the different parameters. Examples illustrating the results are also presented.

**Keywords:** fractional derivative with respect to another function; parameter; positive solution; existence; fixed point theorems of concave operator

**Mathematics Subject Classification:** 34A08, 34A12

## 1. Introduction

In this work, we are concerned with the following fractional boundary value problem

$$D_{0+}^{\alpha, \varphi} u(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \quad (1.1)$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i), \quad (1.2)$$

where  $1 < \alpha \leq 2$ ,  $D_{0+}^{\alpha, \varphi}$  is the fractional derivative operator of order  $\alpha$  with respect to a certain strictly increasing function  $\varphi \in C^2[0, 1]$  with  $\varphi'(x) > 0$  for all  $x \in [0, 1]$  and  $f, \varphi, \beta_i, \eta_i, \lambda$  satisfy:

(H<sub>1</sub>)  $f(t, u) : [0, 1] \times R^+ \rightarrow R^+$  is continuous and increasing in  $u$  for each  $t \in [0, 1]$ , the function  $\varphi : [0, 1] \rightarrow R$  is a strictly increasing function such that  $\varphi \in C^2[0, 1]$ , with  $\varphi'(x) > 0$  for all  $x \in [0, 1]$ ;

(H<sub>2</sub>)  $0 \leq \beta_i < 1$ ,  $0 < \eta_i < 1$  ( $i = 1, 2, \dots, m-2$ ) satisfy  $0 \leq \sum_{i=1}^{m-2} \beta_i < 1$ ,  $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < 1$ ,  $\lambda > 0$  is a parameter.

The fractional calculus first began in 1965. After a long time, this subject was relevant only in pure mathematics, many mathematicians have studied these new fractional operators by presenting new definitions and studying their important properties, but in the last century, this subject showed its applications in the modeling of many phenomena in various fields of science and engineering, such as control theory, electric circuits, viscoelasticity, mechanics, physics, neural networks [1–12].

In all those definitions there is a special kind of a kernel dependency. Therefore, in order to analyze fractional differential equations in a generic way, a fractional derivative with respect to another function called  $\varphi$ -Riemann-Liouville and  $\varphi$ -Caputo derivative was proposed [15] and for particular choices of  $\varphi$ , we obtain some well known fractional derivative. For example, if we choose  $\varphi(t) = t$ , the  $\varphi$ -Riemann-Liouville and  $\varphi$ -Caputo fractional derivative are reduced to the Riemann-Liouville and Caputo derivative respectively in traditional sense. Alternatively, if we choose  $\varphi(t) = \log t$ , where  $\log(\cdot) = \log_e(\cdot)$ , the  $\varphi$ -Riemann-Liouville and  $\varphi$ -Caputo fractional derivative are reduced to the Hadamard and Hadamard-Caputo derivative respectively [13, 14]. Therefore, problem (1.1), (1.2) generates many types and also mixed types of fractional differential equations with boundary conditions.

Let us mention some motivations for studying problem (1.1), (1.2). In the limit case  $\lambda = 1$  and  $\beta_i = \beta, \eta_i = \eta, i = 1, 2, \dots$ , (1.1), (1.2) reduces to

$$D_{0+}^{\alpha, \varphi} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad (1.3)$$

$$u(0) = 0, \quad u(1) = \beta u(\eta). \quad (1.4)$$

In [22], the authors studied the existence of positive solutions for the problem (1.3), (1.4). In [23], using the fixed point theorems of Banach and Schaefer, the authors studied the existence and uniqueness results of the boundary value problems for a fractional differential equation involving Caputo operator with respect to the new function  $\psi$  given by the form

$${}^C D_{a+}^{\alpha, \psi} y(t) = f(t, y(t)), \quad t \in [a, b],$$

$$y_{\psi}^{[k]}(a) = y_a^k, \quad k = 0, 1, \dots, n-2; \quad y_{\psi}^{[n-1]}(b) = y_b,$$

where  ${}^C D_{a+}^{\alpha, \psi}$  is the  $\psi$ -Caputo fractional derivative of order  $n-1 < \alpha \leq n$  ( $n = [\alpha] + 1$ ).

Almeida et al. [24] studied the initial value problem of a fractional differential equation including  $\varphi$ -Caputo fractional derivative

$${}^C D_{a+}^{\alpha, \varphi} y(t) = f(t, y(t)), \quad t \in [a, b],$$

$$y(a) = y_a, \quad y_{\psi}^{[k]}(a) = y_a^k, \quad k = 1, 2, \dots, n-1$$

where  ${}^C D_{a+}^{\alpha, \varphi}$  is the  $\varphi$ -Caputo fractional derivative of order  $n-1 < \alpha < n$ .

On the other hand, most of the existing literature does not consider the properties of the solution. Well, here's the problem: How can we get the properties of the solutions when they are known to exist. So far, few papers can be found in the literature both on the existence and properties of solutions for fractional boundary value problem.

Zhai et al. [16] studied the following nonlinear fractional four-point boundary value problem with a parameter:

$$D_{0+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1,$$

$$u'(0) - \mu_1 u(\xi) = 0, \quad u'(1) + \mu_2 u(\eta) = 0,$$

where  $D_{0+}^{\alpha}$  is the Riemann-Liouville fractional derivative. Unfortunately, the fractional derivative in this paper is just the Caputo derivative in the traditional sense.

Inspired by the work in [16] and many known results, in this paper, we develop the existence and uniqueness of positive solutions to the fractional boundary value problems (1.1), (1.2). Using the fixed point theorems of concave operators in partial ordering Banach spaces, we not only show the existence of positive solution for the problem (1.1), (1.2), but also present some properties of positive solutions to the boundary value problem dependent on the parameter. Some definitions, lemmas and basic results about  $\varphi$ -Riemann-Liouville derivative are presented in Section 2. The main results are given in section 3. Some illustrative examples are constructed in Section 4.

## 2. The preliminary lemmas

Let  $\varphi$  be a function of  $C^2[a, b]$  with  $\varphi'(t) > 0$  for all  $t \in [a, b]$ .

