



Research article**On Uzawa-SSI method for non-Hermitian saddle point problems****Shu-Xin Miao and Jing Zhang***

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Abstract: Based on the single-step iteration (SSI) method for the non-Hermitian positive definite linear systems, we propose a Uzawa-SSI method for solving the saddle point problems with non-Hermitian positive definite $(1, 1)$ block in this paper. The convergence and semi-convergence properties of the Uzawa-SSI method, respectively, for nonsingular and singular cases are analyzed. Numerical examples with experiments are given to show the robustness and the efficiency of Uzawa-SSI method.

Keywords: saddle-point problem; Uzawa-type method; single-step iteration; convergence; semi-convergence

Mathematics Subject Classification: 65F10, 65F50

1. Introduction

The numerical methods for many scientific and engineering problems ultimately lead to the numerical methods for the 2×2 block structure linear systems, see for example, finite element or finite difference methods discretization of some partial differential equations [5, 8, 9, 20], numerical methods for solving weighted least squares problems [23], augmented immersed interface method for Stokes and Darcy or Navier-stokes and Darcy coupling equations [15] and so on. In this paper, we consider the 2×2 block structure linear system arising from mixed finite element discretization Navier-Stokes equation [8], which is called the saddle point problem and has the form

$$\mathcal{A}\mathbf{x} = \begin{bmatrix} A & B \\ -B^* & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix} = b, \quad (1.1)$$

where $A \in \mathbb{C}^{n \times n}$ is a non-Hermitian positive definite matrix, $B \in \mathbb{C}^{n \times m}$ is a matrix with $\text{rank}(B) = r$, here and in the sequence, $\text{rank}(\cdot)$ denotes the rank of a given matrix, $f \in \mathbb{C}^n$ and $g \in \mathbb{C}^m$ are given vectors, $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$ are unknown vectors, with $m \leq n$. Note that when $r = m$, (1.1) is the nonsingular saddle point problem [2] and has a unique solution, and when $r < m$, (1.1) is the singular

saddle point problem, in this case, we suppose that the singular saddle point problem (1.1) is consistent, i.e., $b \in \text{range}(\mathcal{A})$, the range of \mathcal{A} .

Because the block matrices A and B are usually large and sparse, (1.1) is suitable to be solved by the iterative methods. Efficient numerical methods for solving nonsingular and singular saddle point problems have been studied in the literatures, see [13, 25] and the references therein. Due to the advantage of the computational efficiency, Uzawa method [1] for solving (1.1) received wide attention and obtained considerable achievements in recent years. The iteration scheme of Uzawa method can be described as

$$\begin{cases} x_{k+1} = A^{-1}(f - By_k), \\ y_{k+1} = y_k + \tau(B^*x_{k+1} - g), \end{cases}$$

where τ is a positive parameter. The main computation costs of Uzawa method lies in the solution of the linear system $Ax = f - By$ at each step, one prefer to approximate its solution by iteration method since the matrix A is always large and sparse.

Using different iteration methods to approximate x_{k+1} lead to variant forms of the Uzawa method, see [10, 13, 16, 21, 24, 26] for examples. In particular, splitting A as

$$A = H + S, \quad (1.2)$$

where $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$, and approximating x_{k+1} by the efficient HSS method [4], then the Uzawa-HSS method [21, 22] is proposed, the iteration scheme of the Uzawa-HSS method [21, 22] is defined as follows

$$\begin{cases} x_{k+1} = x_k + 2\alpha(\alpha I + S)^{-1}(\alpha I + H)^{-1}(f - Ax_k - By_k), \\ y_{k+1} = y_k + \tau Q^{-1}(B^*x_{k+1} - g), \end{cases}$$

where α and τ are two positive constants, and $Q \in \mathbb{C}^{m \times m}$ is a given Hermitian positive definite matrix. Approximating x_{k+1} by the SHSS method [17], we have the Uzawa-SHSS method [16]

$$\begin{cases} x_{k+1} = x_k + (\alpha I + H)^{-1}(f - Ax_k - By_k), \\ y_{k+1} = y_k + \tau Q^{-1}(B^*x_{k+1} - g). \end{cases}$$

Splitting matrix A into its positive definite and skew-Hermitian parts as $A_p + A_s$, and approximating x_{k+1} by the PSS method [3], then the iteration scheme of the Uzawa-PSS method [10] can be defined as

$$\begin{cases} x_{k+1} = x_k + 2\alpha(\alpha I + A_s)^{-1}(\alpha I + A_p)^{-1}(f - Ax_k - By_k), \\ y_{k+1} = y_k + \tau Q^{-1}(B^*x_{k+1} - g), \end{cases}$$

where $A_p = D_H + 2L_H$, $A_s = L_H^* - L_H + S$, D_H and L_H being the diagonal part and strictly lower triangular part of H . The iteration scheme of the MLHSS method [14] is

$$\begin{cases} x_{k+1} = x_k + (P + H)^{-1}(f - Ax_k - By_k), \\ y_{k+1} = y_k + Q^{-1}(B^*x_{k+1} - g), \end{cases}$$

where $P \in \mathbb{C}^{n \times n}$ and $Q \in \mathbb{C}^{m \times m}$ are Hermitian positive definite matrices.

Recently, Wang et al. [19] proposed the single-step iteration (SSI) method for solving the non-Hermitian positive definite linear systems. In this paper, we will use the SSI method to approximate

x_{k+1} in the first step of the Uzawa method and propose a new Uzawa-type method, named Uzawa-SSI method, to solve the non-Hermitian saddle point problem (1.1).