**Definition 2.1** [21] The  $\varphi$ -Riemann-Liouville fractional integral of order  $\alpha > 0$  of a function  $x : [a, b] \rightarrow R$  is given by

$$I_{a+}^{\alpha, \varphi} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{\alpha-1} x(s) ds.$$

**Definition 2.2** [21] The  $\varphi$ -Riemann-Liouville fractional derivative of order  $\alpha$  of a function  $x$  is defined as

$$D_{a+}^{\alpha, \varphi} x(t) := \left( \frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n I_{a+}^{n-\alpha, \varphi} x(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{n-\alpha-1} x(s) ds,$$

where  $n = [\alpha] + 1$ .

The  $\varphi$ -Caputo fractional derivative of order  $\alpha$  of a function  $x$  is defined as

$${}^C D_{a+}^{\alpha, \varphi} x(t) := D_{a+}^{\alpha, \varphi} \left[ x(t) - \sum_{k=0}^{n-1} \frac{x_{\varphi}^{[k]}(a)}{k!} (\varphi(t) - \varphi(a))^k \right], \quad n = [\alpha] + 1 \text{ for } \alpha \notin N, \quad n = \alpha \text{ for } \alpha \in N,$$

where  $x_{\varphi}^{[k]}(t) := \left( \frac{1}{\varphi'(t)} \frac{d}{dt} \right)^k x(t)$ .

**Theorem 2.1** [15] If  $x \in C^n[a, b]$ , then

$${}^C D_{a+}^{\alpha, \varphi} x(t) = I_{a+}^{n-\alpha, \varphi} \left( \frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n x(t).$$

**Theorem 2.2** [15] Let  $x : [a, b] \rightarrow R$ . Then

1. If  $x \in C[a, b]$ , then  ${}^C D_{a+}^{\alpha, \varphi} I_{a+}^{\alpha, \varphi} x(t) = x(t)$ ;
2. If  $x \in C^{n-1}[a, b]$ , then

$$I_{a+}^{\alpha, \varphi} {}^C D_{a+}^{\alpha, \varphi} x(t) = x(t) - \sum_{k=0}^{n-1} \frac{x_{\varphi}^{[k]}(a)}{k!} (\varphi(t) - \varphi(a))^k.$$

In the following, we give some basic definitions in ordered Banach spaces and two fixed point theorems of concave operators which will be used later, all the materials can be found in [17–20]. We denote  $\theta$  the zero element of  $E$ .

**Definition 2.3** Let  $X$  be a semi-ordered real Banach space. The nonempty convex closed subset  $P$  of  $X$  is called a cone in  $X$  if  $ax \in P$  for all  $x \in P$  and  $a \geq 0$  and  $x \in X$  and  $-x \in X$  imply  $x = \theta$ .

Let the real Banach space  $E$  be partially ordered by a cone  $P$  of  $E$ , that is  $x \leq y$  if and only if  $y - x \in P$ . If  $x \leq y$  and  $x \neq y$ , then we denote  $x < y$  or  $y > x$ .

Recall that cone  $P$  is said to be solid if the interior  $\mathring{P}$  is nonempty and we denote  $x \gg \theta$  if  $x \in \mathring{P}$ .  $P$  is normal if there exists a positive constant  $N$  such that  $\theta \leq x \leq y$  implies  $\|x\| \leq N\|y\|$ ,  $N$  is called the normal constant of  $P$ .

**Definition 2.4** Let  $D$  be a convex subset in  $E$ . An operator  $T : D \rightarrow E$  is a concave operator if

$$T(tx + (1 - t)y) \geq tTx + (1 - t)Ty$$

for all  $x, y \in D$  and  $t \in [0, 1]$ .

**Definition 2.5** Let  $P$  be a cone in real Banach space  $E$  and  $e \in P \setminus \{\theta\}$ . Set

$$E_e = \{x \in E : \text{there exists } \lambda > 0 \text{ such that } -\lambda e \leq x \leq \lambda e\}$$

and define

$$\|x\|_e = \inf\{\lambda > 0 : -\lambda e \leq x \leq \lambda e\}, \quad \forall x \in E_e.$$

It is easy to see that  $E_e$  becomes a normed linear space under the norm  $\|\cdot\|_e$ .  $\|x\|_e$  is called the  $e$ -norm of the element  $x \in E_e$ .

**Lemma 2.3** [17] Suppose  $P$  is a normal cone. Then the following results hold:

(i)  $E_e$  is a Banach space with  $e$ -norm, and there exists a constant  $m > 0$  such that  $\|x\| \leq m\|x\|_e$ ,  $\forall x \in E_e$ ;

(ii)  $P_e = E_e \cap P$  is a normal solid cone of  $E_e$  and  $\mathring{P}_e = \{x \in E_e : \text{there exists } \delta = \delta(x) > 0 \text{ such that } x \geq \delta e\}$ .

**Lemma 2.4** [19] Suppose  $P$  is a normal solid cone and operator  $T : P \rightarrow P$  is a concave operator. Assume that  $T\theta \gg \theta$ . Then the following results hold:

(i) there exists  $0 < \lambda^* \leq \infty$  such that the equation

$$u = \lambda Tu \tag{2.1}$$

has a unique solution  $u(\lambda)$  in  $P$  for  $0 \leq \lambda < \lambda^*$ , when  $\lambda \geq \lambda^*$ , the operator equation (2.1) has no solution in  $P$ ;

(ii) for any  $u_0 \in P$ , let  $u_0(\lambda) = u_0$ ,  $u_n(\lambda) = \lambda T u_{n-1}(\lambda)$ ,  $n = 1, 2, 3, \dots$ . Then for  $0 < \lambda < \lambda^*$ , we have  $u_n(\lambda) \rightarrow u(\lambda)$  as  $n \rightarrow \infty$ ;

(iii)  $u(\cdot) : [0, \lambda^*) \rightarrow P$  is continuous and strongly increasing ( $0 \leq \lambda_1 < \lambda_2 < \lambda^* \Rightarrow u(\lambda_1) \ll u(\lambda_2)$ ). Furthermore,  $u(t\lambda) \leq tu(\lambda)$  for  $0 \leq \lambda < \lambda^*$ ,  $0 \leq t \leq 1$ .