The rest of this paper is organized as follows. In Section 2, the Uzawa-SSI method for solving the non-Hermitian saddle point problems (1.1) is proposed. The convergence for nonsingular saddle point problem and semi-convergence for singular case of the Uzawa-SSI method are discussed in Section 3. Numerical examples are given in Section 4 to show the effectiveness of the proposed method for solving non-Hermitian saddle point problems (1.1). Finally, in Section 5, we make a brief conclusion of this paper.

2. The Uzawa-SSI method

Let $P \in \mathbb{C}^{n \times n}$ be a given Hermitian positive definite matrix, based on the splitting (1.2), the SSI method for solving the non-Hermitian positive linear system $Az = c$ is defined as [19]

$$(P + H)x_{k+1} = (P - S)x_k + c. \quad (2.1)$$

Theoretical as well as numerical results in [19] stated that the SSI method is a more efficient method for solving the non-Hermitian positive definite linear systems. With different choices of the matrix P , the SSI method covers several other methods, for example, if $P = \alpha I$, then the SSI method reduces to the single-step HSS (SHSS) iteration method. The matrix P can also be taken as $P = \alpha H$, $P = \Lambda$ (where $\Lambda = \text{diag}(d_1, d_2, \dots, d_n)$, $d_i > 0$, $i = 1, 2, \dots, n$) or other different Hermitian matrices. Note that (1, 1) block A in saddle point matrix \mathcal{A} is non-Hermitian positive definite, if we adopt the SSI iteration scheme (2.1) to approximate x_{k+1} , we can derive the following Uzawa-SSI method for solving non-Hermitian saddle point problems (1.1).

Method 2.1 (THE UZAWA-SSI METHOD) Given initial vectors $x_0 \in \mathbb{C}^n$, $y_0 \in \mathbb{C}^m$, and positive relaxation parameter τ . For $k = 0, 1, 2, \dots$, until the iteration sequence converges, compute

$$\begin{cases} x_{k+1} = x_k + (P + H)^{-1}(f - Ax_k - By_k), \\ y_{k+1} = y_k + \tau Q^{-1}(B^*x_{k+1} - g), \end{cases} \quad (2.2)$$

where $P \in \mathbb{C}^{n \times n}$ and $Q \in \mathbb{C}^{m \times m}$ are Hermitian positive definite matrices, $\tau > 0$ is a constant.

The iteration scheme of Uzawa-SSI method (2.2) can be rewritten in matrix-vector form as

$$\mathbf{x}_{k+1} = \Gamma \mathbf{x}_k + M^{-1}b, \quad (2.3)$$

where

$$\Gamma = \begin{bmatrix} P + H & 0 \\ -B^* & \frac{1}{\tau}Q \end{bmatrix}^{-1} \begin{bmatrix} P - S & -B \\ 0 & \frac{1}{\tau}Q \end{bmatrix}$$

is the iteration matrix of the Uzawa-SSI method, and

$$M = \begin{bmatrix} P + H & 0 \\ -B^* & \frac{1}{\tau}Q \end{bmatrix}.$$

The Uzawa-SSI method possesses similar iteration schemes as the Uzawa-HSS method [21, 22], the Uzawa-SHSS method [16] and MLHSS method [14]. In fact, the Uzawa-SSI method can be regarded

as a parameterized MLHSS method with the parameter $\frac{1}{\tau}$ in the (2,2) block of the splitting matrix. The Uzawa-SSI method reduces to the MLHSS method when $\tau = 1$, we can choose appropriate parameter τ to make the Uzawa-SSI method have better numerical results. The proposed method (2.2) use iteration (2.1) to approximate x_{k+1} , the solution of the shift skew-Hermitian subsystem is avoided compare with the Uzawa-HSS method. Moreover, if we let $P = \alpha I$, then the Uzawa-SSI method reduces to the Uzawa-SHSS method, we may choose some Hermitian positive definite matrix P instead of αI to improve the computation efficiency of the Uzawa-SSI method.

3. Convergence and semi-convergence analysis

In this section, we will study the convergence and semi-convergence properties of the Uzawa-SSI method when it is used to solving the nonsingular and singular non-Hermitian saddle point problems (1.1), respectively. For this purpose, the following notations, definition and results are needed. $\sigma(E)$ and $\rho(E)$ denote the spectral set and the spectral radius of a square matrix E , respectively. The smallest nonnegative integer i such that $\text{rank}(E^i) = \text{rank}(E^{i+1})$ is called the index of E , and is denoted by $\text{index}(E)$. The range and the null spaces of E are denoted by $R(E)$ and $N(E)$, respectively.

Lemma 3.1. [6] *Both roots of the complex quadratic equation $\lambda^2 - \phi\lambda + \psi = 0$ have modulus less than one if and only if*

$$|\phi - \bar{\phi}\psi| + |\psi|^2 < 1,$$

where $\bar{\phi}$ is the conjugate complex number of ϕ .

Definition 3.1. [7] *The iteration method (2.3) is semi-convergent if for any initial guess $[x_0^*, y_0^*]^*$, the iteration sequence $[x_k^*, y_k^*]^*$ produced by (2.3) converges to a solution $[x_\star^*, y_\star^*]^*$ of linear systems $\mathcal{A}\mathbf{x} = \mathbf{b}$. Moreover, it holds*

$$\begin{bmatrix} x_\star \\ y_\star \end{bmatrix} = (I - \Gamma)^D c + (I - E) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \quad \text{with} \quad E = (I - \Gamma)(I - \Gamma)^D,$$

where I is the identity matrix and $(I - \Gamma)^D$ denotes the Drazin inverse of $I - \Gamma$.

Following lemma describes the sufficient and necessary semi-convergence conditions of the iteration scheme (2.3).

Lemma 3.2. [7] *The iteration scheme (2.3) is semi-convergent if and only if the following two conditions hold true:*

1. $\text{index}(I - \Gamma) = 1$, or equivalently, $\text{rank}(I - \Gamma)^2 = \text{rank}(I - \Gamma)$,
2. $\vartheta(\Gamma) < 1$, where $\vartheta(\Gamma) = \max\{|\lambda|, \lambda \in \sigma(\Gamma), \lambda \neq 1\} < 1$ is called the pseudo-spectral radius of the iteration matrix Γ .

Let λ be an eigenvalue of the Uzawa-SSI iteration matrix Γ and $(u^*, v^*)^* \in \mathbb{C}^{n+m}$ be the corresponding eigenvector, in terms of the expression of Γ , we have

$$\begin{bmatrix} P - S & -B \\ 0 & \frac{1}{\tau}Q \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} P + H & 0 \\ -B^* & \frac{1}{\tau}Q \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

or equivalently,

$$\begin{cases} \lambda(P + H)u = (P - S)u - Bv, \\ \lambda\tau B^*u = (\lambda - 1)Qv. \end{cases} \quad (3.1)$$

3.1. Convergence

For nonsingular saddle point problem (1.1), the Uzawa-SSI method (2.3) is convergent if and only if the spectral radius of the iteration matrix Γ is less than 1, i.e., $\rho(\Gamma) < 1$.

For the convergence of Uzawa-SSI, we have the following results.

Lemma 3.3. *Let A be non-Hermitian positive definite and B be of full column rank. If λ is an eigenvalue of iteration matrix Γ , and $[u^*, v^*]^*$ is the corresponding eigenvector with $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^m$, then $\lambda \neq 1$ and $u \neq 0$.*

Proof. We prove the conclusions by contradiction. If $\lambda = 1$, then it follows from (3.1) that

$$\begin{cases} Au + Bv = 0, \\ -B^*u = 0. \end{cases}$$

It is known that the saddle point matrix

$$\begin{bmatrix} A & B \\ -B^* & 0 \end{bmatrix}$$

is nonsingular when A is non-Hermitian positive definite and B has full column rank, hence we have $u = 0$ and $v = 0$, which contradicts the assumption that $[u^*, v^*]^*$ is an eigenvector of the iteration matrix Γ , so $\lambda \neq 1$.

If $u = 0$, then the first equality in (3.1) reduces to $Bv = 0$. Because B is a matrix of full column rank, we can obtain $v = 0$, which is a contradiction. \square

Theorem 3.1. *Let $A \in \mathbb{C}^{n \times n}$ be non-Hermitian positive definite and $B \in \mathbb{C}^{n \times m}$ be of full column rank, $P \in \mathbb{C}^{n \times n}$ and $Q \in \mathbb{C}^{m \times m}$ are Hermitian positive definite. Then the Uzawa-SSI method (2.3) is convergent if and only if the parameter τ satisfy*

$$\frac{t_2^2 - t_1^2}{2t_1} < 1 \quad \text{and} \quad 0 < \tau < \frac{2t_1(t_1^2 + 2t_1 - t_2^2)}{t_3(t_1^2 + t_2^2)}, \quad (3.2)$$

where $t_1 = \frac{u^*Hu}{u^*Pu}$, $it_2 = \frac{u^*Su}{u^*Pu}$, $t_3 = \frac{u^*BQ^{-1}B^*u}{u^*Pu}$, here i is the imaginary unit.

Proof. It follows from Lemma 3.3 that $\lambda \neq 1$. Using the nonsingularity of A and solving v from the second equality of (3.1), we have

$$v = \frac{\lambda\tau}{\lambda - 1} Q^{-1} B^* u.$$

Substituting it into the first equality of (3.1) yields

$$(P - S)u - \frac{\lambda\tau}{\lambda - 1} BQ^{-1} B^* u = \lambda(P + H)u. \quad (3.3)$$

From Lemma 3.3, we know that $u \neq 0$. Multiplying $u^*/(u^*Pu)$ to the both sides of (3.3) from left gives

$$\frac{u^*(P - S)u}{u^*Pu} + \frac{\lambda\tau}{1 - \lambda} \frac{u^*BQ^{-1}B^*u}{u^*Pu} = \lambda \frac{u^*(P + H)u}{u^*Pu}.$$

Denote $t_1 = \frac{u^*Hu}{u^*Pu}$, $\mathbf{i}t_2 = \frac{u^*Su}{u^*Pu}$, $t_3 = \frac{u^*BQ^{-1}B^*u}{u^*Pu}$. After simple computation, we get $\lambda^2 - \phi\lambda + \psi = 0$, with

$$\phi = \frac{2 + t_1 - \mathbf{i}t_2 - \tau t_3}{1 + t_1}, \quad \psi = \frac{1 - \mathbf{i}t_2}{1 + t_1}.$$

After some careful calculations, we have

$$|\phi - \bar{\phi}\psi| + |\psi|^2 = \frac{1 + t_2^2 + \sqrt{[(2 + t_1 - \tau t_3)t_1 - t_2^2]^2 + (\tau t_2 t_3)^2}}{(1 + t_1)^2}.$$

Therefore, according to Lemma 3.1, $|\phi - \bar{\phi}\psi| + |\psi|^2 < 1$ if and only if

$$\begin{cases} 1 + t_2^2 - (1 + t_1)^2 < 0, \\ \left[1 + t_2^2 - (1 + t_1)^2\right]^2 > [(2 + t_1 - \tau t_3)t_1 - t_2^2]^2 + (\tau t_2 t_3)^2. \end{cases} \quad (3.4)$$

Solving (3.4) gives

$$\frac{t_2^2 - t_1^2}{2t_1} < 1 \quad \text{and} \quad \tau < \frac{2t_1(t_1^2 + 2t_1 - t_2^2)}{t_3(t_1^2 + t_2^2)}.$$