(iv)  $\lambda \rightarrow \lambda^* - 0$  implies  $\|u(\lambda)\| \rightarrow \infty$ ;

(v) if there exist  $v_0 \in P$  and  $\lambda_0 > 0$  such that  $\lambda_0 T v_0 \leq v_0$ , then  $\lambda^* > \lambda_0$ .

**Lemma 2.5** [19] If  $T : P \rightarrow P$  is concave, then  $T$  is increasing.

**Lemma 2.6** [20] Suppose  $P$  is a normal solid cone and operator  $T : \mathring{P} \rightarrow \mathring{P}$  is an increasing operator. We presume there exists a constant  $0 < r < 1$  such that

$$T(tu) \geq t^r Tu, \quad \forall u \in \mathring{P}, \quad 0 < t < 1.$$

Let  $u_\lambda$  be the unique solution in  $\mathring{P}$  of the equation  $Tu = \lambda u$  ( $\lambda > 0$ ). Then

- (i)  $u_\lambda$  is strongly decreasing ( $0 < \lambda_1 < \lambda_2 \Rightarrow u_{\lambda_1} >> u_{\lambda_2}$ );
- (ii)  $u_\lambda$  is continuous ( $\lambda \rightarrow \lambda_0$  ( $\lambda_0 > 0$ )  $\Rightarrow \|u_\lambda - u_{\lambda_0}\| \rightarrow 0$ );
- (iii)  $\lim_{\lambda \rightarrow +\infty} \|u_\lambda\| = 0$ ,  $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = +\infty$ .

### 3. Main results

Denote

$$\mu = (\varphi(1) - \varphi(0))^{\alpha-1} - \sum_{i=1}^{m-2} \beta_i (\varphi(\eta_i) - \varphi(0))^{\alpha-1}.$$

**Lemma 3.1** Assume that  $(H_1), (H_2)$  hold. Then the fractional boundary value problem (1.1), (1.2) is equivalent to the following integral equation

$$u(t) = \int_0^1 G(t, s) \varphi'(s) \lambda f(s, u(s)) ds, \quad (3.1)$$

where

$$G(t, s) = \begin{cases} \frac{\sum_{j=1}^{m-2} \beta_j (\varphi(\eta_j) - \varphi(0))^{\alpha-1} (\varphi(t) - \varphi(s))^{\alpha-1} - (\varphi(t) - \varphi(s))^{\alpha-1} (\varphi(1) - \varphi(0))^{\alpha-1}}{\mu \Gamma(\alpha)} \\ \quad + \frac{(\varphi(t) - \varphi(0))^{\alpha-1} (\varphi(1) - \varphi(s))^{\alpha-1} - \sum_{j=i}^{m-2} \beta_j (\varphi(\eta_j) - \varphi(s))^{\alpha-1} (\varphi(t) - \varphi(0))^{\alpha-1}}{\mu \Gamma(\alpha)} \\ 0 \leq t \leq 1, \quad \eta_{i-1} \leq s \leq \min\{\eta_i, t\}, \quad i = 1, 2, \dots, m-1; \\ \frac{(\varphi(t) - \varphi(0))^{\alpha-1} (\varphi(1) - \varphi(s))^{\alpha-1} - \sum_{j=i}^{m-2} \beta_j (\varphi(\eta_j) - \varphi(s))^{\alpha-1} (\varphi(t) - \varphi(0))^{\alpha-1}}{\mu \Gamma(\alpha)} \\ 0 \leq t \leq 1, \quad \max\{\eta_{i-1}, t\} \leq s \leq \eta_i, \quad i = 1, 2, \dots, m-1. \end{cases} \quad (3.2)$$

Here for the sake of convenience, we write  $\eta_0 = 0, \eta_{m-1} = 1$  and  $\sum_{i=m_1}^{m_2} f_i = 0$  for  $m_2 < m_1$ .

*Proof.* Use  $y(t)$  to replace  $f(t, u)$  in (1.1). Let

$$D_{0^+}^{\alpha, \varphi} u(t) + \lambda y(t) = 0.$$

Then from Theorem 2.2, we have

$$u(t) = c_1 (\varphi(t) - \varphi(0))^{\alpha-1} + c_2 (\varphi(t) - \varphi(0))^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_0^t \varphi'(s) (\varphi(t) - \varphi(s))^{\alpha-1} \lambda y(s) ds. \quad (3.3)$$

Note that  $u(0) = 0$  implies  $c_2 = 0$ . Therefore, we obtain

$$u(t) = c_1(\varphi(t) - \varphi(0))^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t \varphi'(s)(\varphi(t) - \varphi(s))^{\alpha-1} \lambda y(s) ds. \quad (3.4)$$

In particular,

$$u(1) = c_1(\varphi(1) - \varphi(0))^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^1 \varphi'(s)(\varphi(1) - \varphi(s))^{\alpha-1} \lambda y(s) ds$$

and

$$u(\eta_i) = c_1(\varphi(\eta_i) - \varphi(0))^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^{\eta_i} \varphi'(s)(\varphi(\eta_i) - \varphi(s))^{\alpha-1} \lambda y(s) ds.$$

Note that  $u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i)$ , and hence we obtain

$$c_1 = \frac{1}{\mu \Gamma(\alpha)} \int_0^1 \varphi'(s)(\varphi(1) - \varphi(s))^{\alpha-1} \lambda y(s) ds - \frac{\sum_{i=1}^{m-2} \beta_i}{\mu \Gamma(\alpha)} \int_0^{\eta_i} \varphi'(s)(\varphi(\eta_i) - \varphi(s))^{\alpha-1} \lambda y(s) ds.$$

As a result, from (3.4) we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t \varphi'(s)(\varphi(t) - \varphi(s))^{\alpha-1} \lambda y(s) ds \\ &\quad + \frac{(\varphi(t) - \varphi(0))^{\alpha-1}}{\mu \Gamma(\alpha)} \int_0^1 \varphi'(s)(\varphi(1) - \varphi(s))^{\alpha-1} \lambda y(s) ds \\ &\quad - \frac{\sum_{i=1}^{m-2} \beta_i (\varphi(t) - \varphi(0))^{\alpha-1}}{\mu \Gamma(\alpha)} \int_0^{\eta_i} \varphi'(s)(\varphi(\eta_i) - \varphi(s))^{\alpha-1} \lambda y(s) ds \\ &= \int_0^1 G(t, s) \varphi'(s) \lambda y(s) ds. \end{aligned} \quad (3.5)$$