From the above discussion, we obtain (3.2). \square

Corollary 3.1. Assume the condition in Theorem 3.1 are satisfied. Then the Uzawa-SSI method for solving saddle-point problem (1.1) is convergent if the following conditions hold:

$$\frac{\sigma_{\max}^2 - \lambda_{\min}^2}{2\lambda_{\min}} < 1 \quad \text{and} \quad 0 < \tau < \frac{2\lambda_{\min}(\lambda_{\min}^2 + 2\lambda_{\min} - \sigma_{\max}^2)}{\lambda_{\max}(Q^{-1}B^*P^{-1}B)(\lambda_{\min}^2 + \sigma_{\max}^2)}, \quad (3.5)$$

where λ_{\min} is the smallest eigenvalue of matrix \tilde{H} and σ_{\max} is the largest singular-value of matrix \tilde{S} with $\tilde{H} = P^{-\frac{1}{2}}HP^{-\frac{1}{2}}$ and $\tilde{S} = P^{-\frac{1}{2}}SP^{-\frac{1}{2}}$.

In particular, when A is Hermitian positive definite, the above conditions become

$$0 < \tau < \frac{2\lambda_{\min} + 2}{\lambda_{\max}(Q^{-1}B^*P^{-1}B)}. \quad (3.6)$$

Proof. Note that the upper bound of τ in (3.2) can be written as the product of $\zeta_1(t_1, t_2, t_3)$ and $\zeta_2(t_1, t_2, t_3)$ with

$$\zeta_1(t_1, t_2, t_3) = \frac{2t_1^2}{t_3(t_1^2 + t_2^2)} \quad \text{and} \quad \zeta_2(t_1, t_2, t_3) = \frac{t_1^2 + 2t_1 - t_2^2}{t_1}.$$

It is easy to see that both $\zeta_1(t_1, t_2, t_3)$ and $\zeta_2(t_1, t_2, t_3)$ are monotonic increasing with respect to the variable t_1 , and both $\zeta_1(t_1, t_2, t_3)$ and $\zeta_2(t_1, t_2, t_3)$ are monotonic decreasing with respect to t_2^2 .

Let $z = P^{\frac{1}{2}}u$, we have

$$\begin{aligned} t_3 &\leq \max_{Z \in \mathbb{C}^n} \frac{z^* P^{-\frac{1}{2}} B Q^{-1} B^* P^{-\frac{1}{2}} z}{z^* z} \\ &= \lambda_{\max}[(P^{-\frac{1}{2}} B Q^{-\frac{1}{2}})(Q^{-\frac{1}{2}} B^* P^{-\frac{1}{2}})] \\ &= \lambda_{\max}[(Q^{-\frac{1}{2}} B^* P^{-\frac{1}{2}})(P^{-\frac{1}{2}} B Q^{-\frac{1}{2}})] \end{aligned}$$

$$= \lambda_{\max}(Q^{-1}B^*P^{-1}B).$$

Thus, a lower bound of the product of $\zeta_1(t_1, t_2, t_3)$ and $\zeta_2(t_1, t_2, t_3)$ is

$$\begin{aligned} \frac{2t_1(t_1^2 + 2t_1 - t_2^2)}{t_3(t_1^2 + t_2^2)} &\geq \frac{2\lambda_{\min}^2(\lambda_{\min}^2 + 2\lambda_{\min} - \sigma_{\max}^2)}{\lambda_{\max}(Q^{-1}B^*P^{-1}B)\lambda_{\min}(\lambda_{\min}^2 + \sigma_{\max}^2)} \\ &= \frac{2\lambda_{\min}(\lambda_{\min}^2 + 2\lambda_{\min} - \sigma_{\max}^2)}{\lambda_{\max}(Q^{-1}B^*P^{-1}B)(\lambda_{\min}^2 + \sigma_{\max}^2)}. \end{aligned}$$

By making use of Theorem 3.1, inequalities (3.5) and (3.6), it can be seen that the Uzawa-SSI iteration method is convergent if τ satisfy (3.5). Specifically, if A is Hermitian positive definite, the condition (3.6) can be directly from (3.5) as $H = A$ and $S = 0$ in this case. \square

3.2. Semi-convergence

In this subsection, we will analyze the semi-convergence of the Uzawa-SSI method for solving singular saddle point problem (1.1) when $\text{rank}(B) = r < m$. Actually, we only need to verify that the iteration scheme (2.3) satisfies the two conditions in Lemma 3.2.

In the first place, considering the condition index $(I - \Gamma) = 1$, we have the following result.

Theorem 3.2. *Let $A \in \mathbb{C}^{n \times n}$ be non-Hermitian positive definite and $B \in \mathbb{C}^{n \times m}$ be rank deficient. Suppose that P and Q are Hermitian positive definite, and Γ is the iteration matrix of Uzawa-SSI method, parameter $\tau > 0$. Then, $\text{rank}(I_{n+m} - \Gamma)^2 = \text{rank}(I_{n+m} - \Gamma)$.*

Proof. Note that $\Gamma = I_{n+m} - M^{-1}\mathcal{A}$, hence, $\text{rank}(I_{n+m} - \Gamma)^2 = \text{rank}(I_{n+m} - \Gamma)$ hold if

$$\text{null}((M^{-1}\mathcal{A})^2) = \text{null}(M^{-1}\mathcal{A}).$$

Obviously, $\text{null}((M^{-1}\mathcal{A})^2) \supseteq \text{null}(M^{-1}\mathcal{A})$ holds, so we only need to prove

$$\text{null}((M^{-1}\mathcal{A})^2) \subseteq \text{null}(M^{-1}\mathcal{A}).$$

Let $x = [x_1^*, x_2^*]^* \in \mathbb{C}^{n+m}$, with $x_1 \in \mathbb{C}^n$ and $x_2 \in \mathbb{C}^m$, satisfy $(M^{-1}\mathcal{A})^2 x = 0$, and denote $y = M^{-1}\mathcal{A}x$, then we only need to demonstrate $y = 0$. Let $y = [y_1^*, y_2^*]^*$. After simple calculation, we have

$$\begin{aligned} y &= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} (P + H)^{-1}Ax_1 + (P + H)^{-1}Bx_2 \\ \tau Q^{-1}B^*(P + H)^{-1}Ax_1 - \tau Q^{-1}B^*x_1 + \tau Q^{-1}B^*(P + H)^{-1}Bx_2 \end{bmatrix}. \end{aligned} \quad (3.7)$$

Since, $M^{-1}\mathcal{A}y = (M^{-1}\mathcal{A})^2 x = 0$ and M is invertible, $(M^{-1}\mathcal{A})^2 x = 0$ is equality to $\mathcal{A}y = 0$, i.e.,

$$\begin{cases} Ay_1 + By_2 = 0, \\ -B^*y_1 = 0. \end{cases} \quad (3.8)$$

Since A is positive definite, from the first equation of (3.8) we can easily get $y_1 = -A^{-1}By_2$. Then, substituting this relationship into the second equality of (3.8), we obtain $B^*A^{-1}By_2 = 0$. Owing to the

positive definiteness of the matrix A^{-1} , it has $By_2 = 0$. Taking $By_2 = 0$ into $y_1 = -A^{-1}By_2$, we get $y_1 = 0$. Therefore, the first equality of (3.7) becomes $Bx_2 = -Ax_1$, using the second equality of (3.7), we have

$$\begin{aligned} y_2 &= \tau Q^{-1}B^*(P+H)^{-1}Ax_1 - \tau Q^{-1}B^*x_1 - \tau Q^{-1}B^*(P+H)^{-1}Ax_1 \\ &= -\tau Q^{-1}B^*x_1. \end{aligned}$$

Furthermore, using $By_2 = 0$ and $\tau > 0$, we have $x_1^*BQ^{-1}B^*x_1 = 0$, that is $B^*x_1 = 0$, therefore $y_2 = 0$. Hence $y = [y_1^*, y_2^*]^* = 0$. \square

In the following, we verify the iteration scheme (2.3) satisfy $\vartheta(\Gamma) < 1$ of Lemma 3.2. Let $B = U[B_r, 0]V^*$ be the singular value decomposition of the matrix B , where

$$B_r = \begin{bmatrix} \Sigma_r \\ 0 \end{bmatrix} \in \mathbb{C}^{n \times r}, \quad \Sigma_r = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r),$$

where $U \in \mathbb{C}^{n \times n}$ and $V \in \mathbb{C}^{m \times m}$ are two unitary matrices and σ_i ($i = 1, 2, \dots, r$) is a singular value of B .

We partition the matrix V as $V = [V_1, V_2]$ with $V_1 \in \mathbb{C}^{m \times r}$, $V_2 \in \mathbb{C}^{m \times (m-r)}$ and define

$$\Omega = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}.$$

It is obvious that Ω is a $(n+m) \times (n+m)$ unitary matrix and the iteration matrix Γ is unitary similar to the matrix $\widehat{\Gamma} = \Omega^* \Gamma \Omega$. Hence, the matrix Γ has the same spectrum with the matrix $\widehat{\Gamma}$. Thus, we only need to analyze the pseudo-spectral radius of the matrix $\widehat{\Gamma}$ now.

Denoting $\widehat{P} = U^*PU$, $\widehat{H} = U^*HU$, $\widehat{S} = U^*SU$, some careful calculations yields

$$\widehat{\Gamma} = \begin{bmatrix} \widehat{\Gamma}_1 & 0 \\ \widehat{L} & I_{n-r} \end{bmatrix}, \quad (3.9)$$

where

$$\widehat{\Gamma}_1 = \begin{bmatrix} (\widehat{P} + \widehat{H})^{-1}(\widehat{P} - \widehat{S}) & -(\widehat{P} + \widehat{H})^{-1}B_r \\ \tau V_1^* Q^{-1} V_1 B_r^* (\widehat{P} + \widehat{H})^{-1}(\widehat{P} - \widehat{S}) & I_r - \tau V_1^* Q^{-1} V_1 B_r^* (\widehat{P} + \widehat{H})^{-1} B_r \end{bmatrix}$$

and

$$\widehat{L} = \begin{bmatrix} \tau V_2^* Q^{-1} V_1 B_r^* (\widehat{P} + \widehat{H})^{-1}(\widehat{P} - \widehat{S}), & -\tau V_2^* Q^{-1} V_1 B_r^* (\widehat{P} + \widehat{H})^{-1} B_r \end{bmatrix}.$$

Then, from Eq (3.9), $\vartheta(\widehat{\Gamma}) < 1$ hold if and only if $\rho(\widehat{\Gamma}_1) < 1$. Note that $V_1^* Q^{-1} V_1$ is Hermitian positive definite, comparing with the iteration matrix of Uzawa-SSI, $\widehat{\Gamma}_1$ can be viewed actually as the iteration matrix of (2.3) used for solving nonsingular saddle point problem

$$\begin{bmatrix} \widehat{A} & B_r \\ -B_r^* & 0 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix} = \begin{bmatrix} \widehat{f} \\ -\widehat{g} \end{bmatrix},$$

with the preconditioning matrix $V_1^* Q V_1$.