This completes the proof.  $\square$

**Lemma 3.2** Assume that  $(H_1)$ ,  $(H_2)$  hold. Then the Green's function  $G(t, s)$  defined by (3.2) has the following properties:

- (i)  $G(t, s)$  is a continuous function on  $[0, 1] \times [0, 1]$ ;
- (ii)  $G(t, s) > 0$  for all  $(t, s) \in (0, 1)$ ;
- (iii)

$$G(t, s) \leq \frac{(\varphi(1) - \varphi(0))^{\alpha-1} (\varphi(1) - \varphi(s))^{\alpha-1}}{\mu \Gamma(\alpha)}, \quad \forall s \in (0, 1). \quad (3.6)$$

*Proof.* (i) From the expression of  $G(t, s)$ , it is easy to get that  $G(t, s)$  is continuous.

(ii) From  $(H_1)$ ,  $(H_2)$ , we can get  $\mu > 0$ .

For  $0 \leq t \leq 1$ ,  $\eta_{i-1} \leq s \leq \min\{\eta_i, t\}$ ,  $i = 1, 2, \dots, m-1$ ,

$$\begin{aligned} G(t, s) &= \frac{1}{\mu \Gamma(\alpha)} \left[ (\varphi(t) - \varphi(0))^{\alpha-1} (\varphi(1) - \varphi(s))^{\alpha-1} \right. \\ &\quad \left. - \sum_{j=i}^{m-2} \beta_j (\varphi(\eta_j) - \varphi(s))^{\alpha-1} (\varphi(t) - \varphi(0))^{\alpha-1} - \mu (\varphi(t) - \varphi(s))^{\alpha-1} \right] \\ &= \frac{(\varphi(t) - \varphi(0))^{\alpha-1}}{\mu \Gamma(\alpha)} \left[ (\varphi(1) - \varphi(s))^{\alpha-1} - \sum_{j=i}^{m-2} \beta_j (\varphi(\eta_j) - \varphi(s))^{\alpha-1} \right. \\ &\quad \left. - \mu \left( \frac{\varphi(t) - \varphi(s)}{\varphi(t) - \varphi(0)} \right)^{\alpha-1} \right]. \end{aligned}$$

Consider

$$e(t) = (\varphi(1) - \varphi(s))^{\alpha-1} - \sum_{j=i}^{m-2} \beta_j (\varphi(\eta_j) - \varphi(s))^{\alpha-1} - \mu \left( \frac{\varphi(t) - \varphi(s)}{\varphi(t) - \varphi(0)} \right)^{\alpha-1}.$$

Then we have

$$e'(t) = -\mu(\alpha-1) \left( \frac{\varphi(t) - \varphi(s)}{\varphi(t) - \varphi(0)} \right)^{\alpha-2} \frac{\varphi'(t)(\varphi(s) - \varphi(0))}{(\varphi(t) - \varphi(0))^2} \leq 0,$$

which implies that  $e(t)$  is decreasing for  $0 \leq t \leq 1$ ,  $\eta_{i-1} \leq s \leq \min\{\eta_i, t\}$ ,  $i = 1, 2, \dots, m-1$ .

Furthermore, we have

$$\begin{aligned} e(1) &= (\varphi(1) - \varphi(s))^{\alpha-1} - \sum_{j=i}^{m-2} \beta_j (\varphi(\eta_j) - \varphi(s))^{\alpha-1} - \mu \left( \frac{\varphi(1) - \varphi(s)}{\varphi(1) - \varphi(0)} \right)^{\alpha-1} \\ &= \sum_{j=1}^{m-2} \beta_j (\varphi(\eta_j) - \varphi(0))^{\alpha-1} \frac{(\varphi(1) - \varphi(s))^{\alpha-1}}{(\varphi(1) - \varphi(0))^{\alpha-1}} - \sum_{j=i}^{m-2} \beta_j (\varphi(\eta_j) - \varphi(s))^{\alpha-1} \\ &= \sum_{j=1}^{m-2} \beta_j (\varphi(\eta_j) - \varphi(0))^{\alpha-1} \frac{(\varphi(1) - \varphi(s))^{\alpha-1}}{(\varphi(1) - \varphi(0))^{\alpha-1}} - \sum_{j=i}^{m-2} \beta_j (\varphi(\eta_j) - \varphi(0))^{\alpha-1} \frac{(\varphi(\eta_j) - \varphi(s))^{\alpha-1}}{(\varphi(\eta_j) - \varphi(0))^{\alpha-1}}. \end{aligned}$$

As we know,

$$\frac{(\varphi(1) - \varphi(s))^{\alpha-1}}{(\varphi(1) - \varphi(0))^{\alpha-1}} > \frac{(\varphi(\eta_j) - \varphi(s))^{\alpha-1}}{(\varphi(\eta_j) - \varphi(0))^{\alpha-1}},$$

consequently, we have  $e(1) > 0$ .

As a result,  $G(t, s) > 0$  for  $0 \leq t \leq 1$ ,  $\eta_{i-1} \leq s \leq \min\{\eta_i, t\}$ ,  $i = 1, 2, \dots, m-1$ .

For  $0 \leq t \leq 1$ ,  $\max\{\eta_{i-1}, t\} \leq s \leq \eta_i$ ,  $i = 1, 2, \dots, m-1$ ,

$$\begin{aligned} G(t, s) &= \frac{1}{\mu\Gamma(\alpha)} \left[ (\varphi(t) - \varphi(0))^{\alpha-1} (\varphi(1) - \varphi(s))^{\alpha-1} - \sum_{j=i}^{m-2} \beta_j (\varphi(\eta_j) - \varphi(s))^{\alpha-1} (\varphi(t) - \varphi(0))^{\alpha-1} \right] \\ &= \frac{1}{\mu\Gamma(\alpha)} (\varphi(t) - \varphi(0))^{\alpha-1} \left[ (\varphi(1) - \varphi(s))^{\alpha-1} - \sum_{j=i}^{m-2} \beta_j (\varphi(\eta_j) - \varphi(s))^{\alpha-1} \right] > 0. \end{aligned}$$

Therefore,  $G(t, s) > 0$  for all  $t, s \in (0, 1)$ .