Let $\widehat{\lambda}$ be the eigenvalue of $\widehat{\Gamma}_1$ and $[\widehat{u}^*, \widehat{v}^*]^* \in \mathbb{C}^{n+r}$ be the eigenvector of $\widehat{\Gamma}_1$ corresponding to the eigenvalue $\widehat{\lambda}$. Denote

$$\widehat{t}_1 = \frac{\widehat{u}^* \widehat{H} \widehat{u}}{\widehat{u}^* \widehat{P} \widehat{u}}, \quad \widehat{\mathbf{i}t}_2 = \frac{\widehat{u}^* \widehat{S} \widehat{u}}{\widehat{u}^* \widehat{P} \widehat{u}}, \quad \widehat{t}_3 = \frac{\widehat{u}^* B_r V_1^* Q^{-1} V_1 B_r^* \widehat{u}}{\widehat{u}^* \widehat{P} \widehat{u}}, \quad (3.10)$$

where \mathbf{i} is the imaginary unit. Then from Theorem 3.1, we can derive the following result.

Theorem 3.3. *Let A be non-Hermitian positive definite, B be of rank deficient, P and Q are Hermitian positive definite. Then the pseudo-spectral radius of matrix Γ is less than 1, i.e., $\vartheta(\Gamma) < 1$ if and only if parameters τ satisfy*

$$\frac{\widehat{t}_2^2 - \widehat{t}_1^2}{2\widehat{t}_1} < 1 \quad \text{and} \quad 0 < \tau < \frac{2\widehat{t}_1(\widehat{t}_1^2 - \widehat{t}_2^2 + 2\widehat{t}_1)}{\widehat{t}_3(\widehat{t}_1^2 + \widehat{t}_2^2)},$$

where \widehat{t}_1 , \widehat{t}_2 and \widehat{t}_3 are defined in (3.10).

Corollary 3.2. *Assume the conditions in Theorem 3.3 are satisfied. Then, the Uzawa-SSI method for singular saddle-point problem (1.1) is semi-convergent if the parameter τ satisfy:*

$$\frac{\sigma_{\max}^2 - \lambda_{\min}^2}{2\lambda_{\min}} < 1 \quad \text{and} \quad 0 < \tau < \frac{2\lambda_{\min}(\lambda_{\min}^2 + 2\lambda_{\min} - \sigma_{\max}^2)}{\lambda_{\max}(Q^{-1}B^*P^{-1}B)(\lambda_{\min}^2 + \sigma_{\max}^2)},$$

where λ_{\min} is the smallest eigenvalue of matrix \widetilde{H} and σ_{\max} is the largest singular-value of matrix \widetilde{S} with $\widetilde{H} = P^{-\frac{1}{2}}HP^{-\frac{1}{2}}$ and $\widetilde{S} = P^{-\frac{1}{2}}SP^{-\frac{1}{2}}$. In particular, if A is Hermitian positive definite, the above conditions are

$$0 < \tau < \frac{2\lambda_{\min} + 2}{\lambda_{\max}(Q^{-1}B^*P^{-1}B)}.$$

4. Numerical results

In this section, we will verify the efficiency of the Uzawa-SSI method when it is used to solve nonsingular and singular saddle point problem (1.1). The Uzawa-HSS [21, 22], the Uzawa-SHSS [16], the Uzawa-PSS [10] and modified local HSS(MLHSS)[14] methods are compared with the Uzawa-SSI method from the aspects of the number of iteration steps (denoted by ‘IT’) and the elapsed CPU times in seconds (denoted by ‘CPU’).

In the implementation, for the matrices P and Q of the tested methods, we choose $P = \alpha I$ in the MLHSS method, $P = H$ in the Uzawa-SSI method, $Q = \text{diag}(B^*D^{-1}B)$ in the all tested methods, where $D = \text{diag}(A)$. We have proved that the convergence conditions in Corollaries 3.1 and 3.2 are satisfied for the matrices $P = H$ and $Q = \text{diag}(B^*D^{-1}B)$ in the following examples. In addition, all the involved sub-linear system are solved by Cholesky or LU factorization in combination with AMD reordering. The parameters τ for the Uzawa-SSI method, α for the MLHSS method, (α, τ) for the Uzawa-HSS method, Uzawa-SHSS method, and Uzawa-PSS method, are chosen to be the ones resulting in the least iteration step. All the tested iteration methods are started from zero vector and terminated when the current iteration solution \mathbf{x}_k satisfies

$$\text{RES} = \frac{\|b - \mathcal{A}\mathbf{x}_k\|}{\|b\|} < 10^{-6}$$

or the iteration steps exceed $k_{max} = 1500$ (results in this case are denoted by ‘-’). In addition, all runs are performed in MATLAB 2016a on a personal computer with Intel(R) Celeron(R) 3205U @ 1.50GHz (8G RAM) Windows 8 system. Using experimentally found optimal parameters in the interval $(0, 6000]$. In actual computations, we choose right-hand-side vector $[f^*, g^*] \in \mathbb{R}^{3q^2}$ for nonsingular case and $[f^*, g^*]^* \in \mathbb{R}^{3q^2+2}$ for singular case such that the exact solution of (1.1) is x^* with all elements 1.

Example 4.1. [10]. Consider the linearized steady Navier-Stokes equation, i.e., the steady Oseen equation of the following form

$$\begin{cases} -\nu \Delta \mathbf{u} + (\mathbf{w} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega. \\ \operatorname{div} \mathbf{u} = \mathbf{g}, & \text{in } \Omega. \\ \mathbf{u} = \mathbf{g}, & \text{on } \partial\Omega. \end{cases}$$

Where $\nu > 0$ is the kinematic viscosity (inversely proportional to the Reynolds number), Δ is the Laplacian operation, ∇ is the gradient and div is the divergence, the “wind” \mathbf{w} is the velocity field obtained from the previous Picard iteration step.