(iii) From  $(H_1)$ ,  $(H_2)$  and the expression of  $G(t, s)$ , we can easily obtain this property.  $\square$

Let  $E = C[0, 1]$  be a Banach space equipped with the norm

$$\|u\|_E = \max_{t \in [0, 1]} |u(t)|.$$

Define the cone  $K \subset E$  by

$$K = \{u \in E : u(t) \geq 0, 0 \leq t \leq 1\},$$

then  $K$  is a normal solid cone of  $E$ . Let

$$g(t) = \int_0^1 G(t, s) \varphi'(s) ds, \quad 0 \leq t \leq 1. \quad (3.7)$$

From  $(H_1)$  and Lemma 3.2, we have  $g(t) \in E$  and  $g(t) \in K \setminus \{\theta\}$ .

Let  $X = E_e = \{u \in E : \exists \tau > 0, \text{ such that } -\tau g(t) \leq u(t) \leq \tau g(t), 0 \leq t \leq 1\}$  be a Banach space equipped with the norm  $\|u\|_X = \inf\{\tau > 0 : -\tau g(t) \leq u(t) \leq \tau g(t), 0 \leq t \leq 1\}$ .

Define  $\tilde{K} = X \cap K$ , then  $\tilde{K}$  is a normal solid cone of  $X$  and

$\tilde{\tilde{K}} = \{u \in X : \text{there exists } \delta > 0 \text{ such that } u(t) \geq \delta g(t), 0 \leq t \leq 1, \forall u \in X\}$ .

Moreover, there exists a constant  $m > 0$  such that  $\|u\|_E \leq m\|u\|_X$ .

**Theorem 3.3** Assume that  $(H_1), (H_2)$  hold. Furthermore,  $f(t, \cdot)$  is concave and there exist constants  $\chi > 0, \beta > 0$  such that

$$f(t, 0) \geq \chi, \quad f(t, 1) \leq \beta, \quad 0 \leq t \leq 1. \quad (3.8)$$

Then the following results hold:

(i) there exists  $0 < \lambda^* \leq \infty$  such that the fractional boundary value problem (1.1), (1.2) has a unique positive solution  $u_\lambda$  in  $\tilde{K}$  for  $0 < \lambda < \lambda^*$ , the fractional boundary value problem (1.1), (1.2) has no positive solution in  $\tilde{K}$  for  $\lambda \geq \lambda^*$ ;

(ii) for any  $u_0 \in \tilde{K}$ , let

$$u_n(t) = \lambda \int_0^1 G(t, s) \varphi'(s) f(s, u_{n-1}(s)) ds, \quad n = 1, 2, 3, \dots$$

Then for  $0 < \lambda < \lambda^*$ , we have

$$\max_{0 \leq t \leq 1} |u_n(t) - u_\lambda(t)| \rightarrow 0 \text{ as } n \rightarrow \infty;$$

(iii) if  $0 < \lambda < \lambda^*$ ,  $\max_{0 \leq t \leq 1} |u_\lambda(t) - u_{\lambda_0}(t)| \rightarrow 0$  as  $\lambda \rightarrow \lambda_0$  and if  $0 < \lambda_1 < \lambda_2 < \lambda^*$ , then  $u_{\lambda_1}(t) \leq u_{\lambda_2}(t)$ ,  $0 \leq t \leq 1$  and  $u_{\lambda_1}(t) \not\equiv u_{\lambda_2}(t)$ ;

(iv)  $u_{v\lambda}(t) \leq v u_\lambda(t)$  for  $0 < \lambda < \lambda^*$ ,  $0 \leq v \leq 1$ ,  $0 \leq t \leq 1$ .

(v) if  $\lim_{u \rightarrow \infty} \frac{f(t, u)}{u} = 0$  uniformly on  $[0, 1]$ , then  $\lambda^* = \infty$ .

*Proof.* Define an operator  $T$  by

$$(Tu)(t) = \int_0^1 G(t, s) \varphi'(s) f(s, u(s)) ds. \quad (3.9)$$

Thus we know that  $u$  is a solution of (1.1), (1.2) if and only if  $u = \lambda Tu$ . For  $u \in K$ , in view of  $(H_1), (H_2)$  and Lemma 3.2, we can get  $Tu \in K$ . Owing to the concavity of  $f(t, \cdot)$ , for  $u > 1$ , we have

$$f(t, 1) = f\left(t, \frac{1}{u}u + \left(1 - \frac{1}{u}\right)0\right) \geq \frac{1}{u}f(t, u) + \left(1 - \frac{1}{u}\right)f(t, 0),$$

from (3.8), we obtain

$$f(t, u) \leq u f(t, 1) - (u - 1)f(t, 0) \leq u f(t, 1) \leq u \beta.$$

So, for  $u \in K$ , we have

$$0 \leq \int_0^1 G(t, s) \varphi'(s) f(s, u(s)) ds \leq \int_0^1 G(t, s) \varphi'(s) f(s, \|u\|_E) ds \leq N_1 g(t), \quad 0 \leq t \leq 1,$$

where  $N_1 = \max_{0 \leq t \leq 1} f(t, \|u\|_E) \leq \max_{0 \leq t \leq 1} f(t, \|u\|_E + 1) \leq \beta(1 + \|u\|_E)$ , so, we have  $0 \leq (Tu)(t) \leq N_1 g(t)$ , which implies  $Tu \in X$ . Thus,  $Tu \in \tilde{K}$ , this means that  $T : \tilde{K} \rightarrow \tilde{K}$ . In view of the concavity of  $f(t, \cdot)$ , we have the operator  $T$  is concave.



From (3.9), we have  $(T\theta)(t) = \int_0^1 G(t, s)\varphi'(s)f(s, 0)ds \geq \chi g(t)$ ,  $0 \leq t \leq 1$ . So,  $T\theta \in \overset{\circ}{K}$ , this means that  $T\theta \gg \theta$ . Then all the conditions of Lemma 2.4 are satisfied. Thus from Lemma 2.4, there exists  $0 < \lambda^* \leq \infty$  such that the equation  $u = \lambda Tu$  has a unique positive solution  $u_\lambda$  in  $\tilde{K}$  for  $0 < \lambda < \lambda^*$ , the equation  $u = \lambda Tu$  has no positive solution in  $\tilde{K}$  for  $\lambda \geq \lambda^*$ ;

For any  $u_0 \in \tilde{K}$ , let  $u_n = \lambda Tu_{n-1}$ ,  $n = 1, 2, 3, \dots$ , then we have  $u_n \rightarrow u_\lambda$  as  $n \rightarrow \infty$  for  $0 < \lambda < \lambda^*$ ,  $u_\lambda : (0, \lambda^*) \rightarrow \tilde{K}$  is continuous and strongly increasing. Furthermore,  $u_{t\lambda} \leq tu_\lambda$  for  $0 \leq \lambda < \lambda^*$ ,  $0 \leq t \leq 1$ ;  $\lambda \rightarrow \lambda^* - 0$  implies  $\|u_\lambda\|_X \rightarrow \infty$ ; if there exist  $v_0 \in \tilde{K}$  and  $\lambda_0 > 0$  such that  $\lambda_0 Tv_0 \leq v_0$ , then  $\lambda^* > \lambda_0$ . From this results, we can get

(i) the fractional boundary value problem (1.1), (1.2) has a unique positive solution  $u_\lambda$  in  $\tilde{K}$  for  $0 < \lambda < \lambda^*$ , the fractional boundary value problem (1.1), (1.2) has no positive solution in  $\tilde{K}$  for  $\lambda \geq \lambda^*$ ;

(ii) for any  $u_0 \in \tilde{K}$ , let

$$u_n(t) = \lambda \int_0^1 G(t, s)\varphi'(s)f(s, u_{n-1}(s))ds, \quad n = 1, 2, 3, \dots$$

Then for  $0 < \lambda < \lambda^*$ , we have

$$\max_{0 \leq t \leq 1} |u_n(t) - u_\lambda(t)| \rightarrow 0 \text{ as } n \rightarrow \infty;$$

(iii)  $\max_{0 \leq t \leq 1} |u_\lambda(t) - u_{\lambda_0}(t)| \rightarrow 0$  as  $\lambda \rightarrow \lambda_0$ , where  $0 < \lambda_0 < \lambda^*$  and if  $0 < \lambda_1 < \lambda_2 < \lambda^*$ , then there exists  $\delta > 0$  such that  $u_{\lambda_2}(t) - u_{\lambda_1}(t) \geq \delta g(t)$ ,  $0 \leq t \leq 1$ , this means that  $u_{\lambda_1}(t) \leq u_{\lambda_2}(t)$ ,  $0 \leq t \leq 1$  and  $u_{\lambda_1}(t) \not\equiv u_{\lambda_2}(t)$ ;

(iv) for  $0 < \lambda < \lambda^*$ ,  $u_{\delta\lambda}(t) \leq \delta u_\lambda(t)$ ,  $0 \leq \delta \leq 1$ ,  $0 \leq t \leq 1$ .

(v) Denote  $\eta = \|g\|_E$ , for any given  $\lambda > 0$ , by  $\lim_{u \rightarrow \infty} \frac{f(t, u)}{u} = 0$ , there exists a large  $T > 0$ , such that  $f(t, T) \leq (\eta\lambda)^{-1}T$ ,  $0 \leq t \leq 1$ . Let  $v_0(t) = \eta^{-1}Tg(t)$ , from Lemma 2.5, we have

$$\begin{aligned} \lambda(Tv_0)(t) &= \lambda \int_0^1 G(t, s)\varphi'(s)f(s, \eta^{-1}Tg(s))ds \leq \lambda \int_0^1 G(t, s)\varphi'(s)f(s, T)ds \\ &\leq (\eta\lambda)^{-1}T\lambda g(t) = \eta^{-1}Tg(t) = v_0(t), \end{aligned}$$

$$v_0(t) - \lambda(Tv_0)(t) = \int_0^1 G(t, s)\varphi'(s)(\eta^{-1}T - \lambda f(s, \eta^{-1}Tg(s)))ds \leq M_2g(t), \quad 0 \leq t \leq 1,$$

where  $M_2 = \sup_{0 \leq t \leq 1} [\eta^{-1}T + \lambda f(s, T)] \leq \eta^{-1}T + \lambda\beta(T + 1)$ , this means that  $v_0(t) - \lambda Tv_0 \in \tilde{K}$ . That is  $\lambda Tv_0 \leq v_0$ . By Lemma 2.4 (v), we have  $\lambda^* > \lambda$ , because of the arbitrariness of  $\lambda$ , we can get  $\lambda^* = \infty$ .  $\square$

**Theorem 3.4** Assume that  $(H_1)$ ,  $(H_2)$  hold. Furthermore, the following conditions hold:

(i) there exists a number  $0 < r < 1$  such that

$$f(t, \omega u) \geq \omega^r f(t, u), \quad 0 \leq t \leq 1, \quad u \geq 0, \quad 0 < \omega < 1;$$

(ii) there exists a number  $\xi > 0$  such that  $f(t, 1) \leq \xi$ ;

(iii)  $\min_{0 \leq t \leq 1} f(t, g(t)) > 0$ , where  $g(t)$  is given by (3.7).

Then the following results hold:

(i) for any given  $\lambda > 0$ , the fractional boundary value problem (1.1), (1.2) has a unique positive solution  $u_\lambda$  in  $\tilde{K}$ ;

(ii) if  $0 < \lambda_1 < \lambda_2$ , then  $u_{\lambda_1}(t) \leq u_{\lambda_2}(t)$ ,  $0 \leq t \leq 1$  and  $u_{\lambda_1}(t) \not\equiv u_{\lambda_2}(t)$ ;

(iii)  $\max_{0 \leq t \leq 1} |u_\lambda(t) - u_{\lambda_0}(t)| \rightarrow 0$  as  $\lambda \rightarrow \lambda_0$ ;

(iv)  $\max_{0 \leq t \leq 1} |u_\lambda(t)| \rightarrow +\infty$  as  $\lambda \rightarrow +\infty$ ,  $\max_{0 \leq t \leq 1} |u_\lambda(t)| \rightarrow 0$  as  $\lambda \rightarrow 0^+$ .