The test problem is a “leaky” two-dimensional lid-driven cavity problem on the unit square domain. We use the IFISS software package developed by Elman et al. [12] with $Q2 - Q1$ mixed finite element on uniform grids to generate linear system corresponding to 16×16 , 32×32 and 64×64 meshes. The resulting linear system for the discrete solution has the form

$$\mathcal{A} \mathbf{x} = \begin{bmatrix} A & B \\ -B^* & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix} = b, \quad (4.1)$$

where $A \in \mathbb{C}^{n \times n}$ is the discretization of the diffusion and convection terms, $B \in \mathbb{C}^{n \times m}$ is the discrete gradient and B is the negative discrete divergence. In (4.1), A is nonsymmetric positive definite matrix and B is rank deficient, so the \mathcal{A} is singular. The test nonsingular problem is constructed by dropping the last row of the matrix B [11]. For each case, we test viscosity value $\nu = 0.1$.

In Table 1, we list the numerical results of the Uzawa-SSI method, the Uzawa-HSS method, the Uzawa-SHSS method, the Uzawa-PSS method and MLHSS method for nonsingular saddle point problems with $\nu = 0.1$. From Table 1, we can observe that for $\nu = 0.1$, compared with other four methods, the Uzawa-SSI method is the most efficient one, when reaching the stop criterion, it needs the least iteration steps and CPU times.

Table 1. Numerical results for nonsingular case of Example 4.1.

	Method	Parameters	IT	CPU	RES
$q = 16$	Uzawa-SSI	0.4	42	0.0469	6.1802e-07
	Uzawa-HSS	(0.51, 0.1)	153	0.0938	8.2989e-07
	Uzawa-SHSS	(0.042, 0.31)	45	0.0625	9.9330e-07
	Uzawa-PSS	(1.672, 1.08)	323	0.1563	9.7113e-07
	MLHSS	0.56	219	0.1094	9.7869e-07
$q = 32$	Uzawa-SSI	0.37	40	0.0781	7.9691e-07
	Uzawa-HSS	(0.32, 0.06)	325	0.9219	8.3458e-07
	Uzawa-SHSS	(0.036, 0.31)	84	0.2188	9.7044e-07
	Uzawa-PSS	(1.83, 1.06)	992	2.2344	9.4148e-07
	MLHSS	0.58	701	1.5781	8.0074e-07
$q = 64$	Uzawa-SSI	0.31	41	0.6719	9.3715e-07
	Uzawa-HSS	(0.42, 0.08)	1190	23.5	9.3018e-07
	Uzawa-SHSS	(0.022, 0.25)	162	2.8438	9.0700e-07
	Uzawa-PSS	—	—	—	—
	MLHSS	—	—	—	—

In Table 2. We list the numerical results of the Uzawa-SSI method, the Uzawa-HSS method, the Uzawa-SHSS method, the Uzawa-PSS method and MLHSS method for singular saddle point problems with $\nu = 0.1$. We can observe that Uzawa-SSI method is most efficient one, which use least iteration steps and CPU times than the Uzawa-HSS, the Uzawa-SHSS, the Uzawa-PSS and MLHSS methods to achieve stopping criterion.

Table 2. Numerical results for singular case of Example 4.1.

	Method	Parameters	IT	CPU	RES
$q = 16$	Uzawa-SSI	0.39	41	0.0469	9.1261e-07
	Uzawa-HSS	(0.51, 0.1)	153	0.1094	8.3426e-07
	Uzawa-SHSS	(0.05, 0.32)	46	0.0781	7.9389e-07
	Uzawa-PSS	(1.66, 1.08)	322	0.1406	9.8122e-07
	MLHSS	0.56	219	0.0938	7.0314e-07
$q = 32$	Uzawa-SSI	0.37	40	0.0781	7.6433e-07
	Uzawa-HSS	(0.41, 0.1)	369	1.1094	9.0712e-07
	Uzawa-SHSS	(0.01, 0.22)	58	0.7969	9.9234e-07
	Uzawa-PSS	(1.64, 1.02)	940	1.6094	9.5791e-07
	MLHSS	0.58	701	0.8594	7.9512e-07
$q = 64$	Uzawa-SSI	0.35	41	0.4844	9.6804e-07
	Uzawa-HSS	(0.39, 0.1)	1072	22.0938	9.4074e-07
	Uzawa-SHSS	(0.01, 0.2)	89	1.5469	8.7736e-07
	Uzawa-PSS	—	—	—	—
	MLHSS	—	—	—	—

In what follows, we consider two artificially constructed examples.

Example 4.2. Let us consider the nonsingular saddle-point problem (1.1) has the following coefficient sub-matrices:

$$A = \begin{bmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{bmatrix} \in \mathbb{R}^{2q^2 \times 2q^2},$$

$$B = \begin{bmatrix} I \otimes F \\ F \otimes I \end{bmatrix} \in \mathbb{R}^{2q^2 \times q^2},$$

and

$$T = \frac{\nu}{h^2} \cdot \text{tridiag}(-1, 2, -1) + \frac{1}{2h} \cdot \text{tridiag}(-1, 0, 1) \in \mathbb{R}^{q \times q},$$

$$F = \frac{1}{h} \cdot \text{tridiag}(-1, 1, 0) \in \mathbb{R}^{q \times q}.$$

Here, \otimes denotes the Kronecker product, ν is a parameter and $h = \frac{1}{q+1}$ is the discretization meshsize, see [18].

In Table 3, we report the numerical results for Example 4.2, we list the numerical results of the Uzawa-SSI method, the Uzawa-HSS method, the Uzawa-SHSS method, the Uzawa-PSS method and MLHSS method for nonsingular saddle point problems with $\nu = 1$. From Table 3, we can observe that for $\nu = 1$, compared with other four methods, the Uzawa-SSI method is the most efficient one, when reaching the stop criterion, it needs the least iteration steps and CPU times.