*Proof.* For  $u > 1$ , from the condition (i), we have

$$f(t, 1) = f\left(t, \frac{1}{u}\right) \geq \left(\frac{1}{u}\right)^r f(t, u),$$

thus, we get

$$f(t, u) \leq u^r f(t, 1) \leq \xi u^r.$$

So, for  $u \in K$ , we have

$$0 \leq \int_0^1 G(t, s) \varphi'(s) f(s, u(s)) ds \leq \int_0^1 G(t, s) \varphi'(s) f(s, \|u\|_E) ds \leq M_3 g(t), \quad 0 \leq t \leq 1,$$

where  $M_3 = \max_{0 \leq t \leq 1} f(t, \|u\|_E) \leq \max_{0 \leq t \leq 1} f(t, \|u\|_E + 1) \leq \xi(1 + \|u\|_E)^r$ ,

Thus,  $0 \leq Tu(t) \leq M_3 g(t)$ ,  $0 \leq t \leq 1$ , this means that  $T : \tilde{K} \rightarrow \tilde{K}$ .

For  $u \in \tilde{K}$ , there exists  $\delta > 0$  such that  $u(t) \geq \delta g(t) \geq 0$ ,  $0 \leq t \leq 1$ . Let  $\omega \in (0, 1)$  such that  $\omega < \delta$ , we have

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t, s) \varphi'(s) f(s, u(s)) ds \geq \int_0^1 G(t, s) \varphi'(s) f(s, \delta g(s)) ds \\ &\geq \int_0^1 G(t, s) \varphi'(s) f(s, \omega g(s)) ds \geq \omega^r \int_0^1 G(t, s) \varphi'(s) f(s, g(s)) ds. \end{aligned}$$

Set  $a = \min_{0 \leq t \leq 1} f(t, g(t))$ , then  $a > 0$  and  $(Tu)(t) \geq a \omega^r g(t)$ ,  $0 \leq t \leq 1$ , so  $T : \tilde{K} \rightarrow \tilde{K}$ . From  $f(t, \cdot)$  is increasing, we can get that  $T$  is increasing. Moreover, for  $u \in \tilde{K}$  and  $\omega \in (0, 1)$ , we have

$$T(\omega u)(t) = \int_0^1 G(t, s) \varphi'(s) f(s, \omega u(s)) ds \geq \int_0^1 G(t, s) \varphi'(s) \omega^r f(s, u(s)) ds = \omega^r (Tu)(t).$$

In the following, we consider the operator equation

$$(Tu)(t) = \rho u(t). \quad (3.10)$$

Then all the conditions of Lemma 2.6 are satisfied. Thus, from Lemma 2.6, for any given  $\rho > 0$ , (3.10) has a unique solution  $u_\rho$  in  $\tilde{K}$ ,  $u_\rho$  is strongly decreasing,  $u_\rho$  is continuous, this means that  $\|u_\rho - u_{\rho_0}\| \rightarrow 0$  as  $\rho \rightarrow \rho_0$  and furthermore,  $\lim_{\rho \rightarrow +\infty} \|u_\rho\| = 0$ ,  $\lim_{\rho \rightarrow 0^+} \|u_\rho\| = +\infty$ . Set  $\lambda = \frac{1}{\rho}$ ,  $\lambda_0 = \frac{1}{\rho_0}$ ,  $\lambda_1 = \frac{1}{\rho_1}$ ,  $\lambda_2 = \frac{1}{\rho_2}$ . Then (3.10) is equivalent to  $u(t) = \lambda(Tu)(t)$ . As a result, we can get

- (i) for any given  $\lambda > 0$ , the fractional boundary value problem (1.1), (1.2) has a unique positive solution  $u_\lambda$  in  $\tilde{K}$ ;
- (ii) if  $0 < \lambda_1 < \lambda_2$ , there exists  $\delta > 0$  such that  $u_{\lambda_2}(t) - u_{\lambda_1}(t) \geq \delta g(t)$  which implies  $u_{\lambda_1}(t) \leq u_{\lambda_2}(t)$  and  $u_{\lambda_1}(t) \not\equiv u_{\lambda_2}(t)$ ,  $0 \leq t \leq 1$ ;
- (iii)  $\max_{0 \leq t \leq 1} |u_\lambda(t) - u_{\lambda_0}(t)| \rightarrow 0$  as  $\lambda \rightarrow \lambda_0$ ;
- (iv)  $\max_{0 \leq t \leq 1} |u_\lambda(t)| \rightarrow +\infty$  as  $\lambda \rightarrow +\infty$ ,  $\max_{0 \leq t \leq 1} |u_\lambda(t)| \rightarrow 0$  as  $\lambda \rightarrow 0^+$ . □

#### 4. Example

**Example 4.1** Consider the fractional boundary value problem

$$D_{0+}^{\frac{3}{2}, \frac{t}{2}} u(t) + \lambda(t^2 + 1 + u^{\frac{1}{3}}) = 0, \quad 0 < t < 1, \quad (4.1)$$

$$u(0) = 0, \quad u(1) = \frac{1}{2}u\left(\frac{1}{2}\right). \quad (4.2)$$

We note that  $\alpha = \frac{3}{2}, \beta = \frac{1}{2}, \eta = \frac{1}{2}, \varphi(t) = \frac{t}{2}, f(t, u) = t^2 + 1 + u^{\frac{1}{3}}$ . Thus, we have  $f(t, u) : [0, 1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is continuous and increasing in  $u$ ,  $f(t, u)$  is concave. We can check that conditions  $(H_1), (H_2)$  are satisfied. Choose  $\alpha = 1, \beta = 3$  and  $f(t, u)$  satisfy

$$f(t, 0) = t^2 + 1 \geq 1, \quad f(t, 1) = t^2 + 2 \leq 3, \quad 0 \leq t \leq 1,$$

$$\lim_{u \rightarrow \infty} \frac{f(t, u)}{u} = \lim_{u \rightarrow \infty} \frac{t^2 + 1 + u^{\frac{1}{3}}}{u} = 0.$$

Then all the assumptions of Theorem 3.3 are satisfied. Thus from Theorem 3.3, we have  $\lambda^* = \infty$ , and the following conclusions hold:

(i) for  $\lambda > 0$ , the fractional boundary value problem (4.1), (4.2) has a unique positive solution  $u_\lambda$  in  $\tilde{K}$ ;

(ii) for any  $u_0 \in \tilde{K}$ , let

$$u_n(t) = \lambda \int_0^1 G(t, s) \varphi'(s) f(s, u_{n-1}(s)) ds, \quad n = 1, 2, 3, \dots$$

Then for  $\lambda > 0$ , we have

$$\max_{0 \leq t \leq 1} |u_n(t) - u_\lambda(t)| \rightarrow 0 \text{ as } n \rightarrow \infty;$$

(iii)  $\max_{0 \leq t \leq 1} |u_\lambda(t) - u_{\lambda_0}(t)| \rightarrow 0$  as  $\lambda \rightarrow \lambda_0$  and if  $0 < \lambda_1 < \lambda_2 < +\infty$ , then  $u_{\lambda_1}(t) \leq u_{\lambda_2}(t)$ ,  $0 \leq t \leq 1$  and  $u_{\lambda_1}(t) \not\equiv u_{\lambda_2}(t)$ ;

(iv)  $u_{v\lambda}(t) \leq v u_\lambda(t)$  for  $0 < \lambda < +\infty$ ,  $0 \leq v \leq 1$ ,  $0 \leq t \leq 1$ .

#### 5. Conclusions

In this paper, we discussed a class of fractional derivative with respect to another function called  $\varphi$ -Riemann-Liouville fractional derivative. The new derivative herein generalizes the classical definitions of derivatives by choosing particular forms of  $\varphi(t)$ . The main advantage of our paper is we not only develop the existence and uniqueness of positive solutions to the fractional boundary value problem, but also give some properties of positive solutions to the problem.

#### Acknowledgments

We express our sincere thanks to the anonymous reviewers for their valuable comments and suggestions. This work is supported by the Natural Science Foundation of Tianjin (No.(19JCYBJC30700)).

## Conflict of interest

The authors declare no conflict of interest in this paper.

## References

1. S. Corlay, J. Lebovits, J. Vhel, *Multifractional stochastic volatility models*, Math. Finance, **24** (2014), 364–402.
2. C. Fang, H. Sun, J. Gu, *Application of fractional calculus methods to viscoelastic response of amorphous shape memory polymers*, J. Mech., **31** (2015), 427–432.
3. J. Gmez-Aguilar, M. Lpez-Lpez, V. Alvarado-Martnez, et al. *Modeling diffusive transport with a fractional derivative without singular kernel*, Phys. A, **447** (2016), 467–481.
4. S. Pooseh, R. Almeida, D. Torres, *Fractional order optimal control problems with free terminal time*, J. Ind. Manag. Optim., **10** (2014), 363–381.
5. A. Saporá, P. Cornetti, A. Carpinteri, et al. *The use of fractional calculus to model the experimental creep-recovery behavior of modified bituminous binders*, Mater. Struct., **49** (2016), 45–55.
6. D. Tien, *Fractional stochastic differential equations with applications to finance*, J. Math. Anal. Appl., **397** (2013), 334–348.
7. X. Zhang, L. Liu, Y. Wu, *The uniqueness of positive solution for a singular fractional differential system involving derivatives*, Commun. Nonlinear Sci. Numer. Simul., **18** (2013), 1400–1409.
8. X. Zhang, Y. Wu, L. Caccetta, *Nonlocal fractional order differential equations with changing-sign singular perturbation*, Appl. Math. Modell., **39** (2015), 6543–6552.
9. A. Sasso, G. Palmieri, D. Amodio, *Application of fractional derivative models in linear viscoelastic problems*, Mech. Time Depend. Mater., **15** (2011), 367–387.
10. J. Surez, B. Vinagre, A. Caldern, et al. *Using fractional calculus for lateral and longitudinal control of autonomous vehicles*, Springer, Berlin, Heidelberg, (2003), 337–348.
11. V. Tarasov, *Fractional dynamics: Applications of fractional calculus to dynamics of particle*, Springer-Verlag, Fields and Media, Berlin-Heidelberg, 2010.
12. M. Wyrwas, E. Girejko, D. Mozyrska, *Fractional discrete-time consensus models for single- and double-summator dynamics*, Int. J. Syst. Sci., **49** (2018), 1212–1225.
13. J. Wang, Y. Zhang, *On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives*, Appl. Math. Lett., **39** (2015), 85–90.
14. J. Jiang, D. O'Regan, J. Xu, et al. *Positive solutions for a system of nonlinear Hadamard fractional differential equations involving coupled integral boundary conditions*, J. Inequal. Appl., **2019** (2019), 1–18.
15. R. Almeida, *A Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simul., **44** (2017), 460–481.
16. C. Zhai, L. Xu, *Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter*, Commun. Nonlinear Sci. Numer. Simul., **19** (2014), 2820–2827.

17. D. Guo, V. Lakshmikantham, *Nonlinear problems in abstract cones*, Boston and New York: Academic Press, 1988.
18. M. Krasnoselskii, *Positive solutions of operator equations*, Groningen: Noordhoff; 1964.
19. Y. Du, *Fixed points of a class of noncompact operator and its application*, Acta Math. Sin., **32** (1989), 618–627 (in Chinese).
20. D. Guo, *Fixed points and eigenelements of a class of concave and convex operator*, Chin. Sci. Bull., **30** (1985), 1132–1135 (in Chinese).
21. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland mathematics studies, 204. Elsevier Science B.V, Amsterdam, 2006.
22. A. Seemab, M. U. Rehman, J. Alzabut, et al. *On the existence of positive solutions for generalized fractional boundary value problems*, Boundary Value Probl., **2019** (2019), 1–20.
23. M. S. Abdo, S. K. Panchal, A. M. Saeed, *Fractional boundary value problem with  $\psi$ -Caputo fractional Derivative*, Proc. Indian Acad. Sci. (Math. Sci.), **129** (2019), 1–14.
24. R. Almeida, A. B. Malinowska, M. T. Monteiro, *Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications*, Math. Meth. Appl. Sci., **41** (2018), 336–352.



AIMS Press

© 2020 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)