Table 3. Numerical results for Example 4.2 with $\nu = 1$.

	Method	Parameters	IT	CPU	RES
$q = 16$	Uzawa-SSI	2.2	40	0.0156	8.7237e-07
	Uzawa-HSS	(740, 0.58)	162	0.0781	9.9965e-07
	Uzawa-SHSS	(35.5, 1.44)	62	0.0625	9.9653e-07
	Uzawa-PSS	(560, 0.84)	126	0.0469	9.9214e-07
	MLHSS	0.1	79	0.0469	9.3330e-07
$q = 32$	Uzawa-SSI	3.34	44	0.0469	8.5020e-07
	Uzawa-HSS	(910, 0.2)	623	1.2031	9.9177e-07
	Uzawa-SHSS	(20.2, 1.4)	99	0.0938	9.9719e-07
	Uzawa-PSS	(1860, 0.8)	247	0.2969	9.8611e-07
	MLHSS	0.11	123	0.3594	9.9713e-07
$q = 64$	Uzawa-SSI	4.35	70	0.3750	8.9753e-07
	Uzawa-HSS	(4000, 0.2)	1087	11.7344	9.9075e-07
	Uzawa-SHSS	(20.2, 1.448)	147	1.1719	9.9804e-07
	Uzawa-PSS	—	—	—	—
	MLHSS	0.1	189	1.9531	9.7499e-07

Example 4.3. Let us consider the singular saddle-point problem (1.1) has the following coefficient sub-matrices:

$$A = \begin{bmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{bmatrix} \in \mathbb{R}^{2q^2 \times 2q^2},$$

$$B = \begin{bmatrix} \hat{B} & b_1 & b_2 \end{bmatrix} \in \mathbb{R}^{2q^2 \times (q^2+2)},$$

with

$$\hat{B} = \begin{bmatrix} I \otimes F \\ F \otimes I \end{bmatrix} \in \mathbb{R}^{2q^2 \times q^2}, \quad b_1 = \hat{B}^T \begin{bmatrix} e \\ 0 \end{bmatrix}, \quad b_2 = \hat{B}^T \begin{bmatrix} 0 \\ e \end{bmatrix},$$

$$e = [1, 1, \dots, 1] \in \mathbb{R}^{q^2/2},$$

and

$$T = \frac{\nu}{h^2} \cdot \text{tridiag}(-1, 2, -1) + \frac{1}{2h} \cdot \text{tridiag}(-1, 0, 1) \in \mathbb{R}^{q \times q},$$

$$F = \frac{1}{h} \cdot \text{tridiag}(-1, 1, 0) \in \mathbb{R}^{q \times q}.$$

Here, \otimes denotes the Kronecker product, ν is a parameter and $h = \frac{1}{q+1}$ is the discretization meshsize, see [22, 27].

This problem is a technical modification of Example 4.2. Here, matrix B is an augmentation of the full rank matrix \hat{B} with two linearly independent vectors b_1 and b_2 . As b_1 and b_2 are linear combinations of the columns of the matrix \hat{B} , B is a rank-deficient matrix.

In Table 4, we list the numerical results of the Uzawa-SSI method, the Uzawa-HSS method, the Uzawa-SHSS method, the Uzawa-PSS method and MLHSS method for singular saddle point problems with $\nu = 1$. We can observe that Uzawa-SSI method is most efficient one, which use least iteration steps and CPU times than other four methods to achieve stopping criterion.

Table 4. Numerical results for Example 4.3 with $\nu = 1$.

	Method	Parameters	IT	CPU	RES
$q = 16$	Uzawa-SSI	0.41	40	0.0156	8.0268e-07
	Uzawa-HSS	(258, 0.14)	129	0.0625	9.9724e-07
	Uzawa-SHSS	(13.4, 0.27)	58	0.0313	9.9910e-07
	Uzawa-PSS	(230, 0.14)	146	0.1563	9.4478e-07
	MLHSS	—	—	—	—
$q = 32$	Uzawa-SSI	0.295	66	0.0625	8.9038e-07
	Uzawa-HSS	(606, 0.093)	247	0.4844	9.9569e-07
	Uzawa-SHSS	(35.4, 0.214)	82	0.1094	9.6742e-07
	Uzawa-PSS	(510, 0.082)	279	3.3594	9.7925e-07
	MLHSS	—	—	—	—
$q = 64$	Uzawa-SSI	0.16	114	0.5156	9.8078e-07
	Uzawa-HSS	(484, 0.024)	591	6.2031	9.9879e-07
	Uzawa-SHSS	(2.4, 0.06)	120	1.0781	9.9344e-07
	Uzawa-PSS	(1020, 0.04)	545	48.6719	9.9242e-07
	MLHSS	—	—	—	—

From the numerical results, it can be observed that the Uzawa-SSI method is the best choice as it outperforms other four methods for solving the non-Hermitian nonsingular and singular saddle-point problems.

5. Conclusions

In this paper, based on the SSI method for non-Hermitian positive definite linear system, we have proposed the Uzawa-SSI method for solving non-Hermitian nonsingular and singular saddle point problems (1.1). Compared with the Uzawa-HSS, the Uzawa-SHSS, the Uzawa-PSS and the MLHSS methods, the proposed method has more simple convergence conditions, which are easy to be satisfied. Numerical results verified the efficiency of the Uzawa-SSI method.

However, the Uzawa-SSI method involved a parameter τ . It is formidable to find a optimal parameter τ in the actual calculation, therefore, we did not discuss the choice of the parameter τ in this paper. Consider that the validity of the new method depends on the selection of parameters, how to find a easy calculated parameter should be a direction for future research.

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Conflict of interest

The authors declare there is no conflict of interest.

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